Qualifying Exam
Quantum Mechanics – SOLUTIONS
August 2020

Solve <u>one</u> of the two problems in Part A, and <u>one</u> of the two problems in Part B.
Each problem is worth 50 points.

Part A

Problem 1.
Let us consider an electron whose squared orbital angular momentum $\mathbf{L}^2$ is measured to be $6\hbar^2$.

1.a) **[5 points] For each one of the two bases described above, list all the possible states for the electron which are compatible with this measurement.**

We have $s = \frac{1}{2}$ and $l = 2$ (since $l(l+1) = 6$). Since $l + s \leq j \leq |l - s|$, the allowed values for j will be $j = \frac{5}{2}$, $j = \frac{3}{2}$.

As a consequence, there will be ten possible states in the basis $\{|jm_j\rangle\}$: $j = \frac{5}{2}$ with $m_j = \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}$ (six states) and $j = \frac{3}{2}$ with $m_j = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$ (four states).

In the basis $\{|m_l m_s\rangle\}$, we can have all possible combinations of $m_s = -\frac{1}{2}, \frac{1}{2}$ with $m_l = 2, 1, 0, -1, -2$, which is again ten states.

1.b) **[15 points] Find the expression of the following basis states $\{|jm_j\rangle\}$ in terms of the appropriate elements of the basis $\{|m_l m_s\rangle\}$.**

1.b.1 : First state on the ladder, here is only one possible way of satisfying $m_j = m_s + m_l$, therefore

$$|j = \frac{5}{2}\ m_j = \frac{5}{2}\rangle = |m_l = 2\ m_s = \frac{1}{2}\rangle$$

1.b.2 : We should apply the lowering operator on both sides to obtain

$$|j = \frac{5}{2}\ m_j = \frac{3}{2}\rangle = \sqrt{\frac{4}{5}}\ |m_l = 1\ m_s = \frac{1}{2}\rangle + \sqrt{\frac{1}{5}}\ |m_l = 2\ m_s = -\frac{1}{2}\rangle$$

and then apply the orthogonality condition to get

$$|j = \frac{3}{2}\ m_j = \frac{3}{2}\rangle = \sqrt{\frac{1}{5}}\ |m_l = 1\ m_s = \frac{1}{2}\rangle - \sqrt{\frac{4}{5}}\ |m_l = 2\ m_s = -\frac{1}{2}\rangle$$

1.b.3 : Not a possible state. We cannot have $j = \frac{1}{2}$ if $s = \frac{1}{2}$ and $l = 2$.

1.b.4 : We can continue to apply the lowering operator until we reach the minimum of m_j, or start from the bottom of the ladder (which is faster). The only combination of m_s and m_l that allows for $m_j = -\frac{5}{2}$ is

$$|j = \frac{5}{2}\ m_j = -\frac{5}{2}\rangle = |m_l = -2\ m_s = -\frac{1}{2}\rangle$$

1.c) **[15 points] Now assume the electron to be in the state with $j = \frac{3}{2}$, and $m_j = \frac{1}{2}$ (and the value of l is the same as in the previous questions). If one measures the z-components of the electron orbital angular momentum and spin, what are the possible values and their probabilities?**

In our problem, there are only two configurations of m_s and m_l that are compatible with $|j = \frac{3}{2}$, and $m_j = \frac{1}{2}$, namely the states $|m_l = 1\ m_s = -\frac{1}{2}\rangle$ and $|m_l = 0\ m_s = \frac{1}{2}\rangle$.

To obtain the probabilities, we start from the expression that we obtained in question 1.b.2), and we apply once more the lowering operator, to obtain:

$$|j = \frac{3}{2}\ m_j = \frac{1}{2}\rangle = \sqrt{\frac{2}{5}}\ |m_l = 0\ m_s = \frac{1}{2}\rangle - \sqrt{\frac{3}{5}}\ |m_l = 1\ m_s = -\frac{1}{2}\rangle$$

which allows to read the probabilities $\frac{2}{5}$ and $\frac{3}{5}$, respectively for the pair of values $(m_l = 0, m_s = \frac{1}{2})$ and $(m_l = 1, m_s = -\frac{1}{2})$.

1.d) **[15 points] Let us now assume that the electron is in the state with $m_l = 1$ and $m_s = -\frac{1}{2}$ (again, l did not change). What are the possible values of j and their probabilities?**

Here we have the "inverse" problem as 1.c). The state $|m_l = 1 m_s = -\frac{1}{2}\rangle$ must be a linear combination of the states $|j = \frac{3}{2}\ m_j = \frac{1}{2}\rangle$ (which we know already) and $|j = \frac{5}{2}\ m_j = \frac{1}{2}\rangle$ (which we do not have yet).

