# Angular momentum

$$\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$$

$$J_\pm = J_x \pm iJ_y$$

$$J_+ J_- = \boxed{J_x^2 + (iJ_y)^2} = J_x^2 - J_y^2$$

$$J_- J_+ = \boxed{J_x^2 + (iJ_y)^2} = J_x^2 - J_y^2$$

$$\Rightarrow \boxed{J_+ J_- = J_- J_+}$$

This is wrong!

这里都不对!

$$J_+ J_- = (J_x + iJ_y)(J_x - iJ_y) = J_x^2 - iJ_xJ_y + iJ_yJ_x + J_y^2$$

$$= J_x^2 + J_y^2 - \boxed{i[J_x, J_y]}$$

not commute!

$$[J_x, J_y] \neq 0$$

不能消掉

正确形式

$$\begin{cases} J_+ J_- = J_x^2 + J_y^2 - i[J_x, J_y] \\ J_- J_+ = J_x^2 + J_y^2 + i[J_x, J_y] \end{cases}$$

$[J_x, J_y] = i\hbar J_z$

$J_+ J_- = J_x^2 + J_y^2 - i(i\hbar J_z) = J_x^2 + J_y^2 + \hbar J_z$

$J_- J_+ = J_x^2 + J_y^2 - \hbar J_z$

according to $J^2 - J_z^2 = J_x^2 + J_y^2$

$\Rightarrow \quad J^2 - J_z^2 = J_+ J_- - \hbar J_z$

$$J^2 = J_z^2 + J_+ J_- - \hbar J_z$$

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Short :

$J_\pm = J_x \pm i J_y . \Rightarrow J_+ J_- = (J_x + iJ_y)(J_x - iJ_y) = J_x^2 + J_y^2$

$[J_x, J_y] = i\hbar J_z \qquad\qquad - i[J_x, J_y]$

$J_+ J_- = J_x^2 + J_y^2 + \hbar J_z$

$J^2 = J_x^2 + J_y^2 + J_z^2 \Rightarrow J^2 = J_+ J_- - \hbar J_z + J_z^2$

We know that,

$$\begin{cases} J^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle \\ J_z|j,m\rangle = \hbar m|j,m\rangle \end{cases}$$

$$J^2|j,m\rangle = J_+J_-|j,m\rangle + J_z^2|j,m\rangle - \hbar J_z|j,m\rangle$$

$$= J_+J_-|j,m\rangle + (\hbar^2 m^2 - \hbar^2 m)|j,m\rangle$$

$$J_+ = J_-^{\dagger}$$
$$= (J_x - iJ_y)^{\dagger}$$
$$= (J_x + iJ_y)$$

Set $J_-|j,m\rangle = \hbar c_m|j,m-1\rangle$

$$\langle j,m|J_+ = \hbar c_m^* \langle j,m-1|$$

$$\langle j,m|J_+J_-|j,m\rangle = \langle j,m-1|j,m-1\rangle \hbar^2 |c_m|^2$$
$$= \hbar^2|c_m|^2$$

$$\langle j,m|J^2|j,m\rangle = \hbar^2 j(j+1)$$
$$= \hbar^2|c_m|^2 + \hbar^2 m^2 - \hbar^2 m$$

$$|C_m|^2 = J(J+1) - m(m-1)$$

$$C_m = \sqrt{J(J+1) - m(m-1)} \quad \text{for } J_-$$

we may use $x(x-1)$ similar structure on this

expand $J(J+1) - m(m-1) = J^2 - m^2 + J + m$

$(J+m)(J-m)$

thus $C_m = \sqrt{(J+m)(J-m+1)}$

Similay for the $J_+$

we set $J_+ |J, m\rangle = C'_m \hbar |J, m+1\rangle$

$$\langle J, m| J_- J_+ |J, m\rangle = \langle J, m| J^2 - J_z^2 - \hbar J_z |J, m\rangle$$

$$J_- J_+ = J_x^2 + J_y^2 - \hbar J_z$$

$$\hbar^2 |C'_m|^2 = \hbar^2 J(J+1) - \hbar^2 m^2 - \hbar^2 m$$

$$|C'_m|^2 = J(J+1) - m(m+1)$$

$$c'_m = \sqrt{(J-m)(J+m+1)}$$

$\to -m(m+1)$

$\to J(J+1)$

$(J-m)(J+m) = J^2 - m^2$

Thus:

$$\begin{cases} c_m = \sqrt{\{(J+m)(J-m+1)\}} \\ c'_m = \sqrt{\{(J-m)(J+m+1)\}} \end{cases}$$

$$\Rightarrow \quad J_- |J,m\rangle = \sqrt{[(J+m)(J-m+1)]}\, |J, m-1\rangle$$

$$J_+ |J,m\rangle = \sqrt{[(J-m)(J+m+1)]}\, |J, m+1\rangle$$

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Short.

