

Derivation of the recursive pricing equation for Galí (2015), Chapter 3

1 Aggregation

$$N_t = \left(\frac{Y_t}{A_t} \right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\frac{\varepsilon}{1-\alpha}} di \quad (1)$$

From this follows

$$Y_t = A_t \left(\frac{N_t}{\int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\frac{\varepsilon}{1-\alpha}} di} \right)^{1-\alpha} \quad (2)$$

Define the term in the denominator as S_t we get

$$N_{t+k} = \left(\frac{Y_{t+k}}{A_{t+k}} \right)^{\frac{1}{1-\alpha}} S_{t+k} \quad (3)$$

where

$$S_t = \int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\frac{\varepsilon}{1-\alpha}} di \quad (4)$$

$$= (1-\theta) \left(\frac{P_t^*}{P_t} \right)^{-\frac{\varepsilon}{1-\alpha}} + \int_{S(i)} \left(\frac{P_{t-1}(i)}{P_t} \right)^{-\frac{\varepsilon}{1-\alpha}} di \quad (5)$$

$$= (1-\theta) \left(\frac{P_t^*}{P_t} \right)^{-\frac{\varepsilon}{1-\alpha}} + \int_{S(i)} \left(\frac{P_{t-1}}{P_t} \frac{P_{t-1}(i)}{P_{t-1}} \right)^{-\frac{\varepsilon}{1-\alpha}} di \quad (6)$$

$$= (1-\theta) \left(\frac{P_t^*}{P_t} \right)^{-\frac{\varepsilon}{1-\alpha}} + \Pi_t^{\frac{\varepsilon}{1-\alpha}} \int_{S(i)} \left(\frac{P_{t-1}(i)}{P_{t-1}} \right)^{-\frac{\varepsilon}{1-\alpha}} di \quad (7)$$

$$= (1-\theta) \left(\frac{P_t^*}{P_t} \right)^{-\frac{\varepsilon}{1-\alpha}} + \theta \Pi_t^{\frac{\varepsilon}{1-\alpha}} S_{t-1} \quad (8)$$

From the demand function follows:

$$A_{t+k} N_{t+k|t}^{1-\alpha} = Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} Y_{t+k} \quad (9)$$

which implies

$$N_{t+k|t} = \left(\left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} \frac{Y_{t+k}}{A_{t+k}} \right)^{\frac{1}{1-\alpha}} \quad (10)$$

Therefore

$$\frac{N_{t+k|t}}{N_{t+k}} = \frac{\left(\left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} \frac{Y_{t+k}}{A_{t+k}} \right)^{\frac{1}{1-\alpha}}}{\left(\frac{Y_{t+k}}{A_{t+k}} \right)^{\frac{1}{1-\alpha}} S_t} = \frac{\left(\frac{P_t^*}{P_{t+k}} \right)^{\frac{-\varepsilon}{1-\alpha}}}{S_t} \quad (11)$$

Marginal costs are given by:

$$\begin{aligned} MC_{t+k|t} &= \frac{W_{t+k}}{\frac{(1-\alpha)Y_{t+k|t}}{N_{t+k|t}}} = \frac{W_{t+k}}{(1-\alpha)A_{t+k}N_{t+k|t}^{-\alpha}} = \frac{W_{t+k}}{(1-\alpha)A_{t+k}N_{t+k}^{-\alpha}} \frac{N_{t+k}^{-\alpha}}{N_{t+k|t}^{-\alpha}} \\ &= \frac{W_{t+k}}{(1-\alpha)A_{t+k}N_{t+k}^{-\alpha}} \left(\frac{\left(\frac{P_t^*}{P_{t+k}} \right)^{\frac{-\varepsilon}{1-\alpha}}}{S_{t+k}} \right)^{\alpha} = MC_{t+k} \left(\frac{P_t^*}{P_{t+k}} \right)^{\frac{-\alpha\varepsilon}{1-\alpha}} \end{aligned}$$

where

$$MC_t = \frac{W_t}{(1-\alpha)A_tN_t^{-\alpha}S_t^{\alpha}} = \frac{W_{t+k}}{(1-\alpha)\frac{Y_t}{N_t}S_t}$$