Let us first find $|j = \frac{5}{2}\ m_j = \frac{1}{2}\rangle$. We can obtain it by applying the lowering operator to $|j = \frac{5}{2}\ m_j = \frac{3}{2}\rangle$ or simply by using the orthogonality with $|j = \frac{3}{2}\ m_j = \frac{1}{2}\rangle$. In either way, we get:

$$|j = \frac{5}{2}\ m_j = \frac{1}{2}\rangle = \sqrt{\frac{3}{5}}\ |m_l = 0\ m_s = \frac{1}{2}\rangle + \sqrt{\frac{2}{5}}\ |m_l = 1\ m_s = -\frac{1}{2}\rangle$$

We can solve now for $|m_l = 1\ m_s = -\frac{1}{2}\rangle$, to get

$$|m_l = 1\ m_s = -\frac{1}{2}\rangle = \sqrt{\frac{2}{5}}\ |j = \frac{5}{2}\ m_j = \frac{1}{2}\rangle - \sqrt{\frac{3}{5}}\ |j = \frac{3}{2}\ m_j = \frac{1}{2}\rangle$$

. Therefor, we will have $j = \frac{5}{2}$ with probability $\frac{2}{5}$ and $j = \frac{3}{2}$ with probability $\frac{3}{5}$.

Qualifying Exam
Quantum Mechanics
June 2021

Problem 1. Consider two observables $\hat{A}$ and $\hat{B}$ in a three-dimensional Hilbert space. In the basis:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \; ,$$

the observables $\hat{A}$ and $\hat{B}$ are represented, respectively, by the matrices

$$A \to a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix} \quad , \quad B \to b \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

where $b << a$.

a) [**5 points**] Show that the observables $\hat{A}$ and $\hat{B}$ are not compatible.

Check the commutator $[\hat{A}, \hat{B}]$. Since $\hat{A}$ and $\hat{B}$ do not commute, the observables are NOT compatible. in particular, $\hat{A}$ and $\hat{B}$ do not admit a common set of eigenstates.

b) [**10 points**] Compute the possible outcomes and the corresponding probabilities of separate (independent) measurements of $\hat{A}$ and $\hat{B}$ in the state $|\chi\rangle = \sqrt{\frac{1}{2}}\,(|2\rangle - |3\rangle)$.

$\hat{A}$: in general, we have three possible outcomes in the eigenvalues $a_1 = 1a$, $a_2 = 2a$, $a_3 = 6a$, which correspond to the three eigenstates $|1\rangle$, $|2\rangle$, and $|3\rangle$. The probabilities are given by $P_i = |\langle\chi|i\rangle|^2$, where $i = 1, 2, 3$.

Since $|\chi\rangle = \sqrt{\frac{1}{2}}\,(|2\rangle - |3\rangle)$, we will measure $a_2 = 2a$ with a probability of 50% and $a_3 = 6a$ with a probability of 50%.

$\hat{B}$: the possible outcomes are given by the eigenvalues. Let us firts compute them together with the corresponding eigenstates $|B_1\rangle$, $|B_2\rangle$, and $|B_3\rangle$. The probabilities will be given by $P_i = |\langle\chi|B_i\rangle|^2$, where $i = 1, 2, 3$.

The first eigenvalue is $b_1 = -1b$, and $|B_1\rangle = |1\rangle$. By diagonalizing the sub-matrix $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, we find that the additional eigenvalues are $b_2 = -2b$ and $b_3 = +2b$, with respective eigenvectors $|B_2\rangle = \sqrt{\frac{1}{2}}\,(|2\rangle - |3\rangle)$ and $|B_3\rangle\sqrt{\frac{1}{2}}\,(|2\rangle + |3\rangle)$.

Since $|\chi\rangle = |B_2\rangle$, we will measure $b_2 = -2b$ with a probability of 100%.

c) [**10 points**] Compute the possible outcomes and the corresponding probabilities of a measurements of $\hat{B}$ that follows a measurement of $\hat{A}$, if the system is initially in the state $|\chi\rangle$. What about a measurement of $\hat{A}$ that follows a measurements of $\hat{B}$?

After the measurement of $\hat{A}$, the state of the system will be in $|2\rangle$ or $|3\rangle$ with a probability of 50% each. Noting that $|2\rangle = \sqrt{\frac{1}{2}}\,(|B_2\rangle + |B_3\rangle)$ and $|3\rangle = \sqrt{\frac{1}{2}}\,(|B_2\rangle - |B_3\rangle)$, we can conclude that the measurement of $\hat{B}$ will have a 50% probability of $b_2 = -2b$ a 50% probability of $b_3 = +2b$.

