Magnetic Polarizability Tensors for Low Frequency Object Classification and Detection

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***Abstract*—There is considerable interest in obtaining a low-cost mathematical description, which describes the interaction between a low frequency alternating magnetic field and a conducting object. Electrical engineers have proposed that the voltage perturbation in a coil placed in the field can be described in terms of a tensor that characterises its shape and material properties. In previous work [4], [6], we have obtained an explicit expression for the tensor's computation. This presentation extends on this to include a Bayesian inverse solution for the tensor coefficients of different objects. This has important applications in metal detectors for landmine clearance and airport security.**

I. Introduction

Metal detectors are widely used for locating hidden highly conducting objects in landmine clearance, for identifying threat items at airports and public events, locating buried treasure as well as finding items of archaeological significance. While metal detectors offer a low-cost portable means of detection, they are not able to distinguish between different shapes of conducting objects and there are challenges in differentiating between small objects concealed at small depths and larger objects, which might be buried deeper underground.

Location and discriminating between different objects requires the solution of an inverse problem. These are both mathematically and computationally challenging due to limited amounts of noisy measured data and the difficulty of inverting the system (the solution may not exist, be non–unique or not depend continuously on the data and hence is ill–posed). The forward, or direct, problem of determining the fields resulting from a known conducting object, is generally well–posed. In the context of metal detectors, the frequency of operation is low and the objects highly conducting so that eddy current approximation of Maxwell's equations applies.

To help address the challenges of solving the eddy current inverse problem, a low-cost description, which can characterise the shape and material properties of a conducting object in terms of a small number of parameters is used. In the electrical engineering community, it has been proposed that the voltage perturbation due to the presence of a conducting object in a single excitation–measurement coil pair is approximately

$$\boldsymbol{H}_0^{Ms}(\boldsymbol{z}) \cdot (\mathcal{M}\boldsymbol{H}_0^{Ex}(\boldsymbol{z})) \tag{1}$$

where $\boldsymbol{H}_0^{Ex}(\boldsymbol{z})$ is the background magnetic field of the exciting (Ex) coil evaluated at the position of the target, $\boldsymbol{H}_0^{Ms}(\boldsymbol{z})$ is the background field that would result if the measurement (Ms) coil acts as a source and $\mathcal{M}$ is a rank 2 polarizability tensor, which describes the shape and material properties of the object, independent of the background field. However, apart from a spherical object, where $\mathcal{M}$ is known explicitly, it was not clear when this approximation will hold or what the form of $\mathcal{M}$ will be.

II. Asymptotic Expansions

In the applied mathematics literature, asymptotic formulae are available that describe the perturbation in the magnetic field $(\boldsymbol{H}_\alpha - \boldsymbol{H}_0^{Ex})(\boldsymbol{x})$ due to the presence of an object $B_\alpha = \alpha B + \boldsymbol{z}$, which means a unit object B placed at the origin, scaled by the object size α and translated by $\boldsymbol{z}$. Using Einstein's summation convention for implied summation in repeated indices, the ith orthonormal coordinate is

$$\begin{aligned}((\boldsymbol{H}_\alpha - \boldsymbol{H}_0^{Ex})(\boldsymbol{x}))_i =& (\boldsymbol{D}_{\boldsymbol{x}}^2 G(\boldsymbol{x},\boldsymbol{z}))_{ij}\mathcal{A}_{jk}(\boldsymbol{H}_0^{Ex}(\boldsymbol{z}))_k \\ &+ (\boldsymbol{R}(\boldsymbol{x}))_i,\end{aligned} \tag{2}$$

as some suitable limit is taken. In the above, $\boldsymbol{R}(\boldsymbol{x})$ is a residual vector, $G(\boldsymbol{x},\boldsymbol{z}) := 1/(4\pi|\boldsymbol{x}-\boldsymbol{z}|)$, $(\boldsymbol{D}_{\boldsymbol{x}}^2 G(\boldsymbol{x},\boldsymbol{z}))_{ij} = \frac{1}{4\pi r^3}\left(3(\hat{\boldsymbol{r}})_i(\hat{\boldsymbol{r}})_j - \delta_{ij}\right)$, with $\boldsymbol{r} := \boldsymbol{x}-\boldsymbol{z}$, $r = |\boldsymbol{r}|$, $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$, and δ_{ij} the Kronecker delta.

