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Constructive and axiomatic approaches of fuzzy approximation operators

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Abstract

This paper presents a general framework for the study of rough set approximation operators in fuzzy environment in which both constructive and axiomatic approaches are used. In constructive approach, a pair of lower and upper generalized fuzzy rough (and rough fuzzy, respectively) approximation operators is first defined. The representations of both fuzzy rough approximation operators and rough fuzzy approximation operators are then presented. The connections between fuzzy (and crisp, respectively) relations and fuzzy rough (and rough fuzzy, respectively) approximation operators are further established. In axiomatic approach, various classes of fuzzy approximation operators are characterized by different sets of axioms. The minimal axiom sets of fuzzy approximation operators guarantee the existence of certain types of fuzzy or crisp relations producing the same operators.

Keywords: Approximation operators; Binary relations; Fuzzy sets; Fuzzy rough sets; Rough fuzzy sets; Rough sets

1. Introduction

The theory of rough sets, proposed by Pawlak [19], is an extension of set theory for the study of intelligent systems characterized by insufficient and

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incomplete information. Using the concepts of lower and upper approximations in rough set theory, knowledge hidden in information systems may be unravelled and expressed in the form of decision rules.

There are at least two approaches for the development of the rough set theory, the constructive and axiomatic approaches. In constructive approach, binary relations on the universe, partitions of the universe, neighborhood systems, and Boolean algebras are all the primitive notions. The lower and upper approximation operators are constructed by means of these notions [12,19–21,25,26,36–40]. Based on constructive method, extensive research has also been carried out to compare the theory of rough set with other theories of uncertainty such as fuzzy sets, conditional events and Dempster–Shafer theory of evidence [3,17,22–24,32,34,44]. Thus the constructive approach is suitable for practical applications of rough sets. On the other hand, the axiomatic approach, which is appropriate for studying the structures of rough set algebras, takes the lower and upper approximation operators as primitive notions. Under this point of view, rough set theory may be interpreted as an extension theory with two additional unary operators. Lower and upper approximation operators are related to the necessity (box) and possibility (diamond) operators in modal logic, and the interior and closure operators in topological space [4,5,10,13,15,27–31,33,38,42]. Under this approach, a set of axioms is used to characterize approximation operators that are the same as the ones produced by using constructive approach.

The initiations and majority of studies on rough sets have been concentrated on constructive approaches. In Pawlak's rough set model [19,20], an equivalence relation is a key and primitive notion. This equivalence relation, however, seems to be a very stringent condition that may limit the application domain of the rough set model. From both theoretic and practical needs, many authors have generalized the notion of approximation operators by using non-equivalence binary relations. This has lead to various other approximation operators [25,26,39,40,42,43]. On the other hand, by using an equivalence relation on U, one can introduce lower and upper approximations of fuzzy sets to obtain an extended notion called rough fuzzy set [8,17]. Alternatively, a fuzzy similarity relation can be used to replace an equivalence relation. The result is a deviation of rough set theory called fuzzy rough sets [8,9,42]. More general frameworks can be obtained under fuzzy environment based on fuzzy T-similarity relations [2,3,14], weak fuzzy partitions on U [1,11], and Boolean subalgebras of $\mathscr{P}(U)$ [16] etc.

Comparing with the studies on constructive approach, there is less effort needed on axiomatic approach. Zakowski [45] studied a set of axioms on approximation operators. Comer [6,7] investigated axioms on approximation operators in relation to cylindric algebras. The investigation is made within the context of Pawlak information systems [18]. Lin and Liu [13] suggested six axioms on a pair of abstract operators on the power set of universe in the framework of topological spaces. Under these axioms, there exists an equivalence relation such that the derived lower and upper approximations are the same as the abstract operators. The similar result was also stated earlier by Wiweger [33]. A problem arisen is that all these studies are restricted to Pawlak rough set algebra defined by equivalence relations. Wybraniec-Skardowska [37] examined many axioms on various classes of approximation operators. Different constructive methods were suggested to produce such approximation operators. Thiele [27] explored axiomatic characterizations of approximation operators within modal logic for a crisp diamond and box operator represented by an arbitrary binary crisp relation. The most important axiomatic studies for crisp rough sets were made by Yao [38,39,42], Yao and Lin [43], in which various classes of crisp rough set algebras are characterized by different sets of axioms. As to the fuzzy cases, Moris and Yakout [14] studied a set of axioms on fuzzy rough sets, but their studies were restricted to fuzzy T-rough sets defined by fuzzy T-similarity relations which were equivalence crisp relations when they degenerated into crisp ones. Thiele [28–30] investigated axiomatic characterizations of fuzzy rough approximation operators and rough fuzzy approximation operators within modal logic for fuzzy diamond and box operators. Nevertheless, he did not solve the problem under which the minimal axiom set are necessary and sufficient that there exists a fuzzy relation (and a crisp relation, respectively) producing the same fuzzy rough (and rough fuzzy, respectively) approximation operators. Wu et al. examined many axioms on various classes of fuzzy rough approximation operators [35]. A problem with Wu's study is that the constructive definition of lower and upper fuzzy rough approximation operators is represented by a family of crisp approximation operators, which is rather complicated for computing.

The present paper studies generalized rough set approximation operators in fuzzy environment in which both the constructive and axiomatic approaches are used. In the constructive approach, based on an arbitrary fuzzy (or crisp) relation, a pair of dual generalized fuzzy approximation operators is defined. The crisp approximation operators representations of fuzzy approximation operators are exposed, from which we can see that the pair of lower and upper fuzzy rough approximation operators in the present paper is just an equivalent one defined in [35]. The connections between fuzzy or crisp binary relations and generalized fuzzy approximation operators are examined. The resulting generalized fuzzy rough sets and rough fuzzy sets are proper generalizations of generalized rough sets [39], rough fuzzy sets [8,17], and fuzzy rough set [8,9,41]. In the axiomatic approach, various classes of fuzzy approximation operators are characterized by different sets of axioms, the minimal axiom set of fuzzy approximation operators guarantee the existence of certain types of fuzzy relations or crisp relation producing the same operators.

2. Preliminaries

Let X be a finite and nonempty set called the universe of discourse. The class of all subsets (respectively, fuzzy subsets) of X will be denoted by $\mathscr{P}(X)$ (respectively, by $\mathscr{F}(X)$). For any $A \in \mathscr{F}(X)$, the α-level and the strong α-level of A will be denoted by A_α and $A_{\alpha+}$, respectively, that is, $A_\alpha = \{x \in X : A(x) \geqslant \alpha\}$ and $A_{\alpha+} = \{x \in X : A(x) > \alpha\}$, where $\alpha \in I = [0, 1]$, the unit interval, $A_0 = X$, and $A_{1+} = \emptyset$. We denote by $\sim A$ the complement of A.

Definition 1. Let U and W be two finite and nonempty universes of discourse. A subset $R \in \mathscr{P}(U \times W)$ is referred to as a (crisp) binary relation from U to W. The relation R is referred to as serial if for all $x \in U$ there exists $y \in W$ such that $(x, y) \in R$; If $U = W$, R is referred to as a binary relation on U. R is referred to as reflexive if for all $x \in U$, $(x, x) \in R$; R is referred to as symmetric if for all $x, y \in U$, $(x, y) \in R$ implies $(y, x) \in R$; R is referred to as transitive if for all $x, y, z \in U$, $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$; R is referred to as Euclidean if for all $x, y, z \in U$, $(x, y) \in R$ and $(x, z) \in R$ imply $(y, z) \in R$.

Definition 2. A fuzzy subset $R \in \mathscr{F}(U \times W)$ is referred to as a fuzzy binary relation from U to W, $R(x, y)$ is the degree of relation between x and y, where $(x, y) \in U \times W$; If for each $x \in U$, there exists $y \in W$ such that $R(x, y) = 1$, then R is referred to as a serial fuzzy relation from U to W. If $U = W$, then R is referred to as a fuzzy relation on U; R is referred to as a reflexive fuzzy relation if $R(x, x) = 1$ for all $x \in U$; R is referred to as a symmetric fuzzy relation if $R(x, y) = R(y, x)$ for all $x, y \in U$; R is referred to as a transitive fuzzy relation if $R(x, z) \geqslant \vee_{y \in U} (R(x, y) \wedge R(y, z))$ for all $x, z \in U$.

It is easy to see that R is a serial fuzzy relation iff R_α is a serial crisp binary relation for all $\alpha \in I$; R is a reflexive fuzzy relation iff R_α is a reflexive crisp binary relation for all $\alpha \in I$; R is a symmetric fuzzy relation iff R_α is a symmetric crisp binary relation for all $\alpha \in I$; R is a transitive fuzzy relation iff R_α is a transitive crisp binary relation for all $\alpha \in I$.