2 Pricing FOC

The pricing FOC is given by

$$\sum_{k=0}^{\infty} \theta^k E_t \left[\Lambda_{t,t+k} Y_{t+k|t} \frac{1}{P_{t+k}} \left(P_t^* - \frac{\varepsilon}{\varepsilon-1} MC_{t+k|t} P_{t+k} \right) \right] = 0 \quad (12)$$

where $MC_{t+k|t}$ is real marginal costs. Using the demand function, this is

$$\sum_{k=0}^{\infty} \theta^k E_t \left[\Lambda_{t,t+k} \left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} Y_{t+k} \frac{1}{P_{t+k}} \left(P_t^* - \frac{\varepsilon}{\varepsilon-1} MC_{t+k|t} P_{t+k} \right) \right] = 0 \quad (13)$$

or

$$\sum_{k=0}^{\infty} \theta^k E_t \left[\Lambda_{t,t+k} \left(\frac{P_t}{P_{t+k}} \right)^{-\varepsilon} \left(\frac{P_t^*}{P_t} \right)^{-\varepsilon} Y_{t+k} \frac{P_t}{P_{t+k}} \left(\frac{P_t^*}{P_t} - \frac{\varepsilon}{\varepsilon-1} MC_{t+k|t} \frac{P_{t+k}}{P_t} \right) \right] = 0 \quad (14)$$

Now plug in for the stochastic discount factor $\Lambda_{t,t+k}$:

$$\sum_{k=0}^{\infty} \theta^k E_t \left[\beta^k \frac{C_{t+k}^{-\sigma}}{C_t^{-\sigma}} \left(\frac{P_t}{P_{t+k}} \right)^{1-\varepsilon} \left(\frac{P_t^*}{P_t} \right)^{-\varepsilon} Y_{t+k} \left(\frac{P_t^*}{P_t} - \frac{\varepsilon}{\varepsilon-1} MC_{t+k|t} \frac{P_{t+k}}{P_t} \right) \right] = 0 \quad (15)$$

Next, multiply by $C_t^{-\sigma}$ and collect terms:

$$\sum_{k=0}^{\infty} \theta^k E_t \left[\beta^k C_{t+k}^{-\sigma} \left(\frac{P_t}{P_{t+k}} \right)^{1-\varepsilon} \left(\frac{P_t^*}{P_t} \right)^{-\varepsilon} Y_{t+k} \left(\frac{P_t^*}{P_t} - \frac{\varepsilon}{\varepsilon-1} MC_{t+k|t} \frac{P_{t+k}}{P_t} \right) \right] = 0 \quad (16)$$

Using

$$\Pi_t^* = \frac{P_t^*}{P_t} \quad (17)$$

we can write this as

$$\begin{aligned} \sum_{k=0}^{\infty} \theta^k E_t \left[\beta^k C_{t+k}^{-\sigma} \left(\frac{P_t}{P_{t+k}} \right)^{1-\varepsilon} (\Pi_t^*)^{1-\varepsilon} Y_{t+k} \right] = \\ \sum_{k=0}^{\infty} \theta^k E_t \left[\beta^k C_{t+k}^{-\sigma} \left(\frac{P_t}{P_{t+k}} \right)^{-\varepsilon} (\Pi_t^*)^{-\varepsilon} Y_{t+k} \frac{\varepsilon}{\varepsilon - 1} MC_{t+k|t} \right] \end{aligned} \quad (18)$$

Now replace idiosyncratic by average marginal costs:

$$\begin{aligned} \sum_{k=0}^{\infty} \theta^k E_t \left[\beta^k C_{t+k}^{-\sigma} \left(\frac{P_t}{P_{t+k}} \right)^{1-\varepsilon} (\Pi_t^*)^{1-\varepsilon} Y_{t+k} \right] = \\ \frac{\varepsilon}{\varepsilon - 1} \sum_{k=0}^{\infty} \theta^k E_t \left[\beta^k C_{t+k}^{-\sigma} \left(\frac{P_t}{P_{t+k}} \right)^{-\varepsilon} (\Pi_t^*)^{-\varepsilon} Y_{t+k} MC_{t+k} \left(\frac{P_t^*}{P_{t+k}} \right)^{\frac{-\alpha\varepsilon}{1-\alpha}} \right] \end{aligned} \quad (19)$$

Expand the fraction to get Π_t^* :

$$\begin{aligned} \sum_{k=0}^{\infty} \theta^k E_t \left[\beta^k C_{t+k}^{-\sigma} \left(\frac{P_t}{P_{t+k}} \right)^{1-\varepsilon} (\Pi_t^*)^{1-\varepsilon} Y_{t+k} \right] = \\ \frac{\varepsilon}{\varepsilon - 1} \sum_{k=0}^{\infty} \theta^k E_t \left[\beta^k C_{t+k}^{-\sigma} \left(\frac{P_t}{P_{t+k}} \right)^{-\varepsilon} (\Pi_t^*)^{-\varepsilon} Y_{t+k} MC_{t+k} \left(\frac{P_t}{P_{t+k}} \right)^{\frac{-\alpha\varepsilon}{1-\alpha}} \left(\frac{P_t^*}{P_t} \right)^{\frac{-\alpha\varepsilon}{1-\alpha}} \right] \end{aligned} \quad (20)$$

