# M2794.0027 Introduction to Robotics

# Problem Set 3 Solution

## Problem 1

1.

$$T_{ad} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; T_{cd} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2. $T_{ab}T_{bc}T_{cd} = T_{ad}$. Thus $T_{ab} = T_{ad}(T_{bc}T_{cd})^{-1}$.

$$T_{bc}T_{cd} = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(T_{bc}T_{cd})^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -4 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{ab} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -4 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

**Problem 2**

1. Since the platform is rising vertically, $p_{01} = (0, 0, vt)^\intercal$. And $\hat{y}_2$ of the laser is appointing the target. Thus $R_{12} = \mathrm{Rot}(\hat{\mathrm{z}}, -\alpha)\mathrm{Rot}(\hat{\mathrm{x}}, -\gamma)$, where $\alpha$ and $\gamma$ as following:

$$\tan\alpha = \frac{1-\cos\theta}{1-\sin\theta}, \quad \tan\gamma = \frac{Lt}{\sqrt{(1-\cos\theta)^2 + (1-\sin\theta)^2}}.$$

Target is rotating around its $\hat{z}$ axis, $R_{03} = \mathrm{Rot}(\hat{\mathrm{z}}, \frac{\pi}{2} + \omega t)$. Conclusively, we can get $T_{01}, T_{12}, T_{03}$ as follows:

$$T_{01} = \begin{bmatrix} & 0 \\ I & 0 \\ & vt \\ 0 & 1 \end{bmatrix}$$

$$T_{12} = \begin{bmatrix} & 0 \\ \mathrm{Rot}(\hat{\mathrm{z}}, -\alpha)\mathrm{Rot}(\hat{\mathrm{x}}, -\gamma) & 0 \\ & 0 \\ 0 & 1 \end{bmatrix}$$

$$T_{03} = \begin{bmatrix} & 1-\cos\omega t \\ \mathrm{Rot}(\hat{\mathrm{z}}, \frac{\pi}{2} + \omega t) & 1-\sin\omega t \\ & 0 \\ 0 & 1 \end{bmatrix}.$$

2. Following the multiplication rule of SE(3) (i.e.,$T_{ac} = T_{ab}T_{bc}$),

$$\begin{aligned}
T_{23} &= (T_{02})^{-1}T_{03} \\
&= \begin{bmatrix} & 0 \\ \mathrm{Rot}(\hat{\mathrm{z}}, -\alpha)\mathrm{Rot}(\hat{\mathrm{x}}, -\gamma) & 0 \\ & vt \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} & 1-\cos\omega t \\ \mathrm{Rot}(\hat{\mathrm{z}}, \frac{\pi}{2} + \omega t) & 1-\sin\omega t \\ & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} & 0 \\ \mathrm{Rot}(\hat{\mathrm{x}}, \gamma)\mathrm{Rot}(\hat{\mathrm{z}}, \alpha) & vt\sin\gamma \\ & -vt\cos\gamma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} & 1-\cos\omega t \\ \mathrm{Rot}(\hat{\mathrm{z}}, \frac{\pi}{2} + \omega t) & 1-\sin\omega t \\ & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} & \cos\alpha(1-\cos\omega t) - \sin\alpha(1-\sin\omega t) \\ \mathrm{Rot}(\hat{\mathrm{x}}, \gamma)\mathrm{Rot}(\hat{\mathrm{z}}, \frac{\pi}{2} + \omega t + \alpha) & \cos\gamma(\sin\alpha(1-\cos\omega t) + \cos\alpha(1-\sin\omega t)) + vt\sin\gamma \\ & \sin\gamma(\sin\alpha(1-\cos\omega t) + \cos\alpha(1-\sin\omega t)) - vt\cos\gamma \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

**Problem 3**

1. $R = \mathrm{Rot}(\hat{\mathrm{z}}_0, \alpha)\mathrm{Rot}(\hat{\mathrm{y}}_0, \beta)\mathrm{Rot}(\hat{\omega}, \gamma)$, where $\hat{\omega} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$. Hence,

$$R = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} Rot(\hat{w}, \gamma),$$

where

$$\begin{aligned} \mathrm{Rot}(\hat{\omega}, \gamma) &= I + \sin\gamma[\hat{\omega}] + (1-\cos\gamma)[\hat{\omega}]^2 \\ &= \begin{bmatrix} \frac{1}{2}(1+\cos\gamma) & \frac{1}{2}(1-\cos\gamma) & \frac{1}{\sqrt{2}}\sin\gamma \\ \frac{1}{2}(1-\cos\gamma) & \frac{1}{2}(1+\cos\gamma) & -\frac{1}{\sqrt{2}}\sin\gamma \\ -\frac{1}{\sqrt{2}}\sin\gamma & \frac{1}{\sqrt{2}}\sin\gamma & \cos\gamma \end{bmatrix}. \end{aligned}$$

2. (i)

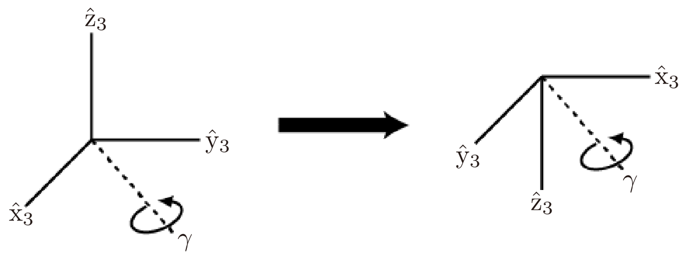


Figure 1: Figure for Exercise 39(b).

From Figure 1, $(\alpha, \beta, \gamma) = (0, 0, \pi)$ or $(-\frac{\pi}{2}, \pi, 0)$.
(ii) By comparing $R_{03}$ derived in $(a)$ with $e^{[\hat{\omega}]\frac{\pi}{2}}$, $(\alpha, \beta, \gamma)$ can be calculated. Denoting $(i, j)$th element of $R_{03}$ by $r_{ij}$,

$$r_{11} + r_{12} = \cos\alpha\cos\beta - \sin\alpha = -\frac{2}{\sqrt{5}} \tag{1}$$

$$r_{21} + r_{22} = \sin\alpha\cos\beta + \cos\alpha = \frac{2}{\sqrt{5}} + \frac{1}{5} \tag{2}$$

$$r_{31} + r_{32} = -\sin\beta = -\frac{1}{\sqrt{5}} + \frac{2}{5}. \tag{3}$$

From Equation (3), $\beta = 2.71°$ or $\beta = 177.29°$.
Case 1. $\beta = 2.71°$.
By (1) and (2), $\alpha = 84.23°$. Substituting $\beta$ into $r_{31}$ and $r_{32}$, $\gamma = 34.91°$.
Case 2. $\beta = 177.29°$.
By (1) and (2), $\alpha = 354°$. Substituting $\beta$ into $r_{31}$ and $r_{32}$, $\gamma = 219°$.