Note that, if we were to perform the measurement of $\hat{B}$ first, it would give a $b_2 = -2b$ with 100% probability and turn the state of the system in — $B_2\rangle$. However, since $|\chi\rangle = |B_2\rangle$, the subsequent measurement of $\hat{A}$ would give the same outcomes as in part b) of the problem.

Let us now construct the Hamiltonian $\hat{H} = \hat{A} + \hat{B}$. Since $b << a$, we can use perturbation theory to study $\hat{H}$.

d) [**10 points**] After writing down the eigenstates and eigenvalues of $\hat{H}_0 = \hat{A}$, compute the first- and second-order corrections to the energy levels due to the correction $\hat{H}_p = \hat{B}$.

Leading Order. (just for defining notation). In the basis defined by

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ,$$

we have $E_1^0 = 1a$, $E_2^0 = 2a$, $E_3^0 = 6a$.
First Order. We use $E_n^1 = \langle n|\hat{H}_p|n\rangle$, to obtain
$E_1^1 = \langle 1|\hat{H}_p|1\rangle = -1b$, $E_2^1 = \langle 2|\hat{H}_p|2\rangle = 0$, $E_3^1 = \langle 3|\hat{H}_p|3\rangle = 0$.
Second Order. Using $E_n^2 = \sum_{k\neq n} \frac{|\langle k|\hat{H}_p|n\rangle|^2}{E_n^0 - E_k^0}$, we get $E_1^2 = 0$, $E_2^2 = -\frac{b^2}{a}$, $E_3^1 = +\frac{b^2}{a}$.
Therefore, up to second order
$E_1 = a - b + \ldots = a\,[1 - X + \mathcal{O}(X^3)]$
$E_2 = 2a - \frac{b^2}{a} + \ldots = a\,[2 - X^2 + \mathcal{O}(X^3)]$
$E_3 = 6a + \frac{b^2}{a} + \ldots = a\,[6 + X^2 + \mathcal{O}(X^3)]$
where $X = b/a$.

e) [**10 points**] Compute the energy eigenstates of $\hat{H}$ up to first order in perturbation theory.

$|p_1^1\rangle = 0$
$|p_2^1\rangle = \sum_{k\neq 2} \frac{\langle k|\hat{H}_p|2\rangle}{E_2^0 - E_k^0}|k\rangle = \frac{\langle 3|\hat{H}_p|2\rangle}{E_2^0 - E_3^0}\ |3\rangle = \frac{2b}{-4a}\ |3\rangle = -\frac{b}{2a}\ |3\rangle$
$|p_3^1\rangle = \sum_{k\neq 3} \frac{\langle k|\hat{H}_p|3\rangle}{E_3^0 - E_k^0}|k\rangle = \frac{\langle 2|\hat{H}_p|3\rangle}{E_3^0 - E_2^0}\ |2\rangle = \frac{2b}{4a}\ |2\rangle = \frac{b}{2a}\ |2\rangle$
Therefore, up to first order:
$|p_1\rangle = |1\rangle$
$|p_2\rangle = |2\rangle - \frac{b}{2a}|3\rangle + \ldots$
$|p_3\rangle = |3\rangle + \frac{b}{2a}|2\rangle + \ldots$

f) [**5 points**] After finding the exact solutions, check that your results for the energy levels obtained in perturbation theory are correct.

By diagonalizing the matrix

$$\hat{H} = a \begin{pmatrix} 1-X & x & 0 \\ x & 2 & 2X \\ 0 & 2X & 6 \end{pmatrix} ,$$

we get the eigenvalues $\lambda_1 = (1 - X)$ and $\lambda_{2,3} = 4 \mp 2\sqrt{1 + x^2}$.
By expanding $\lambda_{2,3}$ for small X, i.e. $\sqrt{1 + X^2} = 1 + \frac{1}{2}X^2 + \mathcal{O}(X^4)$, we obtain:
$\lambda_2 = 4 - 2(1 + \frac{1}{2}X^2) = 2 - X^2$,
and
$\lambda_3 = 4 + 2(1 + \frac{1}{2}X^2) = 6 + X^2$,
in agreement with part a).

Time-independent Perturbation Theory:

$$E_n^1 = \langle n|\hat{H}_p|n\rangle; \quad E_n^i = \langle p_n^0|\hat{H}_p|p_n^{i-1}\rangle; \quad |p_n^1\rangle = \sum_{k\neq n} \frac{\langle k|\hat{H}_p|n\rangle}{E_n^0 - E_k^0}|k\rangle$$

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Quantum Mechanics
August 2021
Solutions

Problem 1.
Consider a spin-1 particle in the state:

$$|\psi\rangle = \sqrt{\frac{1}{3}}\Big(|+\rangle + |0\rangle + |-\rangle\Big)\,,$$

where $|+\rangle \equiv |s=1, s_z=1\rangle$, $|0\rangle \equiv |s=1, s_z=0\rangle$, and $|-\rangle \equiv |s=1, s_z=-1\rangle$ are the eigenstates of $\hat{S}_z$.

