

HW5 continue

Landau Mean Field (Ising Model)

Zhiyan Wang

The core of the Ginzburg-Landau theory lies in treating the free energy as an expansion of the order parameter,

$$f = \frac{\beta F}{N} = f_0 + \frac{r}{2} \Psi^2 + \frac{u}{4} \Psi^4 + \dots \quad (1)$$

- Higher-order terms are usually neglected as required, thus it is a mean field theory.
- The coefficients r, u have constraints that ensure the free energy opens downwards and does not exceed the lower bound, as energy has a lower limit.
- We expect that equation (1) can describe: when the system is in an ordered phase, the order parameter is non-zero, and when the system is in a disordered phase, the order parameter is zero. This can be achieved by solving for the minimum of the free energy where the system is stable.
- f_0 is just a simple consideration, generally constructed by analyzing the symmetry of the system. If the order parameter is a function of space: $\Psi(x)$, then the gradient of the order parameter is used for construction, $f_0 \sim (\nabla \Psi(x))^2$, indicating that the spatial variation of the order parameter consumes energy.

Thermodynamic Phase Transition

Considering the case where the order parameter is spatially uniform.

The simplest case is a thermodynamic phase transition, where we expect the order parameter to be zero at high temperatures and the system to be ordered at low temperatures. Simple order parameters such as specific heat and magnetization can be described by expanding the free energy up to the fourth order.

From equation (1), first, the minimum value requirement:

$$\frac{\partial f}{\partial \Psi} = 0 \rightarrow \Psi(r + u\Psi^2) = 0 \quad (2)$$

This equation inevitably has a trivial solution $\Psi = 0$, but it should be discarded in the ordered phase. Another solution, i.e., $\Psi \neq 0$, must lead to:

$$\Psi = \pm \sqrt{-\frac{r}{u}} \quad (3)$$

Where the coefficient r is:

$$r = a(T - T_c), a > 0 \quad (4)$$

- For high temperatures $T > T_c$, the free energy density curve opens upwards, with no intersections with the horizontal axis other than 0. Therefore, only $\Psi = 0$, which is expected.
- For low temperatures $T < T_c$, the free energy density curve intersects the horizontal axis at three points, in addition to the zero point, the other two points are the solutions we found above $\Psi = \pm \sqrt{-\frac{r}{u}}$.

To solve for the specific critical exponents, return to the free energy density when the order parameter is not zero, equation (1) becomes equation (2):

$$f[\Psi \neq 0] = f_0 - \frac{r^2}{4u} \quad (5)$$

Specific Heat

From the relationship between specific heat (density) and free energy density:

$$C_V = -T \frac{\partial^2 f}{\partial T^2} \quad (6)$$

In the high-temperature disordered phase, the free energy density is zero, leading to zero specific heat; for the low-temperature ordered phase, substituting equation (2),

$$C_V = \frac{T}{2u} \quad (7)$$

Therefore, as $T \rightarrow T_c^-$, the specific heat per unit volume takes a finite value $C_V = \frac{T_c}{2u}$, and once T_c is exceeded, i.e., $T \rightarrow T_c^+$, the specific heat immediately becomes 0. Therefore, the specific heat curve shows a jump at the phase transition point. We know that the strict solution for specific heat diverges logarithmically, and on the high-temperature side approaching the phase transition point is not a step behavior.

Near the phase transition point, we express the specific heat in the form of a power-law

$$C_V \propto |T - T_c|^{-\alpha} \quad (8)$$

Thus, $\alpha = 0$. The specific heat under the Landau mean field is a constant-type jump.

Magnetization

At this point, the order parameter is directly m , so from the roots solved in the previous section $\Psi = \pm \sqrt{-\frac{r}{u}}$, it is directly seen that:

$$m \propto |T_c - T|^{1/2} \quad (9)$$

Therefore, for $m \propto (T_c - T)^\beta$, we get the critical exponent $\beta = 1/2$. This is the result of the mean field. The exact solution is $\beta = 1/8$.

Magnetic Susceptibility

An external field needs to be introduced, and then consider the small field approximation $\lim_{h \rightarrow 0}$. Rewrite the free energy density as:

$$f = f_0 + \frac{r}{2}\Psi^2 + \frac{u}{4}\Psi^4 - h\Psi \quad (10)$$

The solution for the extremum is:

$$r\Psi + u\Psi^3 = h \quad (11)$$

Near the phase transition point, the Ψ^3 term is relatively small, so it is ignored, and $\Psi = \frac{h}{r}$.

$$\chi = \frac{d\Psi}{dh} = \frac{1}{r} \propto |T - T_c|^{-1} \quad (12)$$

Thus, when giving $\chi \propto |T - T_c|^\gamma$, $\gamma = 1$. The exact solution is $\gamma = 7/4$.

