

## Jermann (1998)

### 1 Firms

#### 1.1 Technology growth

Technology  $X_t$  grows deterministically with  $X_t/X_{t-1} = \gamma$

#### 1.2 Firm Problem

Firm maximizes the sum of dividends

$$D_t = A_t K_t^\alpha (X_t N_t)^{1-\alpha} - W_t N_t - I_t$$

subject to the LOM

$$K_{t+1} = (1 - \delta) K_t + \left( \frac{b}{1 - a} \left( \frac{I_t}{K_t} \right)^{1-a} + c \right) K_t$$

The parameters  $a$  and  $c$  need to be determined to satisfy steady state relationships. Detrending yields:

$$D_t = A_t K_t^\alpha N_t^{1-\alpha} - W_t N_t - I_t$$

and

$$\gamma K_{t+1} = (1 - \delta) K_t - \left( \frac{b}{1 - a} \left( \frac{I_t}{K_t} \right)^{1-a} + c \right) K_t$$

Using the households SDF, the firm maximizes

$$\max E_t \sum_{t=0}^{\infty} (\beta^*)^k \frac{\Lambda_{t+k}}{\Lambda_t} \left( A_{t+k} K_{t+k}^\alpha N_{t+k}^{1-\alpha} - W_{t+k} N_{t+k} - I_{t+k} - q_{t+k} \left( \gamma K_{t+k+1} - (1 - \delta) K_{t+k} - \left( \frac{b}{1 - a} \left( \frac{I_{t+k}}{K_{t+k}} \right)^{1-a} + c \right) K_{t+k} \right) \right)$$

FOC Capital

$$\begin{aligned} \frac{\partial L}{\partial K_{t+k+1}} &= (\beta^*)^k \frac{\Lambda_{t+k}}{\Lambda_t} (-q_{t+k} \gamma) \\ &+ E_t (\beta^*)^{k+1} \frac{\Lambda_{t+k+1}}{\Lambda_t} \left[ \alpha A_{t+k+1} K_{t+k+1}^{\alpha-1} N_{t+k+1}^{1-\alpha} - q_{t+k+1} \left( - (1 - \delta) - \frac{ba}{1 - a} I_{t+k+1}^{1-a} K_{t+k+1}^{a-1} - c \right) \right] = 0 \end{aligned}$$

Simplified

$$\gamma \Lambda_{t+k} q_{t+k} = \beta^* \Lambda_{t+k+1} \left( \alpha A_{t+k+1} K_{t+k+1}^{\alpha-1} N_{t+k+1}^{1-\alpha} + E_t q_{t+k+1} \left( (1 - \delta) + \frac{ba}{1 - a} I_{t+k+1}^{1-a} K_{t+k+1}^{a-1} + c \right) \right)$$

Investment FOC

$$\frac{\partial L}{\partial I_{t+k}} = \beta^k \frac{\Lambda_{t+k}}{\Lambda_t} \left( 1 - q_{t+k} \left( -b I_{t+k}^{-a} K_{t+k}^a \right) \right) = 0$$

Simplified

$$1 = q_{t+k} b I_{t+k}^{-a} K_{t+k}^a$$

or

$$q_t = \frac{1}{b} \left( \frac{I_t}{K_t} \right)^a$$

Labor FOC

$$W_t = (1 - \alpha) A_t K_t^\alpha N_t^{1-\alpha}$$

Pricing follows

$$1 = \beta^* E_t \frac{\Lambda_{t+1}}{\Lambda_t} x_t$$

where  $x_t$  is the payoff of any asset. For capital, from (1.2)

$$1 = \beta^* E_t \frac{\Lambda_{t+1}}{\Lambda_t} \frac{1}{\gamma q_t} \left( \alpha A_{t+1} K_{t+1}^{\alpha-1} N_{t+1}^{1-\alpha} + q_{t+1} \left( (1 - \delta) + \frac{ba}{1-a} I_{t+1}^{1-a} K_{t+1}^{a-1} + c \right) \right) = 0$$

## 2 Household

The household maximizes

$$\max E_t \sum_{k=0}^{\infty} \beta^k \left\{ \frac{(C_{t+k} - h C_{t+k-1})^{1-\tau}}{1-\tau} \right\} = E_t \sum_{k=0}^{\infty} \underbrace{(\tilde{\beta} \gamma^{1-\tau})^k}_{\beta^*} \left\{ \frac{(C_{t+k} - \frac{h}{\gamma} C_{t+k-1})^{1-\tau}}{1-\tau} \right\}$$

subject to

$$[W_{t+k} + a'_{t+k} (V_{t+k}^a + D_{t+k}^a) = C_{t+k} + a'_{t+k+1} V_{t+k}^a]$$

where  $a$  is the amount of assets held,  $V$  the value of the asset and  $D$  its dividend. This equation is already in detrended form. The FOCs are

$$\begin{aligned} \frac{\partial L}{\partial C_t} &= \left( C_t - \frac{h}{\gamma} C_{t-1} \right)^{-\tau} - \Lambda_t - \beta^* \frac{h}{\gamma} \left( C_{t+1} - \frac{h}{\gamma} C_t \right)^{-\tau} = 0 \\ \Lambda_t &= \left( C_t - \frac{h}{\gamma} C_{t-1} \right)^{-\tau} - \beta^* \frac{h}{\gamma} \left( C_{t+1} - \frac{h}{\gamma} C_t \right)^{-\tau} \\ \frac{\partial L}{\partial a_{t+k}} &= -\beta^k \Lambda_{t+k} V_{t+k}^a + \beta^{*k+1} E_{t+k} \Lambda_{t+k+1} (V_{t+k+1}^a + D_{t+k+1}^a) = 0 \\ \Lambda_{t+k} V_{t+k}^a &= \beta \Lambda_{t+k+1} (V_{t+k+1}^a + D_{t+k+1}^a) \end{aligned}$$

where  $N = 1$  has been imposed.