1.a) (5 points) Show that the $\hat{S}_z$ operator can be written as $\hat{S}_z = \hbar(|+\rangle\langle+|) - \hbar(|-\rangle\langle-|)$. Write the expression for the operator $\hat{S}_z^2$.

Textbook problem. The simplest way is to use the spectral decomposition $\hat{A} = \sum_i |a_i\rangle\lambda_i\langle a_i|$, where $|a_i\rangle$ and λ_i are the eigenvectors and eigenvalues of the operator $\hat{A}$, respectively.
By squaring $\hat{S}_z$, we obtain $\hat{S}_z^2 = \hbar^2|+\rangle\langle+| + |-\rangle\langle-|$.
In matrix form (if need be) this would be represented by

$$\hat{S}_z \to \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix} \quad , \quad \hat{S}_z^2 \to \hbar^2\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} \,.$$

1.b) (15 points) Beginning with the state $|\psi\rangle$, consider measuring first $\hat{S}_z^2$, then measuring afterwards $\hat{S}_z$. What are the possible measurement outcomes of $\hat{S}_z^2$ and the corresponding probabilities? What are the possible outcomes of a measurement of $\hat{S}_z$ after $\hat{S}_z^2$ has been measured and the corresponding probabilities?

First measurement ($\hat{S}_z^2$): The possible outcomes are given by the eigenvalues of $\hat{S}_z^2$, which are 0 (non degenerate) and $\hbar^2$ (degenerate of degree 2) respectively. We will measure $S_z^2 = 0$ with probability $P_0 = |\langle\psi|0\rangle|^2 = \frac{1}{3}$. We will measure $S_z^2 = \hbar^2$ with probability $P_+ = |\langle\psi|1\rangle|^2 = \frac{2}{3}$, where $|1\rangle = \sqrt{\frac{1}{2}}\Big(|+\rangle + |-\rangle\Big)$ is the eigenstate of eigenvalue $\hbar^2$ and the state of the system after the measurement. Note that the from of $|1\rangle$ can be obtained directly (by observing the form of the operator S_z^2 and the initial state of the system) or by applying the projector $P = |+\rangle\langle+| + |-\rangle\langle-|$ to the initial state of the system and normalizing the outcome.

Second measurement ($\hat{S}_z$): If the outcome of the first measurement was $S_z^2 = 0$, the system is now in the state $|0>$. We will then measure $S_z = 0$ with probability $P_0 = |\langle 0|0\rangle|^2 = 1$.
If the outcome of the first measurement was $S_z^2 = 1$, the system is now in the state $|1\rangle = \sqrt{\frac{1}{2}}\Big(|+\rangle + |-\rangle\Big)$. We will then measure $S_z = \hbar$ with probability $P_+ = |\langle 1|+\rangle|^2 = \frac{1}{2}$ and $S_z = -\hbar$ with probability $P_- = |\langle 1|-\rangle|^2 = \frac{1}{2}$.

Assume the Hamiltonian of the system to be

$$\hat{H} = A\hat{S}_z^2 + B(\hat{S}_x^2 - \hat{S}_y^2)\,.$$

1.c) (5 points) Show that, in the basis of eigenstates of $\hat{S}_z$, the operators $\hat{S}_x$ and $\hat{S}_y$ have the form:

$$\hat{S}_x \to \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad \hat{S}_y \to \frac{i\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} .$$

Textbook calculation. One common way is to use the operators S_+ and S_-, given in the formulae, and then build the linear combinations using $S_\pm = S_x \pm iS_y$.

1.d) (10 points) Find the matrix representation of $\hat{H}$, in the basis of eigenstates of $\hat{S}_z$.

Using the expressions of S_x, S_y, and S_z, which were given in 1.c) and 1.a), one can easily obtain

$$\hat{H} \to \hbar^2 \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix} .$$

1.e) (10 points) Solve the problem exactly, to find the eigenvalues and eigenstates of $\hat{H}$.

The energy eigenstates and eigenvalues are: $|E_\pm\rangle = \sqrt{\frac{1}{2}}\Big(|+\rangle \pm |-\rangle\Big)$ with eigenvalues $E_\pm = \hbar^2(A \pm B)$ and $|E_0\rangle = |0\rangle$ with eigenvalue $E_0 = 0$.

