ALLEN CAREER INSTITUTE
KOTA (RAJASTHAN)
path to success



PRE-MEDICAL

# PHYSICS

ENTHUSIAST | LEADER | ACHIEVER

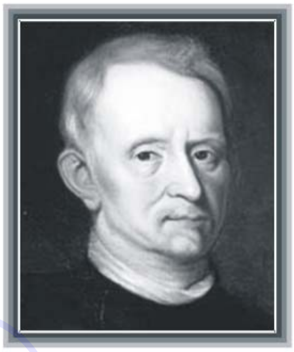
**STUDY MATERIAL**

## Oscillations (SHM)

ENGLISH MEDIUM

# Copyright Statement

**ROBERT HOOKE (1635 – 1703 A.D.)**



Robert Hooke was born on July 18, 1635 in Freshwater, Isle of Wight. He was one of the most brilliant and versatile seventeenth century English scientists. He attended Oxford University but never graduated. Yet he was an extremely talented inventor, instrument-maker and building designer. He assisted Robert Boyle in the construction of Boylean air pump. In 1662, he was appointed as Curator of Experiments to the newly founded Royal Society. In 1665, he became Professor of Geometry in Gresham College where he carried out his astronomical observations. He built a Gregorian reflecting telescope; discovered the fifth star in the trapezium and an asterism in the constellation Orion; suggested that Jupiter rotates on its axis; plotted detailed sketches of Mars which were later used in the 19th century to determine the planet's rate of rotation; stated the inverse square law to describe planetary motion, which Newton modified later etc. He was elected Fellow of Royal Society and also served as the Society's Secretary from 1667 to 1682. In his series of observations presented in Micrographia, he suggested wave theory of light and first used the word 'cell' in a biological context as a result of his studies of cork. Robert Hooke is best known to physicists for his discovery of law of elasticity: Ut tensio, sic vis (This is a Latin expression and it means as the distortion, so the force). This law laid the basis for studies of stress and strain and for understanding the elastic materials.

**HENDRIK ANTOON LORENTZ (1853 - 1928) DUTCH**

Theoretical physicist, professor at Leiden. He Investigated the relationship between electricity, magnetism, and mechanics. In order to explain the observed effect of magnetic fields on emitters of light (Zeeman effect), he postulated the existence of electric charges. In the atom, for which he was awarded the Nobel Prize In 1902. He derived a set of transformation equations (known after him as Lorentz transformation equations) by some tangled mathematical arguments, but he was not aware that these equations hinge on a new concept of space and time.

# OSCILLATIONS
## (SHM, DAMPED AND FORCED OSCILLATIONS & RESONANCE)

## 1. PERIODIC MOTION AND ITS CHARACTERISTICS AND TYPES OF SHM

### 1.1 Periodic Motion

(i) Any motion which repeats itself after regular interval of time is called periodic motion or harmonic motion.

(ii) The constant interval of time after which the motion is repeated is called time period.

Examples : (i) Motion of planets around the sun.
(ii) Motion of the pendulum of wall clock.

### 1.2 Oscillatory Motion

(i) The motion of a body is said to be oscillatory or vibratory motion if it moves back and forth (to and fro) about a fixed point after certain interval of time.

(ii) The fixed point about which the body oscillates is called mean position or equilibrium position.

Examples : (i) Vibration of the wire of 'Sitar'.
(ii) Oscillation of the mass suspended from spring.

### 1.3 Harmonic Functions

The trigonometric function of constant amplitude and single frequency is define as harmonic function. (Among all the trigonometrical functions only "sin" and "cos" functions are taken as harmonic function in basic form")

$$\left.\begin{aligned} y &= A\sin\theta = A\sin\omega t \\ y &= A\cos\theta = A\cos\omega t \end{aligned}\right\}\text{Harmonic function}$$

### 1.4 Some basic terms

**Mean Position**

The point at which the restoring force on the particle is zero and potential energy is minimum, is known as its mean position.

**Restoring Force**

- The force acting on the particle which tends to bring the particle towards its mean position, is known as restoring force.
- This force is always directed towards the mean position.
- Restoring force always acts in a direction opposite to that of displacement. Displacement is measured from the mean position.
- It is given by $F = -kx$ and has dimension $MLT^{-2}$.

**Amplitude**

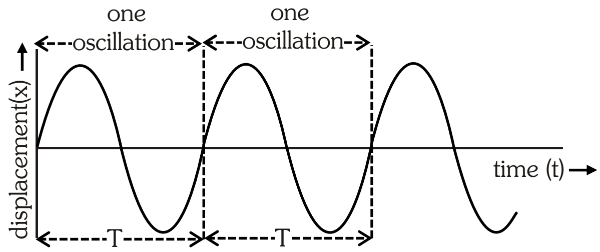
The maximum displacement of particle from mean position is define as amplitude.

**Time period (T)**

- The time after which the particle keeps on repeating its motion is known as time period.
- It is given by $T = \frac{2\pi}{\omega}$ , $T = \frac{1}{n}$ where $\omega$ is angular frequency and n is frequency.

**Oscillation or Vibration**

When a particle goes on one side from mean position and returns back and then it goes to other side and again returns back to mean position, then this process is known as one oscillation.




**Frequency (n or f)**

(a) The number of oscillations per second is define as frequency.

(b) It is given by $n = \frac{1}{T}$, $n = \frac{\omega}{2\pi}$

(c) **SI UNIT :** Hertz (Hz)

1 Hertz = 1 cycle per second (cycle is a number not a dimensional quantity).

(d) **Dimension :** $[M^0 L^0 T^{-1}]$

**Phase**

(a) Phase of a vibrating particle at any instant is the state of the vibrating particle regarding its displacement and direction of vibration at that particular instant.

(b) $y = A\sin(\omega t + \phi)$

The quantity $(\omega t + \phi)$ represents the phase angle at that instant.

(c) The phase angle at time t = 0 is known as **initial phase or epoch.**

(d) The difference of total phase angles of two particles executing S.H.M. with respect to the mean position is known as phase difference.

(e) If the phase angles of two particles executing S.H.M. are $(\omega t + \phi_1)$ and $(\omega t + \phi_2)$ respectively, then the phase difference between two particles is given by

$$\Delta\phi = (\omega t + \phi_2) - (\omega t + \phi_1) \quad \text{or} \quad \Delta\phi = \phi_2 - \phi_1$$

(f) Two vibrating particles are said to be in same phase if the phase difference between them is an even multiple of $\pi$, i.e., $\Delta\phi = 2N\pi \Rightarrow$ Same phase.

(g) Two vibrating particle are said to be in opposite phase if the phase difference between them is an odd multiple of $\pi$ i.e., $\Delta\phi = (2N + 1)\pi \Rightarrow$ opposite phase.

**Angular frequency (ω)**

(a) The rate of change of phase angle of a particle with respect to time is define as its angular frequency.

(b) **SI UNIT :** radian/second

**Dimension :** $[M^0 L^0 T^{-1}]$

**Instantaneous displacement**

(a) The displacement of the particle from mean position in a particular direction at any instant of time is known as instantaneous displacement.

(b) At time t the instantaneous displacement $x = A \sin(\omega t + \phi)$, where $\phi$ is initial phase and A is amplitude.

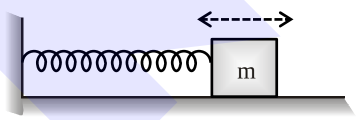
## 1.5 Simple harmonic motion (S.H.M.)

Simple harmonic motion is the simplest form of vibratory or oscillatory motion.

**(i) S.H.M. are of two types**

**(a) Linear S.H.M.**

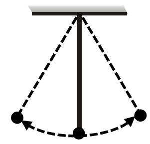
When a particle moves to and fro about a fixed point (called equilibrium position) along a straight line then its motion is called linear simple harmonic motion.




Example : Motion of a mass connected to spring.

**(b) Angular S.H.M.**

When a system oscillates angularly with respect to a fixed axis then its motion is called angular simple harmonic motion.



Example :- Motion of a bob of simple pendulum.

**(ii) Necessary Condition to execute S.H.M.**

**(a)** Motion of particle should be oscillatory.

**(b)** Total mechanical energy of particle should be conserved

(Kinetic energy + Potential energy = constant)

**(c)** Extreme position should be well defined.

**(d) In linear S.H.M.**

The restoring force (or acceleration) acting on the particle should always be proportional to the displacement of the particle and directed towards the equilibrium position

$\therefore \quad F \propto -y \quad$ or $\quad a \propto -y$

Negative sign shows that direction of force and acceleration is towards equilibrium position and y is displacement of particle from equilibrium position.

**(e) In angular S.H.M.**

The restoring torque (or angular acceleration) acting on the particle should always be proportional to the angular displacement of the particle and directed towards the equilibrium position

$\therefore \quad \tau \propto -\theta \quad$ or $\quad \alpha \propto -\theta$

**(iii) Comparison between linear and angular S.H.M.**

| **Linear S.H.M.** | **Angular S.H.M.** |
|---|---|
| $F \propto -x$<br>$F = -kx$<br>Where k is the restoring force constant<br>$a = -\frac{k}{m}x$<br>$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$<br>It is known as differential equation of linear S.H.M.<br>$x = A \sin\omega t$<br>$a = -\omega^2 x$<br>where $\omega$ is the angular frequency<br>$\omega^2 = \frac{k}{m}$<br>$\omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T} = 2\pi n$<br>where T is time period and n is frequency<br>$T = 2\pi\sqrt{\frac{m}{k}}$<br>$n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$<br>This concept is valid for all types of linear S.H.M. | $\tau \propto -\theta$<br>$\tau = -C\theta$<br>Where C is the restoring torque constant<br>$\alpha = -\frac{C}{I}\theta$<br>$\frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0$<br>It is known as differential equation of angular S.H.M.<br>$\theta = \theta_0 \sin \omega t$<br>$\alpha = -\omega^2\theta$<br>$\omega^2 = \frac{C}{I}$<br>$\omega = \sqrt{\frac{C}{I}} = \frac{2\pi}{T} = 2\pi n$<br>$T = 2\pi\sqrt{\frac{I}{C}}$<br>$n = \frac{1}{2\pi}\sqrt{\frac{C}{I}}$<br>This concept is valid for all types of angular S.H.M. |

## GOLDEN KEY POINTS

- Angular oscillatory motion can be treated as an angular simple harmonic motion only in the limit of small amplitude because in this limit the restoring force (or torque) becomes linear.
- Harmonic oscillations is that oscillations which can be expressed in terms of single harmonic function.