Definition 3. Let U and W be two finite universes of discourse. Suppose that R is an arbitrary crisp relation from U to W. We can define a set-valued function $R_s : U \to \mathscr{P}(W)$ by

$$R_s(x) = \{y \in W : (x, y) \in R\}, \quad x \in U.$$

$R_s(x)$ is referred to as the successor neighborhood of x with respect to R. Obviously, any set-valued function F from U to W defines a binary relation from U to W by setting $R = \{(x, y) \in U \times W : y \in F(x)\}$. The triple (U, W, R) is referred to as a generalized crisp approximation space. For any set $A \subseteq W$, a pair of lower and upper approximations, $\underline{R}(A)$ and $\overline{R}(A)$, are defined by

$$\begin{aligned} \underline{R}(A) &= \{x \in U : R_s(x) \subseteq A\}, \\ \overline{R}(A) &= \{x \in U : R_s(x) \cap A \neq \emptyset\}. \end{aligned} \tag{1}$$

The pair $(\underline{R}(A), \overline{R}(A))$ is referred to as a generalized crisp rough set, and $\underline{R}$ and $\overline{R}$ are referred to as lower and upper generalized crisp approximation operators.

From the definition, the following theorem can be easily derived [39].

Theorem 1. *For any relation R from U to W, its lower and upper approximation operators satisfy the following properties: for all $A, B \in \mathscr{P}(W)$,*

$$\begin{array}{ll}
\text{(L1) } \underline{R}(A) = \sim \overline{R}(\sim A), & \text{(U1) } \overline{R}(A) = \sim \underline{R}(\sim A); \\
\text{(L2) } \underline{R}(W) = U, & \text{(U2) } \overline{R}(\emptyset) = \emptyset; \\
\text{(L3) } \underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B), & \text{(U3) } \overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B); \\
\text{(L4) } A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B), & \text{(U4) } A \subseteq B \Rightarrow \overline{R}(A) \subseteq \overline{R}(B); \\
\text{(L5) } \underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B), & \text{(U5) } \overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B).
\end{array}$$

Properties (L1) and (U1) show that the approximation operators $\underline{R}$ and $\overline{R}$ are dual to each other. Properties with the same number may be considered as dual properties. With respect to certain special types, say, serial, reflexive, symmetric, transitive, and Euclidean binary relations on the universe U, the approximation operators have additional properties [38–40].

Theorem 2. *Let R be an arbitrary crisp binary relation on U, and $\underline{R}$ and $\overline{R}$ the lower and upper generalized crisp approximation operators defined by Eq.* (1). *Then*

$$\begin{array}{lll}
(1)\ R \text{ is serial} & \Longleftrightarrow & \text{(L0) } \underline{R}(\emptyset) = \emptyset, \\
 & \Longleftrightarrow & \text{(U0) } \overline{R}(U) = U, \\
 & \Longleftrightarrow & \text{(LU0) } \underline{R}(A) \subseteq \overline{R}(A),\ \forall A \in \mathscr{P}(U), \\
(2)\ R \text{ is reflexive} & \Longleftrightarrow & \text{(L6) } \underline{R}(A) \subseteq A,\ \forall A \in \mathscr{P}(U), \\
 & \Longleftrightarrow & \text{(U6) } A \subseteq \overline{R}(A),\ \forall A \in \mathscr{P}(U), \\
(3)\ R \text{ is symmetric} & \Longleftrightarrow & \text{(L7) } \overline{R}(\underline{R}(A)) \subseteq A,\ \forall A \in \mathscr{P}(U), \\
 & \Longleftrightarrow & \text{(U7) } A \subseteq \underline{R}(\overline{R}(A)),\ \forall A \in \mathscr{P}(U), \\
(4)\ R \text{ is transitive} & \Longleftrightarrow & \text{(L8) } \underline{R}(A) \subseteq \underline{R}(\underline{R}(A)),\ \forall A \in \mathscr{P}(U), \\
 & \Longleftrightarrow & \text{(U8) } \overline{R}(\overline{R}(A)) \subseteq \overline{R}(A),\ \forall A \in \mathscr{P}(U), \\
(5)\ R \text{ is Euclidean} & \Longleftrightarrow & \text{(L9) } \overline{R}(\underline{R}(A)) \subseteq \underline{R}(A),\ \forall A \in \mathscr{P}(U), \\
 & \Longleftrightarrow & \text{(U9) } \overline{R}(A) \subseteq \underline{R}(\overline{R}(A)),\ \forall A \in \mathscr{P}(U).
\end{array}$$

If R is an equivalence relation on U, then the pair (U, R) is the Pawlak approximation space and more interesting properties of lower and upper approximation operators can be derived [19,20].

3. Construction of generalized fuzzy approximation operators

3.1. Definitions of fuzzy approximation operators

Definition 4. Let U and W be two finite non-empty universes of discourse and R a fuzzy relation from U to W. The triple (U, W, R) is called a generalized fuzzy approximation space. For any set $A \in \mathscr{F}(W)$, the lower and upper approximations of A, $\underline{R}(A)$ and $\overline{R}(A)$, with respect to the approximation space (U, W, R) are fuzzy sets of U whose membership functions, for each $x \in U$, are defined, respectively, by

$$\begin{aligned} \overline{R}(A)(x) &= \bigvee_{y \in W} [R(x,y) \wedge A(y)], \quad x \in U, \\ \underline{R}(A)(x) &= \bigwedge_{y \in W} [(1 - R(x,y)) \vee A(y)], \quad x \in U. \end{aligned} \tag{2}$$

The pair $(\underline{R}(A), \overline{R}(A))$ is referred to as a generalized fuzzy rough set, and $\underline{R}$ and $\overline{R} : \mathscr{F}(W) \to \mathscr{F}(U)$ are referred to as lower and upper generalized fuzzy rough approximation operators, respectively.

In special, if R is a crisp relation from U to W, that is, (U, W, R) is a generalized crisp approximation space, the fuzzy rough approximation operators are degenerated to rough fuzzy approximation operators. It can be easily checked that

$$\begin{aligned} \overline{R}(A)(x) &= \bigvee_{y \in R_s(x)} A(y), \quad x \in U, \\ \underline{R}(A)(x) &= \bigwedge_{y \in R_s(x)} A(y), \quad x \in U. \end{aligned} \tag{3}$$

Under such circumstance, the pair $(\underline{R}(A), \overline{R}(A))$ is referred to as a generalized rough fuzzy set.

In sequel, to simplify, we call both of fuzzy rough approximation operators and rough fuzzy approximation operators the fuzzy approximation operators.

3.2. Representations of fuzzy approximation operators

Definition 5. A set-valued mapping $N : I \to \mathscr{P}(U)$ is said to be nested if for all $\alpha, \beta \in I$,

$$\alpha \leqslant \beta \Rightarrow N(\beta) \subseteq N(\alpha). \tag{4}$$

The class of all $\mathscr{P}(U)$-valued nested mappings on I will be denoted by $\mathscr{N}(U)$.

It is well known that the following lemma holds [35]:

Lemma 1. *Let $N \in \mathscr{N}(U)$. Define a function $f : \mathscr{N}(U) \to \mathscr{F}(U)$ by*

$$A(x) := f(N)(x) = \bigvee_{\alpha \in I} \left(\alpha \wedge N(\alpha)(x)\right), \quad x \in U,$$

where $N(\alpha)(x)$ is the characteristic function of $N(\alpha)$. Then f is a surjective homomorphism, and the following properties hold:

$$\begin{aligned}
&(1)\ A_{\alpha+} \subseteq N(\alpha) \subseteq A_{\alpha},\\
&(2)\ A_{\alpha} = \bigcap_{\lambda<\alpha} N(\lambda),\\
&(3)\ A_{\alpha+} = \bigcap_{\lambda>\alpha} N(\lambda),\\
&(4)\ A = \bigvee_{\alpha \in I} \left(\alpha \wedge A_{\alpha+}\right) = \bigvee_{\alpha \in I} \left(\alpha \wedge A_{\alpha}\right).
\end{aligned} \tag{5}$$

Let (U, W, R) be a fuzzy approximation space, i.e. R is a fuzzy relation from U to W, then R_{α} and $R_{\alpha+}$ are two crisp relations from U to W and they induce two crisp approximation spaces (U, W, R_{α}) and $(U, W, R_{\alpha+})$. $\forall X \in \mathscr{P}(W)$, the upper and lower approximations of X with respect to (U, W, R_{α}) and $(U, W, R_{\alpha+})$ are defined respectively as

$$\overline{R_{\alpha}}(X) = \{x \in U : r_{\alpha}(x) \cap X \neq \emptyset\}, \quad \underline{R_{\alpha}}(X) = \{x \in U : r_{\alpha}(x) \subseteq X\},$$
$$\overline{R_{\alpha+}}(X) = \{x \in U : r_{\alpha+}(x) \cap X \neq \emptyset\}, \quad \underline{R_{\alpha+}}(X) = \{x \in U : r_{\alpha+}(x) \subseteq X\},$$

where $r_{\alpha}(x) = \{y \in W : (x, y) \in R_{\alpha}\}$ and $r_{\alpha+}(x) = \{y \in W : (x, y) \in R_{\alpha+}\}$.