1.f) (5 points) Compute the expectation value of the energy in the state $|\psi\rangle$.

Compute $\langle \hat{H} \rangle_\psi = \langle \psi | \hat{H} | \psi \rangle = \sum_i E_i |\langle \psi | E_i \rangle|^2$ to obtain $\langle \hat{H} \rangle_\psi = \frac{2}{3}\hbar^2(A + B)$.

PhD Program in Physics
The Graduate Center CUNY

Qualifying Exam
Quantum Mechanics
SOLUTIONS Problem 1: August 2022

Problem 1. Part A.
In the basis of eigenstates of the angular momentum operator $\hat{S}_z$, namely

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{where} \quad \hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle \, ,$$

the state of a particle of spin $\frac{1}{2}$ is represented by $|\psi\rangle = \begin{pmatrix} 2i \\ -1 \end{pmatrix}$.
A physical observable $\hat{G}$ is represented, in the same basis, by the matrix

$$\hat{G} = g \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} .$$

a) (10 points) After properly normalizing the state $|\psi\rangle$, find the possible outcomes and related probabilities of independent measurements of $\hat{S}_z$ and $\hat{S}^2$.

Since $\langle\psi|\psi\rangle = 5$, the properly normalized initial state is $|\psi\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 2i \\ -1 \end{pmatrix} = \frac{2i}{\sqrt{5}}|+\rangle - \frac{1}{\sqrt{5}}|-\rangle$

The operators $\hat{S}_z$ and $\hat{S}^2$ in the basis of eigenstates of $\hat{S}_z$ are given by

$$\hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \hat{S}^2 = \frac{3\hbar}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

A measurement of $\hat{S}_z$ in $|\psi\rangle$ will give $\lambda_+ = +\frac{\hbar}{2}$ with probability $P_+ = |\langle+|\psi\rangle|^2 = 0.80$ and $\lambda_- = -\frac{\hbar}{2}$ with probability $P_- = |\langle-|\psi\rangle|^2 = 0.20$. A measurement of $\hat{S}^2$ will give $\frac{3\hbar^2}{4}$ with probability 1.

b) (10 points) Compute the expectation values of $\hat{S}_z$ and $\hat{S}^2$ in the state $|\psi\rangle$ and the related uncertainties.

Expectation value of $\hat{S}_z$: $\langle\hat{S}_z\rangle_\psi = \lambda_+ P_+ + \lambda_- P_- = \frac{4}{5}\frac{\hbar}{2} - \frac{1}{5}\frac{\hbar}{2} = \frac{3\hbar}{10}$. (It can be also computed using $\langle\psi|\hat{S}_z|\psi\rangle$...).
Similarly $\langle\hat{S}_z^2\rangle_\psi = \lambda_+^2 P_+ + \lambda_-^2 P_- = \frac{4}{5}\frac{\hbar^2}{4} + \frac{1}{5}\frac{\hbar^2}{4} = \frac{\hbar^2}{4}$.
The related uncertainty is $\Delta\hat{S}_z = \sqrt{\langle\hat{S}_z^2\rangle_\psi - \langle\hat{S}_z\rangle_\psi^2} = \sqrt{\frac{\hbar^2}{4} - \frac{9\hbar^2}{100}} = \sqrt{\frac{4\hbar^2}{25}} = \frac{2\hbar}{5}$.

Expectation value of $\hat{S}^2$: Not much to compute here. $\langle\hat{S}^2\rangle_\psi = \frac{3\hbar^2}{4}$ with $\Delta\hat{S}^2 = 0$.

c) (5 points) Is the observable $\hat{G}$ compatible with $\hat{S}_z$? Is it compatible with $\hat{S}^2$?

Compute the commutators to show that $[\hat{G}, \hat{S}_z] \neq 0$ while $[\hat{G}, \hat{S}^2] = 0$.

d) (15 points) Compute the possible outcomes of an independent measurement of $\hat{G}$ in the state $|\psi\rangle$ and the related probabilities.

First of all we need to "diagonalize" $\hat{G}$, i.e. find eigenvalues and eigenvectors. With standard procedures we can find that the two eigenvalues are $g_1 = g$ and $g_2 = 3g$. The corresponding eigenstates are $|g_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$ and $|g_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$.
A measurement of $\hat{G}$ in $|\psi\rangle$ will give $g_1 = g$ with probability $P_1 = |\langle g_1|\psi\rangle|^2 = \frac{1}{2}$ and $g_2 = 3g$ with probability $P_2 = |\langle g_2|\psi\rangle|^2 = \frac{1}{2}$.

e) (10 points) Suppose the system is initially in the state $|\psi\rangle$. Now consider a measurement of $\hat{S}_z$ performed on the system right after a measurement of $\hat{G}$. List the possible outcomes and their probabilities. Repeat the calculation for a measurement of $\hat{G}$ performed after a measurement of $\hat{S}_z$.