  (i.e. sine function or cosine function)
- The motion of the molecules of a solid, the vibration of the air columns and the vibration of string of musical instruments are either simple harmonic or superposition of simple harmonic motions.

## Illustrations

### Illustration 1

Which of the following functions represent SHM :–

(i) sin 2ωt (ii) sin ωt + 2cos ωt (iii) sin ωt + cos 2ωt

### Solution

(i) As $y = \sin 2\omega t \Rightarrow v = \frac{dy}{dt} = 2\,\omega \cos 2\omega t \Rightarrow$ Acceleration $= \frac{d^2y}{dt^2} = -4\omega^2 \sin 2\omega t = -4\,\omega^2 y$

so y = sin 2ωt represents S.H.M.

(ii) $y = \sin \omega t + 2\cos \omega t \Rightarrow v = \frac{dy}{dt} = \omega\cos\omega t - 2\omega\sin\omega t,$

Acceleration $= \frac{dy}{dt} = -\omega^2\sin\omega t - 2\omega^2\cos\omega t = -\omega^2(\sin\omega t + 2\cos\omega t) = -\omega^2 y$

∴ The given function represents SHM

(iii) $y = \sin \omega t + \cos 2\omega t$

$\Rightarrow \frac{dy}{dt} = \omega\cos\omega t - 2\omega\sin 2\omega t,\ \frac{d^2y}{dt^2} = -\,\omega^2\sin\omega t - 4\omega^2\cos 2\omega t = -\omega^2(\sin\omega t + 4\cos 2\omega t)$

$\frac{d^2y}{dt^2} \not\propto (-y)$ (Oscillatory but S.H.M. not possible)

### Illustration 2

If two S.H.M. are represented by equations $y_1 = 10\sin\left[3\pi t + \frac{\pi}{4}\right]$ and $y_2 = 5\left[\sin(3\pi t) + \sqrt{3}\cos(3\pi t)\right]$ then find the ratio of their amplitudes and phase difference in between them.

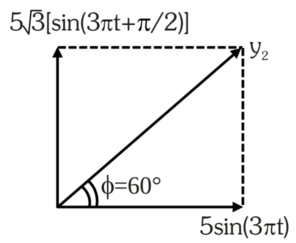
### Solution

As $y_2 = 5\left[\sin(3\pi t) + \sqrt{3}\cos(3\pi t)\right]$ ...(i)

So if $5 = A\cos\phi$ and $5\sqrt{3} = A\sin\phi$

Then $A = \sqrt{5^2 + (5\sqrt{3})^2} = 10$

and $\tan\phi = \frac{5\sqrt{3}}{5} = \sqrt{3}$ so $\phi = \frac{\pi}{3}$




$y_2 = 10\sin(3\pi t + \phi)$

the above equation (i) becomes

$y_2 = A\cos\phi\sin(3\pi t) + A\sin\phi\cos(3\pi t) \Rightarrow y_2 = A\sin(3\pi t + \phi)$

but $y_2 = 10\sin[3\pi t + (\frac{\pi}{3})]$

so, $\frac{A_1}{A_2} = \frac{10}{10} \Rightarrow A_1 : A_2 = 1 : 1,$

Phase difference $= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$ rad.

## BEGINNER'S BOX-1

**1.** Which of the following functions of time represent (a) SHM and (b) Periodic but not SHM (c) Non-periodic motion?

(i) $\sin \omega t + \cos \omega t$ (ii) $\sin \omega t + \cos 2\omega t + \sin 4\omega t$ (iii) $e^{-\omega t}$

(iv) $\log (\omega t)$ (v) $\sin \omega t - \cos \omega t$ (vi) $\sin^2 \omega t$ (vii) $\sin^3 \omega t$

**2.** The equation of motion of a particle executing simple harmonic motion is $a+16\pi^2x = 0$. In this equation, a is the linear acceleration in $m/s^2$ of the particle at a displacement x in metre. Find the time period.

**3.** Displacement of a particle executing SHM is represented by $Y = 0.08\sin\left(3\pi t + \frac{\pi}{4}\right)$ metre. Then calculate:-

(a) Time period (b) Initial phase

(c) Displacement from mean position at $t = \frac{7}{36}$ sec.

## 2. SIMPLE HARMONIC MOTION (SHM) AND ITS EQUATION; VELOCITY, ACCELERATION AND PHASE

### 2.1 Geometrical meaning of S.H.M.

If a particle is moving with uniform speed along the circumference of a circle then the straight line motion of the foot of perpendicular drawn from the particle on the diameter of the circle is called S.H.M.

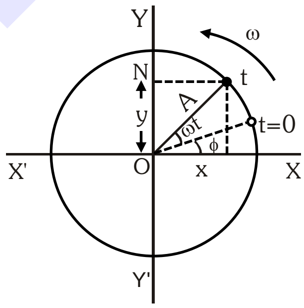
**DESCRIPTION OF S.H.M. BASED ON CIRCULAR MOTION.**

(a) Draw a circle, having radius equal to amplitude (A) of S.H.M.

(b) Suppose particle is moving with uniform speed with angular frequency $\omega$ along the circumference of the circle.

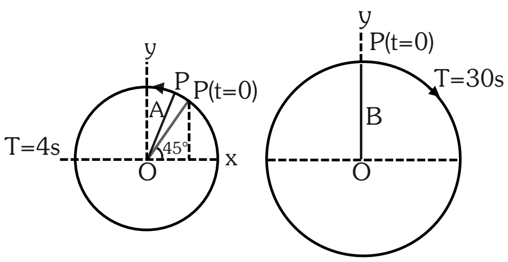
(c) Shadow (foot of the perpendicular from particle position) of particle performs S.H.M. on vertical and horizontal diameter of circle.

(d) Position of particle's shadow can be represented on diameter at t = 0 or any instant and position of particle performing circular motion can be determined by direction of velocity.




(e) By joining centre of circle to particle's position, angle $\theta$ is determined from horizontal or vertical diameter. After time t radius vector will turn $\omega t$. so $\theta = \omega t$.

**Ex.** Depicts two circular motions. The radius of the circle, the period of revolution, the initial position and the sense of revolution are indicated on the figures. Obtain the simple harmonic motions of the x-projection of the radius vector of the rotating particle P in each case.



**Sol.** **(a)** At t = 0, OP makes an angle of 45° = π/4 rad with the (positive direction of) x-axis. After time t, it covers an $\frac{2\pi}{T}t$ in the anticlockwise sense, and makes an angle of $\frac{2\pi}{T}t+\frac{\pi}{4}$ with the x-axis.

The projection of OP on the x-axis at time t is given by,

$$x(t) = A\cos\left(\frac{2\pi}{T}t+\frac{\pi}{4}\right)$$

For T = 4 s,

$$x(t) = A\cos\left(\frac{2\pi}{4}t+\frac{\pi}{4}\right)$$

which is a SHM of amplitude A, period 4 s, and an initial phase = $\frac{\pi}{4}$.

**(b)** In this case at t = 0, OP makes an angle of 90° = $\frac{\pi}{2}$ with the x-axis. After a time t, it t covers an angle of $\frac{2\pi}{T}t$ in the clockwise sense and makes an angle of $\left(\frac{\pi}{2}-\frac{2\pi}{T}t\right)$ with the x-axis. The projection of OP on the x-axis at time t is given by

$$x(t) = B\cos\left(\frac{\pi}{2}-\frac{2\pi}{T}t\right) = B\sin\left(\frac{2\pi}{T}t\right)$$

For T = 30 s,

$$x(t) = B\sin\left(\frac{\pi}{15}t\right)$$

## 2.2 Displacement, Velocity and Acceleration in S.H.M.

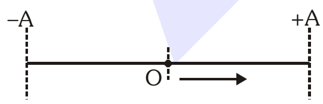
**Displacement in S.H.M.**

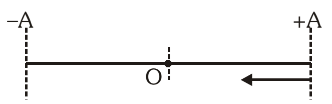
(i) The displacement of a particle executing linear S.H.M. at any instant is defined as the distance of the particle from the mean position at that instant.

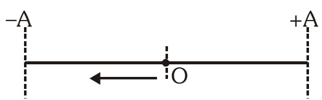
(ii) It can be given by relation x = Asinωt or x = Acosωt.

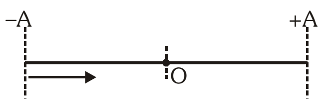
The first relation is valid when the time is measured from the mean position and the second relation is valid when the time is measured from the extreme position of the particle executing S.H.M. along a straight line path.

**Ex.** What will be the equation of displacement in the following different conditions ?

(i)



(ii)



(iii)



(iv)



**Sol.** (i) x = Asinωt

(ii) x = Asin(ωt + $\frac{\pi}{2}$) ⇒ x = Acosωt

(iii) x = Asin(ωt + π) ⇒ x = –Asinωt

(iv) x = Asin(ωt + $\frac{3\pi}{2}$) ⇒ x = –Acosωt

## 2.3 Velocity in S.H.M.

(i) It is define as the time rate of change of the displacement of the particle at a given instant.

(ii) Velocity in S.H.M. is given by $v = \frac{dx}{dt} = \frac{d}{dt}(A\sin\omega t)$ $\Rightarrow$ $v = A\omega\cos\omega t$

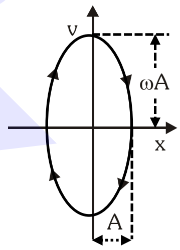
$$v = \pm A\omega\sqrt{1-\sin^2\omega t} \Rightarrow v = \pm A\omega\sqrt{1-\frac{x^2}{A^2}} \quad [\because\ x = A\sin\omega t]$$

$$v = \pm\omega\sqrt{(A^2 - x^2)}$$

Squaring both the sides $v^2 = \omega^2(A^2 - x^2)$ $\Rightarrow$ $\frac{v^2}{\omega^2} = A^2 - x^2$

$$\frac{v^2}{\omega^2 A^2} = 1 - \frac{x^2}{A^2} \Rightarrow \frac{x^2}{A^2} + \frac{v^2}{A^2\omega^2} = 1$$

This is equation of ellipse. So curve between displacement and velocity of particle executing S.H.M. is ellipse.




(iii) The graph between velocity and displacement is shown in figure.

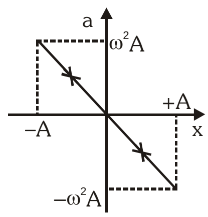
If particle oscillates with unit angular frequency ($\omega = 1$) then curve between v and x will be circle.