It can be easily checked that, $\forall X \in \mathscr{P}(W)$ and $0 \leqslant \alpha \leqslant \beta \leqslant 1$,

$$\begin{aligned}
&(1)\ \underline{R_{\alpha}}(X) \subseteq \underline{R_{\beta}}(X), \quad \underline{R_{\alpha+}}(X) \subseteq \underline{R_{\beta+}}(X),\\
&(2)\ \overline{R_{\beta}}(X) \subseteq \overline{R_{\alpha}}(X), \quad \overline{R_{\beta+}}(X) \subseteq \overline{R_{\alpha+}}(X).
\end{aligned}$$

On the other hand, $\forall A \in \mathscr{F}(W)$ and $0 \leqslant \beta \leqslant 1$, the upper and lower approximations of A_{β} and $A_{\beta+}$ with respect to (U, W, R_{α}) and $(U, W, R_{\alpha+})$ are defined respectively as

$$\overline{R_{\alpha}}(A_{\beta}) = \{x \in U : r_{\alpha}(x) \cap A_{\beta} \neq \emptyset\}, \quad \underline{R_{\alpha}}(A_{\beta}) = \{x \in U : r_{\alpha}(x) \subseteq A_{\beta}\},$$

$$\overline{R_{\alpha}}(A_{\beta+}) = \{x \in U : r_{\alpha}(x) \cap A_{\beta+} \neq \emptyset\}, \quad \underline{R_{\alpha}}(A_{\beta+}) = \{x \in U : r_{\alpha}(x) \subseteq A_{\beta+}\},$$

$$\overline{R_{\alpha+}}(A_{\beta}) = \{x \in U : r_{\alpha+}(x) \cap A_{\beta} \neq \emptyset\}, \quad \underline{R_{\alpha+}}(A_{\beta}) = \{x \in U : r_{\alpha+}(x) \subseteq A_{\beta}\},$$

$$\overline{R_{\alpha+}}(A_{\beta+}) = \{x \in U : r_{\alpha+}(x) \cap A_{\beta+} \neq \emptyset\}, \quad \underline{R_{\alpha+}}(A_{\beta+}) = \{x \in U : r_{\alpha+}(x) \subseteq A_{\beta+}\}.$$

It is easily verified that

(1) $\overline{R_{\alpha+}}(A_{\beta+}) \subseteq \overline{R_{\alpha+}}(A_{\beta}) \subseteq \overline{R_{\alpha}}(A_{\beta})$,

(2) $\overline{R_{\alpha+}}(A_{\beta+}) \subseteq \overline{R_{\alpha}}(A_{\beta+}) \subseteq \overline{R_{\alpha}}(A_{\beta})$,

(3) $\underline{R_{\alpha}}(A_{\beta+}) \subseteq \underline{R_{\alpha+}}(A_{\beta+}) \subseteq \underline{R_{\alpha+}}(A_{\beta})$,

(4) $\underline{R_{\alpha}}(A_{\beta+}) \subseteq \underline{R_{\alpha}}(A_{\beta}) \subseteq \underline{R_{\alpha+}}(A_{\beta})$.

Therefore it can be observed that all of following classes are $\mathscr{P}(U)$ valued nested mapping on I: $\{\underline{R_{1-\alpha}}(A_{\alpha}) : \alpha \in I\}$, $\{\underline{R_{1-\alpha}}(A_{\alpha+}) : \alpha \in I\}$, $\{\underline{R_{(1-\alpha)+}}(A_{\alpha}) : \alpha \in I\}$, $\{\underline{R_{(1-\alpha)+}}(A_{\alpha+}) : \alpha \in I\}$, $\{\overline{R_{\alpha}}(A_{\alpha}) : \alpha \in I\}$, $\{\overline{R_{\alpha}}(A_{\alpha+}) : \alpha \in I\}$, $\{\overline{R_{\alpha+}}(A_{\alpha}) : \alpha \in I\}$, $\{\overline{R_{\alpha+}}(A_{\alpha+}) : \alpha \in I\}$. By Lemma 1, each of them is associated with a fuzzy subset of U. The following theorem shows that the fuzzy rough approximation operators can be represented by the generalized crisp approximation operators.

Theorem 3. *Let (U, W, R) be a fuzzy approximation space and $A \in \mathscr{F}(W)$, then*

$$(1)\ \overline{R}(A) = \bigvee_{\alpha\in I}\left[\alpha \wedge \overline{R_{\alpha}}(A_{\alpha})\right] = \bigvee_{\alpha\in I}\left[\alpha \wedge \overline{R_{\alpha}}(A_{\alpha+})\right] = \bigvee_{\alpha\in I}\left[\alpha \wedge \overline{R_{\alpha+}}(A_{\alpha})\right] = \bigvee_{\alpha\in I}\left[\alpha \wedge \overline{R_{\alpha+}}(A_{\alpha+})\right],$$

$$(2)\ \underline{R}(A) = \bigvee_{\alpha\in I}\left[\alpha \wedge \underline{R_{1-\alpha}}(A_{\alpha})\right] = \bigvee_{\alpha\in I}\left[\alpha \wedge \underline{R_{1-\alpha}}(A_{\alpha+})\right] = \bigvee_{\alpha\in I}\left[\alpha \wedge \underline{R_{(1-\alpha)+}}(A_{\alpha})\right] = \bigvee_{\alpha\in I}\left[\alpha \wedge \underline{R_{(1-\alpha)+}}(A_{\alpha+})\right],$$

and

(3) $[\overline{R}(A)]_{\alpha+} \subseteq \overline{R_{\alpha+}}(A_{\alpha+}) \subseteq \overline{R_{\alpha+}}(A_{\alpha}) \subseteq \overline{R_{\alpha}}(A_{\alpha}) \subseteq [\overline{R}(A)]_{\alpha}$,

(4) $[\overline{R}(A)]_{\alpha+} \subseteq \overline{R_{\alpha+}}(A_{\alpha+}) \subseteq \overline{R_{\alpha}}(A_{\alpha+}) \subseteq \overline{R_{\alpha}}(A_{\alpha}) \subseteq [\overline{R}(A)]_{\alpha}$,

(5) $[\underline{R}(A)]_{\alpha+} \subseteq \underline{R_{1-\alpha}}(A_{\alpha+}) \subseteq \underline{R_{(1-\alpha)+}}(A_{\alpha+}) \subseteq \underline{R_{(1-\alpha)+}}(A_{\alpha}) \subseteq [\underline{R}(A)]_{\alpha}$,

(6) $[\underline{R}(A)]_{\alpha+} \subseteq \underline{R_{1-\alpha}}(A_{\alpha+}) \subseteq \underline{R_{1-\alpha}}(A_{\alpha}) \subseteq \underline{R_{(1-\alpha)+}}(A_{\alpha}) \subseteq [\underline{R}(A)]_{\alpha}$.

Proof. $\forall A \in \mathscr{F}(W)$ and $x \in U$, it should be noted that for all α, $\overline{R_{\alpha}}(A_{\alpha})(x) \in \{0, 1\}$, then

$$\begin{aligned}\bigvee_{\alpha\in I}\left[\alpha\wedge\overline{R_\alpha}(A_\alpha)\right](x) &= \bigvee_{\alpha\in I}\left[\alpha\wedge(\overline{R_\alpha}(A_\alpha)(x))\right] = \sup\{\alpha\in I:\overline{R_\alpha}(A_\alpha)(x)=1\}\\ &= \sup\{\alpha\in I: x\in\overline{R_\alpha}(A_\alpha)\} = \sup\{\alpha\in I: r_\alpha(x)\cap A_\alpha\neq\emptyset\}\\ &= \sup\{\alpha\in I:\exists y\in W[R(x,y)\geqslant\alpha, A(y)\geqslant\alpha]\}\\ &= \sup\{\alpha\in I:\exists y\in W[R(x,y)\wedge A(y)\geqslant\alpha]\}\\ &= \sup\left\{\alpha\in I:\bigvee_{y\in W}\left[R(x,y)\wedge A(y)\right]\geqslant\alpha\right\}\\ &= \bigvee_{y\in W}\left[R(x,y)\wedge A(y)\right] = \overline{R}(A)(x).\end{aligned}$$