Measurement of $\hat{S}_z$ performed on the system right after a measurement of $\hat{G}$: after the first measurement ($\hat{G}$) the system will not be in $|\psi\rangle$ anymore, but in the state $|g_1\rangle$ or $|g_2\rangle$, according to the outcome of the measurement of $\hat{G}$. We now measure $\hat{S}_z$. If the measurement of $\hat{G}$ gave g_1, we will find $\lambda_+ = +\frac{\hbar}{2}$ with probability $P_+ = |\langle +|g_1\rangle|^2 = \frac{1}{2}$ and $\lambda_- = -\frac{\hbar}{2}$ with probability $P_- = |\langle -|g_1\rangle|^2 = \frac{1}{2}$. Similarly, ff the measurement of $\hat{G}$ gave g_2, we will find $\lambda_+ = +\frac{\hbar}{2}$ with probability $P_+ = |\langle +|g_2\rangle|^2 = \frac{1}{2}$ and $\lambda_- = -\frac{\hbar}{2}$ with probability $P_- = |\langle -|g_2\rangle|^2 = \frac{1}{2}$.

Measurement of $\hat{G}$ performed on the system right after a measurement of $\hat{S}_z$: after the first measurement ($\hat{S}_z$) the system will not be in $|\psi\rangle$ anymore, but in the state $|+\rangle$ or $|-\rangle$, according to the outcome of the measurement of $\hat{S}_z$. We now measure $\hat{G}$. If the measurement of $\hat{S}_z$ gave $+\frac{\hbar}{2}$, we will find $g_1 = g$ with probability $P_1 = |\langle g_1|+\rangle|^2 = \frac{1}{2}$ and $g_2 = 3g$ with probability $P_2 = |\langle g_2|+\rangle|^2 = \frac{1}{2}$. Similarly, ff the measurement of $\hat{S}_z$ gave $-\frac{\hbar}{2}$ we will find $g_1 = g$ with probability $P_1 = |\langle g_1|-\rangle|^2 = \frac{1}{2}$ and $g_2 = 3g$ with probability $P_2 = |\langle g_2|-\rangle|^2 = \frac{1}{2}$.

Qualifying Exam – June 2023
Quantum Mechanics

Solutions

The operators **L** and **S** (such that $[\mathbf{L}, \mathbf{S}] = 0$) are associated with angular momenta $l = 3$ and $s = 1$, respectively. The operator **J** is defined as the the sum of **L** and **S**, namely $\mathbf{J} = \mathbf{L} + \mathbf{S}$.

1.a) **[10 pts] How many states belong to each of the two o.n.c. sets $\{|l, s, m_l, m_s\rangle\} = \{|m_l, m_s\rangle\}$ and $\{|l, s, j, m_j\rangle\} = \{|j, m_j\rangle\}$? List all the elements of each set.**

We have $s = 1$ and $l = 3$. Since $l + s \leq j \leq |l - s|$, the allowed values for j will be $j = 2$, $j = 3$, and $j = 4$.
As a consequence, there will be 21 possible states in the basis $\{|jm_j\rangle\}$: $j = 4$ with $m_j = 4, 3, 2, 1, 0, -1, -2, -3, -4$ (9 values); $j = 3$ with $m_j = 3, 2, 1, 0, -1, -2, -3$ (7 values); $j = 2$ with $m_j = 2, 1, 0, -1, -2$ (5 values).

In the basis $\{|m_l m_s\rangle\}$, we can have all possible combinations of $m_s = 1, 0, -1$ with $m_l = 3, 2, 1, 0, -1, -2, -3$, namely $(2m_s + 1) * (2m_l + 1)$, which is again 21 states.

1.b) **[15 pts] Find the expression of the following states in terms of the appropriate elements of the basis $\{|m_l, m_s\rangle\}$. This part requires the direct construction of all CG coefficients relevant to the solution of the problem.**

$$|j = 4,\ m_j = 4\rangle$$
$$|j = 3,\ m_j = 3\rangle$$
$$|j = 4,\ m_j = -4\rangle$$