Nonlinear Response

(From the region where magnetization is not linearly changing with the external field: $m \rightarrow h^{1/\delta}$.) When the system is exactly at the phase transition point, $r = 0$, then

$$u\Psi^3 = h \quad (13)$$

Solving gives:

$$m = \left(\frac{h}{u} \right)^{1/3} \quad (14)$$

Then giving $m \propto h^{1/\delta}$, $\delta = 3$.

Two-Point Correlation Function

Note that to distinguish from the distance r in the following discussion, we use a, b, c , and the r, u above as expansion factors, do not confuse them with the following discussion.

The order parameter is Fourier transformed:

$$\Psi(r) = \frac{1}{\sqrt{V}} \sum_k e^{ikr} \psi_k \quad (15)$$

Note that since the order parameter is real, there must be $\psi(k) = \psi^*(-k)$, which means we do not need to sum over the entire space of k , but can clearly separate the parts of $k > 0$ and $k < 0$, and then take the part of $k > 0$.

$$\begin{aligned} \Psi(r) &= \frac{1}{\sqrt{V}} \sum_{k>0} e^{ikr} \psi_k + \frac{1}{\sqrt{V}} \sum_{k<0} e^{ikr} \psi_k \\ &= \frac{1}{\sqrt{V}} \sum_{k>0} e^{ikr} \psi_k + \frac{1}{\sqrt{V}} \sum_{k>0} e^{-ikr} \psi_{-k} \\ &= \frac{1}{\sqrt{V}} \sum_{k>0} (e^{ikr} \psi_k + e^{-ikr} \psi_k^*) \end{aligned}$$

On the other hand, for the term $(\nabla\Psi(r))^2$ in the free energy density, taking the derivative of the e^{ikr} factor gives: $(\nabla\Psi(r))^2 \rightarrow k^2|\Psi(r)|^2$. Next, we ignore the higher-order terms (because it is assumed that $T \rightarrow T_c$ when the order parameter is a small quantity, which is reasonable for the correlation function assumption), that is, do not consider the $\Psi^4(r)$ term. Organize the free energy density form as:

$$f = \frac{1}{V} \sum_{k>0} \frac{ak^2 + b}{2} \psi_k \psi_{-k} = \frac{1}{V} \sum_{k>0} \frac{ak^2 + b}{2} \psi_k \psi_k^* \quad (16)$$

This reminds us to handle the real and imaginary parts separately,

$$f = \sum_{k>0} \frac{ak^2 + b}{2} \left(\text{Re}^2 + \text{Im}^2 \right) \quad (17)$$

Next, solve for the expectation value of the correlation function. From the path integral, it is known that,

$$\langle \dots \rangle = \frac{1}{Z} \int D\Psi(\dots) e^{-\beta F} \quad (18)$$

Therefore, the expectation value of the correlation function in k-space is:

$$\langle \psi_k \psi_{-k} \rangle = \frac{1}{Z} \int D\Psi \psi_k \psi_{-k} e^{-\beta F} \quad (19)$$

Where the free energy $F = \frac{1}{\beta} \int d^d x f$.

For the integral:

$$\int D\Psi(r) \rightarrow \int \prod_k d\psi_k \rightarrow \int \prod_{k>0} d\psi_k d\psi_k^* \rightarrow \int \prod_{k>0} d\text{Re} d\text{Im} \quad (20)$$

Still considering the parts of $k > 0$ and $k < 0$, it can be decomposed into real and imaginary parts.

Then the partition function is,

$$\begin{aligned} Z &= \int D\Psi e^{-\int d^d r \left[\frac{a}{2} (\nabla \Psi(r))^2 + \frac{b}{2} \Psi^2(r) \right]} \\ &= \int D\Psi e^{-\sum_{k>0} \frac{ak^2+b}{2} (\text{Re}^2 + \text{Im}^2)} \\ &= \int \prod_{k>0} d\psi_k^* d\psi_k e^{-\sum_{k>0} \frac{ak^2+b}{2} (\text{Re}^2 + \text{Im}^2)} \\ &= \prod_{k>0} \int d\psi_k^* d\psi_k e^{-\frac{ak^2+b}{2} (\text{Re}^2 + \text{Im}^2)} \\ &= \prod_{k>0} \int d\text{Re} d\text{Im} e^{-\frac{ak^2+b}{2} (\text{Re}^2 + \text{Im}^2)} \end{aligned}$$

