# Analytical derivations for the average productivity of exporters $\tilde{z}_{X,t}$ in Ghironi and Melitz (2005, QJE)

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March 6, 2018

## Method 1

The first method makes use of the definitions for $\tilde{d}_{X,t}$ and $\tilde{\rho}_{X,t}$ on page 875. We are told that:

$$\tilde{d}_{X,t} = d_{X,t}(\tilde{z}_{X,t}) \quad \text{and}$$
$$\tilde{\rho}_{X,t} = \rho_{X,t}(\tilde{z}_{X,t})$$

From page 873, profits of an exporting firm are given by:

$$d_{X,t}(z) = \frac{Q_t}{\theta}(\rho_{X,t}(z))^{1-\theta} C_t^* - \frac{w_t f_{X,t}}{Z_t} \tag{1}$$

Given the Pareto distribution for $G(z)$, we are told in page 876 that:

$$d_{X,t}(z_{X,t}) = 0 \quad \Leftrightarrow \quad \tilde{d}_{X,t} = (\theta - 1)\left(\frac{\nu^{\theta-1}}{k}\right)\frac{w_t f_{X,t}}{Z_t}$$

Using the fact that $\nu \equiv \left(\frac{k}{k-(\theta-1)}\right)^{\frac{1}{\theta-1}}$, we get:

$$\tilde{d}_{X,t} = \left(\frac{\theta - 1}{k - \theta + 1}\right)\frac{w_t f_{X,t}}{Z_t} \tag{2}$$

Equating (2) with (1) for $\rho(\tilde{z}_{X,t})$, and using the definition for $\rho_{X,t}(z)$ from page 873 we get:

$$\left(\frac{\theta - 1}{k - \theta + 1}\right)\frac{w_t f_{X,t}}{Z_t} = \frac{Q_t}{\theta}\left[\frac{\tau_t}{Q_t}\left(\frac{\theta}{\theta - 1}\right)\frac{w_t}{Z_t \tilde{z}_{X,t}}\right]^{1-\theta} C_t^* - \frac{w_t f_{X,t}}{Z_t}$$

Solving for $\tilde{z}_{X,t}$, we get:

$$\boxed{\tilde{z}_{X,t} = \left[\left(\frac{w_t}{Z_t}\right)^{\theta} f_{X,t}\left(\frac{\theta - 1}{k - \theta + 1} + 1\right) Q_t^{-\theta}\tau^{\theta-1}\theta\left(\frac{\theta}{\theta - 1}\right)^{\theta-1}\frac{1}{C_t^*}\right]^{\frac{1}{\theta-1}}}$$

The average productivity of exporters in the foreign market $\tilde{z}^*_{X,t}$ follows the same expression as above, using the relevant foreign economy parameters and variables ($w_t^*$,$f_{X,t}^*$, etc.).

## Method 2

The second method is quicker and makes use of the fact that $\tilde{z}_{X,t} = \nu z_{X,t}$. From the zero profit condition:

$$d_{X,t}(z_{X,t}) = 0 \quad \Leftrightarrow \quad \frac{Q_t}{\theta}[\rho_{X,t}(z)]^{1-\theta} C_t^* = \frac{w_t f_{X,t}}{Z_t}$$

Plugging in the expression for $\rho_{X,t}(z)$ (page 873), and simplifying, we get the cut-off productivity $z_{X,t}$:

$$z_{X,t} = \tau_t \frac{\theta}{\theta - 1} \left( \frac{w_t}{Q_t Z_t} \right)^{\frac{\theta}{\theta-1}} \left( \frac{1}{\theta} \frac{C_t^*}{f_{X,t}} \right)^{\frac{1}{1-\theta}}$$

Given the Pareto distribution for $G(z)$, we are told that: $\tilde{z}_{X,t} = \nu z_{X,t}$ (p.876). Plugging in the expression for the cut-off productivity, and, since we know that $\nu \equiv \left( \frac{k}{k-(\theta-1)} \right)^{\frac{1}{\theta-1}}$, we get:

$$\boxed{\tilde{z}_{X,t} = \left( \frac{k}{k - (\theta - 1)} \right)^{\frac{1}{\theta-1}} \left( \frac{\theta}{\theta - 1} \right) \tau_t \left( \frac{w_t}{Q_t Z_t} \right)^{\frac{\theta}{\theta-1}} \left( \frac{1}{\theta} \frac{C_t^*}{f_{X,t}} \right)^{\frac{1}{1-\theta}}}$$

## References

Ghironi, F. and Melitz, M. J. (2005). International trade and macroeconomic dynamics with heterogeneous firms. *The Quarterly Journal of Economics*, 120(3):865–915.