Optimum Depth of Investigation and Conductivity Response Rejection of the Different Electromagnetic Devices Measuring Apparent Magnetic Susceptibility

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ABSTRACT Electromagnetic susceptibility surveys are valuable for archaeological prospection owing to their ability to cover large areas of land. Their use, however, is often compromised by the conductivity influence of the soil and the limited investigation depth of the susceptibility response. To examine these constraints further, we compared the characteristics of two types of apparatus: coincident loop (e.g. Bartington MS2 field coil) and 'Slingram' instruments (EM38, SH3, CS60 and CS150). Theoretical considerations suggest that in contrast to coincident loop apparatus, Slingram instruments are less influenced by the soil conductivity and offer a greater maximum depth of investigation. The experimental results presented in this paper collected over an artificial structure at the Centre de Recherches Géophysiques (CRG) of Garchy (France), confirm the theoretical results and indicate that Slingram instruments are preferable for field susceptibility measurements.

Keywords: magnetic susceptibility; investigation depth; 'Slingram', coincident loop.

INTRODUCTION

The first applications of electromagnetic (EM) prospection to archaeology began about 30 years ago (Howell, 1966, 1968; Tite and Mullins, 1969; Tabbagh, 1971). This method offers the advantage over magnetometer surveys of allowing a direct measurement of the absolute value of soil magnetic susceptibility. This latter property is often influenced by anthropogenic activity and has been used in archaeological prospecting at both fine and coarse sample intervals to locate and define site limits. For example, in Great Britain, topsoil magnetic susceptibility survey (Clark, 1990) has been applied to the evaluation of major developments such as roads schemes, by utilizing a coarse interval to identify areas of possible archaeological activity when enhanced magnetic material is presumed to have been translocated into the current ploughed soil horizon. The electromagnetic method is most suitable for this type of survey owing to its high speed and ability to map near-surface distributions over large areas. The different apparatus that can be applied, however, have varying depths of maximum investigation and a precise determination of this parameter is necessary in order to define the range of depth to be investigated and the archaeological significance of the results obtained. The present study investigates the most appropriate apparatus for both wide-mesh susceptibility surveys and close sample interval mapping without limiting the investigation to ploughed soil.

Two types of EM apparatus are available that permit the measurement of apparent magnetic susceptibility: `Slingram' type apparatus (SCM, EM15, SH3, CS60 and CS150) composed of two separate magnetic dipoles (a transmitter and a receiver), and apparatus with one (coincident loop) or two loops, such as the apparatus manufactured by Bartington (Ltd) used preferentially by British teams for topsoil susceptibility surveys. The measurement of apparent magnetic susceptibility is possible with both types of apparatus owing to the operational frequency and the geometric dimensions of the apparatus with respect to the low induction number (LIN) condition (Nabighian, 1991). When this condition applies, the secondary field in phase with the primary field is for its main part proportional to the magnetic susceptibility of the subsurface.

The ground electrical conductivity, however, can contribute a significant component of the inphase response, which may lead to the erroneous determination of the apparent magnetic susceptibility response. Results may be compromised further by the limited maximum investigation depth of certain EM instruments in comparison with other geophysical methods.

The present study comprises two parts: (i) a theoretical study examines the in-phase response of the different apparatus as a function of soil conductivity and investigation depths and (ii) experimental confirmation of its conclusion was performed over a test site at CRG of Garchy, France.

THEORETICAL STUDY

Apparatus

"Slingram" apparatus

The `Slingram' apparatus consists of two separate magnetic dipoles (coils), separation and orientation of which determine the investigation depth of the instrument.

One of the first apparatus used for the archaeological prospecting was the EM15 manufactured by Geonics Ltd, Canada (coil separation: 0.83 m, orientation PARA (358 from the vertical)). A new apparatus, the SH3 (Parchas and Tabbagh, 1978; Parchas, 1979) was constructed at the Centre de Recherches Géophysiques (CRG), Garchy with a 1.5 m coil separation to enhance the investigation depth to an estimated 0.8mmaximum. The instrument also allowed the simultaneous measurement of apparent conductivity.

A theoretical study (Tabbagh, 1986) comparing the different types of coil orientation showed that the vertical coplanar (VCP) and perpendicular (PERP) configurations achieved the greatest depths of investigation depth for susceptibility measurements.

These predictions were confirmed through field results obtained over archaeological sites, conducted with the CS60 (Eurocim S.A.) (VCP or horizontal coplanar (HCP) orientation, a 0.6 m coil separation) and the EM38 (Geonics Ltd) (VCP or HCP orientation, a 1 m coil separation) instruments. It should be noted that both these instruments were designed primarily for the measurements of soil conductivity (using the quadrature component) rather than soil magnetic susceptibility.

The last apparatus proposed by the CRG team, CS150 (Eurocim S.A.), was designed with a 1.5 m coil separation similar to the SH3. A greater distance between the coils would increase the investigation depth but would reduce lateral resolution. The coils of the CS150 are perpendicular, which provides the advantage of zero coupling between the coils, in contrast to HCP or VCP orientations. This instrument also allows the simultaneous measurement of the in-phase and quadrature components, thus providing both the apparent magnetic susceptibility and the apparent conductivity of the soil. Two frequencies (4.4 kHz and 10 kHz) are used in this apparatus, which permits the magnetic viscosity to be determined under appropriate conditions (Beaussillon et al, 1996).

Coincident loop apparatus

The MS2 apparatus (Bartington Ltd) comprises a single loop of 0.185 m diameter and operates at a frequency of 958 Hz. It has been widely used in England and has become a reference for coarse sample interval topsoil susceptibility survey in archaeology.