$|j = 4\ m_j = 4\rangle$. As the first state on the ladder, here there is only one possible way of satisfying $m_j = m_s + m_l$, therefore

$$|j = 4\ m_j = 4\rangle = |m_l = 3\ m_s = 1\rangle\,.$$

$|j = 3\ m_j = 3\rangle$. We should first apply the lowering operator on both sides to obtain

$$|j = 4\ m_j = 3\rangle = \sqrt{\frac{3}{4}}\,|m_l = 2\ m_s = 1\rangle + \sqrt{\frac{1}{4}}\,|m_l = 3\ m_s = 0\rangle\,,$$

and then apply the orthogonality condition to get

$$|j = 3\ m_j = 3\rangle = \sqrt{\frac{1}{4}}\,|m_l = 2\ m_s = 1\rangle - \sqrt{\frac{3}{4}}\,|m_l = 3\ m_s = 0\rangle\,.$$

$|j = 4\ m_j = -4\rangle$. We can continue to apply the lowering operator until we reach the minimum of m_j, or start from the bottom of the ladder (which is more advisable!). The only combination of m_s and m_l that allows for $m_j = -4$ is

$$|j = 4\ m_j = -4\rangle = |m_l = -3\ m_s = -1\rangle\,.$$

1.c) **[15 pts] A quantum system is prepared in the state**

$$|\phi\rangle = |j = 3, m_j = 3\rangle \, .$$

Compute the expectation values of $\hat{S}_z$, $\hat{L}_z$, $\hat{J}_z$, $\hat{L}^2$, $\hat{S}^2$, and $\hat{J}^2$ in the state $|\phi\rangle$.

$|\phi\rangle$ is an eigenstate of $\hat{L}^2$, $\hat{S}^2$, $\hat{J}^2$, $\hat{J}_z$, with quantum numbers $l = 3$, $s = 1$, $j = 3$ and $m_j = 3$, respectively. Therefore, the corresponding expectation values are:

$$\langle\phi|\hat{L}^2|\phi\rangle = \hbar^2 l(l+1) = 12\hbar^2$$

$$\langle\phi|\hat{S}^2|\phi\rangle = \hbar^2 s(s+1) = 2\hbar^2$$

$$\langle\phi|\hat{J}^2|\phi\rangle = \hbar^2 j(j+1) = 12\hbar^2$$

$$\langle\phi|\hat{J}_z|\phi\rangle = \hbar m_j = 3\hbar \, .$$

In order to compute the expectation values of $\hat{S}_z$ and $\hat{L}_z$, we express $|\phi\rangle$ in the basis $\{|m_l m_s\rangle\}$. As found previously, we have

$$|\phi\rangle = |j = 3 \; m_j = 3\rangle = \sqrt{\frac{1}{4}}\, |m_l = 2 \; m_s = 1\rangle - \sqrt{\frac{3}{4}}\, |m_l = 3 \; m_s = 0\rangle \, .$$

We can now compute $\langle\phi|\hat{L}_z|\phi\rangle$ and $\langle\phi|\hat{S}_z|\phi\rangle$ to find:

$$\langle\phi|\hat{L}_z|\phi\rangle = \frac{11}{4}\hbar$$

$$\langle\phi|\hat{S}_z|\phi\rangle = \frac{1}{4}\hbar \, .$$

1.d) **[10 pts] List the probabilities of all possible outcomes in a measurement of m_s performed in the quantum state $|\phi\rangle$ (and check that they sum to one). Repeat the same calculation for independent measurements of m_l and m_j in the state $|\phi\rangle$.**

Since

$$|\phi\rangle = |j = 3 \; m_j = 3\rangle = \sqrt{\frac{1}{4}}\, |m_l = 2 \; m_s = 1\rangle - \sqrt{\frac{3}{4}}\, |m_l = 3 \; m_s = 0\rangle \, ,$$

we would measure $m_s = 1\hbar$ with a probability $P_1 = \frac{1}{4}$ and $m_s = 0$ with a probability $P_0 = \frac{3}{4}$ (while $P_{-1} = 0$). Similarly for m_l, we would measure $m_l = 3\hbar$ with a probability $P_3 = \frac{3}{4}$ and $m_l = 2\hbar$ with a probability $P_2 = \frac{1}{4}$ (all other values with probability zero). Finally, a measurement of m_j would always give $3\hbar$ ($P_3 = 1$, all other probabilities are zero).

Qualifying Exam – August 2023
Quantum Mechanics

Solutions

Problem 1. Consider two observables $\hat{A}$ and $\hat{B}$ in a four-dimensional Hilbert space. In the basis $\{|n\rangle\}$:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad , \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad |4\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} ,$$

the physical observables $\hat{A}$ and $\hat{B}$ are represented, respectively, by the matrices

$$A \to \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & 10 & 5i \\ 0 & 0 & -5i & 10 \end{pmatrix} \quad , \quad B \to \begin{pmatrix} 2 & 4 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & b_4 \end{pmatrix} ,$$

where the parameters a_1, a_2, b_3, b_4 are real numbers different from zero.

