Question 1

Prove the following formula using Resolution Calculus:

$$\exists X.\forall Y.\exists Z.\exists W.((\neg P(Z) \land \neg R(b,a)) \lor \neg R(a,b) \lor R(W,a) \lor (P(Y) \land R(X,b)))$$

First, We label the formula with F and apply CNF rules.

$$(\exists X.\forall Y.\exists Z.\exists W.((\neg P(Z) \land \neg R(b,a)) \lor \neg R(a,b) \lor R(W,a) \lor (P(Y) \land R(X,b))))^F$$

We apply $\exists^F$

$$X1 \notin \text{free}(\forall Y.\exists Z.\exists W.((\neg P(Z) \land \neg R(b,a)) \lor \neg R(a,b) \lor R(W,a) \lor (P(Y) \land R(X,b)))). \quad X1 \notin \{X\}$$

$$(\forall Y.\exists Z.\exists W.((\neg P(Z) \land \neg R(b,a)) \lor \neg R(a,b) \lor R(W,a) \lor (P(Y) \land R(X1,b))))^F$$

We apply $\forall^F$

$$\text{free}(\forall Y.\exists Z.\exists W.((\neg P(Z) \land \neg R(b,a)) \lor \neg R(a,b) \lor R(W,a) \lor (P(Y) \land R(X1,b)))) = \{X1\}$$

$$(\exists Z.\exists W.((\neg P(Z) \land \neg R(b,a)) \lor \neg R(a,b) \lor R(W,a) \lor (P(f(X1)) \land R(X1,b))))^F$$

We apply $\exists^F$

$$Z1 \notin \text{free}(\exists W.((\neg P(Z) \land \neg R(b,a)) \lor \neg R(a,b) \lor R(W,a) \lor (P(f(X1)) \land R(X1,b)))). \quad Z1 \notin \{Z, X1\}$$

$$(\exists W.((\neg P(Z1) \land \neg R(b,a)) \lor \neg R(a,b) \lor R(W,a) \lor (P(f(X1)) \land R(X1,b))))^F$$

We apply $\exists^F$

$$W1 \notin \text{free}((\neg P(Z1) \land \neg R(b,a)) \lor \neg R(a,b) \lor R(W,a) \lor (P(f(X1)) \land R(X1,b))). \quad W1 \notin \{Z1, W, X1\}$$

This gives us:

$$((\neg P(Z1) \land \neg R(b,a)) \lor \neg R(a,b) \lor R(W1,a) \lor (P(f(X1)) \land R(X1,b)))^F$$

The formula is a bunch of ORs. To be False, we need all terms to be False.

$$(\neg P(Z1) \land \neg R(b,a))^F \land (\neg R(a,b))^F \land (R(W1,a))^F \land (P(f(X1)) \land R(X1,b))^F$$

For an AND to be False we need either one term to be False. And remember, $(\neg A)^F \equiv A^T$

$$(P(Z1)^T \lor R(b,a)^T) \land R(a,b)^T \land R(W1,a)^F \land (P(f(X1))^F \lor R(X1,b)^F)$$

Final clause set:

$P(Z1)^T \lor R(b,a)^T$	1
$R(a,b)^T$	2
$R(W1,a)^F$	3
$P(f(X1))^F \lor R(X1,b)^F$	4

Now we start resolving:

- We resolve 2 with 4 under $\sigma = [a/X1]$ we get $P(f(a))^F$. (clause 5)
- We resolve 5 with 1 under $\sigma = [a/Z1]$ we get $R(b,a)^T$ (clause 6)
- We resolve 6 with 3 under $\sigma = [b/W1]$ we get $\{\phi\}$

Explaining the Trick:

$$\exists X.\forall Y.\exists Z.\exists W.((\neg P(Z) \wedge \neg R(b,a)) \vee \neg R(a,b) \vee R(W,a) \vee (P(Y) \wedge R(X,b)))$$

First, We negate.

$$\neg(\exists X.\forall Y.\exists Z.\exists W.((\neg P(Z) \wedge \neg R(b,a)) \vee \neg R(a,b) \vee R(W,a) \vee (P(Y) \wedge R(X,b))))$$

We push the negation inside:

$$\forall X.\exists Y.\forall Z.\forall W.\neg((\neg P(Z) \wedge \neg R(b,a)) \vee \neg R(a,b) \vee R(W,a) \vee (P(Y) \wedge R(X,b)))$$

A universally bounded Variable we leave as it is. An existentially bounded Variable is Skolemized.