Influence of the electrical conductivity on the in-phase response

For both Slingram and coincident loop apparatus, it is possible to calculate the theoretical response of a layered ground with

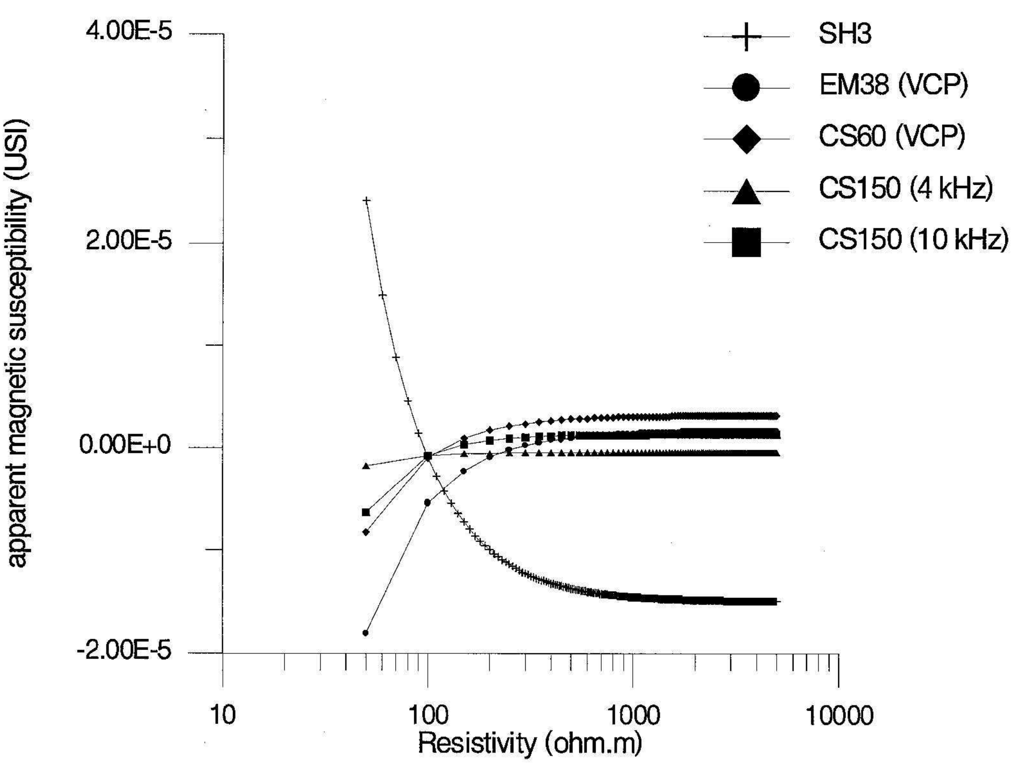


Figure 1. Influence of the conductivity on the in-phase response for Slingram instruments.

differing electrical conductivities and magnetic susceptibilities (see Tabbagh (1982) for the Slingram and the Appendix for the coincident loop). Thus it is possible to determine the influence of the inphase response generated by electrical conductivity of the soil. This response is expressed as an apparent magnetic susceptibility even when the actual magnetic susceptibility of the soil equals zero. This spurious apparent susceptibility originates from the soil conductivity and will vary with the magnitude of this parameter, which we will consider in more detail below.

In-phase conductivity response for Slingram apparatus (Figure 1)

The influence of conductivity for any `Slingram' apparatus is low and reaches a maximum equivalent susceptibility value of 2.5 10^{-5} SI in the case of the SH3 for a 50 ohm.m resistivity homogeneous ground. We can distinguish the EM38, CS60 and CS150 curves from the SH3, which exhibits an inverse response. The curves of the first three instruments reach an asymptotic upper limit at 200 ohm.m, beyond which the influence of the soil conductivity is negligible. In contrast, the response of the SH3 decreases to a constant value of approximately -1.5 10^{-5} SI.

In-phase conductivity response for coincident loop apparatus (Figure 2)

The in-phase response, expressed in apparent susceptibility, is more than 10 times greater than the Slingram case and produces a significant error. Furthermore, the curve does not reach a maximum constant value for higher resistivity values. For low resistivities (<100 ohm.m), the values of apparent magnetic susceptibility become negative. In Figure 2, where the apparent magnetic susceptibility is determined by reference to a 100 ohm.m homogeneous soil (in other words the zero susceptibility level is arbitrarily fixed for a 100 ohm.m resistivity), the influence of conductivity for a 500 ohm.m soil is twice as high as that for a 50 ohm.m soil.

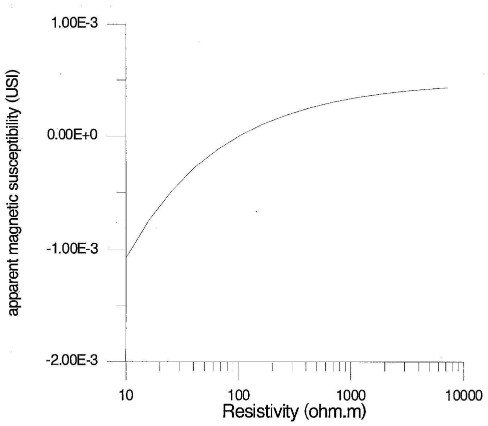


Figure 2. Influence of the conductivity on the in-phase response for the Bartington apparatus.

Curves for a 0.185 m (e.g. Bartington MS2) and for a 0.5 m diameter loop apparatus are almost identical. The influence of conductivity on the inphase response in the case of coincident loop apparatus is, for this range of diameters, practically independent of the loop size.

Investigation depth

Two models were calculated, each representing a three-layer ground, in order to evaluate the investigation depths for the different types of apparatus (Figures 3 and 4). The first model consists of a high-resistance intermediate magnetic layer within a conductive medium and the second model a more conductive intermediate magnetic layer within a high resistant medium. These models allow the influence of soil conductivity over investigation depth to be evaluated for two extreme cases.

The curves obtained (Fig. 5) show the response of the apparent magnetic susceptibility as a function of the depth to top of the intermediate layer. The depth of investigation can be determined from the maximum depth at which the intermediate layer remains detectable.

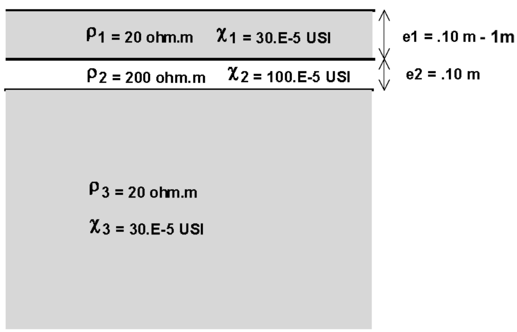


Figure 3. Model 1. High-resistance magnetic layer in a conductive soil.

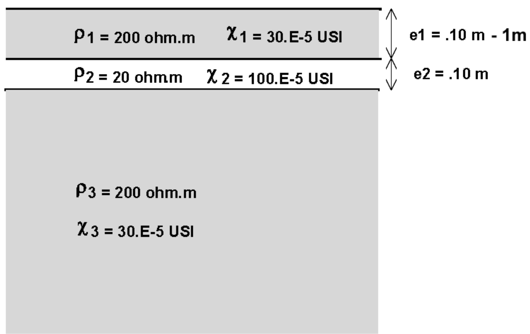


Figure 4. Model 2. Conductive magnetic layer in a resistant soil.