For the magnetostatic response of a small object [3], [5] then (2) holds as $\alpha \to 0$ with $\mathcal{A} = \mathcal{T}(\mu_*/\mu_0)$ being the Poylá–Szegö tensor and $\boldsymbol{R}(\boldsymbol{x}) = O(\alpha^4)$. The 6 real components of the symmetric Poylá–Szegö rank 2 tensor can be computed by solving a transmission problem [2]. This tensor is parameterised by the constant in permeability (μ_*/μ_0) and describes the response of magnetic, but not highly conducting objects.

For the eddy current case, Ammari *et al.* [1], have obtained an asymptotic formula for $(\boldsymbol{H}_\alpha - \boldsymbol{H}_0^{Ex})(\boldsymbol{x})$, which shows the response for highly conducting objects, with conductivity σ_* and permeability μ_*, can be expressed in terms of a rank 4 tensor, as $\alpha \to 0$. This, at first sight, appears to disagree with (1) however, in [4] we have shown that, for the case of orthonormal coordinates, their asymptotic expansion reduces to the form stated in (2) with $\boldsymbol{R}(\boldsymbol{x}) = O(\alpha^4)$ as $\alpha \to 0$. We provide an explicit expression for the complex symmetric rank 2 tensor $\mathcal{A} = \widecheck{\widecheck{\mathcal{M}}}$, which is different to $\mathcal{T}(\mu_*/\mu_0)$, and, if (2) is multiplied by the ith component of the dipole moment of

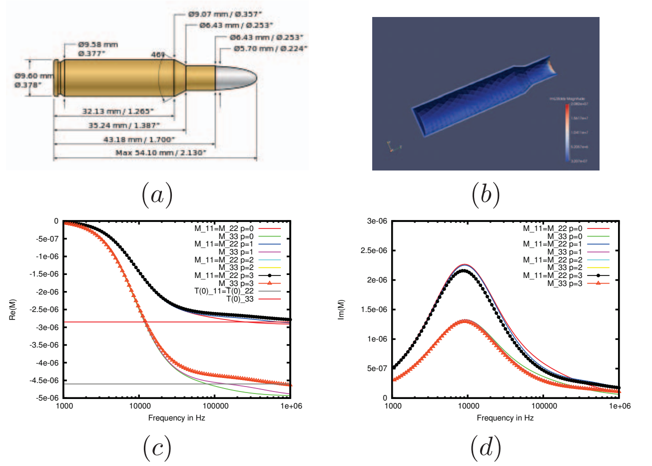

Fig. 1. Remington rifle shell casing showing: (a) geometry [7], (b) contours of $|\boldsymbol{\theta}_1|$, (c) frequency sweep of $\mathrm{Re}(\breve{\breve{\mathcal{M}}})$ and (d) frequency sweep of $\mathrm{Im}(\breve{\breve{\mathcal{M}}})$

the Ms coil, it reduces to the form in (1) thus providing a theoretical framework for the engineering prediction [4], [6].

III. Polarizability Tensor Computation

In [4], [6] we describe a numerical procedure for computing the coefficients of our $\breve{\breve{\mathcal{M}}} := \mathcal{N} - \check{\mathcal{C}}$ where

$$\check{\mathcal{C}}_{ij} := -\frac{\mathrm{i}\omega\sigma_*\mu_0\alpha^5}{4}\hat{\boldsymbol{e}}_i \cdot \int_B \boldsymbol{\xi} \times (\boldsymbol{\theta}_j + \hat{\boldsymbol{e}}_j \times \boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi},$$

$$\mathcal{N}_{ij} := \alpha^3\left(1 - \frac{\mu_0}{\mu_*}\right)\int_B \left(\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j + \frac{1}{2}\hat{\boldsymbol{e}}_i \cdot \nabla \times \boldsymbol{\theta}_j\right)\mathrm{d}\boldsymbol{\xi},$$

$\hat{\boldsymbol{e}}_i$ denotes the unit basis vector for the ith coordinate and $\boldsymbol{\theta}_i$ solves the vector valued transmission problem

$$\begin{aligned}
\nabla \times \mu_*^{-1}\nabla \times \boldsymbol{\theta}_i - \mathrm{i}\omega\sigma_*\alpha^2\boldsymbol{\theta}_i &= \mathrm{i}\omega\sigma_*\alpha^2\hat{\boldsymbol{e}}_i \times \boldsymbol{\xi} && \text{in } B,\\
\nabla \times \mu_0^{-1}\nabla \times \boldsymbol{\theta}_i &= \boldsymbol{0},\ \nabla \cdot \boldsymbol{\theta}_i = 0 && \text{in } B^c,\\
[\boldsymbol{\theta}_i \times \hat{\boldsymbol{n}}]_\Gamma &= \boldsymbol{0}, && \text{on } \Gamma,\\
\left[\mu^{-1}\nabla \times \boldsymbol{\theta}_i \times \hat{\boldsymbol{n}}\right]_\Gamma &= -2\left[\mu^{-1}\right]_\Gamma \hat{\boldsymbol{e}}_i \times \hat{\boldsymbol{n}} && \text{on } \Gamma,\\
\vec{\theta}_i(\boldsymbol{\xi}) &= O(|\boldsymbol{\xi}|^{-1}) \text{ as } |\boldsymbol{\xi}| \to \infty && .
\end{aligned}$$