$$J_- |J,m\rangle = \hbar c_m |J, m-1\rangle \xrightarrow{J_+ = J_-^{\dagger}} \langle J, m| J_+ = \hbar c_m^* \langle J, m-1|$$

$$\langle J,m| J^2 |J,m\rangle = \langle J,m| J_+ J_- + J_z^2 - \hbar J_z |J,m\rangle$$

$$\hbar^2 J(J+1) = \hbar^2 |c_m|^2 + \hbar^2 m^2 - \hbar^2 m$$

$$c_m = \sqrt{\underbrace{(J-m)(J-m+1)}}$$

① 先写成 $(J-m)(J-m\underbrace{+1})$

② $J(J+1)$ 决定这里是 $+1$

③ $J^2-m^2$ 决定必然是一个 $(J-m)$ 一个 $(J+m)$

④ $(J \; ? \; m)(\cdots + 1)$ 由m的符号决定

$\hbar^2 m$ 项移到 $J(J+1)$ 同一边时的符号

在这里 $J(J+1) \cdots + m$ 意味着

$$(J + m)(J - m + 1)$$

$$\Rightarrow \quad C_m = \sqrt{(J+m)(J-m+1)}$$

$J_+$ is similar.

# 自旋自由度耦合

总自旋 = 自旋取值之差 到 自旋取值之和

$l=1,\quad s=\frac{1}{2} \quad \Rightarrow \quad |l, m_l; s, m_s\rangle \quad \Rightarrow \quad |J, m\rangle$

$m_l = -1, 0, 1$

$m_s = -\frac{1}{2}, \frac{1}{2} \quad \Rightarrow \quad 2\times 3 = 6$ states.

$$V_{l=1} \otimes V_{s=\frac{1}{2}} = V_{J=\frac{1}{2}} \oplus V_{J=\frac{1}{2}}$$

$$\underbrace{|l-s|}_{\frac{1}{2}} \leq J \leq \underbrace{l+s}_{\frac{3}{2}}$$

① $J=\frac{3}{2}$ Top state: $|J, m\rangle = |\frac{3}{2}, +\frac{3}{2}\rangle$

C-G coefficient is 1.

Lowing operator

Then, $\underset{\sim}{J_-}\, |J, m\rangle = \sqrt{(J+m)(J-m+1)}\ |J, m-1\rangle$

L.H.S. $J_-|\frac{3}{2}, \frac{3}{2}\rangle = \sqrt{(\frac{3}{2}+\frac{3}{2})(\frac{3}{2}-\frac{3}{2}+1)}\ |\frac{3}{2}, \frac{1}{2}\rangle$

$$= \sqrt{3}\,|\tfrac{3}{2},\tfrac{1}{2}\rangle$$

R.H.S $\quad J_-|l=1, m_l=1; s=\tfrac{1}{2}, m_s=\tfrac{1}{2}\rangle$

$$=(J_-^{(l)}+J_-^{(s)})\,|1,1;\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$= L_-^{(l)}\,|1,1;\tfrac{1}{2},\tfrac{1}{2}\rangle + S_-\,|1,1;\tfrac{1}{2},\tfrac{1}{2}\rangle$$

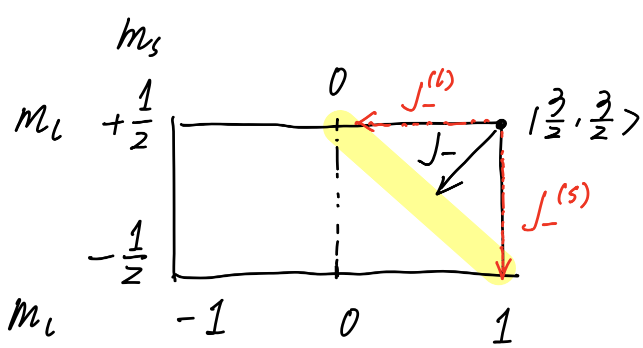
$$=\sqrt{(1+1)(1-1+1)}\,|1,0;\tfrac{1}{2},\tfrac{1}{2}\rangle+\sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}-\tfrac{1}{2}+1)}\,|1,1;\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$=\sqrt{2}\,|1,0;\tfrac{1}{2},\tfrac{1}{2}\rangle+\sqrt{1}\,|1,1;\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$\Rightarrow \quad$ L.H.S. = R.H.S. $\downarrow$ 归一化.