`Slingram' apparatus (Figure 5)

The curves of the CS60 (VCP), SH3 and EM38 (VCP) show maxima at about 0.06 m, 0.1 m and 0.16 m respectively. The maximum investigation depth is governed by the decay of the response and is least pronounced for the EM38 in VCP configuration than for the SH3 and CS60. At 0.1 m, the value of CS60 in-phase response (about 5 10^{-4} SI) is greater than that of the EM38 (4 10^{-4} SI). The CS60 (VCP) and the SH3 are thus more sensitive to the superficial effects than the EM38 (VCP).

The CS150 curves, which overprint on the figure, show a maximum at about 0.4 m and decrease with a more gentle slope than the other instruments. The investigation depth is thus better and a greater signal magnitude is maintained. Furthermore, superficial near-surface anomalies will not mask the response of deeper structures owing to the lower sensitivity of the CS150 to the first few centimetres of soil. The influence of soil conductivity on the in-phase response is also very small and does

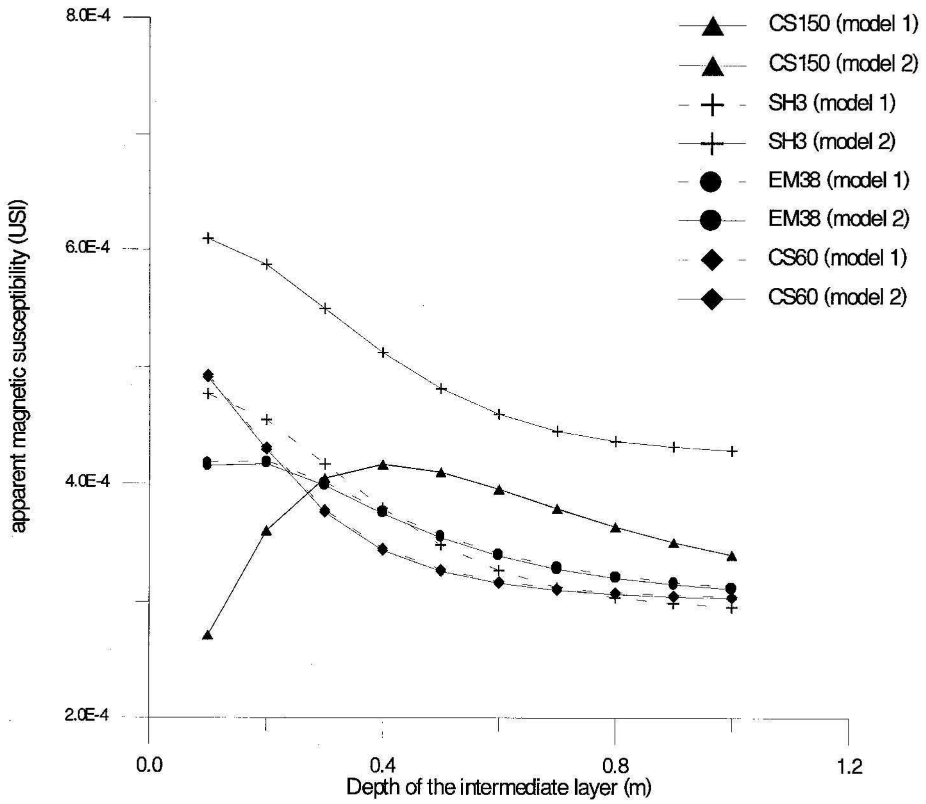


Figure 5. Assessment of the maximum investigation depth by varying the depth of the top of a thin magnetic layer (Slingram apparatus).

not vary with the thickness of the superficial layer.

Coincident loop apparatus

In the case of resistive layer (model 1: Figure 6a) for the MS2, the values of apparent magnetic susceptibility are negative but only the magnitudes must be taken into account. The magnitude curve is highly decreasing. In the preceding section, we demonstrated that soil conductivity has a significant influence on the MS2 in-phase response and a similar effect is observed here.

The curve calculated from model 2 (Figure 6b) decreases rapidly and illustrates the reduced investigation depth of the MS2. The instrument is most sensitive to the near-surface layers, to a depth of 20 cm. Below this depth the response decreases rapidly to a value of 4.4 10^{-4} SI, which consists of two components derived from the magnetic susceptibility (3 10^{-4} SI) and electrical conductivity (about 1.5 10^{-4} SI) of the nearsurface layer.

The curves for a 0.5 m diameter loop apparatus exhibit the same behaviour to that of the MS2, but with a weaker magnitude. Thus a larger diameter loop does not compensate for the rapid decrease of the in-phase signal in the first centimetres of the ground.

The `Slingram' apparatus has a small maximum followed by a gentle decrease, whereas the decrease in response of the MS2 is extremely rapid. For the coincident loop apparatus the rapid decrease of the in-phase response exists for the first 10 cm and below this depth the response will be dominated by the conductivity component. Thus the MS2 is sensitive to the superficial layer only and has a very limited maximum investigation depth. The soil electrical conductivity have a weak influence on the apparent magnetic susceptibility response for `Slingram' instruments in contrast to the MS2.