Using the Gaussian integral:

$$\int_{-\infty}^{\infty} dx dy e^{-a(x^2+y^2)} = \frac{\pi}{a}, \quad \int_{-\infty}^{\infty} d\bar{z} dz (\bar{z}z) e^{-a(\bar{z}z)} = \frac{2\pi}{a^2} \quad (21)$$

Here, $x = \text{Re}$, $y = \text{Im}$, $a = \frac{ak^2+b}{2}$, therefore,

$$Z = \prod_{k>0} \frac{2\pi}{ak^2 + b} \quad (22)$$

Similarly, the numerator can be written as,

$$\begin{aligned} &= \prod_{k>0} \int d\psi_k^* d\psi_k |\psi_k \psi_k^*| e^{-\frac{ak^2+b}{2}(\psi_k \psi_k^*)} \\ &= \delta_{k+k'=0} \prod_{k' \neq k>0} \frac{2\pi}{ak'^2 + b} \frac{2\pi}{(ak^2 + b)^2} \end{aligned}$$

So, we can see that as a pair of correlation functions in k-space:
 $\langle \psi_k \psi_{-k} \rangle = \langle \psi_k \psi_k^* \rangle = \langle \text{Re}^2 \rangle + \langle \text{Im}^2 \rangle$

$$\langle \psi_k \psi_{-k} \rangle = \frac{1}{ak^2 + b} \quad (23)$$

To obtain the correlation function in real space, we perform an inverse Fourier transform on the correlation function in k-space

$$G(r) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\vec{k} \cdot \vec{r}}}{ak^2 + b} \quad (24)$$

Using the Bessel function, this Gaussian integral is transformed into:

$$\frac{1}{ak^2 + b} = \int_0^\infty ds \exp(-(\alpha^2 k^2 + \mu^2)s) \quad (25)$$

$$\rightarrow G(r) = \int_0^\infty ds \int \frac{d^d k}{(2\pi)^d} e^{(-(ak^2+b)s + i\vec{k} \cdot \vec{r})} \quad (26)$$

$$k \rightarrow k + ir/2as,$$

$$G(r) = \int_0^\infty ds e^{-bs - r^2/4as} \int \frac{d^d k}{(2\pi)^d} e^{-ask^2}. \quad (27)$$

$$\rightarrow G(r) = \frac{1}{(4\pi a)^{d/2}} \int_0^\infty ds s^{-d/2} e^{-bs - r^2/4as}. \quad (28)$$

$$G(r) \sim \frac{K_{d/2-1} \left(\frac{\sqrt{br}}{\sqrt{a}} \right)}{r^{d/2-1}} \quad (29)$$

Where the Bessel function $K_{d/2-1} \left(\frac{\mu r}{\alpha} \right) \sim \frac{e^{-\mu r/\alpha}}{\sqrt{2\pi\mu r/\alpha}}$.

Then, (this part of the derivation refers to: [http://chimera.roma1.infn.it/ENZO/FC/TESTI/vanweert_statfieldth.pdf](<http://chimera.roma1.infn.it/ENZO/FC/TESTI/vanweert>

$$G(r) \sim \frac{e^{-r/\xi}}{r^{\frac{1}{2}(d-1)}} \quad (30)$$

The correlation length $\xi = \sqrt{\frac{a}{b}}$, it can be seen that the correlation function decays exponentially. For the correlation length $\xi = \sqrt{\frac{a}{b}} \propto \sqrt{\frac{a}{|T-T_c|}}$, thus giving the critical exponent for the correlation length, $\xi \propto |T - T_c|^{-\nu}$, $\nu = 1/2$.

However, when $T = T_c$, $b \rightarrow 0$, causing the correlation length to diverge, at this point we should return to the expression of $G(r)$ about the inverse Fourier transform.

For the vector dot product, we use the angle to represent, calculate the integral,

$$G(r) \sim \int_{-1}^{+1} d \cos \theta \int_{-\infty}^{+\infty} \frac{k^{d-1} dk}{(2\pi)^d} \frac{e^{ikr \cos \theta}}{ak^2 + b} \sim \int_{-\infty}^{+\infty} k^{d-1} dk \frac{1}{ikr} \frac{e^{ikr} - e^{-ikr}}{ak^2 + b} \sim \quad (31)$$

Perform a variable substitution, $kr = l \rightarrow k = \frac{l}{r}$,

$$\sim r^{-d+2} \int l^{d-1} dl \frac{e^{il} - e^{-il}}{l^2} \quad (32)$$

The integral is independent of r . Therefore,

$$G(r) \sim r^{-d+2} \quad (33)$$

For $G(r) \sim r^{-d+2+\eta}$, then giving $\eta = 0$.