- If before $\exists$ there is a $\forall$ We Skolemize by new function taking the universally quantified variables before the $\exists$
- If no $\forall$ before $\exists$, we just skolemize with a new Skolem constant

In our case, this leaves us with:

$$\neg((\neg P(Z) \wedge \neg R(b,a)) \vee \neg R(a,b) \vee R(W,a) \vee (P(f(X)) \wedge R(X,b)))$$

Which is equiv to

$$((\neg P(Z) \wedge \neg R(b,a)) \vee \neg R(a,b) \vee R(W,a) \vee (P(f(X)) \wedge R(X,b)))^F$$

We continue as usual

Another Example:

$$\exists X.\forall Y.\exists W.\exists Z.\neg((R(Z,Y) \vee \neg P(Z)) \wedge (\neg Q(d) \vee P(c)) \wedge (Q(d) \vee \neg P(c)) \wedge (\neg R(Z,Y) \vee \neg P(W) \vee \neg Q(X)) \wedge P(c))$$

We negate:

$$\neg(\exists X.\forall Y.\exists W.\exists Z.\neg((R(Z,Y) \vee \neg P(Z)) \wedge (\neg Q(d) \vee P(c)) \wedge (Q(d) \vee \neg P(c)) \wedge (\neg R(Z,Y) \vee \neg P(W) \vee \neg Q(X)) \wedge P(c)))$$

We push negation inside:

$$\forall X.\exists Y.\forall W.\forall Z.((R(Z,Y) \vee \neg P(Z)) \wedge (\neg Q(d) \vee P(c)) \wedge (Q(d) \vee \neg P(c)) \wedge (\neg R(Z,Y) \vee \neg P(W) \vee \neg Q(X)) \wedge P(c))$$

Skolemization:

$$((R(Z,f(X)) \vee \neg P(Z)) \wedge (\neg Q(d) \vee P(c)) \wedge (Q(d) \vee \neg P(c)) \wedge (\neg R(Z,f(X)) \vee \neg P(W) \vee \neg Q(X)) \wedge P(c))^T$$

For a bunch of ANDs to be true. All terms must be true.

$$(R(Z,f(X)) \vee \neg P(Z))^T \wedge (\neg Q(d) \vee P(c))^T \wedge (Q(d) \vee \neg P(c))^T \wedge (\neg R(Z,f(X)) \vee \neg P(W) \vee \neg Q(X))^T \wedge P(c)^T$$

The CNF form is:

$$(R(Z,f(X))^T \vee P(Z)^F) \wedge (Q(d)^F \vee P(c)^T) \wedge (Q(d)^T \vee P(c)^F) \wedge (R(Z,f(X))^F \vee P(W)^F \vee Q(X)^F) \wedge P(c)^T$$

Question 2 (Solved using the Trick)

Prove Using Resolution:

$$\forall X \forall Y \forall Z \exists Z \exists U \exists V \exists W [(P(X,Y) \Rightarrow (P(Z,a) \Rightarrow R(a))) \Rightarrow ((P(U,V) \wedge P(W,a)) \Rightarrow R(a))]$$

We Negate and push negation inside

$$\neg(\forall X \forall Y \forall Z \exists Z \exists U \exists V \exists W [(P(X,Y) \Rightarrow (P(Z,a) \Rightarrow R(a))) \Rightarrow ((P(U,V) \wedge P(W,a)) \Rightarrow R(a))])$$

$$\exists X \exists Y \exists Z \forall Z \forall U \forall V \forall W \neg[(P(X,Y) \Rightarrow (P(Z,a) \Rightarrow R(a))) \Rightarrow ((P(U,V) \wedge P(W,a)) \Rightarrow R(a))]$$

We Skolemize:

$$[(P(c_x,c_y) \Rightarrow (P(c_z,a) \Rightarrow R(a))) \Rightarrow ((P(U,V) \wedge P(W,a)) \Rightarrow R(a))]^F$$
$$(P(c_x,c_y) \Rightarrow (P(c_z,a) \Rightarrow R(a)))^T \wedge ((P(U,V) \wedge P(W,a)) \Rightarrow R(a))^F$$

$$(P(c_x,c_y)^F \vee (P(c_z,a) \Rightarrow R(a))^T) \wedge ((P(U,V) \wedge P(W,a))^T \wedge R(a)^F)$$

$$(P(c_x,c_y)^F \vee P(c_z,a)^F \vee R(a)^T) \wedge (P(U,V)^T \wedge P(W,a)^T \wedge R(a)^F)$$

The clauses are:

$P(c_x,c_y)^F \vee P(c_z,a)^F \vee R(a)^T$	1
$P(U,V)^T$	2
$P(W,a)^T$	3
$R(a)^F$	4

- 1 with 2 with $\sigma = [c_x/U], [c_y/V] \rightarrow P(c_z,a)^F \vee R(a)^T$ (clause 5)
- 5 with 3 with $\sigma = [c_z/W] \rightarrow R(a)^T$ (clause 6)
- 6 with 4 $\rightarrow \{\phi\}$