Prereqs: Mathematics and Classical Computing

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Course Overview:

- Lecture 1: Pre-requisite overview
- Lecture 2: Qubits, Quantum Logic Gates and Quantum Circuits
- Lecture 3: Teleportation, No Cloning Theorem, Superdense Coding, and BB84
- Lecture 4: Review on Quantum Circuits, Oracle, Deutsch
- Lecture 5: Practical Workshop
- Weekend: Hackathon!

Goals of this lecture

- Get you comfortable with the basic tools of mathematics commonly used in quantum computing
- Get you comfortable with Dirac notation, used in quantum mechanics and therefore in quantum computing
- Look at the inner works of classical computers so we can compare them to those of quantum computers later

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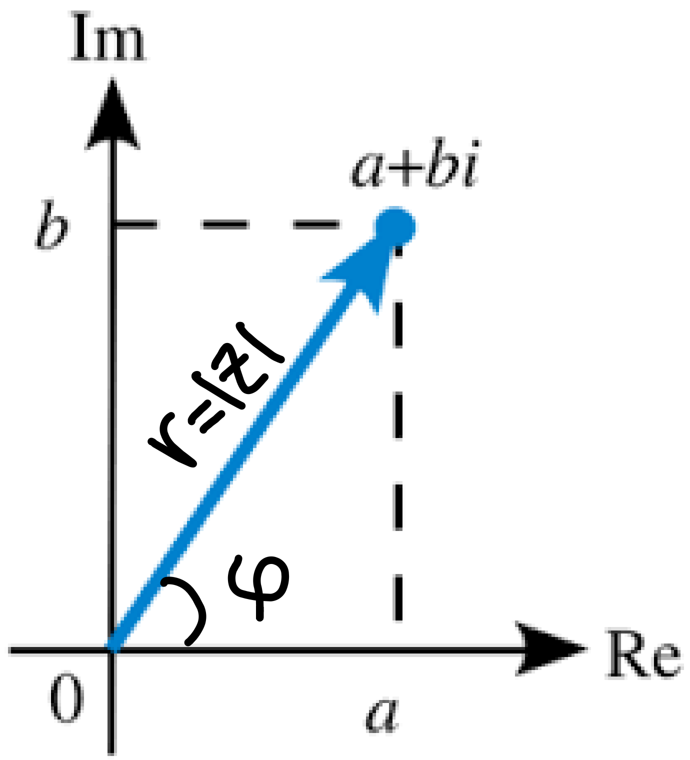
Contents

- Complex Numbers
  - Basic operations
- Linear algebra
  - Vector spaces
  - Operators
  - Eigenvalues, eigenvectors
- Classical computing
  - Gates and universality

Complex numbers

- $z = a + bi$, where $i^2 = -1$
- $z = r\cos(\varphi) + i\, r\sin(\varphi)$



- Magnitude of z is $|z| = \sqrt{a^2 + b^2}$
- Addition:
- Subtraction:
- Multiplication:

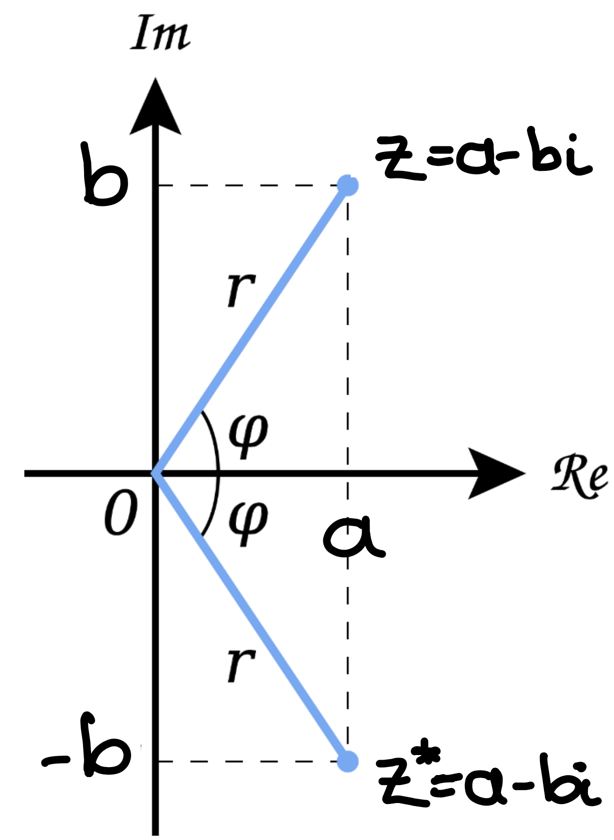
Complex conjugation

- All it does is turn i into $-i$

- If we have a complex number $z = a + bi$, its complex conjugate is $z^* = a - bi$