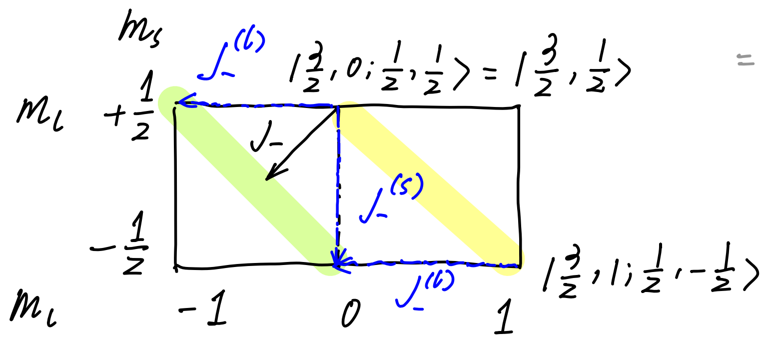
$$\sqrt{3}\,|J=\tfrac{3}{2}, m=\tfrac{1}{2}\rangle=\sqrt{2}\,|1,0;\tfrac{1}{2},\tfrac{1}{2}\rangle+\sqrt{1}\,|1,1;\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$\Rightarrow |\tfrac{3}{2},\tfrac{1}{2}\rangle=\sqrt{\tfrac{2}{3}}\,|1,0;\tfrac{1}{2},\tfrac{1}{2}\rangle+\sqrt{\tfrac{1}{3}}\,|1,1;\tfrac{1}{2},-\tfrac{1}{2}\rangle$$




图像示意

Then,




In this way.

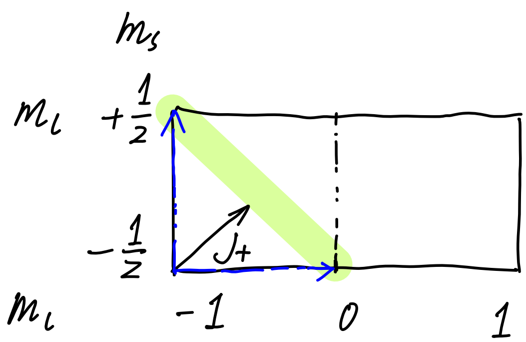
$J_-|\frac{3}{2},\frac{1}{2}\rangle$

$= (J_-^{(l)}+J_-^{(s)})\left[\sqrt{\frac{2}{3}}\cdots+\sqrt{\frac{1}{3}}\cdots\right]$

继续携带系数

This is may not very convinient.

This is better to calculate!




$$|\frac{3}{2},-\frac{1}{2}\rangle = \frac{\sqrt{3}}{2}|1,0,\frac{1}{2},-\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}|1,-1;\frac{1}{2},\frac{1}{2}\rangle$$

如何通过正交关系算其它 $|J,m\rangle$, $|m_l,m_s\rangle$?

② Notice!! For $J=\frac{1}{2}$

$|J=\frac{1}{2}, m=\frac{1}{2}\rangle$ must be orthogonal with :

$$|\tfrac{3}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|1,0;\tfrac{1}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,|1,1;\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$\Rightarrow \quad \langle \tfrac{2}{3},\tfrac{1}{2}|\tfrac{1}{2},\tfrac{1}{2}\rangle = 0$$

set $|\frac{1}{2},\frac{1}{2}\rangle = a|1,0;\frac{1}{2},\frac{1}{2}\rangle + b\,|1,1;\frac{1}{2},-\frac{1}{2}\rangle$

$$\Rightarrow \quad \sqrt{\tfrac{2}{3}}\,a + \sqrt{\tfrac{1}{3}}\,b = 0$$

$$b = \sqrt{2}\,a$$

$$\Rightarrow \quad |\tfrac{1}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|1,0;\tfrac{1}{2},\tfrac{1}{2}\rangle - \tfrac{\sqrt{2}}{3}\,|1,1;\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

Similar for $|\frac{1}{2},-\frac{1}{2}\rangle$

Prove $\begin{cases} [J_z, J_\pm] = \pm J_\pm \\ [J_+, J_-] = 2J_z \end{cases}$

$$[J_z, J_+] = [J_z, J_x + iJ_y] = [J_z, J_x] + i[J_z, J_y]$$

$$= iJ_y + i(-iJ_x)$$

$$= iJ_y + J_x = J_+$$

$$[J_z, J_-] = [J_z, J_x] - i[J_z, J_y] = iJ_y - i(iJ_x)$$

$$= iJ_y - J_x = -(J_x - iJ_y)$$

$$= -J_-$$

$$\Rightarrow [J_z, J_\pm] = \pm J_\pm$$

$$[J_+, J_-] = [J_x + iJ_y, J_x - iJ_y] = [J_x, J_x]_{=0}$$

$$+ [J_x, -iJ_y]_{= -i(iJ_z)}$$

$$+ [+iJ_y, J_x]_{= i(-iJ_z)}$$

$$+ [+iJ_y, -iJ_y]_{=0}$$

$$= +J_z + J_z = 2J_z$$

$$\Rightarrow [J_+, J_-] = 2J_z$$

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Short

$$[J_z, J_\pm] = [J_z, J_x \pm iJ_y] = iJ_y \pm i(iJ_x) = \pm J_\pm$$

$$[J_+, J_-] = [J_x + iJ_y, J_x - iJ_y] = i(-iJ_z) - i(iJ_z) = 2J_z$$

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