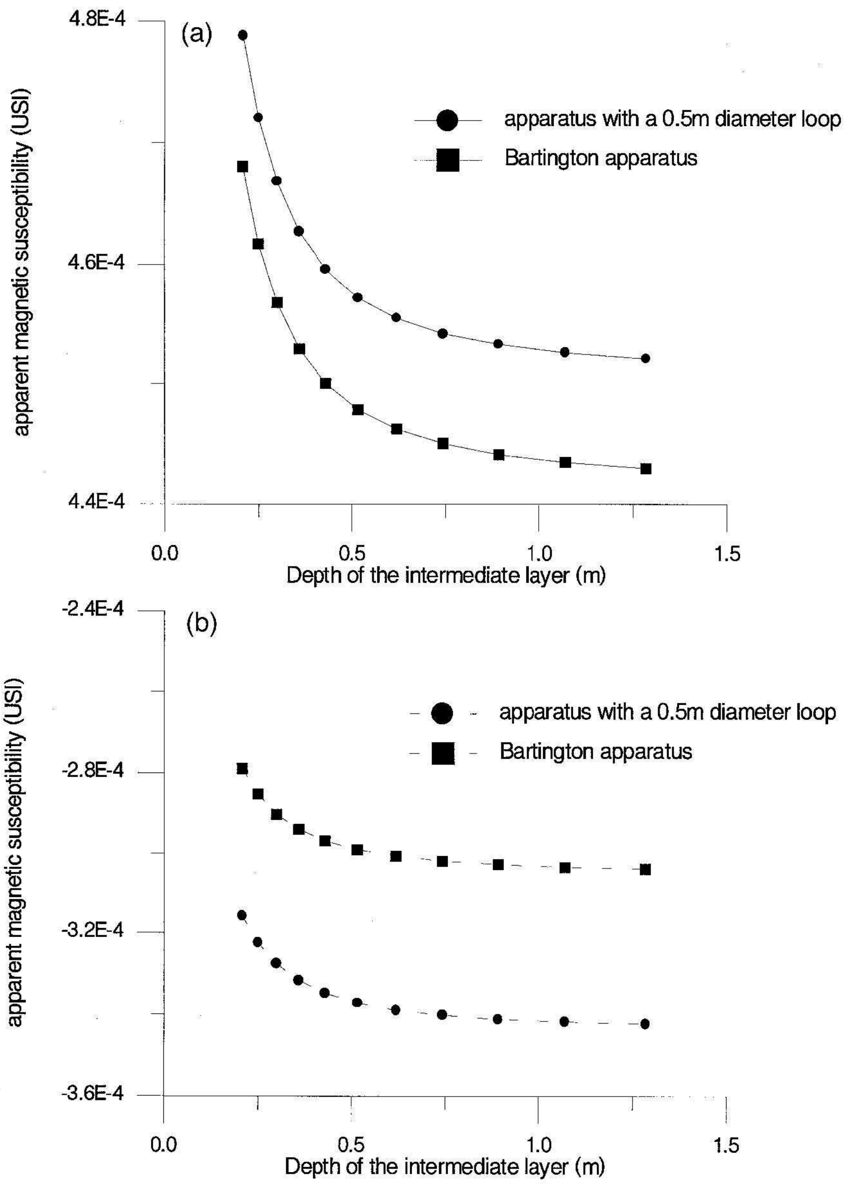


Figure 6. Assessment of the investigation depth by varying the depth of the top of a thin magnetic layer (coincident loop): (a) model 1 and (b) model 2.

Conclusion of the theoretical studies

The theoretical study demonstrates the advantage of `Slingram' apparatus (EM38, SH3, CS60 and CS150), mainly the CS150, over coincident loop (MS2) instruments with respect to the spurious response to the electrical conductivity of the soil and maximum investigation depth. The influence of conductivity to the response of the MS2 is significant and may be dominant when surveying over highly conductive near-surface layers, whereas it can be neglected for the `Slingram' apparatus. Even under more favourable conditions (resistive near-surface layer) for the MS2, the rapid decrease of the response with depth in the first 10 cm may prove restrictive. `Slingram' apparatus demonstrate greater maximum investigation depth and are less sensitive to the near-surface layers. Of all apparatus considered in this study the CS150 would be expected to produce the greatest magnitude of response with increasing depth.

EXPERIMENTAL STUDY

The experimental results were collected over a test site at CRG, Garchy. The site consists of a buried L-shaped ditch 1 m wide cut into a calcareous substrate. The ditch is filled with a mixture of clay subsoil and topsoil (Figure 7). This mixture is less magnetic than normal topsoil and more magnetic than limestone. The northern branch of the ditch is 10 m long and has a depth of 0.6 m, and the southern branch is 20 m long and has a depth of 1.2 m.

The results of field survey with the EM instrumentation are compared with magnetic susceptibility measurements made at the laboratory of St Maur on soil samples recovered from the site.

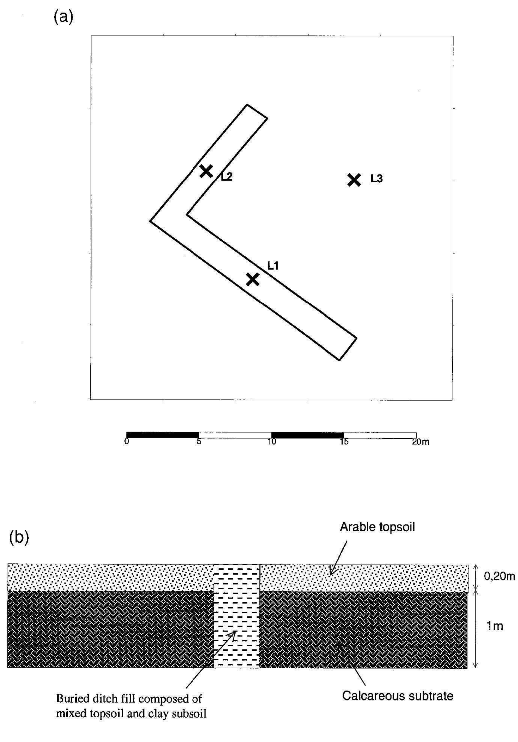


Figure 7. Map (a) and vertical section (b) of the L-shaped structure used for the field tests.

Apparent magnetic susceptibility survey

MS2

The two branches of the ditch are clearly visible on the map produced with the MS2 (Figure 8). They appear less magnetic (apparent susceptibility about 30 10^{-5} SI) than the surrounding ground (apparent susceptibility 70 10^{-5} SI) and measurements have recorded the contrast between the undisturbed topsoil and the less magnetic fill of the buried ditch (see Laboratory measurements below). The maximum depth of investigation is thus limited to that of the immediate top soil (20 cm), where the filling is less magnetic than its surrounding. The area of increased magnetism to the north of the map results from the introduction of more magnetic topsoil to fill ruts near the structure.

The results from the MS2 can be compared with those obtained over the same site with the DECCO instrument. This is a TDEM apparatus with two square concentric loops (0.52 m and 0.26 m) designed to measure magnetic viscosity (Tabbagh and Dabas, 1996). The map obtained with this instrument also demonstrates a negative contrast and a low viscosity anomaly over the ditch feature, as the DECCO has a similar, very limited depth of investigation (Figure 9).