2.a) [**15 pts**] What are the possible outcomes in a measurement of $\hat{A}$? What about a measurement of $\hat{B}$?

In order to find the possible outcomes in a measurement, we should diagonalize the operators $\hat{A}$ and $\hat{B}$, in the 2×2 sector that is not already diagonal.
Regarding $\hat{A}$, by solving

$$\det \begin{pmatrix} 10 & 5i \\ -5i & 10 \end{pmatrix} = 0 ,$$

we find the eigenvalues $\lambda_{A,3} = 5$ and $\lambda_{A,4} = 15$. **The possible outcomes for a measurement of $\hat{A}$ are therefore a_1, a_2, 5, and 15.** We also compute right away the corresponding eigenvectors (this will useful later on) to find

$$|3'\rangle = \frac{1}{\sqrt{2}} (|3\rangle - i|4\rangle)$$

$$|4'\rangle = \frac{1}{\sqrt{2}} (|3\rangle + i|4\rangle)$$

for $\lambda_{A,3}$ and $\lambda_{A,4}$, respectively. For $\hat{B}$, in a similar manner, we solve

$$\det \begin{pmatrix} 2 & 4i \\ 4 & 2 \end{pmatrix} = 0 ,$$

to find the eigenvalues $\lambda_{B,1} = -2$ and $\lambda_{B,2} = 6$ and the corresponding eigenvectors

$$|1'\rangle = \frac{1}{\sqrt{2}} (|1\rangle - |2\rangle)$$

$$|2'\rangle = \frac{1}{\sqrt{2}} (|3\rangle + |4\rangle)$$

for $\lambda_{B,1}$ and $\lambda_{B,2}$, respectively. **The possible outcomes for a measurement of $\hat{B}$ are therefore -2, 6, b_3 and b_4.**

2.b) [**15 pts**] We perform a measurement of $\hat{A}$ for a quantum system that is in the physical state represented by $|s\rangle = \frac{1}{\sqrt{2}}|2\rangle + \frac{1}{2}|3\rangle + \frac{1}{2}|4\rangle$. List the probabilities for each one of the possible outcomes of the measurement and compute the expectation value of $\hat{A}$ in the state $|s\rangle$.

The probabilities for each outcome are given by the inner products $|\langle\lambda|s\rangle|^2$, where we denote with $|\lambda\rangle$ the appropriate eigenstate. Using the results obtained above, we find:

$$P(a_1) = |\langle 1|s\rangle|^2 = 0$$

$$P(a_2) = |\langle 2|s\rangle|^2 = \frac{1}{2}$$

$$P(5) = |\langle 3'|s\rangle|^2 = \frac{1}{4}$$

$$P(15) = |\langle 4'|s\rangle|^2 = \frac{1}{4}$$

which sum correctly to 1. Regarding the expectation value, while we could compute directly $\langle s|\hat{A}|s\rangle$ using the matrix representation (this is also a correct path!), we opt to use the information already available as $\langle s|\hat{A}|s\rangle = \sum_i \lambda_i P_i$. Either way, we obtain

$$\langle s|\hat{A}|s\rangle = \frac{a_2}{2} + \frac{5}{4} + \frac{15}{4} = 5 + \frac{a_2}{2}\,.$$

2.c) [**15 pts**] What are the conditions that a_1, a_2, b_3, b_4 should satisfy in order for the two observables $\hat{A}$ and $\hat{B}$ to be compatible? Under these conditions, find a set $\{|n'\rangle\}$ of common eigenstates.

Two observables are compatible if they commute. In our problem, $\hat{A}$ and $\hat{B}$ commute if $a_1 = a_2$ and $b_3 = b_4$. A basis of common eigenstates is readily provides by $|1'\rangle$, $|2'\rangle$, $|3'\rangle$, $|4'\rangle$, provided as part of the answer to the first question.

2.d) [**5 pts**] If we choose the values $a_1 = 2$, $a_2 = 2$, $b_3 = 5$, $b_4 = 5$, what is the correct statement about the obsevables $\hat{A}$ and $\hat{B}$? a) they are NOT compatible; b) they are compatible but DO NOT form a *complete set of compatible observables*; c) they are compatible and form a *complete set of compatible observables*. Explain your answer.

The choice of the parameters $a_1 = 5$, $a_2 = 5$, $b_3 = 6$, $b_4 = 6$ satisfies the condition of compatibility, namely $a_1 = a_2$ and $b_3 = b_4$. However, the states $|2'\rangle$ and $|3'\rangle$ would not be distinguishable based on measurements of $\hat{A}$ and $\hat{B}$ alone, thus $\hat{A}$ and $\hat{B}$ do not form a *complete set of compatible observables*.