- z times its conjugate:

- Product of conjugates:



Complex conjugate

- Sum of conjugates:

- Conjugate of conjugate:

- Magnitude of z:

Linear algebra

- Since quantum states can be represented as vectors, we need to get into linear algebra
- Linear algebra allows us to study vectors, how they interact with each other, and how they are transformed about matrices
- The field is much broader than this, but we only need some basic concepts to get started

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Vectors

- A vector has the form $\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ or $\vec{a}^T = (a_1 \cdots a_n)$
- The dimension of a vector is its number of entries
  - $n = 1$
  - $n = 2$
  - $n = 3$

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Vectors

- Vector addition

- Scalar multiplication

- Conjugate transpose:

- Magnitude:

Vector space

- A vector space is a collection of vectors, a field of scalars, and two operations: vector addition and scalar multiplication

- There are real vector spaces and complex vector spaces

- In quantum computing, we work with **complex vector spaces**

Vector space

- Imagine a vector space V and two vectors $\vec{u}$ and $\vec{v}$ in it. The following are true:
  - $\vec{u} + \vec{v}$ is in V
  - $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
  - $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{w} + \vec{v})$, where $\vec{w}$ is also in V
  - There is a $\vec{0}$ vector in V, where $\vec{u} + \vec{0} = \vec{u}$
  - For every $\vec{u}$ there is a $-\vec{u}$ such that $\vec{u} - \vec{u} = \vec{0}$

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Vector space

- Imagine a vector space V with scalar field $\mathbb{C}$, two vectors $\vec{u}$ and $\vec{v}$ in V, and scalars c and d in $\mathbb{C}$. The following are true:
  - $c\vec{u}$ is in V
  - $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
  - $c(d\vec{u}) = d(c\vec{u})$
  - $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
  - $1(\vec{u}) = \vec{u}$

Dot (Inner) product

- Vector space → Hilbert Space

- The dot product of two vectors $\vec{u}, \vec{v} \in \mathbb{C}^n$ is denoted by:

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i^* v_i$$

- We can write the magnitude of a vector in terms of an inner product:

$$|\vec{u}| = \sqrt{\sum_{i=1}^{n} u_i^* u_i} = \sqrt{\sum_{i=1}^{n} |u_i|^2}$$

Dot (Inner) product

- Geometrically, the dot product is defined as
$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\ cos\theta$$

- If we are working with unit vectors,
$$|\vec{u}| = |\vec{v}| = 1$$

- If the vectors are orthogonal, $\theta = \pi/2$ and thus $\vec{u} \cdot \vec{v} = 0$

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Orthonormal basis

- Vectors can be represented as a sum of other vectors, i.e. a linear combination

- Particularly, we are interested in representing a vector as a sum of linearly independent vectors of unit magnitude that are orthogonal to each other

- $\begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Orthonormal basis

- Given a set of basis vectors $\vec{u}_i$, we can write any vector as a linear combination of them

$$\vec{a} = \sum_i \alpha_i \vec{u}_i$$

Dirac notation

- Makes working with the concepts presented easier
- Uses bras and kets: $\langle \cdot \,|$ and $|\, \cdot \rangle$
- $|v\rangle = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ and $\langle v| = |v\rangle^\dagger = (v_1^* \quad \cdots \quad v_n^*)$
- A bra-ket takes an inner product!

$$\langle v|u\rangle = (v_1^* \quad \cdots \quad v_n^*) \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \sum_{i=1}^{n} v_i^* u_i$$

Qubits as quantum states

- We can represent a quantum state as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
- $|0\rangle$ and $|1\rangle$ are distinguishable quantum states
- $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- Therefore, we can write $|\psi\rangle = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

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Qubit representation

- A qubit is represented as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
- Values α and β are called probability amplitudes. Furthermore, $|\alpha|^2$ is the probability of finding the qubit in state $|0\rangle$ and $|\beta|^2$ is the same but for state $|1\rangle$
- Therefore, $|\alpha|^2 + |\beta|^2 = 1$ (Statevector) In other words, $\big||\psi\rangle\big| = 1$