**Note:** (i) The direction of velocity of a particle in S.H.M. is either towards or away from the mean position.

(ii) At mean position ($x = 0$), velocity is maximum ($=A\omega$) and at extreme position ($x = \pm A$), the velocity of particle executing S.H.M. is zero.

## 2.4 Acceleration in S.H.M.

(i) It is define as the time rate of change of the velocity of the particle at given instant.




(ii) Acceleration in S.H.M. is given by $a = \frac{dv}{dt} = \frac{d}{dt}(A\omega\cos\omega t)$

$$a = -\omega^2 A\sin\omega t \Rightarrow a = -\omega^2 x$$

(iii) The graph between acceleration and displacement is a straight line as shown in figure.

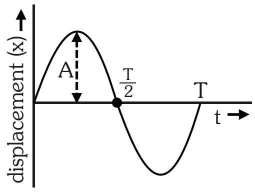
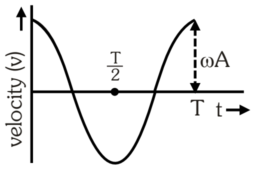
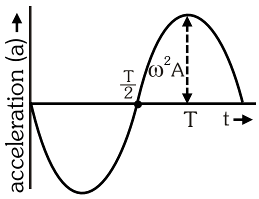
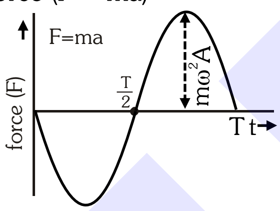
**Note :**

(i) The acceleration of a particle executing S.H.M. is always directed towards the mean position.

(ii) The acceleration of the particle executing S.H.M. is maximum at extreme position ($= \omega^2 A$) and minimum at mean position (= zero).

## 2.5 Graphical Representation

**Graphical study of displacement, velocity, acceleration and force in S.H.M.**

| S. No. | Graph | In form of t | In form of x | Maximum value |
|---|---|---|---|---|
| **1.** | Displacement  | $x = A\sin\omega t$ | $x = x$ | x = IA |
| **2.** | Velocity  | $v = A\omega\cos\omega t$ | $v = \pm\omega\sqrt{A^2 - x^2}$ | $v = \pm\omega A$ |
| **3.** | Acceleration  | $a = -\omega^2 A\sin\omega t$ | $a = -\omega^2 x$ | $a = \pm\omega^2 A$ |
| **4.** | Force (F = ma)  | $F = -m\omega^2 A\sin\omega t$ | $F = -m\omega^2 x$ | $F = \pm m\omega^2 A$ |

### GOLDEN KEY POINTS

- The direction of displacement is always away from the mean position whether the particle is moving from or coming towards the mean position.
- In linear S.H.M., the length of S.H.M. path = 2A
- In S.H.M., the total work done and displacement in one complete oscillation is zero but total travelled length is 4A.
- In S.H.M., the velocity and acceleration varies simple harmonically with the same frequency as displacement.
- Velocity is always ahead of displacement by phase angle $\frac{\pi}{2}$ radian
- Acceleration is ahead of displacement by phase angle $\pi$ radian i.e., opposite to displacement.
- Acceleration leads the velocity by phase angle $\frac{\pi}{2}$ radian.

## Illustrations

**Illustration 3**

An object performs S.H.M. of amplitude 5 cm and time period 4 s. If timing is started when the object is at the centre of the oscillation i.e., x = 0 then calculate –

(i) Frequency of oscillation

(ii) The displacement at 0.5 sec.

(iii) The maximum acceleration of the object.

(iv) The velocity at a displacement of 3 cm.

**Solution**

(i) Frequency $f = \frac{1}{T} = \frac{1}{4} = 0.25\ \text{Hz}$

(ii) The displacement equation of object x = Asinωt

so at t = 0.5 s $\quad x = 5\sin(2\pi \times 0.25 \times 0.5) = 5\sin\frac{\pi}{4} = \frac{5}{\sqrt{2}}$ cm

(iii) Maximum acceleration $a_{max} = \omega^2 A = (0.5\ \pi)^2 \times 5 = 12.3\ \text{cm/s}^2$

(iv) Velocity at x = 3 cm is $v = \pm\omega\sqrt{A^2 - x^2} = \pm 0.5\pi\sqrt{5^2 - 3^2} = \pm 6.28\ \text{cm/s}$

**Illustration 4**

A particle executes S.H.M. from extreme position and covers a distance equal to half of its amplitude in 1 s. Determine the time period of motion.

**Solution**

For particle starting S.H.M. from extreme position

$y = A\cos\omega t \quad \Rightarrow \quad \frac{A}{2} = A\cos(\omega \times 1)$

$\Rightarrow \quad \cos\omega = \cos\frac{\pi}{3} \quad \Rightarrow \quad \omega = \frac{\pi}{3} \quad \Rightarrow \quad T = \frac{2\pi}{\omega} = \frac{2\pi \times 3}{\pi} = 6\,s$

**Illustration 5**

Amplitude of a harmonic oscillator is A, when velocity of particle is half of maximum velocity, then determine position of particle.

**Solution**

$v = \omega\sqrt{A^2 - x^2} \quad$ but $\quad v = \frac{v_{max}}{2} = \frac{A\omega}{2}$

$\frac{A\omega}{2} = \omega\sqrt{A^2 - x^2} \quad \Rightarrow \quad A^2 = 4[A^2 - x^2]$

$\Rightarrow \quad x^2 = \frac{4A^2 - A^2}{4}$

$\Rightarrow \quad x = \pm\frac{\sqrt{3}A}{2}$

**Illustration 6**

The velocity of a particle in S.H.M. at position $x_1$ and $x_2$ are $v_1$ and $v_2$ respectively. Determine value of time period and amplitude.

**Solution**

$v = \omega\sqrt{A^2 - x^2} \quad \Rightarrow \quad v^2 = \omega^2 (A^2 - x^2)$

At position $x_1$ velocity $v_1^2 = \omega^2 (A^2 - x_1^2)$ ... (i)

At position $x_2$ velocity $v_2^2 = \omega^2 (A^2 - x_2^2)$ ... (ii)

Subtracting (ii) from (i) $v_1^2 - v_2^2 = \omega^2 (x_2^2 - x_1^2) \Rightarrow \omega = \sqrt{\dfrac{v_1^2 - v_2^2}{x_2^2 - x_1^2}}$

Time period $T = \dfrac{2\pi}{\omega} \Rightarrow T = 2\pi\sqrt{\dfrac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$

Dividing (i) by (ii) $\dfrac{v_1^2}{v_2^2} = \dfrac{A^2 - x_1^2}{A^2 - x_2^2} \Rightarrow v_1^2A^2 - v_1^2x_2^2 = v_2^2A^2 - v_2^2x_1^2$

So $A^2(v_1^2 - v_2^2) = v_1^2x_2^2 - v_2^2x_1^2 \Rightarrow A = \sqrt{\dfrac{v_1^2x_2^2 - v_2^2x_1^2}{v_1^2 - v_2^2}}$

**Illustration 7**

A particle executing S.H.M. having amplitude 0.01 m and frequency 60 Hz. Determine maximum acceleration of particle.

**Solution**

Maximum acceleration $a_{max.} = \omega^2A = 4\pi^2n^2A$

$= 4\pi^2(60)^2 \times (0.01)$

$= 144\ \pi^2\ m/s^2$

**Illustration 8**

A particle performing SHM is found at its equilibrium position at t = 1 sec and it is found to have a speed of 0.25 m/s at t = 2 sec. If the period of oscillation is 8 sec. Calculate the amplitude of oscillations.

**Solution**

$x = A\sin(\omega t + \phi)$

at t = 1 sec. particle at mean position

$0 = A\sin\left(\dfrac{2\pi}{8} \times 1 + \phi\right) \Rightarrow \boxed{\phi = -\dfrac{\pi}{4}}$

at t = 2 sec. velocity of particle is 0.25 m/s

$0.25 = A\omega\cos\left(\dfrac{\pi}{4} \times 2 - \dfrac{\pi}{4}\right)$

$0.25 = \dfrac{A\omega}{\sqrt{2}} \Rightarrow A = \dfrac{\sqrt{2}}{\pi}m$

## BEGINNER'S BOX-2

**1.** A particle executing simple harmonic motion completes 1200 oscillations per minute and passes through the mean position with a velocity of $3.14 ms^{-1}$. Determine the maximum displacement of the particle from its mean position. Also obtain the displacement equation of the particle if its displacement be zero at the instant t = 0.

**2.** A particle oscillates along the x–axis according to equation $x = 0.05 \sin\left(5t - \frac{\pi}{6}\right)$ where x is in metre and t is in second. Find its velocity at t = 0 second.

**3.** A particle is executing SHM given by $x = A \sin(\pi t + \phi)$. The initial displacement of particle is 1 cm and its initial velocity is π cm/sec. Find the amplitude of motion and initial phase of the particle.

**4.** A body executing S.H.M. has its velocity 10 cm/sec and 7 cm/sec when its displacement from the mean positon are 3 cm and 4 cm respectively. Calculate the length of the path.

**5.** A particle is executing S.H.M. of time period 4s. What is the time taken by it to move from the

(a) Mean position to half of the amplitude.

(b) Extreme position to half of the amplitude.

**6.** A particle undergoes simple harmonic motion having time period T. Find the time taken to complete $\frac{3}{8}$ oscillation.

**7.** The displacement of a particle executing simple harmonic motion is given by $y = 10\sin(6t + \frac{\pi}{3})$. Here y is in metre and t is in second. Find initial displacement & velocity of the particle.

**8.** For a particle executing S.H.M. In which part of a complete oscillation

(a) Acceleration supports the velocity. (b) Acceleration opposes the velocity.

**9.** A particle is in linear simple harmonic motion between two points A and B, 10 cm apart. Take the direction from A to B as the positive direction and given the signs of velocity, acceleration and force on the particle when it is

(a) at the end A. (b) at the end B.

(c) at the mid-point of AB going towards A. (d) at 2 cm away from B going towards A.

(e) at 3 cm away from A going towards B, and (f) at 4 cm away from A going towards A.

**10.** The piston in the cylinder head of a locomotive has a stroke (twice of the amplitude) of 1.0 m. If the piston moves with simple harmonic motion with an angular frequency of 200 rad/min., what is its maximum speed?