`Slingram' apparatus

In the case of SH3 (Figure 10) and EM38 in VCP configuration (Figure 11), only the southern

branch of the ditch is visible. The anomaly appears more magnetic (apparent susceptibility 100 10^{-5} SI and 70 10^{-5} SI, respectively) for the two instruments than the surrounding ground (apparent susceptibility ca. 50 10^{-5} SI and 30 10^{-5} SI). A similar response is generated by the CS150 instrument (Figure 12), which also fails to detect the northern branch of the buried ditch. For the northern branch, the sum of the response generated by the first 20 cm (negative) and of the response generated by the following 40 cm (positive) is nul. For the deeper southern branch the depth of the ditch is greater (1.20 m) and the second response corresponding to 0.20-1.20 m thickness dominates. In contrast to the MS2 results, the southern branch anomaly appears as a high susceptibility anomaly with respect to the surrounding ground. This confirms the greater depth of investigation of the `Slingram' apparatus.

In the EM38 data the imported topsoil is detected but is not visible in the results

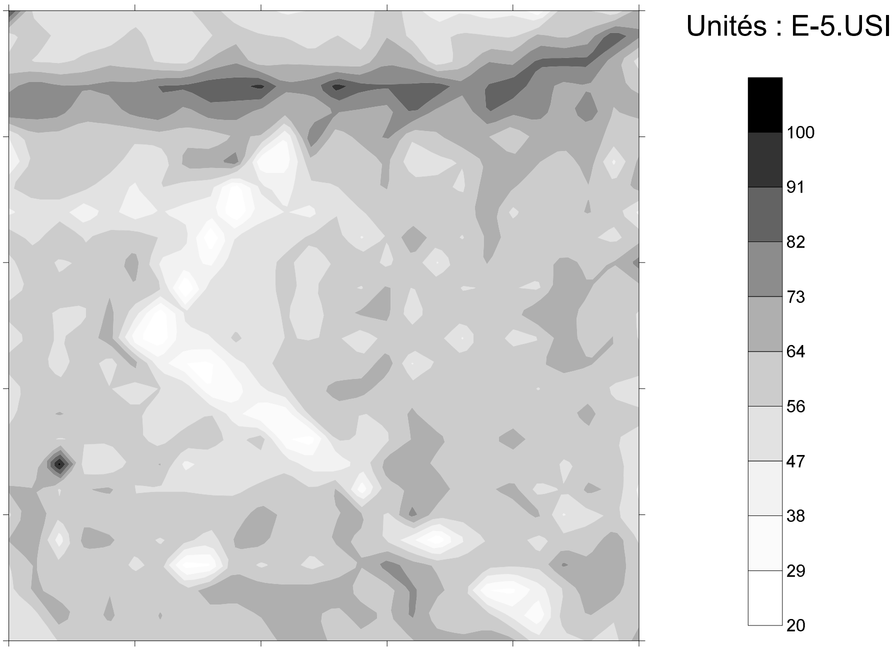


Figure 8. MS2 Bartington survey over the L-shaped structure (1x1 m² sample interval).

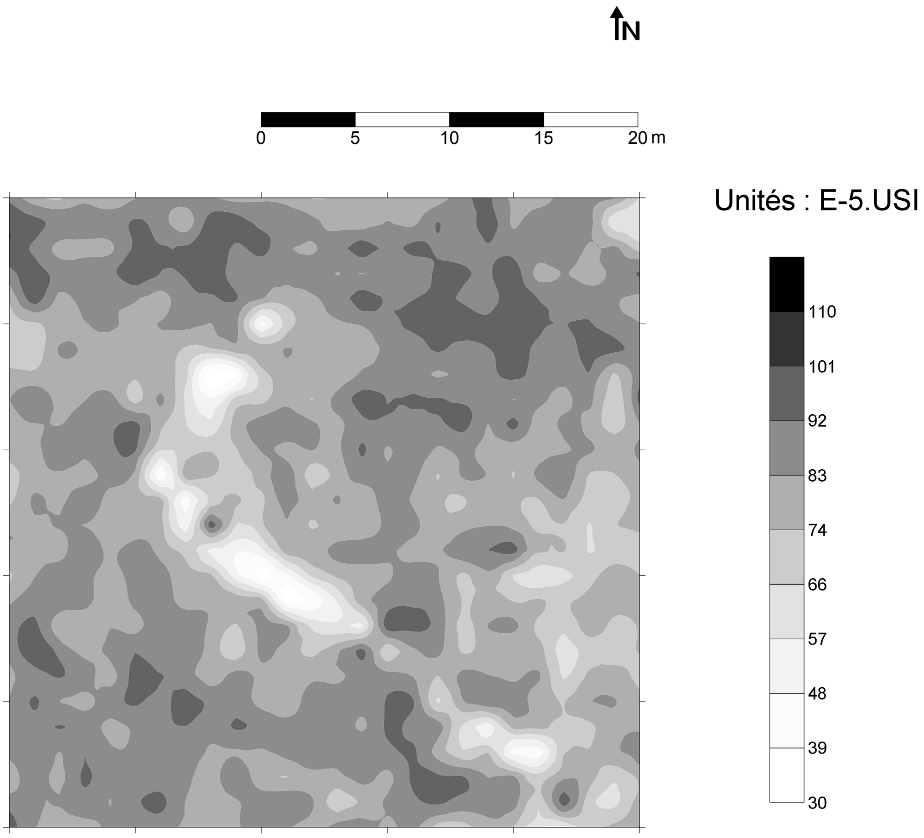


Figure 9. DECCO survey over the L-shaped structure (1 _1 m2 sample interval).