It follows that

$$\bigvee_{\alpha\in I}[\alpha\wedge\overline{R_\alpha}(A_\alpha)] = \overline{R}(A).$$

Similarly, it can be proved that

$$\overline{R}(A) = \bigvee_{\alpha\in I}\left[\alpha\wedge\overline{R_\alpha}(A_{\alpha+})\right] = \bigvee_{\alpha\in I}\left[\alpha\wedge\overline{R_{\alpha+}}(A_\alpha)\right] = \bigvee_{\alpha\in I}\left[\alpha\wedge\overline{R_{\alpha+}}(A_{\alpha+})\right].$$

On the other hand, $\forall x\in U$, we have

$$\begin{aligned}\bigvee_{\alpha\in I}\left[\alpha\wedge\underline{R_{1-\alpha}}(A_\alpha)\right](x) &= \bigvee_{\alpha\in I}\left[\alpha\wedge(\underline{R_{1-\alpha}}(A_\alpha)(x))\right] = \sup\{\alpha\in I:\underline{R_{1-\alpha}}(A_\alpha)(x)=1\}\\ &= \sup\{\alpha\in I: r_{1-\alpha}(x)\subseteq A_\alpha\}\\ &= \sup\{\alpha\in I:\forall y\in W[y\in r_{1-\alpha}(x)\Rightarrow y\in A_\alpha]\}\\ &= \sup\{\alpha\in I:\forall y\in W[R(x,y)\geqslant 1-\alpha\Rightarrow A(y)\geqslant\alpha]\}\\ &= \sup\{\alpha\in I:\forall y\in W[1-R(x,y)\leqslant\alpha\Rightarrow A(y)\geqslant\alpha]\}\\ &= \sup\{\alpha\in I:\forall y\in W[(1-R(x,y))\vee A(y)\geqslant\alpha]\}\\ &= \sup\left\{\alpha\in I:\bigwedge_{y\in W}\left[(1-R(x,y))\vee A(y)\right]\geqslant\alpha\right\}\\ &= \bigwedge_{y\in W}\left[(1-R(x,y))\vee A(y)\right].\end{aligned}$$

Thus

$$\underline{R}(A) = \bigvee_{\alpha\in I}\left[\alpha\wedge\underline{R_{1-\alpha}}(A_\alpha)\right].$$

Likewise, it can be derived that

$$\underline{R}(A) = \bigvee_{\alpha\in I}\left[\alpha\wedge\underline{R_{1-\alpha}}(A_{\alpha+})\right] = \bigvee_{\alpha\in I}\left[\alpha\wedge\underline{R_{(1-\alpha)+}}(A_\alpha)\right] = \bigvee_{\alpha\in I}\left[\alpha\wedge\underline{R_{(1-\alpha)+}}(A_{\alpha+})\right].$$

Combining (1), (2) and Lemma 1, we can conclude that (3)–(6) hold. $\square$

If the fuzzy relation in Theorem 3 is replaced by a crisp relation, then we get the representation theorem for rough fuzzy approximation operators.

Theorem 4. *Let* (U, W, R) *be a generalized crisp approximation space and* $A \in \mathscr{F}(W)$, *then*

(1) $\underline{R}(A) = \bigvee_{\alpha \in I} [\alpha \wedge \underline{R}(A_\alpha)] = \bigvee_{\alpha \in I} [\alpha \wedge \underline{R}(A_{\alpha+})]$,

(2) $\overline{R}(A) = \bigvee_{\alpha \in I} [\alpha \wedge \overline{R}(A_\alpha)] = \bigvee_{\alpha \in I} [\alpha \wedge \overline{R}(A_{\alpha+})]$

and

(3) $[\underline{R}(A)]_{\alpha+} \subseteq \underline{R}(A_{\alpha+}) \subseteq \underline{R}(A_\alpha) \subseteq [\underline{R}(A)]_\alpha$,

(4) $[\overline{R}(A)]_{\alpha+} \subseteq \overline{R}(A_{\alpha+}) \subseteq \overline{R}(A_\alpha) \subseteq [\overline{R}(A)]_\alpha$.

3.3. Properties of fuzzy approximation operators

By the representation theorem of fuzzy approximation operators we can see that the pair of upper and lower fuzzy rough approximation operators defined by Eq. (2) is just as the one defined in [35].

Theorem 5. *The lower and upper fuzzy rough (rough fuzzy, respectively) approximation operators,* $\underline{R}$ *and* $\overline{R}$, *defined by Eq.* (2) (*by Eq.* (3), *respectively*) *satisfy the properties*: $\forall A, B \in \mathscr{F}(W)$, $\forall \alpha \in I$,

(FL1) $\underline{R}(A) = \sim \overline{R}(\sim A)$, (FU1) $\overline{R}(A) = \sim \underline{R}(\sim A)$,
(FL2) $\underline{R}(A \vee \widehat{\alpha}) = \underline{R}(A) \vee \widehat{\alpha}$, (FU2) $\overline{R}(A \wedge \widehat{\alpha}) = \overline{R}(A) \wedge \widehat{\alpha}$,
(FL3) $\underline{R}(A \wedge B) = \underline{R}(A) \wedge \underline{R}(B)$, (FU3) $\overline{R}(A \vee B) = \overline{R}(A) \vee \overline{R}(B)$,

where $\widehat{\alpha}$ *is the constant fuzzy set*: $\widehat{\alpha}(x) = \alpha$, *for all* $x \in U$ *and* $x \in W$.

Properties (FL1) and (FU1) show that the fuzzy approximation operators $\underline{R}$ and $\overline{R}$ are dual to each other. Properties with the same number may be regarded as dual properties. It can be checked that

(FL4) $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$, (FU4) $A \subseteq B \Rightarrow \overline{R}(A) \subseteq \overline{R}(B)$,
(FL5) $\underline{R}(A \vee B) \supseteq \underline{R}(A) \vee \underline{R}(B)$, (FU5) $\overline{R}(A \wedge B) \subseteq \overline{R}(A) \wedge \overline{R}(B)$.

Evidently, properties (FL2) and (FU2) imply the following properties (FL2)$'$ and (FU2)$'$:

(FL2)$'$ $\underline{R}(W) = U$, (FU2)$'$ $\overline{R}(\emptyset) = \emptyset$.

Theorem 6. *If* R *is an arbitrary fuzzy relation* (*and crisp relation*, *respectively*) *from* U *to* W, $\underline{R}$ *and* $\overline{R}$ *are the fuzzy rough* (*and rough fuzzy*, *respectively*) *approximation operators defined by Eq.* (2) (*and Eq.* (3), *respectively*). *Then* R *is serial iff one of the following properties holds*:

(FL0) $\underline{R}(\emptyset) = \emptyset$,

(FU0) $\overline{R}(W) = U$,

(FL0)$'$ $\underline{R}(\widehat{\alpha}) = \widehat{\alpha},\ \forall \alpha \in I$,

(FU0)$'$ $\overline{R}(\widehat{\alpha}) = \widehat{\alpha},\ \forall \alpha \in I$,

(FLU0) $\underline{R}(A) \subseteq \overline{R}(A),\ \forall A \in \mathscr{F}(W)$.

Proof. First, by [35, Theorem 3.8] we know that

R is serial $\iff$ (FL0) $\iff$ (FU0) $\iff$ (FLU0).

Second, by the dual properties (FL1) and (FU1) we can observed that

(FU0)$'$ $\iff$ (FL0)$'$.

It is clear that

(FL0)$'$ $\Rightarrow$ (FL0).

It is only to prove that

(FL0) $\Rightarrow$ (FL0)$'$.

In fact, if we assume that (FL0) holds, then by setting $A = \emptyset$ and invoking property (FL2) we have

$$\underline{R}(\widehat{\alpha}) = \underline{R}(\emptyset \vee \widehat{\alpha}) = \underline{R}(\emptyset) \vee \widehat{\alpha} = \emptyset \vee \widehat{\alpha} = \widehat{\alpha}, \quad \forall \alpha \in I.$$

From which we conclude (FL0)$'$. $\square$

In the case of connections between other special fuzzy relations and fuzzy rough approximation operators, according to Theorem 3 and the results in [35], we obtain the following theorem.

