Postulates of Quantum Mechanics

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The four postulates of Quantum Mechanics

- Quantum computing devices take advantage of laws of Quantum Mechanics to provide non-trivial speed up over classical problems
- The four postulates of QM revolve around the following ideas
  - How can a quantum system be described?
  - How to operate and perform actions on the quantum system?
  - How to describe multiple quantum systems working together?
  - How to use and process information from quantum systems?

Postulate 1: Individual Quantum Systems

- In classical computing we have a bit which can have the value either 0 or 1. In the quantum world, there is a radical shift in this concept.
- A quantum bit or a *'qubit'* can assume both values 0 and 1 at the same time!
- The bit values 0 and 1 are encoded in the standard basis vectors $|0\rangle$ and $|1\rangle$
- And their superposition looks like this

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{1}$$

where the amplitudes α and β determine the extent to which the qubit is in state $|0\rangle$ or $|1\rangle$

Figure 1: A spinning coin is a probabilistic classical system

Postulate 1: Individual Quantum Systems

- There is a restriction that $|\psi\rangle$ must be a unit vector.

$$|\alpha|^2 + |\beta|^2 = 1 \tag{2}$$

- And here, $|\alpha|^2$ and $|\beta|^2$ determines the probability with which we will get the state $|0\rangle$ or $|1\rangle$ after measuring the qubit or the vector $|\psi\rangle$
- Qubit is a 2-dimensional quantum system. We can also describe an n-dimensional quantum system called *'qudit'* as follows

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + ...\alpha_{d-1}|d-1\rangle = \sum_{i=0}^{d-1} \alpha_i|i\rangle \tag{3}$$

Representing a Qubit

There are 3 primary ways to represent a qubit

- **Vector Notation**

$$\vec{\psi} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{4}$$

- **Dirac Notation**

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5}$$

The superposition is defined as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{6}$$

Representing a Qubit

- **Bloch Sphere** Bloch Sphere lets us represent the qubit geometrically. It helps us see the state of a qubit visually. We represent the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ on the sphere as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\varphi}|1\rangle \tag{7}$$

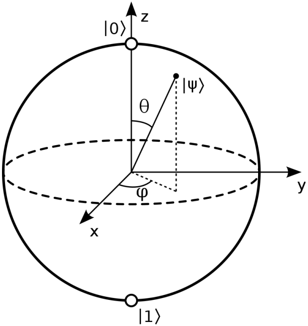

Figure 2: Bloch Sphere

Postulate 2: Quantum Operations

- Since a qubit can be represented as a vector, we can perform multiplication by a matrix to perform operations on a qubit
- But there are restrictions on the type of matrices we can use
  - As from equation 2, $|\psi\rangle$ must remain a unit vector after the operation
  - The operation must be reversible in nature i.e. if you can determine the input from the output and vice versa, the operation is reversible!
- Matrices satisfying the above conditions are called Unitary matrices, usually represented by U. A matrix is unitary if $UU^\dagger = U^\dagger U = I$

Basic Quantum Gates

The most common single qubit gates are know as the *Pauli* Gates after Austrian-Swiss physicist Wolfgang Pauli

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8}$$

- The X gate acts as the 'quantum' NOT gate. It is also called the 'bit flip' gate.

$$X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle \tag{9}$$

$$X|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle \tag{10}$$

Basic Quantum Gates

- The Z gate has no classical analogue. It is called the 'phase flip' gate. You'll see it injects a 'phase' of -1 in front of $|1\rangle$.

$$Z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle \tag{11}$$

$$Z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -|1\rangle \tag{12}$$

- Z-gate is special because it injects a *'relative phase'* into the quantum state.

$$\begin{aligned} Z\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) &= \frac{1}{\sqrt{2}} Z|0\rangle + \frac{1}{\sqrt{2}} Z|1\rangle \\ &= \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \end{aligned} \tag{13}$$

Basic Quantum Gates

- Another important unitary gate is the *Hadamard Gate*:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{14}$$

- Hadamard gate is special because it can create and erase superpositions.

$$H|0\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big) = |+\rangle\,, \quad H|+\rangle = |0\rangle \tag{15}$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle - |1\rangle\big) = |-\rangle\,, \quad H|-\rangle = |1\rangle \tag{16}$$

