M2794.0027 Introduction to Robotics

Problem Set 2 Solution

Problem 1

(a) The conditions for force closure of Figure 1(a) can be arranged into a linear equation of the form $Ax = b$:

$$f_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} c_1, f_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} c_2, f_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} c_3, f_4 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} c_4, f_5 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} c_5$$

$$\begin{bmatrix} 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \\ M_z \end{bmatrix}$$

Matrix A can be reduced to the following row echelon form via Gauss-Jordan elimination:

$$[I \quad S] = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

There exists w, satisfied $Sw < 0$. Thus, it is a force closure.

(b) The conditions for force closure of Figure 1(b) can be arranged into a linear equation of the form $Ax = b$:

$$f_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} c_1, f_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} c_2, f_3 = \begin{bmatrix} -\mu \\ 1 \end{bmatrix} c_3, f_4 = \begin{bmatrix} \mu \\ 1 \end{bmatrix} c_4$$

$$\begin{bmatrix} 1 & -1 & -\mu & \mu \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

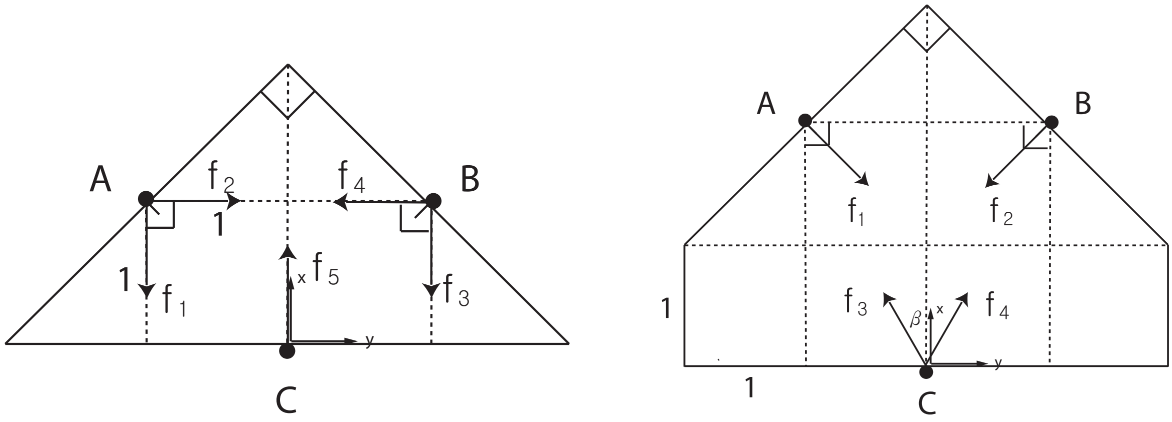
Matrix A can be reduced to the following row echelon form via Gauss-Jordan elimination:

$$\begin{aligned} \begin{bmatrix} 1 & -1 & -\mu & \mu \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & \frac{\mu}{2} - \frac{1}{2} & -\frac{\mu}{2} - \frac{1}{2} \\ 0 & 0 & 1 & -1 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{aligned}$$

$$[I \quad S] = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Regardless of μ, this grasp is force closure except when $\mu = 0$. When $\mu = 0$, there are only 3 columns in the matrix, so it is unable to make the fourth column with all elements negative.

$$\therefore 0 < \beta \leq \frac{\pi}{2}$$



(a) (b)

Figure 1: Problem 4

Problem 2

(a) The conditions for form closure can be arranged into a linear equation of the form $Ax = b$:

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The third row elements are all zeros. Thus, it is NOT a form closure. To construct a force closure grasp, we should slide the position of one of the frictionless point contacts as shown in Figure 2.

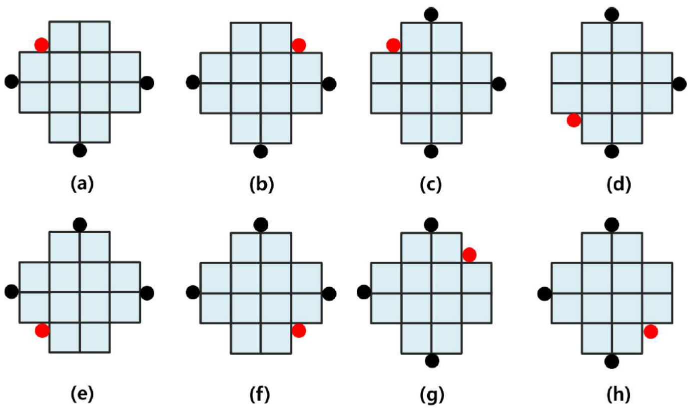


Figure 2: Eight possible positions to construct a force closure grasp

(b) From Nguyen's theorem, the grasp of Figure 3 is a form closure grasp.

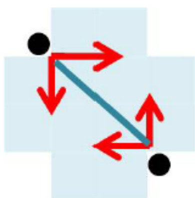


Figure 3: A force closure grasp with two frictionless point contacts at the corners

(c) We have one point contact with friction and two frictionless point contacts at corners. Thus, the grasp matrix should be 3 x 6 matrix.
Since open set near the origin is included in the convex hull consisting of the columns of grasp matrix, the grasp is force-closure.