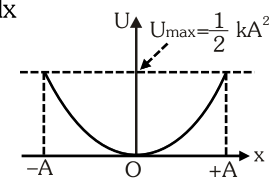
---

## 3. ENERGY IN SHM – POTENTIAL & KINETIC ENERGIES

### 3.1 Potential Energy (U or P.E.)

#### (i) In terms of displacement

The potential energy is related to force by the relation $F = -\frac{dU}{dx} \Rightarrow \int dU = -\int F dx$

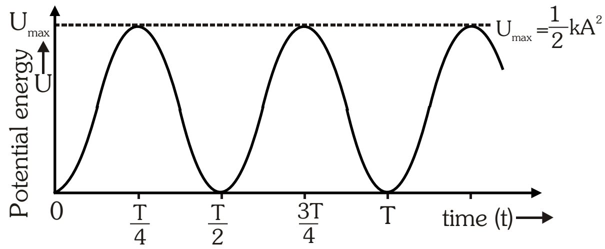



For S.H.M. F = – kx so $\int dU = -\int(-kx)dx = \int kx\,dx \Rightarrow U = \frac{1}{2}kx^2 + C$

At x = 0, $U = U_0 \Rightarrow C = U_0$ So $U = \frac{1}{2}kx^2 + U_0$

Where the potential energy at equilibrium position = $U_0$ when $U_0 = 0$ then $U = \frac{1}{2}kx^2$

**(ii) In terms of time**




Since $x = A\sin(\omega t + \phi)$, $U = \frac{1}{2}kA^2\sin^2(\omega t + \phi)$

If initial phase ($\phi$) is zero then $U = \frac{1}{2}kA^2\sin^2\omega t = \frac{1}{2}m\omega^2A^2\sin^2\omega t$

**Note :**

(i) In S.H.M. the potential energy is a parabolic function of displacement, the potential energy is minimum at the mean position ($x = 0$) and maximum at extreme position ($x = \pm A$)

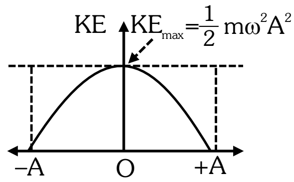
(ii) The potential energy is the periodic function of time.

It is minimum at $t = 0, \frac{T}{2}, T, \frac{3T}{2}$... and maximum at $t = \frac{T}{4}, \frac{3T}{4}, \frac{5T}{4}$...

## 3.2 Kinetic Energy (KE)

**(i) In terms of displacement**

If mass of the particle executing S.H.M. is m and Its velocity is v then kinetic energy at any instant.

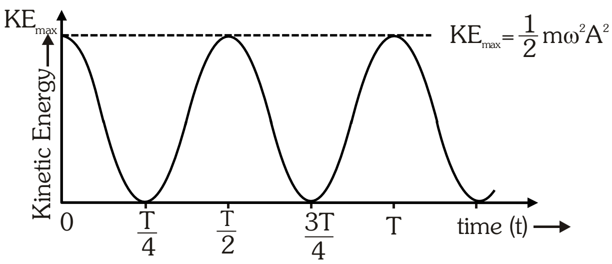



$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(A^2 - x^2) = \frac{1}{2}k(A^2 - x^2)$$

**(ii) In terms of time**

$\because v = A\omega\cos(\omega t + \phi)$

$\therefore K = \frac{1}{2}m\omega^2A^2\cos^2(\omega t + \phi)$




If initial phase $\phi$ is zero

$K = \frac{1}{2}m\omega^2A^2\cos^2\omega t$

**Note :**

(i) In S.H.M. the kinetic energy is a inverted parabolic function of displacement. The kinetic energy is maximum ($\frac{1}{2}kA^2$) at mean position ($x = 0$) and minimum (zero) at extreme position ($x = \pm A$)

(ii) The kinetic energy is the periodic function of time. It is maximum at $t = 0, \frac{T}{2}, T, \frac{3T}{2}$.............and minimum at $t = \frac{T}{4}, \frac{3T}{4}, \frac{5T}{4}$...

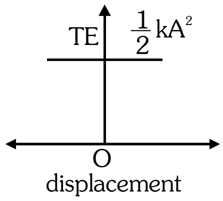
## 3.3 Total energy (E)

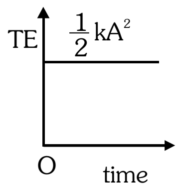
Total energy in S.H.M. is given by ; E = potential energy + kinetic energy = U + K

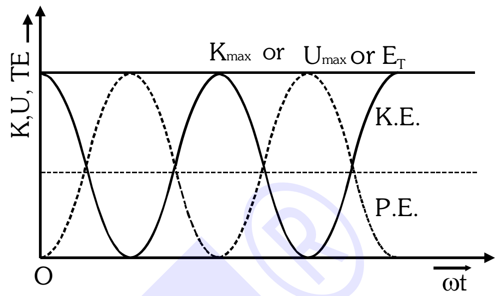
**(i)** w.r.t. position $E = \frac{1}{2}kx^2 + \frac{1}{2}k(A^2 - x^2)$ $\Rightarrow$ $E = \frac{1}{2}kA^2$ = constant

**(ii)** w.r.t. time

$$E = \frac{1}{2}m\omega^2A^2\sin^2\omega t + \frac{1}{2}m\omega^2A^2\cos^2\omega t = \frac{1}{2}m\omega^2A^2(\sin^2\omega t + \cos^2\omega t) = \frac{1}{2}m\omega^2A^2 = \frac{1}{2}kA^2 = \text{constant}$$




**Note :**

(i) Total energy of a particle in S.H.M. is same at all instant and at all displacement.

(ii) Total energy depends upon mass, amplitude and frequency of vibration of the particle executing S.H.M.

## 3.4 Average energy in S.H.M.

**(i)** The time average of P.E. and K.E. over one cycle is

(a) $\langle KE\rangle_t = \langle \frac{1}{2}m\omega^2A^2\cos^2\omega t\rangle = \frac{1}{2}m\omega^2A^2\langle\cos^2\omega t\rangle = \frac{1}{2}m\omega^2A^2\left(\frac{1}{2}\right) = \frac{1}{4}m\omega^2A^2 = \frac{1}{4}kA^2$

(b) $\langle PE\rangle_t = \langle \frac{1}{2}m\omega^2A^2\sin^2\omega t\rangle = \frac{1}{2}m\omega^2A^2\langle\sin^2\omega t\rangle = \frac{1}{2}m\omega^2A^2\left(\frac{1}{2}\right) = \frac{1}{4}m\omega^2A^2 = \frac{1}{4}kA^2 + U_0$

(c) $\langle TE\rangle_t = \langle \frac{1}{2}m\omega^2A^2 + U_0\rangle = \frac{1}{2}m\omega^2A^2 + U_0 = \frac{1}{2}kA^2 + U_0$

### GOLDEN KEY POINTS

- The frequency of oscillation of potential energy and kinetic energy is twice as that of displacement or velocity or acceleration of a particle executing S.H.M.
- Frequency of total energy is zero because it remains constant.

## Illustrations

**Illustration 9**

In case of simple harmonic motion –

(a) What fraction of total energy is kinetic and what fraction is potential when displacement is one half of the amplitude.

(b) At what displacement the kinetic and potential energies are equal.

**Solution**

In S.H.M.

K. E. $= \frac{1}{2}k(A^2 - x^2)$ P. E. $= \frac{1}{2}kx^2$ T.E. $= \frac{1}{2}kA^2$

(a) $f_{K.E.} = \frac{K.E.}{T.E.} = \frac{A^2 - x^2}{A^2}$ $f_{P.E.} = \frac{P.E.}{T.E} = \frac{x^2}{A^2}$

at $x = \frac{A}{2}$ $f_{K.E.} = \frac{A^2 - A^2/4}{A^2} = \frac{3}{4}$ and $f_{P.E.} = \frac{A^2/4}{A^2} = \frac{1}{4}$

(b) K. E. = P. E. $\Rightarrow$ $\frac{1}{2}k(A^2 - x^2) = \frac{1}{2}kx^2$ $\Rightarrow$ $2x^2 = A^2$ $\Rightarrow$ $x = \pm\frac{A}{\sqrt{2}}$

**Illustration 10**

A particle starts oscillating simple harmonically from its equilibrium position with time period T. Determine ratio of K.E. and P.E. of the particle at time $t = \frac{T}{12}$.

**Solution**

at $t = \frac{T}{12}$ $\qquad x = A\sin\frac{2\pi}{T}\times\frac{T}{12} = A\sin\frac{\pi}{6} = \frac{A}{2}$

so $\quad K.E. = \frac{1}{2}k(A^2 - x^2) = \frac{3}{4}\times\frac{1}{2}kA^2$ $\quad$ and $P.E. = \frac{1}{2}kx^2 = \frac{1}{4}\times\frac{1}{2}kA^2$

$\therefore \quad \frac{K.E.}{P.E.} = \frac{3}{1}$

**Illustration 11**

The potential energy of a particle executing S.H.M. is 2.5 J, when its displacement is half of the amplitude, then determine total energy of particle.

**Solution**

$P.E. = \frac{1}{2}kx^2 \quad \Rightarrow \quad \frac{1}{2}k\frac{A^2}{4} = 2.5 \quad \Rightarrow \quad$ Total energy $= \frac{1}{2}kA^2 = 2.5 \times 4 = 10$ J

**Illustration 12**

A harmonic oscillator of force constant $4 \times 10^6$ Nm$^{-1}$ and amplitude 0.01 m has total energy 240 J. What is maximum kinetic energy and minimum potential energy ?

**Solution**

$k = 4 \times 10^6$ N/m, $a = 0.01$ m, T.E. = 240 J, $\qquad$ As $\omega^2 = \frac{k}{m}$

Maximum kinetic energy $= \frac{1}{2}m\omega^2a^2 = \frac{1}{2}ka^2 = \frac{1}{2}\times 4\times 10^6\times(0.01)^2 = 200$ J

Minimum potential energy = Total energy – Maximum kinetic energy = 40 J

**Illustration 13**

The potential energy of a particle oscillating on x-axis is $U = 20 + (x - 2)^2$. Here U is in joules and x in meters. Total mechanical energy of the particle is 36 J.

(a) State whether the motion of the particle is simple harmonic or not.

(b) Find the mean position.

(c) Find the maximum kinetic energy of the particle.