For an one-period risk free bond, this implies

$$1 = \beta^* E_t \frac{\Lambda_{t+1}}{\Lambda_t} (1 + r^*)$$

For a firm share, this implies

$$\Lambda_t = \beta^* E_t \Lambda_{t+1} \frac{(V_{t+1}^k + D_{t+1})}{V_t^k}$$

while for a consol that in perpetuity pays the risk free rate, we have

$$\Lambda_t = \beta^* E_t \Lambda_{t+1} \frac{(V_{t+1}^c + r^*)}{V_t^c}$$

### 3 Market Clearing

As usual

$$Y = C + I$$

All assets except for firm shares are in zero net supply, i.e.  $a=0$  except for  $a^k$  which must be 1  $\forall t$ , implying

$$W_t + (V_t^k + D_t^k) = C_t + V_t^k \Rightarrow W_t + D_t = C_t$$

### 4 Steady State

In steady state, we have

$$K = \frac{1}{\gamma} (1 - \delta) K + \underbrace{\frac{1}{\gamma} \left( \frac{b}{1-a} \left( \frac{I}{K} \right)^{1-a} + c \right)}_I K$$

Thus, we need

$$\frac{I}{K} = \left( 1 - \frac{1}{\gamma} (1 - \delta) \right)$$

i.e. the investment capital ratio can be computed. From the investment FOC with Tobin's  $q = 1$  imposed, we have

$$b I_t^{-a} K_t^a = 1 \Rightarrow b = \left( \frac{I}{K} \right)^a$$

Moreover, the LOM implies

$$\frac{1}{\gamma} \left( \frac{b}{1-a} \left( \frac{I}{K} \right)^{1-a} + \text{const} \right) = \frac{I}{K}$$

which allows solving for *const*:

$$\text{const} = \gamma \frac{I}{K} - \frac{b}{1-a} \left( \frac{I}{K} \right)^{1-a}$$

Finally, the capital FOC implies

$$\begin{aligned}
\Lambda_t q_t \gamma &= \beta^* \Lambda_{t+1} \left( \alpha A_{t+1} K_{t+1}^{\alpha-1} + q_{t+1} \left( (1-\delta) + c + \frac{ba}{1-a} I_{t+1}^{1-a} K_{t+1}^{a-1} \right) \right) \\
\gamma &= \beta^* \left( \alpha K^{\alpha-1} + \left( (1-\delta) + c + \frac{ba}{1-a} \left( \frac{I}{K} \right)^{1-a} \right) \right) \\
\frac{\gamma}{\beta^*} &= \left( \alpha K^{\alpha-1} + \left( (1-\delta) + c + \frac{ba}{1-a} \left( \frac{I}{K} \right)^{1-a} \right) \right) \\
\frac{\gamma}{\beta^*} - \left( (1-\delta) + c + \frac{ba}{1-a} \left( \frac{I}{K} \right)^{1-a} \right) &= \alpha K^{\alpha-1} \\
\left[ \frac{1}{\alpha} \left( \frac{\gamma}{\beta^*} - \left( (1-\delta) + c + \frac{ba}{1-a} \left( \frac{I}{K} \right)^{1-a} \right) \right) \right]^{\frac{1}{\alpha-1}} &= K
\end{aligned}$$

From  $I/K$  investment immediately follows. Moreover,

$$\begin{aligned}
W &= (1-\alpha) K^\alpha \\
D &= K^\alpha - W - I
\end{aligned}$$

and

$$\begin{aligned}
\Lambda_{t+k} V_{t+k}^a &= \beta^* \Lambda_{t+k+1} (V_{t+k+1}^a + D_{t+k+1}^a) \\
\frac{1}{\beta} &= \left( 1 + \frac{D}{V} \right) \\
\left( \frac{1}{\beta^*} - 1 \right) V &= D \\
\Lambda &= \left( C - \frac{h}{\gamma} C \right)^{-\tau} \left( 1 - \beta \frac{h}{\gamma} \right) \\
W_t + D_t &= C_t \\
\left( C_t \left( 1 - \frac{h}{\gamma} \right) \right)^{-\tau} &= \Lambda_t
\end{aligned}$$

## 5 Mapping $a$ and $\xi$

$\xi$  is the elasticity of

$$\begin{aligned}
q &= \frac{1}{b} \left( \frac{I}{K} \right)^\alpha \\
\log q &= \log \left( \frac{1}{b} \right) + a \log \left( \frac{I}{K} \right) \\
\log \left( \frac{I}{K} \right) &= +\frac{1}{a} \log \left( \frac{1}{b} \right) + \frac{1}{a} \log q \\
\frac{\partial \log \left( \frac{I}{K} \right)}{\partial \log q} &= \frac{1}{a} = \xi
\end{aligned}$$