Theorem 7. *Let R be an arbitrary fuzzy relation on U and $\underline{R}$ and $\overline{R}$ the lower and upper fuzzy rough approximation operators defined by Eq.* (2). *Then*:

$$\begin{aligned}
&(1)\ R \textit{ is reflexive} \iff \text{(FL6) } \underline{R}(A) \subseteq A,\ \forall A \in \mathscr{F}(U),\\
&\qquad\qquad\qquad\quad \iff \text{(FU6) } A \subseteq \overline{R}(A),\ \forall A \in \mathscr{F}(U),\\
&(2)\ R \textit{ is symmetric} \iff \text{(FL7)}' \ \underline{R}(1_{U-\{x\}})(y) = \underline{R}(1_{U-\{y\}})(x),\ \forall (x,y) \in U \times U,\\
&\qquad\qquad\qquad\quad \iff \text{(FU7)}' \ \overline{R}(1_x)(y) = \overline{R}(1_y)(x),\ \forall (x,y) \in U \times U,\\
&(3)\ R \textit{ is transitive} \iff \text{(FL8) } \underline{R}(A) \subseteq \underline{R}(\underline{R}(A)),\ \forall A \in \mathscr{F}(U),\\
&\qquad\qquad\qquad\quad \iff \text{(FU8) } \overline{R}(\overline{R}(A)) \subseteq \overline{R}(A),\ \forall A \in \mathscr{F}(U).
\end{aligned}$$

Remark 1. Just as pointed out by Wu et al. [35], the results in Theorem 7 with respect to fuzzy rough approximation operators may be seen as the counterparts of Theorem 2 with respect to generalizations of crisp approximation operators corresponding to reflexive, symmetric, and transitive relations. When R is a symmetric fuzzy relation, an example given by Wu et al. [35] illustrated that properties $A \subseteq \underline{R}(\overline{R}(A))$ and $\overline{R}(\underline{R}(A)) \subseteq A$ do not hold for fuzzy rough approximation operators.

A fuzzy relation R is referred to as an Euclidean fuzzy relation on U iff

$$R(y,z) \geqslant \bigvee_{x \in U} \left[R(x,y) \wedge R(x,z)\right], \ \forall (y,z) \in U \times U.$$

It can be easily justified that a fuzzy relation R on U is Euclidean iff R_α is an Euclidean crisp relation for all $\alpha \in I$. It is unfortunate that the counterparts of properties (L9) and (U9) for fuzzy rough approximation operators do not hold.

Example 1. Let $U = \{1,2,3\}$ and a fuzzy relation R on U defined by the fuzzy matrix

$$R = \begin{pmatrix} 0.1 & 1 & 0.6 \\ 0.1 & 0 & 0.6 \\ 0.1 & 0.6 & 0 \end{pmatrix}.$$

It can be easily checked that R is an Euclidean fuzzy relation. Let $A = 1/1 + 1/2 + 0/3 \in \mathscr{F}(U)$, then it can be calculated that

$$\overline{R}(A) = 1/1 + 0.1/2 + 0.6/3, \quad \underline{R}(\overline{R}(A)) = 0.1/1 + 0.6/2 + 0.4/3,$$

thus $\overline{R}(A) \subseteq \underline{R}(\overline{R}(A))$ does not hold.

What properties the Euclidean fuzzy rough approximation operators posses is still a problem to be solved.

In the case of relationships between special crisp relations and rough fuzzy approximation operators, we summarize as the following theorem.

Theorem 8. *Let R be an arbitrary crisp relation on U and $\underline{R}$ and $\overline{R}$ the lower and upper rough fuzzy approximation operators defined by Eq.* (3). *Then*

$$\begin{aligned}
&(1)\ R\ is\ reflexive \iff (\mathrm{FL6})\ \underline{R}(A) \subseteq A,\ \forall A \in \mathscr{F}(U),\\
&\qquad\qquad\qquad\ \iff (\mathrm{FU6})\ A \subseteq \overline{R}(A),\ \forall A \in \mathscr{F}(U),\\
&(2)\ R\ is\ symmetric \iff (\mathrm{FL7})\ \overline{R}(\underline{R}(A)) \subseteq A,\ \forall A \in \mathscr{F}(U),\\
&\qquad\qquad\qquad\ \iff (\mathrm{FU7})\ A \subseteq \underline{R}(\overline{R}(A)),\ \forall A \in \mathscr{F}(U),\\
&\qquad\qquad\qquad\ \iff (\mathrm{FL7})'\ \underline{R}(1_{U-\{x\}})(y) = \underline{R}(1_{U-\{y\}})(x),\ \forall (x,y) \in U \times U,\\
&\qquad\qquad\qquad\ \iff (\mathrm{FU7})'\ \overline{R}(1_x)(y) = \overline{R}(1_y)(x),\ \forall (x,y) \in U \times U,\\
&(3)\ R\ is\ transitive \iff (\mathrm{FL8})\ \underline{R}(A) \subseteq \underline{R}(\underline{R}(A)),\ \forall A \in \mathscr{F}(U),\\
&\qquad\qquad\qquad\ \iff (\mathrm{FU8})\ \overline{R}(\overline{R}(A)) \subseteq \overline{R}(A),\ \forall A \in \mathscr{F}(U),\\
&(4)\ R\ is\ Euclidean \iff (\mathrm{FL9})\ \overline{R}(\underline{R}(A)) \subseteq \underline{R}(A),\ \forall A \in \mathscr{F}(U),\\
&\qquad\qquad\qquad\ \iff (\mathrm{FU9})\ \overline{R}(A) \subseteq \underline{R}(\overline{R}(A)),\ \forall A \in \mathscr{F}(U),
\end{aligned}$$

where 1_y denotes the fuzzy singleton with value 1 at y and 0 elsewhere.

Proof. Since a crisp relation may be treated as a special fuzzy relation, we need only prove (2) and (4) thanks to Theorems 2 and 7.

(2) By the dual properties of rough fuzzy approximation operators and according to Theorem 7, it is only to prove that

$$R \text{ is symmetric} \iff (\mathrm{FU7}).$$

Assume that R is a symmetric crisp relation, for any $\alpha \in I$, in terms of Theorems 2 and 4 we have

$$A_\alpha \subseteq \underline{R}(\overline{R}(A_\alpha)) \subseteq \underline{R}((\overline{R}(A))_\alpha),\ \forall A \in \mathscr{F}(U).$$

Then by the representation theorem we can conclude that

$$A = \bigvee_{\alpha\in I} (\alpha \wedge A_\alpha) \subseteq \bigvee_{\alpha\in I} (\alpha \wedge \underline{R}((\overline{R}(A))_\alpha)) = \underline{R}(\overline{R}(A)),\ \forall A \in \mathscr{F}(U),$$

that is, property (FU7) holds.

Conversely, since property (FU7) implies property (U7), by Theorem 2 it is clear that

$$(\mathrm{FU7}) \Rightarrow R \text{ is symmetric.}$$

(4) By the duality it is only to prove that

$$R \text{ is Euclidean} \iff (\mathrm{FU9}).$$

Assume that R is an Euclidean crisp relation, for any $\alpha \in I$, in terms of Theorems 2 and 4 we have

$$\overline{R}(A_\alpha) \subseteq \underline{R}(\overline{R}(A_\alpha)) \subseteq \underline{R}((\overline{R}(A))_\alpha),\ \forall A \in \mathscr{F}(U).$$

Then by the representation theorem we can conclude that

$$\overline{R}(A) = \bigvee_{\alpha\in I} (\alpha \wedge \overline{R}(A_\alpha)) \subseteq \bigvee_{\alpha\in I} (\alpha \wedge \underline{R}((\overline{R}(A))_\alpha)) = \underline{R}(\overline{R}(A)),\ \forall A \in \mathscr{F}(U),$$

from which property (FU9) holds.

On the other hand, since property (FU9) implies property (U9), by Theorem 2 we can observe that

$$(\text{FU9}) \Rightarrow R \text{ is Euclidean.} \qquad \square$$

4. Axiomatic characterization of fuzzy approximation operators

In an axiomatic approach, rough sets are axiomatized by abstract operators. For the case of fuzzy rough sets and rough fuzzy sets, the primitive notion is a system $(\mathscr{F}(U), \mathscr{F}(W), \wedge, \vee, \sim, L, H)$, where $L, H : \mathscr{F}(W) \to \mathscr{F}(U)$ are operators from $\mathscr{F}(W)$ to $\mathscr{F}(U)$. In this section, we show that fuzzy rough approximation operators and rough fuzzy approximation operators can be characterized by axioms, the results may be viewed as the generalized counterparts of Yao [38,39,42].

Definition 6. Let $L, H : \mathscr{F}(W) \to \mathscr{F}(U)$ be two operators. They are referred to as dual operators if for all $A \in \mathscr{F}(W)$,

(fl1) $L(A) = \sim H(\sim A)$,

(fu1) $H(A) = \sim L(\sim A)$.

By the dual properties of lower and upper approximation operators, we only need to define one operator.

According to Theorem 3 and [35, Theorem 4.2] we can easily obtain Theorem 9.

