

The steps I have taken to solve equation (1) are outlined below:

Solve for spin accumulation

$$\frac{\partial \vec{m}(x,t)}{\partial t} = -\frac{\partial \vec{j}_m}{\partial x} - \frac{\vec{m}(x,t) \times \hat{M}(x)}{\lambda_J^2} - \frac{\hat{M}(x) \times (\vec{m}(x,t) \times \hat{M}(x))}{\lambda_\phi} - \frac{\vec{m}(x,t) - m_\infty}{\lambda_{sf}^2} \quad (1)$$

The derivatives can be expressed via the difference notation given below ↓

$$\frac{\partial u(x,t)}{\partial t} \approx \frac{U_i^{n+1} - U_i^n}{\Delta t}$$

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = -\frac{\partial \vec{j}_m}{\partial x} - \frac{\vec{m}(x,t) \times \hat{M}(x)}{\lambda_J^2} - \frac{\hat{M}(x) \times (\vec{m}(x,t) \times \hat{M}(x))}{\lambda_\phi} - \frac{\vec{m}(x,t) - m_\infty}{\lambda_{sf}^2} \quad (2)$$

Step Taken → multiply by $-\Delta t$

$$U_i^{n+1} - U_i^n = -\Delta t \left(\frac{\partial \vec{j}_m}{\partial x} + \frac{\vec{m}(x,t) \times \hat{M}(x)}{\lambda_J^2} + \frac{\hat{M}(x) \times (\vec{m}(x,t) \times \hat{M}(x))}{\lambda_\phi} + \frac{\vec{m}(x,t) - m_\infty}{\lambda_{sf}^2} \right)$$

Step Taken → take knowns to RHS

$$U_i^{n+1} = U_i^n - \Delta t \left(\frac{\partial \vec{j}_m}{\partial x} + \frac{\vec{m}(x,t) \times \hat{M}(x)}{\lambda_J^2} + \frac{\hat{M}(x) \times (\vec{m}(x,t) \times \hat{M}(x))}{\lambda_\phi} + \frac{\vec{m}(x,t) - m_\infty}{\lambda_{sf}^2} \right) \quad (3)$$

Step Taken → expand out the cross products

$$U_i^{n+1} = U_i^n - \Delta t \left[\frac{\partial \vec{j}_m}{\partial x} + \frac{1}{\lambda_J^2} \begin{pmatrix} -M_y U_z^n + M_z U_y^n \\ M_x U_z^n - M_z U_x^n \\ -M_x U_y^n + M_y U_x^n \end{pmatrix} + \frac{1}{\lambda_\phi^2} \begin{pmatrix} -M_x M_y U_y^n - M_x M_z U_z^n + M_y^2 U_x^n + M_z^2 U_x^n \\ M_x^2 U_y^n - M_x M_y U_x^n - M_y M_z U_z^n + M_z^2 U_y^n \\ M_x^2 U_z^n - M_x M_z U_x^n + M_y^2 U_z^n - M_y M_z U_y^n \end{pmatrix} + \frac{U_i^n - |m_\infty| \hat{M}_i}{\lambda_{sf}^2} \right]$$

Use spin accumulation to solve for spin current

$$\vec{j}_m = \beta(x) \hat{M} j_e - 2D \left[\frac{\partial \vec{m}}{\partial x} - \beta(x) \beta' \hat{M} \left(\hat{M} \cdot \frac{\partial \vec{m}}{\partial x} \right) \right]$$