Qubit representation

- What if $|\alpha|^2 + |\beta|^2 \neq 1$? We need to normalize our state!

$$|\hat{\psi}\rangle = \frac{|\psi\rangle}{||\psi\rangle|}$$

Multi-qubit representation

- Most of the times, we will be working with more than 1 qubit

- Suppose you have two qubits $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\phi\rangle = \delta|0\rangle + \gamma|1\rangle$. The combined state is $|\psi\rangle \otimes |\phi\rangle$

- Dimension of vector representing n qubits = 2^n.

- A statevector in a Hilbert space $\mathcal{H}$ of dimension 2^n describes an n qubit system

Operators

- Operators act on qubits to transform them. $A|v\rangle = |w\rangle$
- Operators in quantum computing need to be linear and unitary to preserve probability:
  - $A(|v\rangle + |w\rangle) = A|v\rangle + A|w\rangle$
  - $A(c|v\rangle) = c(A|v\rangle)$, where c is a complex scalar
  - $\big||v\rangle\big| = |A|v\rangle|$, i.e., magnitude is conserved

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Operators

- What is the outer product ($|a\rangle\langle b|$)?

- Pauli Operators:

  - $I = |0\rangle\langle 0| + |1\rangle\langle 1|$

  - $X = |1\rangle\langle 0| + |0\rangle\langle 1|$

  - $Y = i|1\rangle\langle 0| - i|0\rangle\langle 1|$

  - $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

Operators

- Let’s apply the Pauli operators on a qubit defined by

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Operators as matrices

- To convert from outer product to matrix, just perform the operation they encode

$$|a\rangle\langle b| = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \begin{pmatrix} b_1^* & \dots & b_n^* \end{pmatrix}$$

- Now, we can write the Pauli operators as matrices
- $|0\rangle\langle 0|$
- $|0\rangle\langle 1|$
- $|1\rangle\langle 0|$
- $|1\rangle\langle 1|$

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Operators as matrices

- Let’s apply multiple Pauli matrices to the same qubit

Operators as matrices

Multi-qubit operators

- The CNOT **always** acts on two qubits. It has a control qubit and a target qubit. If the control is set to $|1\rangle$, it'll apply the X operator to the target qubit.

- Outer product representation:

$$|00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|$$

- We can define it as $CNOT|x, y\rangle = |x, x \oplus y\rangle$, where $\oplus$ denotes addition modulo 2

Unitary matrices

- In quantum computing, all operators **need** to be unitary matrices

- This means that $U^\dagger U = UU^\dagger = I$

Eigenvalues and eigenvectors

- The eigenvectors of a matrix are those that remain unchanged, up to a constant factor, when the matrix acts on them

- $A|u_\lambda\rangle = \lambda|u_\lambda\rangle$, A is an operator, $|u_\lambda\rangle$ is an eigenvector, and λ is the corresponding eigenvalue

- Characteristic equation:

Eigenvalues and eigenvectors

- Let’s get the eigenvalues and eigenvectors of some of the Pauli matrices

Eigenvalues and eigenvectors

- The spectral decomposition of an operator is

$$A = \sum_i \lambda_i |u_{\lambda_i}\rangle\langle u_{\lambda_i}|$$

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Classical computing

- In classical computing, we also use gates!
  - NOT:
  - AND:
  - OR:

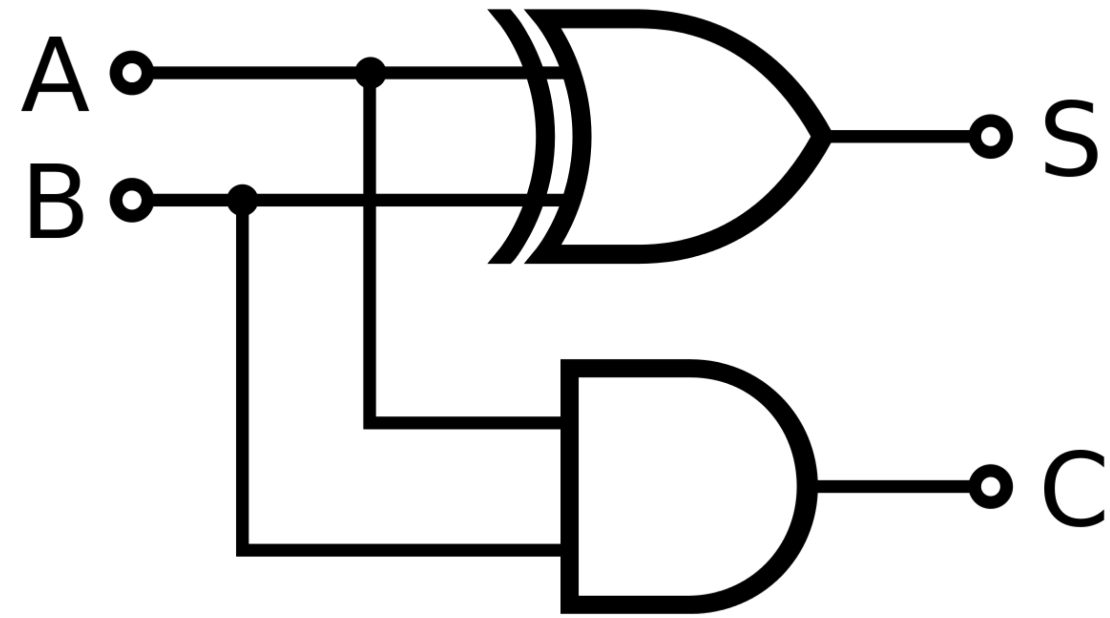
Classical computing

- NAND:

- XOR:

Classical computing

- Let’s build a half adder



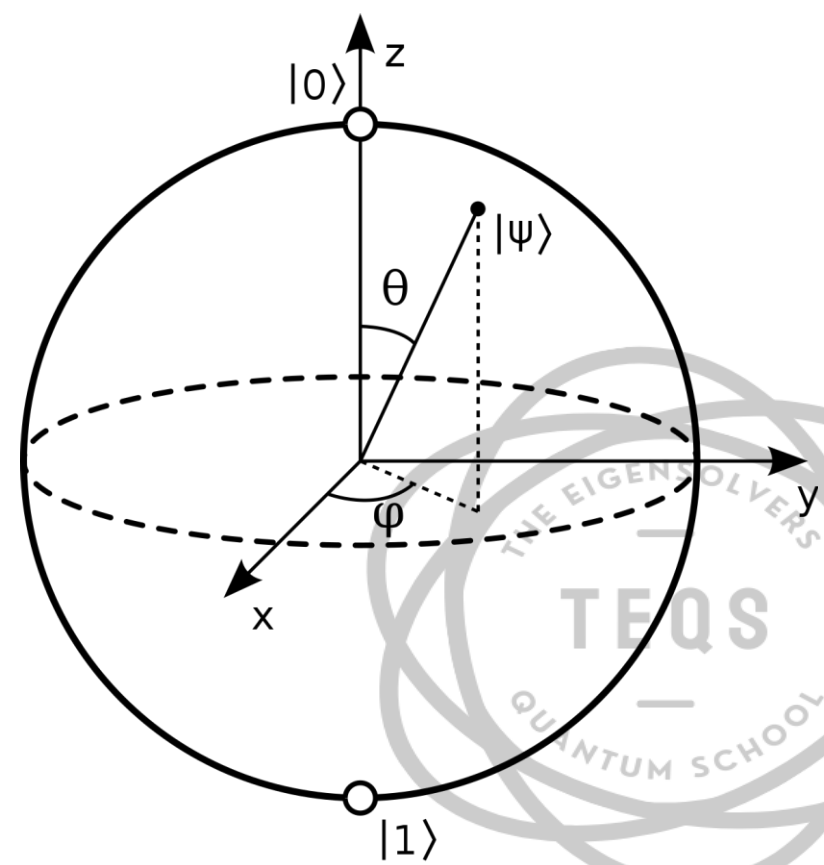
Universality of NAND

- In classical computing, you can build any gate out of NAND gates. Therefore, any computation can be done with only NAND gates
- Let’s build the AND, OR, and NOT from NAND gates

Universal quantum gates

- Like in classical computing, we also have a set of universal quantum gates that can compute any function

- These are the rotation gates $R_{i\in\{x,y,z\}}$, phase shift gate $P(\theta)$, and the $CNOT$ gate



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Thanks for listening!

Good luck on Lecture 2!