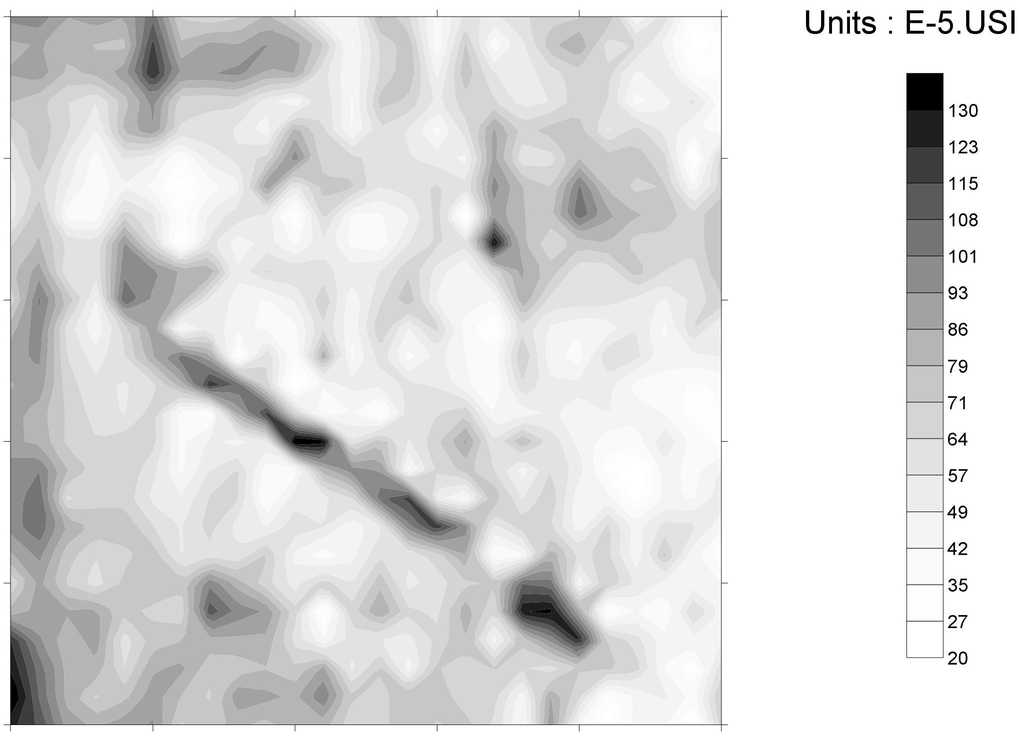
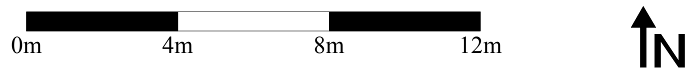


Figure 10. SH3 survey over the L-shaped structure (1x1 m² sample interval).



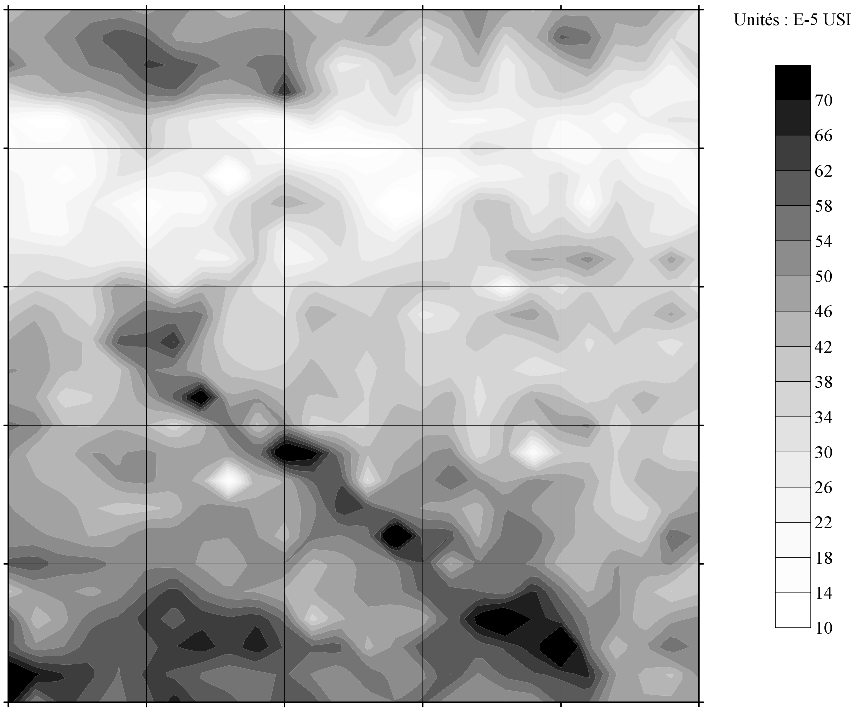


Figure 11. EM38 survey on the L-shaped structure (1x1 m² sample interval).

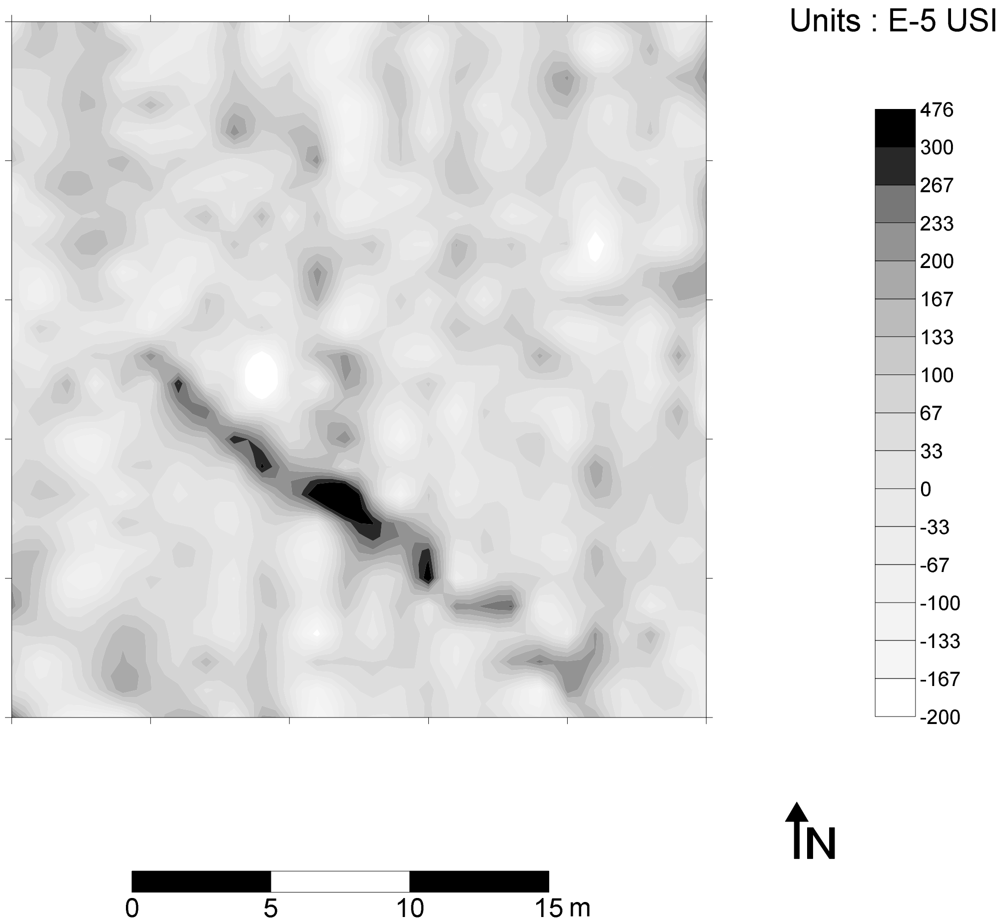


Figure 12. CS150 survey on the L-shaped structure (1x1 m² sample interval).

collected with the SH3 and the CS150, which are less sensitive to very near-surface material.