and bring Π_t^* to the left:

$$\begin{aligned} \sum_{k=0}^{\infty} \theta^k E_t \left[\beta^k C_{t+k}^{-\sigma} \left(\frac{P_t}{P_{t+k}} \right)^{1-\varepsilon} (\Pi_t^*)^{1+\frac{\alpha\varepsilon}{1-\alpha}} Y_{t+k} \right] = \\ \frac{\varepsilon}{\varepsilon - 1} \sum_{k=0}^{\infty} \theta^k E_t \left[\beta^k C_{t+k}^{-\sigma} Y_{t+k} MC_{t+k} \left(\frac{P_t}{P_{t+k}} \right)^{-\varepsilon - \frac{\alpha\varepsilon}{1-\alpha}} \right] \end{aligned} \quad (21)$$

Factoring out, we get

$$\begin{aligned} (\Pi_t^*)^{1+\frac{\alpha\varepsilon}{1-\alpha}} \sum_{k=0}^{\infty} \theta^k E_t \left[\beta^k C_{t+k}^{-\sigma} \left(\frac{P_t}{P_{t+k}} \right)^{1-\varepsilon} Y_{t+k} \right] = \\ \frac{\varepsilon}{\varepsilon - 1} \sum_{k=0}^{\infty} \theta^k E_t \left[\beta^k C_{t+k}^{-\sigma} Y_{t+k} MC_{t+k} \left(\frac{P_t}{P_{t+k}} \right)^{-\varepsilon - \frac{\alpha\varepsilon}{1-\alpha}} \right] \end{aligned} \quad (22)$$

which, defining two auxiliary variables we write this as:

$$(\Pi_t^*)^{1+\frac{\alpha\varepsilon}{1-\alpha}} x_{2,t} = \frac{\varepsilon}{\varepsilon - 1} x_{1,t} \quad (23)$$

where

$$x_{2,t} = E_t \sum_{k=0}^{\infty} \theta^k \left[\beta^k C_{t+k}^{-\sigma} \left(\frac{P_t}{P_{t+k}} \right)^{1-\varepsilon} Y_{t+k} \right] \quad (24)$$

$$= C_t^{-\sigma} Y_t + E_t \sum_{k=1}^{\infty} \theta^k \left[\beta^k C_{t+k}^{-\sigma} \left(\frac{P_t}{P_{t+k}} \right)^{1-\varepsilon} Y_{t+k} \right] \quad (25)$$

$$= C_t^{-\sigma} Y_t + E_t \left(\frac{P_t}{P_{t+1}} \right)^{1-\varepsilon} \sum_{k=1}^{\infty} \theta^k \left[\beta^k C_{t+k}^{-\sigma} \left(\frac{P_{t+1}}{P_{t+k}} \right)^{1-\varepsilon} Y_{t+k} \right] \quad (26)$$

$$= C_t^{-\sigma} Y_t + \beta \theta E_t \Pi_{t+1}^{\varepsilon-1} x_{2,t+1} \quad (27)$$

and

$$x_{1,t} = \sum_{k=0}^{\infty} \theta^k E_t \left[\beta^k C_{t+k}^{-\sigma} Y_{t+k} M C_{t+k} \left(\frac{P_t}{P_{t+k}} \right)^{-\varepsilon - \frac{\alpha\varepsilon}{1-\alpha}} \right] \quad (28)$$

$$= C_t^{-\sigma} M C_t Y_t + E_t \sum_{k=1}^{\infty} \theta^k \left[\beta^k C_{t+k}^{-\sigma} Y_{t+k} M C_{t+k} \left(\frac{P_t}{P_{t+k}} \right)^{-\varepsilon - \frac{\alpha\varepsilon}{1-\alpha}} \right] \quad (29)$$

$$= C_t^{-\sigma} M C_t Y_t + E_t \left(\frac{P_t}{P_{t+1}} \right)^{-\varepsilon - \frac{\alpha\varepsilon}{1-\alpha}} \sum_{k=1}^{\infty} \theta^k \left[\beta^k C_{t+k}^{-\sigma} Y_{t+k} M C_{t+k} \left(\frac{P_{t+1}}{P_{t+k}} \right)^{-\varepsilon - \frac{\alpha\varepsilon}{1-\alpha}} \right] \quad (30)$$

$$= C_t^{-\sigma} M C_t Y_t + \beta \theta E_t \Pi_{t+1}^{\varepsilon + \frac{\alpha\varepsilon}{1-\alpha}} x_{1,t+1} \quad (31)$$

$$= C_t^{-\sigma} M C_t Y_t + \beta \theta E_t \Pi_{t+1}^{\frac{\varepsilon}{1-\alpha}} x_{1,t+1} \quad (32)$$