**Solution**

(a) $F = -\frac{dU}{dx} = -2(x - 2)$ $\qquad$ By assuming $x - 2 = X$, we have $F = -2X$

Since, $F \propto -X$ $\qquad$ The motion of the particle is simple harmonic

(b) The mean position of the particle is $X = 0 \Rightarrow x - 2 = 0$, which gives $x = 2$m

(c) Maximum kinetic energy of the particle is, $K_{max} = E - U_{min} = 36 - 20 = 16$ J

**Note :** $U_{min}$ is 20 J at mean position or at x = 2m.

**BEGINNER'S BOX-3**

**1.** A point particle of mass 0.1 Kg is executing SHM with amplitude of 0.1m. When the particle passes through the mean position, its kinetic energy is $8 \times 10^{-3}$ Joule. Obtain the equation of motion of this particle if the initial phase of oscillation is 45°.

**2.** In case of SHM what fraction of total energy is kinetic and what fraction is potential, when displacement is one fourth of amplitude.

**3.** A particle execute S.H.M. with frequency f. Find frequency with which its kinetic energy oscillates?

**4.** A particle of mass 10g is placed in potential field given by $V = (50x^2 + 100)$ erg/g. What will be frequency of oscillation of particle ?

## 4. OSCILLATIONS OF A SPRING BLOCK SYSTEM

### 4.1 Spring Block System

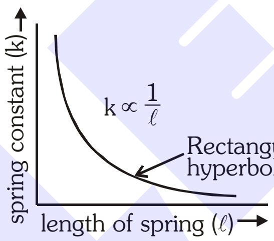
(i) When spring is given small displacement by stretching or compressing it, then restoring elastic force is developed in it because it obeys Hook's law.

$F \propto -x \quad \Rightarrow F = -kx \quad$ Here k is spring constant

(ii) Spring is assumed massless, so restoring elastic force in spring is assumed same everywhere.

(iii) Spring constant (k) depends on length ($\ell$), radius and material of wire used in spring.

For spring, $k\ell$ = constant

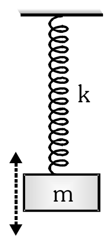



### 4.2 Spring Pendulum

(i) When a small mass is suspended from a mass-less spring then this arrangement is known as spring pendulum.

For small linear displacement the motion of spring pendulum is simple harmonic.

(ii) For a spring pendulum




$$F = -kx \Rightarrow m\frac{d^2x}{dt^2} = -kx \quad [\because F = ma = m\frac{d^2x}{dt^2}]$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

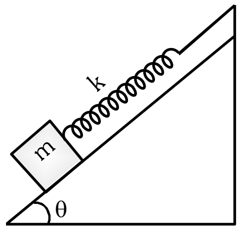
$$\because \frac{d^2x}{dt^2} = -\omega^2 x \Rightarrow \omega^2 = \frac{k}{m}$$

This is standard equation of linear S.H.M.

Time period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$, Frequency $n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

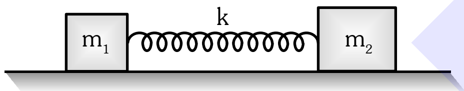
(iii) Time period of a spring pendulum is independent of acceleration due to gravity. This is why a clock based on oscillation of spring pendulum will keep proper time everywhere on a hill or moon or in a satellite or different places of earth.

(iv) If a spring pendulum oscillates in a vertical plane is made to oscillate on a horizontal surface or on an inclined plane then time period will remain unchanged.



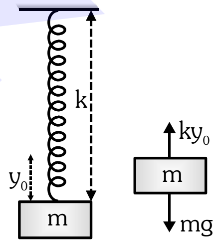

(v) By increasing the mass, time period of spring pendulum increases $(T \propto \sqrt{m})$, but by increasing the force constant of spring (k). Its time period decreases $\left[T \propto \frac{1}{\sqrt{k}}\right]$ whereas frequency increases $(n \propto \sqrt{k})$

(vi) If two masses $m_1$ and $m_2$ are connected by a spring and made to oscillate then time period $T = 2\pi\sqrt{\frac{\mu}{k}}$




Here, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ = reduced mass

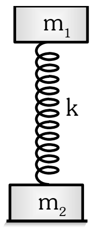
(vii) If the stretch in a vertically loaded spring is $y_0$ then for equilibrium of mass m.




$ky_0 = mg$ i.e., $\frac{m}{k} = \frac{y_0}{g}$

So, time period $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{y_0}{g}}$

But remember time period of spring pendulum is independent of acceleration due to gravity.



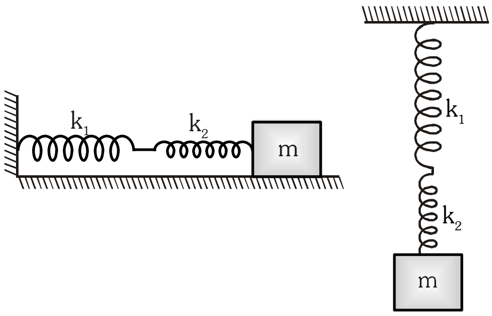

(viii) If two particles are attached with spring in which only one is oscillating then the

Time period $= 2\pi\sqrt{\frac{\text{mass of oscillating particle}}{\text{force constant}}} = 2\pi\sqrt{\frac{m_1}{k}}$

## 4.3 Various Spring Arrangements

- **Series combination of springs**

In series combination same restoring force exerts in all springs but extension will be different.




Total displacement $x = x_1 + x_2$

Force acting on both springs $F = -k_1x_1 = -k_2x_2$

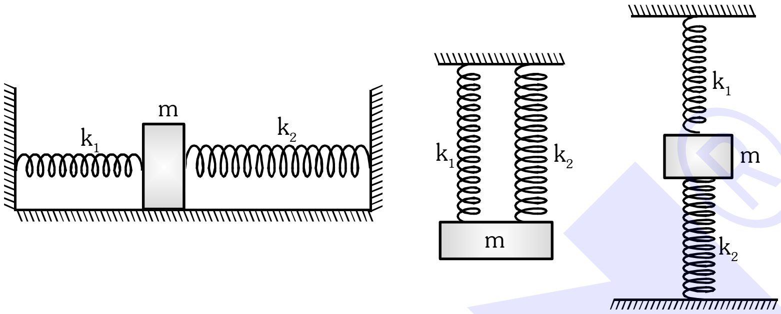
$\because\ x_1 = -\dfrac{F}{k_1}$ and $x_2 = -\dfrac{F}{k_2}$ $\quad\therefore\ x = -\left[\dfrac{F}{k_1}+\dfrac{F}{k_2}\right]$ ....(i)

If equivalent force constant is $k_s$ then $F = -k_s x$

so by equation (i) $-\dfrac{F}{k_s} = -\dfrac{F}{k_1}-\dfrac{F}{k_2} \Rightarrow \dfrac{1}{k_s}=\dfrac{1}{k_1}+\dfrac{1}{k_2} \Rightarrow k_s = \dfrac{k_1 k_2}{k_1+k_2}$

Time period $T = 2\pi\sqrt{\dfrac{m}{k_s}}$ Frequency $n = \dfrac{1}{2\pi}\sqrt{\dfrac{k_s}{m}}$, Angular frequency $\omega = \sqrt{\dfrac{k_s}{m}}$

- **Parallel Combination of springs**




In parallel combination displacement on each spring is same but restoring force is different.

Force acting on the system $F = F_1 + F_2 \Rightarrow F = -k_1x - k_2x$ ...(i)

If equivalent force constant is $k_P$ then, $F = -k_Px$, so by equation (i) $-k_Px = -k_1x - k_2x \Rightarrow k_P = k_1 + k_2$

Time period $T = 2\pi\sqrt{\dfrac{m}{k_P}} = 2\pi\sqrt{\dfrac{m}{k_1+k_2}}$ ; Frequency $n = \dfrac{1}{2\pi}\sqrt{\dfrac{k_P}{m}}$ ; Angular frequency $\omega = \sqrt{\dfrac{k_1+k_2}{m}}$

## GOLDEN KEY POINTS

- If the length of the spring is made n times then effective force constant becomes $\dfrac{1}{n}$ times and the time period becomes $\sqrt{n}$ times.
- If a spring of spring constant k is divided into n equal parts, the spring constant of each part becomes nk and time period becomes $\dfrac{1}{\sqrt{n}}$ times.
- In case of a loaded spring the time period comes out to be the same in both horizontal and vertical arrangement of spring system.
- The force constant k of a stiffer spring is higher than that of a soft spring. So the time period of a stiffer spring is less than that of a soft spring.

## Illustrations

**Illustration 14**

A body of mass m attached to a spring which is oscillating with time period 4 seconds. If the mass of the body is increased by 4 kg, its timer period increases by 2 sec. Determine value of initial mass m.

**Solution**

In I[st] *case* : $T = 2\pi\sqrt{\dfrac{m}{k}}\quad 4 = 2\pi\sqrt{\dfrac{m}{k}}$ ...(i) and in II[nd] *case:* $6 = 2\pi\sqrt{\dfrac{m+4}{k}}$ ...(ii)

Divide (i) by (ii) $\dfrac{4}{6} = \sqrt{\dfrac{m}{m+4}} \Rightarrow \dfrac{16}{36} = \dfrac{m}{m+4} \Rightarrow m = 3.2$ kg

**Illustration 15**

One body is suspended from a spring of length $\ell$, spring constant k and has time period T. Now if spring is divided in two equal parts which are joined in parallel and the same body is suspended from this arrangement then determine new time period.

**Solution**

Spring constant in parallel combination k' = 2k + 2k = 4k

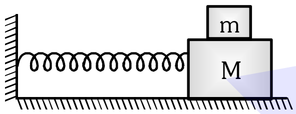
$$\therefore T' = 2\pi\sqrt{\frac{m}{k'}} = 2\pi\sqrt{\frac{m}{4k}} = 2\pi\sqrt{\frac{m}{k}} \times \frac{1}{\sqrt{4}} = \frac{T}{\sqrt{4}} = \frac{T}{2}$$

**Illustration 16**

A block of mass m is on a horizontal slab of mass M which is moving horizontally and executing S.H.M. The coefficient of static friction between block and slab is μ. If block is not separated from slab then determine angular frequency of oscillation.

**Solution**

If block is not separated from slab then restoring force due to S.H.M. should be less than frictional force between slab and block.




$$F_{restoring} \le F_{friction} \Rightarrow m\,a_{max.} \le \mu mg \Rightarrow a_{max.} \le \mu g \Rightarrow \omega^2 A \le \mu g \Rightarrow \omega \le \sqrt{\frac{\mu g}{A}}$$

**Illustration 17**

A block of mass m is attached from a spring of spring constant k and dropped from its natural length. Find the amplitude of S.H.M.