Rotation Gates

These gates performs counter-clockwise rotations around their respective axis on the Bloch sphere.

$$R_x(\theta) = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \tag{17}$$

$$R_y(\theta) = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \tag{18}$$

$$R_z(\theta) = \begin{bmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{bmatrix} \tag{19}$$

Spectral Decomposition

- Spectral decomposition is a way to represent the matrix in terms of its eigenvalues and eigenvectors. Only normal matrices can be represented in this way.
- A matrix M is normal when $MM^\dagger = M^\dagger M$
- In terms of outer product representation a matrix M can be written via spectral decomposition as:

$$M = \sum_i \lambda_i |i\rangle\langle i| \tag{20}$$

where λ_i are the eigenvalues of M and $|i\rangle$ forms an orthonormal basis where each $|i\rangle$ is an eigenvector of M for value λ_i

Postulate 3: Multiple Quantum Systems

How can we mathematically describe a joint state for two qubits?

- Tensor product denoted by $\otimes$ is used to perform this task.
- Tensor product allows us to represent two vectors, $|\psi\rangle, |\phi\rangle \in \mathbb{C}^2$, together to form a 4-dimensional vector

$$|\psi\rangle \otimes |\phi\rangle \in \mathbb{C}^4$$

- Formally, $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^{2\times 2}$. Thus you can say

$$|\psi\rangle \otimes |\phi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

and we can think of $|\psi\rangle \otimes |\phi\rangle \in \mathbb{C}^4$ as a vector with four rows.

Tensor Product Examples

We will take all possible combinations of the computational basis vectors $|0\rangle, |1\rangle$ as $|\psi\rangle, |\phi\rangle$

$$|0\rangle \otimes |0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 \cdot \begin{pmatrix}1\\0\end{pmatrix} \\ 0 \cdot \begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \tag{21}$$

$$|0\rangle \otimes |1\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad |1\rangle \otimes |0\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad |1\rangle \otimes |1\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{22}$$

Tensor Product Examples

- Observe the above examples and you'll notice that the results form a orthonormal basis for $\mathbb{C}^4$ and it is the computational basis for $\mathbb{C}^4$
- Thus, for two orthonormal bases $B_1 = \{|\psi_0\rangle, |\psi_1\rangle\}$ and $B_2 = \{|\phi_0\rangle, |\phi_1\rangle\}$ for $\mathbb{C}^2$, we can construct an orthonornmal basis for $\mathbb{C}^4$ by tensoring together elements of B_1 and B_2 in all possible combinations.
- $\{|\psi_0\rangle \otimes |\phi_0\rangle,\ |\psi_0\rangle \otimes |\phi_1\rangle,\ |\psi_1\rangle \otimes |\phi_0\rangle,\ |\psi_1\rangle \otimes |\phi_1\rangle\}$ forms an orthonormal basis for $\mathbb{C}^4$
- To generalize, for $|\psi\rangle \in \mathbb{C}^{d_1}$ and $|\phi\rangle \in \mathbb{C}^{d_2}$, we have $|\psi\rangle \otimes |\phi\rangle \in \mathbb{C}^{d_1 d_2}$
- For the sake of brevity, we shall drop the notation $\otimes$ and simply write $|\psi\rangle \otimes |\phi\rangle = |\psi\rangle|\phi\rangle$

Properties of Tensor Product

For $|a\rangle, |b\rangle \in \mathbb{C}^{d_1}$ and $|c\rangle, |d\rangle \in \mathbb{C}^{d_2}$

$$\begin{aligned}
\Big(|a\rangle + |b\rangle\Big) \otimes |c\rangle &= |a\rangle \otimes |c\rangle + |b\rangle \otimes |c\rangle \\
|a\rangle \otimes \Big(|c\rangle + |d\rangle\Big) &= |a\rangle \otimes |c\rangle + |a\rangle \otimes |d\rangle \\
s\Big(|a\rangle \otimes |c\rangle\Big) &= \Big(s|a\rangle\Big) \otimes |c\rangle = |a\rangle \otimes \Big(s|c\rangle\Big) \qquad (23) \\
\Big(|a\rangle \otimes |c\rangle\Big)^\dagger &= |a\rangle^\dagger \otimes |c\rangle^\dagger = \langle a| \otimes \langle c| \\
\Big(\langle a| \otimes \langle c|\Big)\Big(|b\rangle \otimes |d\rangle\Big) &= \langle a|b\rangle\langle c|d\rangle
\end{aligned}$$

Two Qubit Gates

We have seen tensoring two single qubit states lets us describe a two qubit system. Let us use this similar intuition to describe at least a type of two qubit gates.