Theorem 9. *Suppose that $L, H : \mathscr{F}(W) \to \mathscr{F}(U)$ are two dual operators. Then there exists a fuzzy binary relation R from U to W such that $L(A) = \underline{R}(A)$ and $H(A) = \overline{R}(A)$ for all $A \in \mathscr{F}(W)$ iff L satisfies the axioms* (fl2) *and* (fl3), *or equivalently H satisfies axioms* (fu2) *and* (fu3):

(fl2) $L(A \vee \widehat{\alpha}) = L(A) \vee \widehat{\alpha},\ \forall A \in \mathscr{F}(W),\ \forall \alpha \in I$;

(fl3) $L(A \wedge B) = L(A) \wedge L(B),\ \forall A, B \in \mathscr{F}(W)$;

(fu2) $H(A \wedge \widehat{\alpha}) = H(A) \wedge \widehat{\alpha},\ \forall A \in \mathscr{F}(W),\ \forall \alpha \in I$;

(fu3) $H(A \vee B) = H(A) \vee H(B),\ \forall A, B \in \mathscr{F}(W)$.

Theorem 10. *Suppose that $L, H : \mathscr{F}(W) \to \mathscr{F}(U)$ are two dual operators. Then there exists a crisp binary relation R from U to W such that $L(A) = \underline{R}(A)$ and $H(A) = \overline{R}(A)$ for all $A \in \mathscr{F}(W)$ iff L satisfies axioms* (flc), (fl2), *and* (fl3), *or equivalently H satisfies axioms* (fuc), (fu2), *and* (fu3):

(flc) $L(1_{W-\{y\}}) \in \mathscr{P}(U),\ \forall y \in W,$
(fl2) $L(A \vee \widehat{\alpha}) = L(A) \vee \widehat{\alpha},\ \forall A \in \mathscr{F}(W),\ \forall \alpha \in I;$
(fl3) $L(A \wedge B) = L(A) \wedge L(B),\ \forall A, B \in \mathscr{F}(W);$
(fuc) $H(1_y) \in \mathscr{P}(U),\ \forall y \in W;$
(fu2) $H(A \wedge \widehat{\alpha}) = H(A) \wedge \widehat{\alpha},\ \forall A \in \mathscr{F}(W),\ \forall \alpha \in I;$
(fu3) $H(A \vee B) = H(A) \vee H(B),\ \forall A, B \in \mathscr{F}(W).$

Proof. "$\Rightarrow$" follows immediately from Theorem 2.

"$\Leftarrow$" Suppose that the operator H obeys the axioms (fuc), (fu2) and (fu3). Using H and axiom (fuc), we can define a crisp relation R from U to W by

$$(x, y) \in R \Longleftrightarrow R(x, y) = 1 \Longleftrightarrow H(1_y)(x) = 1, \quad (x, y) \in U \times W,$$

$$(x, y) \notin R \Longleftrightarrow R(x, y) = 0 \Longleftrightarrow H(1_y)(x) = 0, \quad (x, y) \in U \times W.$$

It is evident that for all $A \in \mathscr{F}(W)$,

$$A = \bigvee_{y \in W} \left(1_y \wedge \widehat{A(y)}\right).$$

Then for any $x \in U$, by (fu2) and (fu3) we have

$$\begin{aligned}
\overline{R}(A)(x) &= \overline{R}\left(\bigvee_{y \in W} \left(1_y \wedge \widehat{A(y)}\right)\right)(x) = \bigvee_{y \in W} \left[\overline{R}(1_y \wedge \widehat{A(y)})\right](x) \\
&= \bigvee_{y \in W} \left[\overline{R}(1_y) \wedge \widehat{A(y)}\right](x) = \bigvee_{y \in W} \left[\overline{R}(1_y)(x) \wedge A(y)\right] \\
&= \bigvee_{y \in W} \left[R(x, y) \wedge A(y)\right] = \bigvee_{y \in W} \left(H(1_y)(x) \wedge A(y)\right) \\
&= H\left(\bigvee_{y \in W} \left(1_y \wedge \widehat{A(y)}\right)\right)(x) = H(A)(x),
\end{aligned}$$

which implies that $H(A) = \overline{R}(A)$.

$L(A) = \underline{R}(A)$ follows immediately from the conclusion $H(A) = \overline{R}(A)$ and the dual axioms (fl1) and (fu1). □

Remark 2. According to Theorem 9, axioms (fl1), (fu1), (fl2), and (fl3) or equivalently, axioms (fl1), (fu1), (fu2), and (fu3) are considered to be basic axioms (minimal axiom set) of fuzzy rough approximation operators. Analogically, in terms of Theorem 10, axioms (flc), (fl1), (fu1), (fl2), and (fl3), or equivalently, axioms (fuc), (fl1), (fu1), (fu2), and (fu3) are considered to be basic axioms of rough fuzzy approximation operators. These lead to the following definitions of fuzzy rough set algebra and rough fuzzy set algebra.

Definition 7. Let $L, H : \mathscr{F}(W) \to \mathscr{F}(U)$ be a pair of dual operators. If L satisfies axioms (fl2) and (fl3), or equivalently H satisfies axioms (fu2) and (fu3), then the system $(\mathscr{F}(W), \mathscr{F}(U), \wedge, \vee, \sim, L, H)$ is referred to as a fuzzy rough set algebra, and L and H are referred to as lower and upper fuzzy rough approximation operators respectively. If L satisfies axioms (flc), (fl2), and (fl3), or equivalently H satisfies axioms (fuc), (fu2), and (fu3), then the system $(\mathscr{F}(W), \mathscr{F}(U), \wedge, \vee, \sim, L, H)$ is referred to as a rough fuzzy set algebra, and L and H are referred to as lower and upper rough fuzzy approximation operators respectively.

Theorem 11. *Suppose that $L, H : \mathscr{F}(W) \to \mathscr{F}(U)$ is a pair of dual fuzzy rough approximation operators, i.e. L satisfies axioms* (fl1), (fl2), *and* (fl3), *and H satisfies axioms* (fu1), (fu2), *and* (fu3). *Then there exists a serial fuzzy relation R from U to W such that $L(A) = \underline{R}(A)$ and $H(A) = \overline{R}(A)$ for all $A \in \mathscr{F}(W)$ iff L satisfies axiom* (fl0), *or equivalently H satisfies axiom* (fu0):

(fl0) $L(\widehat{\alpha}) = \widehat{\alpha}, \ \forall \alpha \in I,$

(fu0) $H(\widehat{\alpha}) = \widehat{\alpha}, \ \forall \alpha \in I.$

Analogically, if $L, H : \mathscr{F}(W) \to \mathscr{F}(U)$ is a pair of dual rough fuzzy approximation operators, i.e. L satisfies axioms (flc), (fl1), (fl2), *and* (fl3), *and H satisfies* (fuc), (fu1), (fu2), *and* (fu3). *Then there exists a serial crisp relation R from U to W such that $L(A) = \underline{R}(A)$ and $H(A) = \overline{R}(A)$ for all $A \in \mathscr{F}(W)$ iff L satisfies axiom* (fl0), *or equivalently H satisfies axiom* (fu0).

Proof. "$\Rightarrow$" follows immediately from Theorem 6, and "$\Leftarrow$" follows immediately from Theorem 9, Theorem 10, and Theorem 6. □

Remark 3. By Theorem 6, it can be easily seen that axioms (fl0) and (fu0) can be replaced by one of the following axioms:

(fl0)$'$ $L(\emptyset) = \emptyset,$

(fu0)$'$ $H(W) = U,$

(flu0)$'$ $L(A) \subseteq H(A), \ \forall A \in \mathscr{F}(W).$

Theorem 12. *Suppose that $L, H : \mathscr{F}(U) \to \mathscr{F}(U)$ is a pair of dual fuzzy rough approximation operators, i.e. L satisfies axioms* (fl1), (fl2), *and* (fl3), *and H satisfies axioms* (fu1), (fu2), *and* (fu3). *Then there exists a reflexive fuzzy relation R on U such that $L(A) = \underline{R}(A)$ and $H(A) = \overline{R}(A)$ for all $A \in \mathscr{F}(U)$ iff L satisfies axiom* (fl6), *or equivalently H satisfies axiom* (fu6):

(fl6) $L(A) \subseteq A, \forall A \in \mathscr{F}(U),$

(fu6) $A \subseteq H(A), \forall A \in \mathscr{F}(U).$

Similarly, if $L, H : \mathscr{F}(U) \to \mathscr{F}(U)$ *is a pair of dual rough fuzzy approximation operators, i.e. L satisfies axioms* (flc), (fl1), (fl2), *and* (fl3), *and H satisfies axioms* (fuc), (fu1), (fu2), *and* (fu3). *Then there exists a reflexive crisp relation R on U such that* $L(A) = \underline{R}(A)$ *and* $H(A) = \overline{R}(A)$ *for all* $A \in \mathscr{F}(U)$ *iff L satisfies axiom* (fl6), *or equivalently H satisfies axiom* (fu6).