**Solution**

Let amplitude of S.H.M. be A then by work energy theorem W = ΔKE

$$mgx_0 - \frac{1}{2}kx_0^2 = 0 \Rightarrow x_0 = \frac{2mg}{k}$$

So amplitude $A = \frac{mg}{k}$

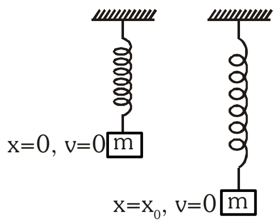



**Illustration 18**

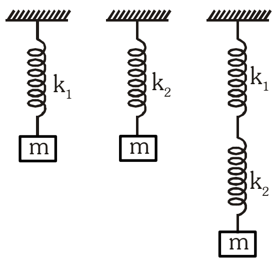
Periodic time of oscillation $T_1$ is obtained when a mass is suspended from a spring. If another spring is used with same mass then periodic time of oscillation is $T_2$. Now if this mass is suspended from series combination of above springs then calculate the time period.

**Solution**

$$T_1 = 2\pi\sqrt{\frac{m}{k_1}} \Rightarrow T_1^2 = 4\pi^2\frac{m}{k_1} \Rightarrow k_1 = \frac{4\pi^2 m}{T_1^2} \text{ and } T_2 = 2\pi\sqrt{\frac{m}{k_2}} \Rightarrow T_2^2 = 4\pi^2\frac{m}{k_2} \Rightarrow k_2 = \frac{4\pi^2 m}{T_2^2}$$

$$K_{eq.} = \frac{4\pi^2 m}{T_{eq.}^2}$$

In series combination –




$$\frac{1}{K_{eq.}} = \frac{1}{K_1} + \frac{1}{K_2}$$

$$\frac{T_{eq.}^2}{4\pi^2 m} = \frac{T_1^2}{4\pi^2 m} + \frac{T_2^2}{4\pi^2 m}$$

$$T_{eq.} = \sqrt{T_1^2 + T_2^2}$$

**Illustration 19**

Infinite springs with force constants k, 2k, 4k, 8k, ..... respectively are connected in series. Calculate the effective force constant of the spring.

**Solution**

$$\frac{1}{k_{eff}}=\frac{1}{k}+\frac{1}{2k}+\frac{1}{4k}+\frac{1}{8k}+............\infty \quad \text{(For infinite G.P. } S_\infty=\frac{a}{1-r} \text{ where a = First term, r = common ratio)}$$

$$\frac{1}{k_{eff}}=\frac{1}{k}\left[1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+..........\right]=\frac{1}{k}\left[\frac{1}{1-\frac{1}{2}}\right]=\frac{2}{k} \text{ so } k_{eff}=k/2$$

**Illustration 20**

Frequency of oscillation of a body is 6 Hz when force $F_1$ is applied and 8 Hz when $F_2$ is applied. If both forces $F_1$ & $F_2$ are applied together then find out the frequency of oscillation. **[AIPMT 2004]**

**Solution**

According to question $F_1=-K_1x$ & $F_2=-K_2x$

so $n_1=\frac{1}{2\pi}\sqrt{\frac{K_1}{m}}=6\text{Hz}$ ; $n_2=\frac{1}{2\pi}\sqrt{\frac{K_2}{m}}=8\text{ Hz}$

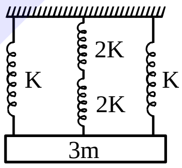
Now $F=F_1+F_2=-(K_1+K_2)x$ Therefore $n=\frac{1}{2\pi}\sqrt{\frac{K_1+K_2}{m}}$

$$\Rightarrow \quad n=\frac{1}{2\pi}\sqrt{\frac{4\pi^2n_1^2m+4\pi^2n_2^2m}{m}}=\sqrt{n_1^2+n_2^2}=\sqrt{8^2+6^2}=10\text{ Hz}$$

**BEGINNER'S BOX-4**

**1.** A man with a wrist watch on his hand falls from the top of a tower. Does the watch give correct time during the free fall ?

**2.** Calculate the time period of following system.




**3.** A body of mass 'm' is suspended from a vertical spring of length 'L'. It execute S.H.M. with a time period T. Find the time period if

(a) Length of spring is made half.

(b) Mass of body is made half.

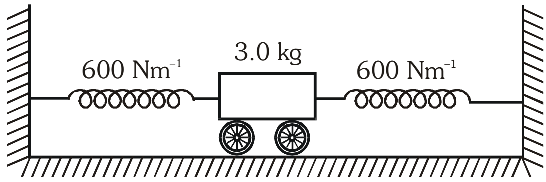
**4.** When a mass of 1 kg is suspended from a vertical spring, its length increases by 0.98 m. If this mass is pulled downwards and then released, what will be periodic time of vibration of spring ? ($g=9.8\ m/s^2$).

**5.** A block of mass 'm' moving with velocity (v) collides perfectly inelastically with another identical block attached to spring of force constant K. What will be the amplitude of resulting SHM ?

**6.** A spring having a spring constant 1200 $Nm^{-1}$ is mounted on a horizontal table. A mass of 3kg is attached to the free end of the spring. The mass is then pulled to a distance of 2.0 cm and released. Determine Maximum acceleration of mass.

**7.** A spring balance has a scale that reads from 0 to 50 kg. The length of the scale is 20 cm. A body suspended from this balance, when displaced and released, oscillates with a period of 0.6 s. What is the weight of the body?

**8.** A trolley of mass 3.0 kg, as shown in Fig., is connected to two springs, each of spring has spring constant 600 $Nm^{-1}$. If the trolley is displaced from its equilibrium position by 5.0 cm and released, what is (a) the period of ensuing oscillations, and (b) the maximum speed of the trolley ? How much energy is dissipated as heat by the time the trolley comes to rest due to damping forces?



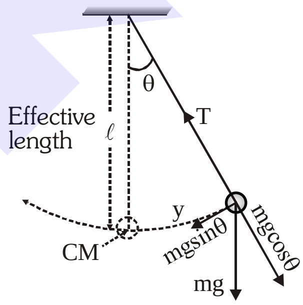

## 5. SIMPLE PENDULUM

If a heavy point mass is suspended by a weightless, inextensible and perfectly flexible string from a rigid support, then this arrangement is called a simple pendulum

**Expression for time period**

**Method-I (Force Method)**

For small angular displacement, $\sin\theta \approx \theta$, so that




$F = -mg\sin\theta$

$= -mg\,\theta$

$= -\left(\dfrac{mg}{\ell}\right)y$

$= -ky \quad \Rightarrow \quad K = \dfrac{mg}{\ell}$

(because $y = \ell\theta$), Thus, the time period of the simple pendulum is

$$T = 2\pi\sqrt{\frac{m}{k}} \Rightarrow \quad \text{or } T = 2\pi\sqrt{\frac{\ell}{g}}$$

**Method-II (Torque Method)**

Bob of pendulum moves along the arc of circle in vertical plane. Here motion involved is angular and oscillatory where restoring torque is provided by gravitational force.

$\tau = -(mg)(\ell\sin\theta)$ (Negative sign shows opposite direction of $\tau$ and angular displacement)

$\tau = -mg\ell\theta$ (If angular displacement is small, then $\sin\theta \approx \theta$)

$I\alpha = -mg\ell\theta$

$\alpha = -\dfrac{mg\ell}{I}\theta$ (where I = moment of inertia of bob about point of suspension so $I = m\ell^2$)

Now $\alpha = -\dfrac{mg\ell}{m\ell^2}\theta \Rightarrow \alpha = -\dfrac{g}{\ell}\theta \Rightarrow \dfrac{d^2\theta}{dt^2} + \dfrac{g}{\ell}\theta = 0$

It is differential equation of angular SHM of simple pendulum.

Comparing with standard differential equation of angular SHM $\left(\dfrac{d^2\theta}{dt^2} + \omega^2\theta = 0\right)$ :-

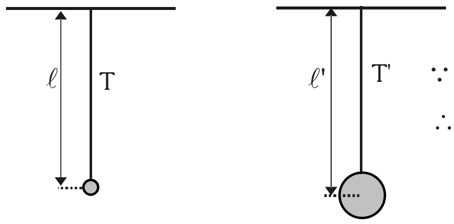
Therefore $\quad T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{\ell}{g}}$

**Note :** Simple pendulum is the example of SHM but only when its angular displacement is very small.

**Important points :**

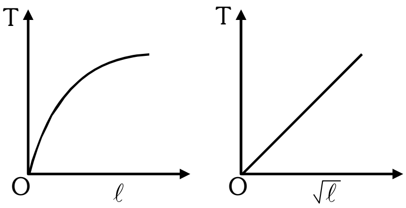
**1.** The time period of simple pendulum is independent from mass of the bob but it depends on size of bob (position of centre of mass). So in simple pendulum when a solid iron bob is replaced by light aluminium bob of same radius then time period remains unchanged.

**2.** Time period of simple pendulum is directly proportional to square root of length.




$\because \ \ell' > \ell$

$\therefore \ T' > T$




When a person sitting on an oscillating swing comes in standing position then centre of mass raises upwards and length decreases, so time period decreases and frequency increases means swing oscillates faster.

**3.** When a hollow spherical bob of simple pendulum is completely filled with water and a small hole is made in bottom of it, then as water drainout, at first its time period increases, after that it decrease and when sphere becomes empty then finally it becomes as before (T).

**4.** If simple pendulum is shifted to poles, equator or hilly areas, then its time period may be different $\left(T \propto \frac{1}{\sqrt{g}}\right)$

**5.** If a clock based on oscillation of simple pendulum is shifted from earth to moon then it becomes slow because its time period increases and becomes $\sqrt{6}$ times compare to earth. $\frac{g_M}{g_E} = \frac{1}{6} \Rightarrow T_M = \sqrt{6}T_E$

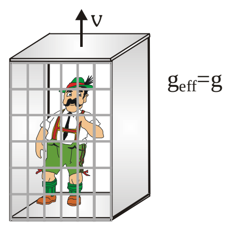
**6. Periodic time of simple pendulum in reference system**

$$T = 2\pi\sqrt{\frac{\ell}{g_{eff}}}$$

where, $g_{eff.}$ = effective gravity acceleration in reference system

or total downward acceleration.