- Result of tensor product of two one qubit gates is a two qubit gate

$$X \otimes I = \begin{pmatrix} 0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{24}$$

- The tensor product for matrices share the properties for vectors with addition of two rules
  - $(A \otimes B)(C \otimes D) = AC \otimes BD$
  - $\mathrm{Tr}\,(A \otimes B) = \mathrm{Tr}(A)\mathrm{Tr}(B)$
- There also are two qubit gates which are not tensor product of one qubit gates. One such important gate is the CNOT gate.

CNOT gate

- CNOT treats one qubit as the control qubit, and the other as the target qubit
- It applies the Pauli X gate to the target qubit only if the control qubit is $|1\rangle$.
- CNOT acts on two qubit states as follows:

$$\begin{aligned} \text{CNOT}|00\rangle &= |00\rangle, \quad \text{CNOT}|01\rangle = |01\rangle \\ \text{CNOT}|10\rangle &= |11\rangle, \quad \text{CNOT}|11\rangle = |10\rangle \end{aligned} \tag{25}$$

- The CNOT gate as a matrix is

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} \tag{26}$$

Quantum Entanglement

- There are states that cannot be expressed as a tensor product of two individual qubit states.
- These states demonstrate a phenomenon which is called *entanglement*

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{27}$$

- If a pair of qubit q_0 and q_1 is entangled, they are so tightly bound that their state cannot be represented individually, only their joint state can be described accurately.
- Formally, there do not exist $|\psi_1\rangle, |\psi_2\rangle \in \mathbb{C}^2$ such that $|\Phi^+\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$

Bell States

The two qubit entangled state $|\Phi^+\rangle$ is also called a *Bell State*. There are three other such bell states: (**Note**: $|0\rangle|0\rangle \equiv |00\rangle$)

$$
\begin{aligned}
|\Phi^-\rangle &= \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}
\end{aligned}
\tag{28}
$$

Postulate 4: Measurements

- How do we extract information after performing operations on single qubit and multi qubit systems?
- What happens when you observe/measure the quantum system?
- How do we measure quantum systems?

Postulate 4: Measurements

- We need to observe the states of qubits to get the values they hold after the operations performed on them.
- Like the unique phenomenon of superposition and entanglement, measuring a qubit alters the state of the qubit irreversibly.
- To model the phenomenon of measurement, we will use the notion of a *projective* or *von Neumann* measurement.
- Let us define a set of three operators that will be necessary going forward

Linear Operators

- **Hermitian Operators:** An operator M is Hermitian if $M = M^{\dagger}$. They have a vital property that all of that all of its eignevalues are *real* All Pauli X, Y and Z gates are not only unitary but also Hermitian.
- **Positive semi-definite operators:** : If a Hermitian operator has only non-negative eigenvalues, then it is called positive semi-definite.
- **Orthogonal projection operators:** A Hermitian matrix Π is an orthogonal projection operator or just projector if $\Pi^2 = \Pi$. They have 0 and 1 as eigenvalues.

Projectors

- Since eigenvalues of projectors can be only 0 and 1, its spectral decomposition is of the form $\Pi = \sum_i |\psi_i\rangle\langle\psi_i|$ where $\{|\psi_i\rangle\}$ are an orthonormal set.
- Similarly you can get a projector by summing any orthonormal set $\{|\psi_i\rangle\}$ as above
- Let us see what a projector $\Pi = \sum_i |\psi_i\rangle\langle\psi_i|$ does on state $|\phi\rangle$ to be measured

$$\Pi|\phi\rangle = \left(\sum_i |\psi_i\rangle\langle\psi_i|\right)|\phi\rangle = \sum_i (\langle\psi_i|\phi\rangle)|\psi_i\rangle \in \mathrm{Span}(\{\psi_i\}) \tag{29}$$