Proof. "⇒" follows immediately from Theorems 7 and 8, and "⇐" follows immediately from Theorems 7–10. □

Theorem 13. *Suppose that* $L, H : \mathscr{F}(U) \to \mathscr{F}(U)$ *is a pair of dual fuzzy rough approximation operators, i.e. L satisfies axioms* (fl1), (fl2), *and* (fl3), *and H satisfies axioms* (fu1), (fu2), *and* (fu3). *Then there exists a symmetric fuzzy relation R on U such that* $L(A) = \underline{R}(A)$ *and* $H(A) = \overline{R}(A)$ *for all* $A \in \mathscr{F}(U)$ *iff L satisfies axiom* (fl7)′, *or equivalently H satisfies axiom* (fu7)′:

(fl7)′ $L(1_{U-\{x\}})(y) = L(1_{U-\{y\}})(x), \ \forall (x, y) \in U \times U,$

(fu7)′ $H(1_x)(y) = H(1_y)(x), \ \forall (x, y) \in U \times U.$

Moreover if $L, H : \mathscr{F}(U) \to \mathscr{F}(U)$ *is a pair of dual rough fuzzy approximation operators, i.e. L satisfies axioms* (flc), (fl1), (fl2), *and* (fl3), *and H satisfies axioms* (fuc), (fu1), (fu2), *and* (fu3). *Then there exists a symmetric crisp relation R on U such that* $L(A) = \underline{R}(A)$ *and* $H(A) = \overline{R}(A)$ *for all* $A \in \mathscr{F}(U)$ *iff L satisfies axiom* (fl7), *or equivalently H satisfies axiom* (fu7):

(fl7) $A \subseteq L(H(A)), \ \forall A \in \mathscr{F}(U),$

(fu7) $H(L(A)) \subseteq A, \ \forall A \in \mathscr{F}(U).$

Proof. "⇒" follows immediately from Theorems 7 and 8, and "⇐" follows immediately from Theorems 7–10. □

Remark 4. For rough fuzzy approximation operators, axioms (fl7) and (fu7) can be replaced by axioms (fl7)′ and (fu7)′ respectively. Nevertheless, axioms (fl7)′ and (fu7)′ of fuzzy rough approximation operators cannot be replaced by axioms (fl7) and (fu7). In fact, the next theorem shows that, under the axioms (fl2), (fl3), (fu2), (fu3), axioms (fl7) and (fu7) imply axioms (flc) and (fuc), that is, the axiom set (fl1), (fu1), (fl2), (fl3), (fl7),(or equivalently the axiom set (fl1), (fu1), (fu2), (fu3), (fu7)) is the minimal axiom set to characterize the symmetric rough fuzzy set algebra.

Theorem 14. *Let* $L, H : \mathscr{F}(U) \to \mathscr{F}(U)$ *be a pair of dual operators. If L satisfies axioms* (fl2) *and* (fl3), *or equivalently H satisfies axioms* (fu2) *and* (fu3), *then axiom* (fu7) *implies axiom* (fuc), *i.e.*

$$H(L(A)) \subseteq A, \ \forall A \in \mathscr{F}(U) \Rightarrow H(1_y) \in \mathscr{P}(U), \ \forall y \in U. \tag{6}$$

Proof. It should be noted that $\forall A \in \mathscr{F}(U)$ we have

$$A = \bigvee_{y \in U} \left[1_y \wedge \underline{\underline{A(y)}}\right], \tag{7}$$

where $\underline{\underline{\alpha}} = \widehat{\alpha}$ denotes the constant fuzzy set.

By axioms (fu2) and (fu3) we can obtain that

$$H(A) = \bigvee_{y \in U} \left[H(1_y) \wedge \underline{\underline{A(y)}}\right]. \tag{8}$$

According to the dual properties of L and H, we can easily conclude that

$$L(A) = \bigwedge_{y \in U} \left[(1 - H(1_y)) \vee \underline{\underline{A(y)}}\right]. \tag{9}$$

$\forall z \in U$, let $A = 1_{U-\{z\}}$, then by Eq. (9) we have

$$\begin{aligned} L(A) &= \bigwedge_{y \in U} \left[(1 - H(1_y)) \vee \underline{\underline{A(y)}}\right] \\ &= \bigwedge_{y \in U} \left[(1 - H(1_y)) \vee \underline{\underline{1_{U-\{z\}}(y)}}\right] = 1 - H(\sim A) = 1 - H(1_z). \end{aligned}$$

As a result, by using Eq. (8) we have

$$H(L(A)) = \bigvee_{y \in U} \left[H(1_y) \wedge \underline{\underline{(1 - H(1_z)(y))}}\right].$$

By the assumption we derive

$$H(L(A))(x) = \bigvee_{y \in U} \left[H(1_y)(x) \wedge (1 - H(1_z)(y))\right] \leqslant A(x) = 1_{U-\{z\}}(x), \ \forall x \in U.$$

Let $x = z$, then

$$\bigvee_{y \in U} \left[H(1_y)(z) \wedge (1 - H(1_z)(y))\right] = 0.$$

It means that

$$H(1_y)(z) \wedge (1 - H(1_z)(y)) = 0, \ \forall (y, z) \in U \times U. \tag{10}$$

Therefore either $H(1_y)(z) = 0$ or $H(1_z)(y) = 1$. If $H(1_y)(z) \neq 0$, then $H(1_z)(y) = 1$. It follows by Eq. (10) that

$$H(1_z)(y) \wedge (1 - H(1_y)(z))] = 0,$$

from which we can observe that $H(1_y)(z) = 1$.

Thus we have proved that $\forall (y, z) \in U \times U$, $H(1_y)(z) \in \{0, 1\}$, i.e. $H(1_y) \in \mathscr{P}(U)$. $\square$

Theorem 15. *Suppose that* $L, H : \mathscr{F}(U) \to \mathscr{F}(U)$ *is a pair of dual fuzzy rough approximation operators, i.e. L satisfies axioms* (fl1), (fl2), *and* (fl3), *and H satisfies axioms* (fu1), (fu2), *and* (fu3). *Then there exists a transitive fuzzy relation R on U such that* $L(A) = \underline{R}(A)$ *and* $H(A) = \overline{R}(A)$ *for all* $A \in \mathscr{F}(U)$ *iff L satisfies axiom* (fl8), *or equivalently H satisfies axiom* (fu8):

(fl8) $L(A) \subseteq L(L(A)),\ \forall A \in \mathscr{F}(U)$,

(fu8) $H(H(A)) \subseteq H(A),\ \forall A \in \mathscr{F}(U)$.

Similarly, if $L, H : \mathscr{F}(U) \to \mathscr{F}(U)$ *is a pair of dual rough fuzzy approximation operators, i.e. L satisfies axioms* (flc), (fl1), (fl2), *and* (fl3), *and H satisfies axioms* (fuc), (fu1), (fu2), *and* (fu3). *Then there exists a transitive crisp relation R on U such that* $L(A) = \underline{R}(A)$ *and* $H(A) = \overline{R}(A)$ *for all* $A \in \mathscr{F}(U)$ *iff L satisfies axiom* (fl8), *or equivalently H satisfies axiom* (fu8).

Proof. "⇒" follows immediately from Theorems 7 and 8, and "⇐" follows immediately from Theorems 7–10. □

Theorem 16. *Suppose that* $L, H : \mathscr{F}(U) \to \mathscr{F}(U)$ *is a pair of dual rough fuzzy approximation operators, i.e. L satisfies axioms* (flc), (fl1), (fl2), *and* (fl3), *and H satisfies axioms* (fuc), (fu1), (fu2), *and* (fu3). *Then there exists an Euclidean crisp relation R on U such that* $L(A) = \underline{R}(A)$ *and* $H(A) = \overline{R}(A)$ *for all* $A \in \mathscr{F}(U)$ *iff L satisfies axiom* (fl9), *or equivalently H satisfies axiom* (fu9):

(fl9) $H(L(A)) \subseteq L(A),\ \forall A \in \mathscr{F}(U)$,

(fu9) $H(A) \subseteq L(H(A)),\ \forall A \in \mathscr{F}(U)$.

Proof. "⇒" follows immediately from Theorem 8, and "⇐" follows immediately from Theorems 8 and 10. □

5. Conclusion

There are at least two aspects in the study of rough set theory: constructive and axiomatic approaches. In constructive approaches, the lower and upper approximation operators are defined in terms of binary relations, partitions of the universe, neighborhood systems or Boolean subalgebras of $\mathscr{P}(U)$. The axiomatic approaches consider the reverse problem, namely, the lower and upper approximation operators are taken as primitive notions. A set of axioms is used to characterize approximation operators that are the same as those derived by using constructive approaches.