Laboratory magnetic susceptibility measurements

Mass magnetic susceptibility measurements (Tables 1-3) were carried out on soil samples collected by auger from the test site. The measurements were made at the St Maur Geomagnetism Laboratory (France), using a KLY-2 (Geofysika) susceptibility meter. The magnetic susceptibilities of the superficial layers (between 0 and 10 cm) are about 30 10^{-5} SI to 50 10^{-5} SI (with a density of 800 kg m^{-3}) in the L-shaped ditch and about 100 10^{-5} SI in the neighbouring topsoil. The magnetic susceptibilities of the lower layers inside this feature are about 70±90 10^{-5} SI (with a density about 1500 kg m^{-3}). The magnetic susceptibility of the limestone is equal to or lower than 20 10^{-5} SI. The magnetic susceptibility of the superficial layers in the L-shaped ditch is thus weaker than the magnetic susceptibility in the lower layers. The values measured during the MS2 survey, between 30 10^{-5} SI and 70 10^{-5} SI, are in agreement with those measured in the laboratory. This result confirms the weak investigation depth of the MS2 apparatus, and explains the `negative' response obtained when using this apparatus.

In contrast, the values obtained during the EM38 survey (about 80 10^{-5} SI) correspond mainly to lower layer values (20-40 cm) if we consider the laboratory result of 70-90 10^{-5} SI in the L-shaped ditch. This result confirms that the `positive' response obtained with EM38 measurements corresponds to an investigation depth clearly greater than that of the MS2 apparatus.

Table 1. Samples taken in the L shaped ditch (fig.7a)

. L1 drill:

	Mass susceptibility (10^{-8} m^3/kg)	Volume susceptibility (10^{-5} SI)
0-6cm	49	40
12-21cm	64	70
21-32cm	60	90

. L2 drill :

	Mass susceptibility ($10^{-8}m^3$/kg)	Volume susceptibility (10^{-5} SI)
0-8cm	33	30
8-18cm	34	40
18-24cm	45	70

. L3 drill : samples taken near the L shaped ditch (the limestone below 20 cm was impossible to sample using auger) (fig.7a):

	Mass susceptibility ($10^{-8}m^3$/kg)	Volume susceptibility (10^{-5} SI)
0-7cm	120	100
7-14cm	90	100
14-20cm	87	100

Conclusion of the experimental studies

The experimental study corroborates the theoretical results and confirms the advantages of `Slingram' instruments (with a dipole separation greater than 1 m) over coincident loop apparatus in terms of investigation depth. In the field test, where a negative contrast in the first 20 cm overrides a positive contrast of 40 cm thickness (northern branch) and of 1 m thickness (southern branch), the MS2 response is always negative, which proves its limited investigation depth, whereas the `Slingram' apparatus response is dominated by the lower part of the filling when its thickness is sufficient.

The influence of the conductivity on the inphase response is more limited in the `Slingram' instrument than in coincident loop instruments. `Slingram' apparatus therefore appears to be able to detect undisturbed archaeological features and are convenient for recording by fine sample interval surveys.

For coarse sample interval survey they are also preferable because they are sensitive to both ploughed and undisturbed subsoil susceptibility and thus extend the detection ability to undisturbed archaeological layers.

REFERENCES

Beausillon, R., Benech, Ch. and Tabbagh, A. (1996). A two frequency Slingram apparatus for archaeological and pedological application. 2nd Environmental and Engineering Geophysical Society Meeting, Nantes, 2-5 September, pp. 15-18.

Clark, A. J. (1990). *Seeing Beneath the Soil*. London: Batsford.

Howell, B. (1966). A soil conductivity meter. *Archaeometry* 9: 20-23.

Howell, B. (1968). The soil conductivity anomaly detector (SCM) in archaeological prospection. *Prospezioni Archeologiche* 3: 101-104.

Nabighian, M. N. (1991). *Electromagnetics Methods in Applied Geophysics*. Vol. I, Theory 523 pp., Vol. II, Applications, 972 pp. Society of Exploration Geophysicists, Tulsa.

Parchas, C. (1979). Mesures simultanées de la conductivité électrique et de la susceptibilité magnétique du sol. Paris: Thèse de 3ème cycle, Université Paris 6.

Parchas, A. and Tabbagh, A. (1978). Simultaneous measurement of electrical conductivity and magnetic susceptibility of the ground in the EM prospecting. *Archaeophysika* 10: 682-691.

Tabbagh, A. (1971). Définition des caractéristiques d'un appareil EM classique adapté à la prospection archéologique. Paris: Thèse de 3ème cycle, Université Paris.

Tabbagh, A. (1982). L'interprétation des données en prospection électromagnétique avec les appareils SH3 et EM15. *Revue d'Achéométrie* 6: 1-11.

Tabbagh, A. (1986). What is the best coil orientation in the Slingram electromagnetic prospection methods? *Archaeometry* 28: 185-196.

Tabbagh, A. and Dabas, M. (1996). Absolute magnetic viscosity determination using time-domain electromagnetic devices. *Archaeological Prospection* 3: 199-208.

Tite, M. S. and Mullins, C. (1969). Electromagnetic prospecting: a preliminary investigation. *Prospezioni Archeologiche* 9: 20-23.

APPENDIX

Theoretical response of a coincident loop apparatus for a layered ground

Consider a horizontal circular loop with radius a, at distance d, from the ground. In this configuration, the rotational symmetry suggests the use of cylindrical coordinates (r,φ,z) and an EM field comprised of three components H_r, H_z and E_φ. These three components can be deduced from the potential vector A ($\vec{B} = \vec{rot}\vec{A}$), which has a single component A_φ.