**(a)** If reference system is lift

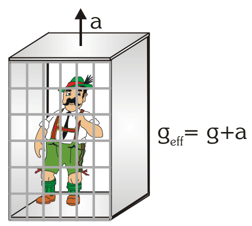



**(i)** If velocity of lift v = constant

acceleration a = 0 and $g_{eff.} = g$

$$\therefore \quad T = 2\pi\sqrt{\frac{\ell}{g}}$$

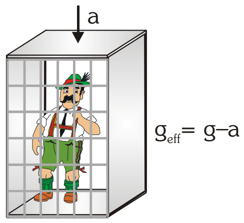
**(ii)** If lift is moving upwards with acceleration a




$g_{eff.} = g + a$

$$T = 2\pi\sqrt{\frac{\ell}{g+a}} \quad \Rightarrow \quad T \text{ decreases}$$

**(iii)** If lift is moving downwards with acceleration a




$$g_{eff.} = g - a$$

$$\therefore \quad T = 2\pi\sqrt{\frac{\ell}{g-a}} \quad \Rightarrow \quad \text{T increases}$$

**(iv)** If lift falls downwards freely

$$g_{eff.} = g - g = 0 \quad \Rightarrow \quad T = \infty \quad \text{simple pendulum will not oscillate}$$

If simple pendulum is shifted to the centre of earth, freely falling lift, in artificial satellite then it will not oscillate and its time period is infinite ($\because g_{eff.} = 0$).

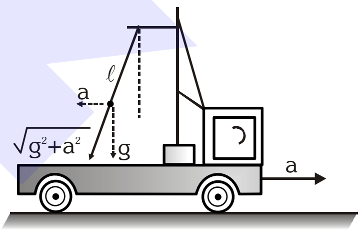
**(b)** A simple pendulum is mounted on a moving truck

**(i)** If truck is moving with constant velocity, no pseudo force acts on the pendulum and time period remains same $T = 2\pi\sqrt{\frac{\ell}{g}}$

**(ii)** If truck accelerates forward with acceleration f then a pseudo force acts in opposite direction.

So effective acceleration, $g_{eff.} = \sqrt{g^2 + a^2}$ and $T' = 2\pi\sqrt{\frac{\ell}{g_{eff.}}}$

Time period $T' = 2\pi\sqrt{\frac{\ell}{\sqrt{g^2+a^2}}} \Rightarrow$ T' decreases

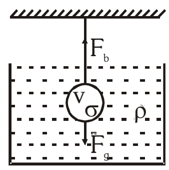



**7.** If a simple pendulum of density σ is made to oscillate in a liquid of density ρ then its time period will increase as compare to that of air and is given by

$$F_{net} = F_g - F_b$$

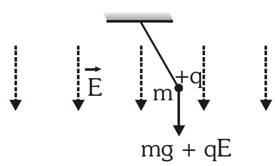
$$\frac{mg_{net}}{m} = \frac{mg}{m} - \frac{V\rho g}{m}, \qquad g_{net} = g - \frac{V\rho g}{V\sigma} = g\left(1 - \frac{\rho}{\sigma}\right)$$

$$T = 2\pi\sqrt{\frac{\ell}{\left[1 - \frac{\rho}{\sigma}\right]g}}$$



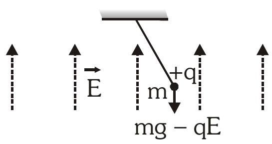

**8.** **(a)** If the bob of simple pendulum has positive charge q and pendulum is placed in uniform electric field which is in downward direction then time period decreases

$$T = 2\pi\sqrt{\frac{\ell}{g + \frac{qE}{m}}}$$




**(b)** If the bob of simple pendulum has positive charge q and is made is oscillate in uniform electric field acting in upward direction then time period increases

$$T = 2\pi\sqrt{\frac{\ell}{g - \frac{qE}{m}}}$$



**9.** $T = 2\pi\sqrt{\frac{\ell}{g}}$ is valid when length of simple pendulum ($\ell$) is negligible as compare to radius of earth ($\ell << R$)

but if $\ell$ is comparable to radius of earth

then time period $T = 2\pi\sqrt{\frac{R}{\left[1+\frac{R}{\ell}\right]g}} = 2\pi\sqrt{\frac{\ell}{g\left[1+\frac{\ell}{R}\right]}} = 2\pi\sqrt{\frac{1}{\left[\frac{1}{\ell}+\frac{1}{R}\right]g}}$

The time period of oscillation of simple pendulum of infinite length

$T = 2\pi\sqrt{\frac{R}{g}} \simeq 84.6$ minute $\approx 1\frac{1}{2}$ hour     It is maximum time period.

**10. Second's pendulum**

If the time period of a simple pendulum is 2 second then it is called second's pendulum. Second's pendulum take one second to go from one extreme position to other extreme position.

For second's pendulum, time period $T = 2 = 2\pi\sqrt{\frac{\ell}{g}}$

At the surface of earth $g = 9.8\ m/s^2 \approx \pi^2 m/s^2$,

So length of second pendulum at the surface of earth $\ell \approx 1$ metre

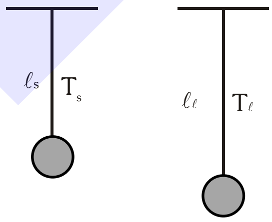
**11.** When a long and short pendulum start oscillation simultaneously then both will be in same phase in minimum time when short pendulum complete one more oscillation compare to long pendulum.

After starting, if long pendulum completes N oscillation to come in same phase in minimum time then short will complete (N+1) oscillation.

$t = NT_\ell = (N+1)T_S$

$N\left(2\pi\sqrt{\frac{\ell_\ell}{g}}\right) = (N+1)\left(2\pi\sqrt{\frac{\ell_s}{g}}\right)$

$\boxed{N\sqrt{\ell_\ell} = (N+1)\sqrt{\ell_s}}$




## GOLDEN KEY POINTS

- Simple pendulum performs angular S.H.M. but due to small angular displacement, it is considered as linear S.H.M.
- If time period of clock based on simple pendulum increases then clock will be slow and if time period decreases then clock will be fast.
- If $\Delta\ell$ is change in length and $\Delta g$ is the change in acceleration then for small variation (up to 5%) change in time period ($\Delta T$) will be

$$\frac{\Delta T}{T}\times 100 = \left[\frac{1}{2}\frac{\Delta\ell}{\ell} - \frac{1}{2}\frac{\Delta g}{g}\right]\times 100$$

- Due to change in shape of earth (not spherical but elliptical) gravitational acceleration is different at different places. So time period of simple pendulum varies with variation of g.
- The time period of simple pendulum is independent of mass of bob.

## Illustrations

**Illustration 21**

A simple pendulum is suspended from the ceiling of a lift. When the lift is at rest, its time period is T. With what acceleration should lift be accelerated upwards in order to reduce its time period to $\frac{T}{2}$?

**Solution**

In stationary lift $T = 2\pi\sqrt{\frac{\ell}{g}}$ ...(i)

In accelerated lift $\frac{T}{2} = T' = 2\pi\sqrt{\frac{\ell}{g+a}}$ ...(ii)

Divide (i) by (ii) $2 = \sqrt{\frac{g+a}{g}}$ or $g + a = 4g$ or $a = 3g$

**Illustration 22**

The length of a second's pendulum at the surface of earth is 1m. Determine the length of second's pendulum at the surface of moon.

**Solution**

For second's pendulum at the surface of earth $2 = 2\pi\sqrt{\frac{\ell_e}{g_e}}$ ...(i)

For second's pendulum at the surface of moon $2 = 2\pi\sqrt{\frac{\ell_m}{g_m}}$ ...(ii)

From (i) and (ii) $\frac{\ell_e}{g_e} = \frac{\ell_m}{g_m}$ $\Rightarrow$ $\ell_m = \left[\frac{g_m}{g_e}\right]\ell_e$ $\Rightarrow$ $\ell_m = \frac{\ell_e}{6}$ $\left[\because \quad g_m = \frac{g_e}{6}\right]$

$\Rightarrow$ $\ell_m = \frac{1}{6}m$

**Illustration 23**

If length of a simple pendulum is increased by 4%. Then determine percentage change in time period.

**Solution**

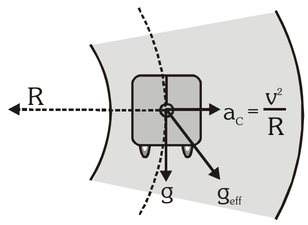
Percentage change in time period $\frac{\Delta T}{T}\times 100\% = \frac{1}{2}\frac{\Delta\ell}{\ell}\times 100$ $[\because \quad \Delta g = 0]$

According to question $\frac{\Delta\ell}{\ell}\times 100 = 4\%$ $\therefore \frac{\Delta T}{T}\times 100\% = \frac{1}{2}\times 4\% = 2\%$

**Illustration 24**

A simple pendulum of length L and mass M is suspended in a car. The car is moving on a circular track of radius R with a uniform speed v. If the pendulum makes oscillation in a radial direction about its equilibrium position, then calculate its time period .

**Solution**




Centripetal acceleration $a_c = \frac{v^2}{R}$ & Acceleration due to gravity = g

So $g_{eff} = \sqrt{g^2 + \left(\frac{v^2}{R}\right)^2}$ $\Rightarrow$ Time period $T = 2\pi\sqrt{\frac{L}{g_{eff}}} = 2\pi\sqrt{\frac{L}{\sqrt{g^2 + \frac{v^4}{R^2}}}}$

**BEGINNER'S BOX-5**

**1.** The angle made by the string of a simple pendulum with the vertical depends upon time as $\theta = \frac{\pi}{90} \sin \pi t$. Find the length of the pendulum if $g = \pi^2 \, ms^{-1}$

**2.** The bob of a simple pendulum is a hollow sphere filled with water. Explain how will the period of oscillation change if the water begins to drain out of the hollow sphere through a small hole in the bottom ?

**3.** A girl is swinging on the swing in the sitting position. What shall be effect on the frequency of oscillation if she stands up ?

**4.** Why a pendulum clock does not work during free fall or in an artificial satellite ?

**5.** Find the time period and frequency of a simple pendulum of 1·000 m at a location where $g = 9{\cdot}800 \ m/s^2$.

**6.** What is the length of a simple pendulum whose time period of oscillation for small amplitudes equals 2·0 seconds

**7.** What is the frequency of oscillation of a simple pendulum mounted in a cabin that is freely falling under gravity ?