In this paper, we have developed a general framework for the study of rough set approximation operators in fuzzy environment in which both constructive and axiomatic approaches are considered. In our constructive method, generalized fuzzy rough sets and generalized rough fuzzy sets are derived from a fuzzy approximation space and a crisp approximation space respectively. The fuzzy approximation operators may be also composed by a family of crisp approximation operators, from which the definition of the pair of lower and upper fuzzy rough approximation operators defined in our previous paper [35] is simplified. In axiomatic approach, fuzzy approximation operators can be characterized by axioms. Minimal axiom sets of fuzzy approximation operators guarantee the existence of certain types of fuzzy relations or crisp relations producing the same operators. This work may be viewed as the extension of Yao [38,39,42], and it may be also taken as the completeness of Thiele [28–30] and Wu et al. [35]. We believe that the constructive approaches we offer here will turn out to be more useful for practical applications of the rough set theory while the axiomatic approaches will help us to gain much more insights into the mathematical structures of fuzzy approximation operators.

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References

[1] S. Bodjanova, Approximation of a fuzzy concepts in decision making, Fuzzy Sets and Systems 85 (1997) 23–29.
[2] D. Boixader, J. Jacas, J. Recasens, Upper and lower approximations of fuzzy sets, International Journal of General Systems 29 (4) (2000) 555–568.
[3] K. Chakrabarty, R. Biswas, S. Nanda, Fuzziness in rough sets, Fuzzy Sets and Systems 110 (2000) 247–251.
[4] M. Chuchro, On rough sets in topological Boolean algebras, in: W. Ziarko (Ed.), Rough Sets, Fuzzy Sets and Knowledge Discovery, Springer-Verlag, Berlin, 1994, pp. 157–160.
[5] M. Chuchro, A certain conception of rough sets in topological Boolean algebras, Bulletin of the Section of Logic 22 (1) (1993) 9–12.
[6] S. Comer, An algebraic approach to the approximation of information, Fundamenta Informaticae 14 (1991) 492–502.
[7] S. Comer, On connections between information systems, rough sets, and algebraic logic, in: C. Rauszer (Ed.), Algebraic Methods in Logic and Computer Science, vol. 28, Banach Center Publisher, Polish Academy of Sciences, 1993, pp. 117–127.
[8] D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, International Journal of General Systems 17 (1990) 191–208.
[9] D. Dubois, H. Prade, Twofold fuzzy sets and rough sets—some issues in knowledge representation, Fuzzy Sets and Systems 23 (1987) 3–18.
[10] J. Kortelainen, On relationship between modified sets, topological space and rough sets, Fuzzy Sets and Systems 61 (1994) 91–95.

[11] L.I. Kuncheva, Fuzzy rough sets: Application to feature selection, Fuzzy Sets and Systems 51 (1992) 147–153.
[12] T.Y. Lin, Neighborhood systems—application to qualitative fuzzy and rough sets, in: P.P. Wang (Ed.), Advances in Machine Intelligence and Soft-Computing, Department of Electrical Engineering, Duke University, Durham, NC, USA, 1997, pp. 132–155.
[13] T.Y. Lin, Q. Liu, Rough approximate operators: axiomatic rough set theory, in: W. Ziarko (Ed.), Rough Sets, Fuzzy Sets and Knowledge Discovery, Springer, Berlin, 1994, pp. 256–260.
[14] N.N. Morsi, M.M. Yakout, Axiomatics for fuzzy rough sets, Fuzzy Sets and Systems 100 (1998) 327–342.
[15] A. Nakamura, J.M. Gao, On a KTB-modal fuzzy logic, Fuzzy Sets and Systems 45 (1992) 327–334.
[16] S. Nanda, S. Majumda, Fuzzy rough sets, Fuzzy Sets and Systems 45 (1992) 157–160.
[17] S.K. Pal, Roughness of a fuzzy set, Information Sciences 93 (1996) 235–246.
[18] Z. Pawlak, Information systems, theoretical foundations, Information Systems 6 (1981) 205–218.
[19] Z. Pawlak, Rough sets, International Journal of Computer and Information Science 11 (1982) 341–356.
[20] Z. Pawlak, Rough Sets: Theoretical Aspects of Reasoning about Data, Kluwer Academic Publishers, Boston, 1991.
[21] J.A. Pomykala, Approximation operations in approximation space, Bulletin of the Polish Academy of Sciences: Mathematics 35 (1987) 653–662.
[22] A. Skowron, The relationship between the rough set theory and evidence theory, Bulletin of Polish Academy of Science: Mathematics 37 (1989) 87–90.
[23] A. Skowron, The rough sets theory and evidence theory, Fundamenta Informatica XIII (1990) 162–245.
[24] A. Skowron, J. Grzymala-Busse, From rough set theory to evidence theory, in: R.R. Yager, M. Fedrizzi, J. Kacprzyk (Eds.), Advances in the Dempster–Shafer Theory of Evidence, Wiley, New York, 1994, pp. 193–236.
[25] R. Slowinski, D. Vanderpooten, Similarity relation as a basis for rough approximations, in: P.P. Wang (Ed.), Advances in Machine Intelligence and Soft-Computing, Department of Electrical Engineering, Duke University, Durham, NC, USA, 1997, pp. 17–33.
[26] R. Slowinski, D. Vanderpooten, A Generalized definition of rough approximations based on similarity, IEEE Transactions on Knowledge and Data Engineering 12 (2) (2000) 331–336.
[27] H. Thiele, On axiomatic characterisations of crisp approximation operators, Information Sciences 129 (2000) 221–226.
[28] H. Thiele, On axiomatic characterisation of fuzzy approximation operators I, the fuzzy rough set based case, RSCTC 2000, Banff Park Lodge, Bariff, Canada, 19–19 October 2000, Conference Proceedings, pp. 239–247.
[29] H. Thiele, On axiomatic characterisation of fuzzy approximation operators II, the rough fuzzy set based case, Proceedings of the 31st IEEE International Symposium on Multiple-Valued Logic, 2001, pp. 330–335.
[30] H. Thiele, On axiomatic characterization of fuzzy approximation operators III, the fuzzy diamond and fuzzy box cases, The 10th IEEE International Conference on Fuzzy Systems 2001, vol. 2, pp. 1148–1151.
[31] D. Vakarelov, A modal logic for similarity relations in Pawlak knowledge representation systems, Fundamenta Informaticae 15 (1991) 61–79.
[32] A. Wasilewska, Conditional knowledge representation systems—model for an implementation, Bulletin of the Polish Academy of Sciences: Mathematics 37 (1990) 63–69.
[33] R. Wiweger, On topological rough sets, Bulletin of Polish Academy of Sciences: Mathematics 37 (1989) 89–93.

[34] W.-Z. Wu, Y. Leung, W.-X. Zhang, Connections between rough set theory and Dempster–Shafer theory of evidence, International Journal of General Systems 31 (4) (2002) 405–430.
[35] W.-Z. Wu, J.-S. Mi, W.-X. Zhang, Generalized fuzzy rough sets, Information Sciences 151 (2003) 263–282.
[36] W.-Z. Wu, W.-X. Zhang, Neighborhood operator systems and approximations, Information Sciences 144 (1–4) (2002) 201–217.
[37] U. Wybraniec-Skardowska, On a generalization of approximation space, Bulletin of the Polish Academy of Sciences: Mathematics 37 (1989) 51–61.
[38] Y.Y. Yao, Constructive and algebraic methods of the theory of rough sets, Journal of Information Sciences 109 (1998) 21–47.
[39] Y.Y. Yao, Generalized rough set model, in: L. Polkowski, A. Skowron (Eds.), Rough Sets in Knowledge Discovery 1. Methodology and Applications, Physica-Verlag, Heidelberg, 1998, pp. 286–318.
[40] Y.Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, Information Sciences 111 (1998) 239–259.
[41] Y.Y. Yao, Combination of rough and fuzzy sets based on α-level sets, in: T.Y. Lin, N. Cercone (Eds.), Rough Sets and Data Mining: Analysis for Imprecise Data, Kluwer Academic Publishers, Boston, 1997, pp. 301–321.
[42] Y.Y. Yao, Two views of the theory of rough sets in finite universes, International Journal of Approximate Reasoning 15 (1996) 291–317.
[43] Y.Y. Yao, T.Y. Lin, Generalization of rough sets using modal logic, Intelligent Automation and Soft Computing, an International Journal 2 (1996) 103–120.
[44] Y.Y. Yao, P.J. Lingras, Interpretations of belief functions in the theory of rough sets, Information Sciences 104 (1998) 81–106.
[45] W. Zakowski, On a concept of rough sets, Demonstratio Mathematica XV (1982) 1129–1133.