- This tells us that Π projects $|\phi\rangle$ onto the span of orthonormal vectors $\{|\psi_i\rangle\}$

Projective measurements

- A projector Π has a rank 1 if and only if $\Pi = |\psi\rangle\langle\psi|$
- Rank of a matrix Π is equal to the number of non-zero eigenvalues of Π only when Π is a normal matrix.
- $\Pi = |\psi\rangle\langle\psi|$ is the spectral decomposition of Π with a rank 1.
- A projective measurement is defined as a set of projectors $B = \{\Pi_i\}_{i=0}^{m}$ such that $\sum_{i=0}^{m} \Pi_i = I$
- $\sum_{i=0}^{m} \Pi_i = I$ is called the *completeness relation*
- If each Π_i in B is rank 1 that is $\Pi_i = |\psi_i\rangle\langle\psi_i|$ then we say B models a measurement in basis $\{|\psi_i\rangle\}$
- $B = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ is called the computational basis.

Projective measurements

- If the quantum system is in state $|\psi\rangle$, the probability of obtaining the outcome $i = \{0, 1, ...m\}$ using projective measurement B is given by

$$\begin{aligned} \Pr(\text{outcome } i) &= \mathrm{Tr}(\Pi_i|\psi\rangle\langle\psi|\Pi_i) = \mathrm{Tr}(\Pi_i^2|\psi\rangle\langle\psi|) \\ &= \mathrm{Tr}(\Pi_i|\psi\rangle\langle\psi|) \end{aligned} \tag{30}$$

- The second equality follows due to the cyclic property of Trace and third follows because $\Pi_i^2 = \Pi_i$ as Π_i is a projector.

Projective measurements

- Let us measure the probabilities of outcomes 0 and 1 in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ using the computational basis, i.e. using $B = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$

$$\begin{aligned}
\Pr(\text{outcome } i) &= \operatorname{Tr}(|0\rangle\langle 0|\psi\rangle\langle\psi|) \\
&= \operatorname{Tr}(|0\rangle\langle 0|(\alpha|0\rangle + \beta|1\rangle)(\alpha\langle 0| + \beta\langle 0|)) \\
&= \operatorname{Tr}(\langle 0|(\alpha|0\rangle + \beta|1\rangle)(\alpha\langle 0| + \beta\langle 1|)|0\rangle) \\
&= \operatorname{Tr}((\alpha\langle 0|0\rangle + \beta\langle 0|1\rangle)(\alpha\langle 0|0\rangle + \beta\langle 1|0\rangle)) \\
&= \operatorname{Tr}((\alpha \times 1 + \beta \times 0)(\alpha \times 1 + \beta \times 0)) \\
&= \operatorname{Tr}(\alpha^2) \\
&= |\alpha|^2
\end{aligned} \tag{31}$$

Projective measurements

- As we discussed before, after performing the measurement the state of the system changes irreversibly. The state it collapses to assuming we obtained outcome i

$$|\psi'\rangle = \frac{\Pi_i|\psi\rangle}{||\Pi_i|\psi\rangle||_2} = \frac{\Pi_i|\psi\rangle}{\sqrt{\langle\psi|\Pi_i\Pi_i|\psi\rangle}} = \frac{\Pi_i|\psi\rangle}{\sqrt{\langle\psi|\Pi_i|\psi\rangle}} \tag{32}$$

- Lets calculate the final state assuming the outcome to be 0

$$\begin{aligned} |\psi'\rangle &= \frac{\Pi_i|\psi\rangle}{\sqrt{\langle\psi|\Pi_i|\psi\rangle}} = \frac{|0\rangle\langle 0|(\alpha|0\rangle + \beta|1\rangle)}{\sqrt{(\alpha\langle 0| + \beta\langle 1|)|0\rangle\langle 0|(\alpha|0\rangle + \beta|1\rangle)}} \\ &= \frac{|0\rangle(\alpha \times 1 + \beta \times 0)}{\sqrt{(\alpha \times 1 + \beta \times 0)(\alpha \times 1 + \beta \times 0)}} \\ &= |0\rangle \end{aligned} \tag{33}$$

Any doubts?

Thank You for listening