---

## 6. DIFFERENT TYPES OF OSCILLATIONS

**Different types of oscillations**

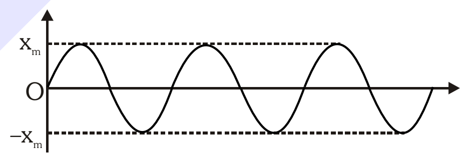
**Free, Damped, Forced oscillations and Resonance**

**(a) Free oscillation**

(i) The oscillations of a particle with fundamental frequency under the influence of restoring force are defined as free oscillations.

(ii) The amplitude, frequency and energy of oscillations remain constant.

(iii) The oscillator which keeps on oscillating with constant amplitude for infinite time is known as free oscillator.




**(b) Damped oscillations :**

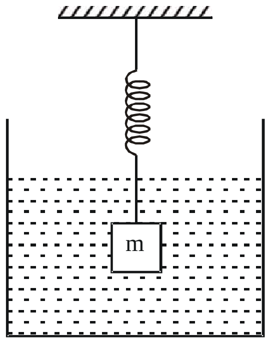
(i) The oscillations in which the amplitude decreases gradually with the passage of time are called damped oscillations.

(ii) In many real systems, non-conservative forces such as friction retard the motion. Consequently the mechanical energy of the system diminishes in time, and the motion is said to be damped. The lost mechanical energy is transformed into internal energy of the object & the retarding medium.

(iii) The retarting force can be expressed as $\vec{F} = -b\vec{v}$ (where b is a constant called the damping coefficient) and restoring force on the system is –kx, we can write Newton's second law as

$$\Sigma f_x = -kx - bv = ma$$

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$




This is the differential equation of damped oscillation, solution of this equation is given by

$$x = Ae^{(-b/2m)t}\cos(\omega' t + \phi)$$

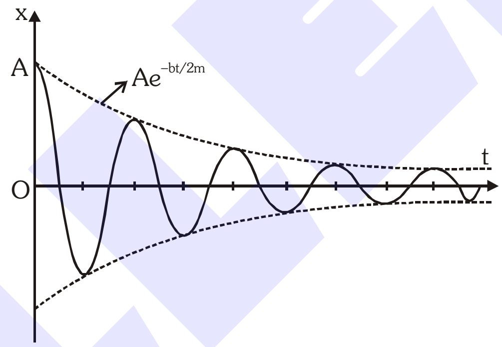
where angular frequency of oscillation is

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = \sqrt{\omega^2 - \left(\frac{b}{2m}\right)^2}$$

where $\omega = \sqrt{\frac{k}{m}}$ represents the angular frequency in the absence of retarting force (the undamped oscillator) & is called natural frequency.

**(c) Effect of damped oscillation :**

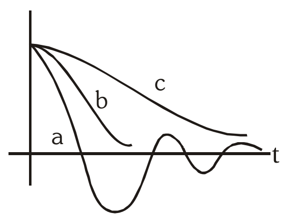
(i) When the retarding force is small, the oscillatory character of the motion is preserved but amplitude decreases in time, and it decays exponentially with time




(a) When the magnitude of retarding force is small such that $b/2m < \omega$, the system is said to be underdamped

(b) When b reaches a critical value $b_c$ such that $\frac{b_c}{2m} = \omega$, then system does not oscillate & is said to be critically damped.

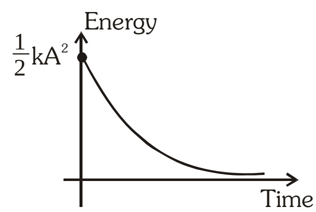
(c) When retarding force is large as compared to restoring force, i.e., $\frac{b}{2m} > \omega$ then system is overdamped.




a = underdamped oscillator
b = critically damped oscillator
c = overdamped oscillator

(ii) Mechanical energy of undamped oscillator is $\frac{1}{2}kA^2$. For a damped oscillator amplitude is not constant but depend on time, so total energy is

$$E(t) = \frac{1}{2}k(Ae^{-bt/2m})^2 = \frac{1}{2}kA^2e^{-bt/m}$$



**(d) Forced oscillations :**

(i) All free oscillations eventually die out because of ever present damping force, However, an external agency can maintain these oscillations. These are called forced or driven oscillations.

(ii) Under forced periodic oscillation, system does not oscillate with its natural frequency ($\omega$) but with driven frequency ($\omega_d$).

(iii) Suppose an external force F(t) of amplitude $F_0$ that varies periodically with time is applied to a damped oscillator. Such a force is $F(t) = F_0 \cos \omega_d t$

The equation of particle under combined force is

$$ma(t) = -kx - bv + F_0 \cos \omega_d t$$

$$\frac{md^2x}{dt^2} + \frac{bdx}{dt} + kx = F_0 \cos \omega_d t$$

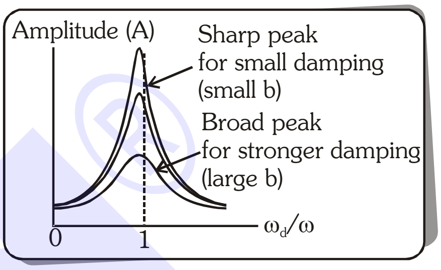
After solving

$$x = A' \cos(\omega_d t + \phi)$$

where

$$A' = \frac{F_0}{m\sqrt{(\omega_d^2 - \omega^2)^2 + \left(\frac{b\omega_d}{m}\right)^2}}$$

($\omega$ is natural frequency, $\omega = \sqrt{\frac{k}{m}}$)




(iv) The amplitude of driven oscillator decreases due to damping forces but on account of the energy gained from external source (driver) it remains constant.

(v) The amplitude of forced vibration is determined by the difference between the frequency of applied force & the natural frequency. If difference between frequency is small then amplitude will be large.

**Resonance :**

(i) For small damping, the amplitude is large when the frequency of the driving force is near the natural frequency of oscillation ($\omega_d \approx \omega$). The increase in amplitude near the natural frequency is called resonance, & the natural frequency $\omega$ is also called the resonance frequency of the system.

(ii) In the state of resonance, there occurs maximum transfer of energy from the driver to the driven.

**GOLDEN KEY POINTS**

- When a tunning fork is struck against a rubber pad, the prongs begin to execute free vibration.
- When the stem of vibrating tunning fork is pressed against the top of tabla, then the tabla will suffer forced vibration.
- Soldiers are asked to break step while crossing a bridge, If the soldiers march in step, there is a possibility that the frequency of the foot steps may match the natural frequency of the bridge. Due to resonance, the bridge may start oscillating violently, thereby damaging itself.
- The oscillations of simple pendulum in air are damped oscillations.

## Illustrations

**Illustration 25**

The amplitude of a damped oscillator becomes half in one minute. The amplitude after 3 minutes will be $\frac{1}{x}$ times of the original. Determine the value of x.

**Solution**

Amplitude of damped oscillation is $A = A_0e^{-\gamma t}$ [from $x = x_me^{-\gamma t}$]

at t = 1 min $A = \frac{A_0}{2}$ so $\frac{A_0}{2} = A_0e^{-\gamma}$ or $e^{\gamma} = 2$

After 3 minutes $A = \frac{A_0}{x}$ so $\frac{A_0}{x} = A_0e^{-\gamma\times3}$ or $x = e^{3\gamma} = (e^{\gamma})^3 = 2^3 = 8$

**Illustration 26**

In damped oscillations, the amplitude after 50 oscillations is 0.8 $a_0$, where $a_0$ is the initial amplitude, then determine amplitude after 150 oscillations **[AIPMT 2008]**

**Solution**

The amplitude, a, at time t is given by $a = a_0 \exp(-\alpha t)$

$a_{50} = a_0 \exp(-\alpha \times 50T) = 0.80\ a_0$ where T is the period of oscillation

$a_{150} = a_0 \exp(-\alpha \times 150T) = a_0\ (0.8)^3 = 0.512\ a_0$

**Illustration 27**

Which agency provides the restoring force in the following case ?

(a) A spring compressed and then free to vibrate.

(b) Water pressed in U tube and then left free to vibrate.

(c) Ball released in a diametric tunnel of earth.

(d) Pendulum pulled from its mean position and released.

**Solution**

(a) Elasticity of spring

(b) Hydrostatic pressure of water

(c) Gravitational force of earth.

(d) Weight of bob of pendulum

# ANSWER KEY

## BEGINNER'S BOX-1

**1.** (i) – (a), (ii) – (b), (iii) – (c), (iv) – (c), (v) – (a), (vi) – (a), (vii) – (b)

**2.** 0.5 s

**3.** (a) 0.67 s, (b) $\frac{\pi}{4}$rad , (c) 0.04 metre

## BEGINNER'S BOX-2

**1.** 0.025m, y = 0.025 sin (40πt) m

**2.** $\frac{\sqrt{3}}{8}$ ms$^{-1}$ **3.** $\sqrt{2}$ cm and $\frac{\pi}{4}$ rad

**4.** 9.52 cm **5.** (a) $\frac{1}{3}$ s, (b) $\frac{2}{3}$s

**6.** $\frac{5T}{12}$ **7.** $5\sqrt{3}$ m,30 m/s

**8.** (a) When particle moves from extreme to mean position

(b) When particle moves from mean to extreme position

**9.** (a) 0, +, +; (b) 0, –, –; (c) –, 0, 0;

(d) –, –, – ; (e) +, +, +; (f) –, +, +

**10.** 100 m/min

## BEGINNER'S BOX-3

**1.** $Y = 0.1\sin\left[4t + \frac{\pi}{4}\right]$

**2.** $\frac{15}{16}$ and $\frac{1}{16}$ **3.** 2f **4.** $\frac{5}{\pi}$ s$^{-1}$

## BEGINNER'S BOX-4

**1.** watch will give correct time because it depends on spring action and does not depend on gravity.

**2.** $2\pi\sqrt{\frac{m}{K}}$ **3.** (a) $T' = \frac{T}{\sqrt{2}}$, (b) $T' = \frac{T}{\sqrt{2}}$

**4.** 2s **5.** $A = v\sqrt{\frac{m}{2K}}$ **6.** 8 m/s$^2$

**7.** 218.7 N **8.** (a) 0.314 s, (b) 1 m/s, 1.5 J

## BEGINNER'S BOX-5

**1.** 1 m

**2.** The time period will increase at first, then decrease until the sphere is empty acquire its initial value.

**3.** increase

**4.** In both cases effective acceleration due to gravity becomes zero. In absence of '$g_{eff}$' there is no restoring force and pendulum does not oscillate.

**5.** 2 s, 0.5 Hz

**6.** 0·99 m

**7.** zero