Calculation of the primary field and potential

For a point P located on the axis, the expression of the primary field emitted by the loop is :

$$H_{zp} = \frac{1}{2}\frac{a^2.I}{\left(a^2 + (z+d)^2\right)^{3/2}} \qquad (1)$$

where I is the current. It is possible to write H_{zp} by means of a Hankel transform:

$$\begin{aligned} H_{zp} &= \frac{aI}{2}\frac{a}{(a^2+(z+d)^2)^{3/2}} \\ &= \frac{aI}{2}\left(-\frac{\partial}{\partial a}\right)\left(\frac{1}{\sqrt{(z+d)^2+a^2}}\right) \qquad (2) \\ &= \frac{aI}{2}\int_0^{+\infty}\lambda e^{-\lambda|z+d|}J_1(\lambda r)d\lambda \end{aligned}$$

where J1 is the Bessel function of the first kind and order 1. We have the following relation between H_{zp} and $A_{\varphi p}$:

$$H_{zp} = \frac{1}{\mu_0}\frac{1}{r}\frac{\partial}{\partial r}(rA_{\varphi p}) \qquad (3)$$

As $A_{\varphi p}$ satisfies the equation :

$$\frac{\partial^2 A_{\varphi p}}{\partial r^2} + \frac{1}{r}\frac{\partial A_{\varphi p}}{\partial r} - \frac{A_{\varphi p}}{r^2} + \frac{\partial^2 A_{\varphi p}}{\partial z^2} = 0 \quad (4)$$

a general solution of $A_{\varphi p}$ can be written :

$$A_{\varphi p} = \int_0^{+\infty}\alpha_p(\lambda)e^{-\lambda|z+d|}J_1(\lambda a)d\lambda \qquad (5)$$

By deriving $A_{\varphi p}$ and considering r=0, we can deduce the expression of $\alpha_p(\lambda)$ from the expression of (3) and obtain :

$$\alpha_p(\lambda) = \frac{Ia\mu_0}{2}J_1(\lambda a) \qquad (6)$$

. *Calculation of the secondary field and potential*

In the air, $A_{\varphi s}$ satisfies the equation :

$$\frac{\partial^2 A_{\varphi s}}{\partial r^2}+\frac{1}{r}\frac{\partial A_{\varphi s}}{\partial r}-\frac{A_{\varphi s}}{r^2}+\frac{\partial^2 A_{\varphi s}}{\partial z^2}=0 \quad (7)$$

the general solution of which can be expressed by : $A_{\varphi s}=\int_0^{+\infty}\alpha_0(\lambda)e^{-\lambda|z+d|}J_1(\lambda r)d\lambda$

In the p[th] layer, we have :

$$\frac{\partial^2 A_{\varphi s}}{\partial r^2}+\frac{1}{r}\frac{\partial A_{\varphi s}}{\partial r}-\frac{A_{\varphi s}}{r^2}+\frac{\partial^2 A_{\varphi s}}{\partial z^2}-i\sigma_p\mu_p\omega A_{\varphi s}=0 \quad (8)$$

the general solution of which can be expressed by:

$$A_{\varphi s}=\int_0^{+\infty}\left(\alpha_p(\lambda)e^{-u_p(z+d)}+\beta_p(\lambda)e^{u_p(z+d)}\right)J_1(\lambda r)d\lambda \quad (9)$$

with $\gamma_p{}^2=i\sigma_p\mu_p\omega$ and $u_p=\sqrt{\lambda^2+\gamma_p{}^2}$

. Determination of $\alpha_0(\lambda)$

In this part, to determine $\alpha_0(\lambda)$, one considers the boundary conditions at each interface. The 2N $\alpha_p(\lambda)$ and $\beta_p(\lambda)$ will be calculated by a iteration process.

At the interface, H_r, $\mu.H_z$ et E_φ are continuous. Thus :

$\mu.H_z$ continue implies by integration that A_φ is continuous

$$H_r=\frac{1}{\mu}\frac{\partial A_\varphi}{\partial z} \text{ then } \frac{\partial A_\varphi}{\partial z} \text{ continuous}$$

$$E_\varphi=-i\omega A_\varphi \text{ then } A_\varphi \text{ continuous}$$

We define R_N as the Hankel transform of the ratio :

$$\frac{\frac{1}{\mu_N}\frac{\partial A_\varphi}{\partial z}}{A_\varphi}$$

for N[th] layer we have $R_N=\frac{u_N}{\mu_N}$ (10)

For the p[th] layer :

$$R_p=\frac{-\frac{u_p}{\mu_p}\left(\alpha_p e^{-u_p h_p}-\beta_p e^{u_p h_p}\right)}{\alpha_p e^{-u_p h_p}+\beta_p e^{u_p h_p}} \quad (11)$$

And

$$R_{p+1}=\frac{-\frac{u_p}{\mu_p}\left(\alpha_p e^{-u_p h_{p+1}}-\beta e^{u_p h_{p+1}}\right)}{\alpha_p e^{-u_p h_{p+1}}+\beta_p e^{u_p h_{p+1}}} \quad (12)$$

If we calculate the factor $\frac{\alpha_p}{\beta_p}$ in (11) and replace it in (12), we obtain a relation of recurrence between R_p and R_{p+1} :

$$R_{p+1}=-\frac{u_p}{\mu_p}\frac{th(u_p e_p)-\frac{\mu_p}{u_p}R_p}{1-\frac{\mu_p}{u_p}R_p th(u_p e_p)}$$

avec $e_p=h_{p+1}-h_p$ (13)

Thanks to this relation, we can calculate R_1 at z=0 and by the mean of the expression of the potential in the air, we obtain :

$$R_1 = \frac{\frac{\lambda}{\mu_0}\alpha_0 - \frac{Ia\mu_0}{2}\frac{\lambda}{\mu_0}e^{-\lambda d}}{\alpha_0 + \frac{Ia\mu_0}{2}J_1(\lambda a)e^{-\lambda d}} \qquad (14)$$

Then the expression of α_0 is:

$$\alpha_0 = \frac{\frac{\lambda}{\mu_0} + R_1}{\frac{\lambda}{\mu_0} - R_1}\frac{Ia\mu_0}{2}e^{-\lambda d} \qquad (15)$$

Having the expression of the potential $A_{\varphi s}$ of a coincident loop of radius a, we obtain the e.m.f.:

$$\begin{aligned} e(t) &= -\frac{\partial \Phi}{\partial t} = -\frac{\partial}{\partial t}\oint_{Cercle} r\vec{o}t\vec{A}_{\varphi}.d\vec{s} \\ &= -\frac{\partial}{\partial t}\left(2\pi a A_{\varphi s}\right) = -2\pi a i \omega A_{\varphi s}(a) \end{aligned} \qquad (16)$$