Our approach is based on the application of high order edge-finite elements and we have previously applied it to a series of simply and multiply connected three dimensional objects. In Fig 1 we show illustrative results for a Remington rifle shell casing with $\mu_* = \mu_0$ and $\sigma_* = 1.5\times 10^7$S/m with the truncated $B \cup B^c$ region discretised by 23 551 tetrahedra. The figure shows the geometry, the computed contours of a typical $|\boldsymbol{\theta}_1|$ and the convergence of $\mathrm{Re}(\breve{\breve{\mathcal{M}}})$ and $\mathrm{Im}(\breve{\breve{\mathcal{M}}})$ with polynomial refinement of the finite elements over a wide frequency range.

Depending on the symmetries of an object, we have shown that the number of independent coefficients of $\breve{\breve{\mathcal{M}}}$ can be fewer than 6 (it is 2 for the object in Fig. 1). We have obtained results for low frequency, high conductivity limiting cases of $\breve{\breve{\mathcal{M}}}$, which relate it to $\mathcal{T}(\mu_*/\mu_0)$ and $\mathcal{T}(0)$, respectively [4], [6]. We will summarise these results in our presentation.

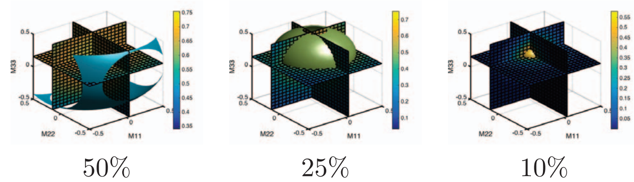

Fig. 2. Posterior probability density distribution $\pi(\boldsymbol{p}|\boldsymbol{V})$ interrogated by the MAP estimate when $\boldsymbol{p} = \{\breve{\breve{\mathcal{M}}}_{ii}, i = (1, 2, 3)\}$ for different noise levels

IV. Bayesian Inversion

Given a set of parameters used to describe the object, $\boldsymbol{p}$, an approach to describing the metal detection inverse problem is

$$\min_{\boldsymbol{p}} \|\boldsymbol{V} - \boldsymbol{F}(\boldsymbol{p})\|_P^2 + \lambda\|\boldsymbol{p}\|_Q^2 \tag{3}$$

where $\boldsymbol{V}$ are the measured voltages and $\boldsymbol{F}(\boldsymbol{p})$ represents the direct problem and describes how the voltages depend on the parameters. To overcome measurement noise the regularisation term $\lambda\|\boldsymbol{p}\|_Q$ is added. For $\boldsymbol{F}(\boldsymbol{p})$ we have seen that we can use (1) and (2), which provides a rigorous way of describing objects in terms of $\breve{\breve{\mathcal{M}}}$ and its position $\boldsymbol{z}$. Typically P, Q are set as the 2-norm and Tikhonov regularisation is applied. However, there are difficulties in choosing λ as it can can dampen out or pollute the solution to (3).

In the Bayesian approach, the likelihood $\boldsymbol{V} - \boldsymbol{F}(\boldsymbol{p})$ and the previous knowledge about $\boldsymbol{p}$, known as the prior, are expressed as probability distributions, $\pi(\boldsymbol{V}|\boldsymbol{p})$ and π_{prior}, respectively. An application of Bayes' theorem leads to the result that the posterior distribution is $\pi(\boldsymbol{p}|\boldsymbol{V}) \sim \pi(\boldsymbol{V}|\boldsymbol{p})\pi_{prior}$ and, in the case of Gaussian distributions, as $\exp\{-\|\boldsymbol{V} - \boldsymbol{F}(\boldsymbol{p})\|_P^2 - \|\boldsymbol{p}\|_Q^2\}$ with P, Q being informed by the covariance of the likelihood and prior distributions, respectively. Then $\pi(\boldsymbol{p}|\boldsymbol{V})$ quantifies the uncertainty and can be interrogated to obtain different interpretations for the set of parameters $\boldsymbol{p}$ without the need to provide regularisation. Illustrative results are included in Fig. 2 and we will discuss this further in our presentation.

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