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ -4 & 4 & -2 & 2 & -2 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \\ M_z \end{bmatrix}$$

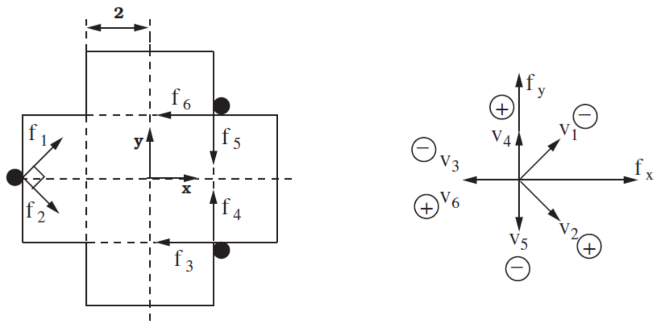


Figure 4: Two frictionless point contacts and one frictional point contact

Problem 3

Consider the any rotation matrix $A, B \in SO(3)$ have rotation axes and angles such that

$$A = \text{Rot}(\hat{\omega}_a, \alpha)$$
$$B = \text{Rot}(\hat{\omega}_b, \beta).$$

If there exists $R \in SO(3)$ satisfying $AR = RB$, then $A = RBR^T$. We get then

$$\begin{aligned} e^{[\hat{\omega}_a]\alpha} &= Re^{[\hat{\omega}_b]\beta}R^T \\ &= e^{[R\hat{\omega}_b]\beta}. \end{aligned}$$

R should satisfy the following condition: $\hat{\omega}_a \alpha = R\hat{\omega}_b \beta$.

1. If $\hat{\omega}_a = \hat{\omega}_b = \hat{z}$ and $\alpha = \beta$, then $\hat{z} = R\hat{z}$. Then the matrix R should be a $\hat{z}$ axis rotation with any angles.

2. If $\hat{\omega}_a = \hat{\omega}_b = \hat{z}$ and $\alpha \neq \beta$, then we should consider two cases:
   Case 1. $|\alpha| \neq |\beta|$. No solutions.
   Case 2. $|\alpha| = |\beta|$. From $\alpha = -\beta$, we get $\hat{z} = -R\hat{z}$.

   $$R = \text{Rot}(\hat{\omega}_t, \theta),$$

   where $\hat{\omega}_t$ is a unit vector on x-y plane and $\theta = \pm\pi$.

3. If $\hat{\omega}_a = \hat{\omega}_b$, then we can similarly solve this problem in the same way with (a) and (b). If $\hat{\omega}_a \neq \hat{\omega}_b$, then we separate this problem to three cases:

Case 1. $|\alpha| \neq |\beta|$. No solutions.
Case 2. $\alpha = \beta$. From $\hat{\omega}_a = R\hat{\omega}_b$,

$$R = \mathrm{Rot}(\hat{\omega}_t, \theta),$$

where $\hat{\omega}_t$ is a unit vector and its direction is $\pm(\hat{\omega}_a + \hat{\omega}_b)$, and $\theta = \pm\pi$.
Case 3. $\alpha = -\beta$. The direction of $\hat{\omega}_t$ in R should be $\pm(\hat{\omega}_a - \hat{\omega}_b)$.

Problem 4

1. As a sequence of rotations about the axes of the fixed frame, rotation matrix should be multiplied on the left. Beginning with $R_{01} = \mathrm{I}$,
   Step 1. $R_{02} = \mathrm{Rot}(\hat{\mathrm{x}}_0, \alpha)R_{01} = \mathrm{Rot}(\hat{\mathrm{x}}_0, \alpha)$.
   Step 2. $R_{03} = \mathrm{Rot}(\hat{\mathrm{y}}_0, \beta)R_{02} = \mathrm{Rot}(\hat{\mathrm{y}}_0, \beta)\mathrm{Rot}(\hat{\mathrm{x}}_0, \alpha)$.
   Step 3. $R_{04} = \mathrm{Rot}(\hat{\mathrm{z}}_0, \gamma)R_{03} = \mathrm{Rot}(\hat{\mathrm{z}}_0, \gamma)\mathrm{Rot}(\hat{\mathrm{y}}_0, \beta)\mathrm{Rot}(\hat{\mathrm{x}}_0, \alpha)$.

$$R_{04} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}.$$

2. As this rotation is about the axes of the moving frame, rotation matrix should be multiplied on the right.

$$\begin{aligned} R_{04} &= R_{03}\mathrm{Rot}(\hat{\mathrm{z}}_3, \gamma) \\ &= \mathrm{Rot}(\hat{\mathrm{y}}_0, \beta)\mathrm{Rot}(\hat{\mathrm{x}}_0, \alpha)\mathrm{Rot}(\hat{\mathrm{z}}_3, \gamma) \\ &= \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

3.

$$\begin{aligned} T_{ca} = T_{cb}T_{ba} = T_{cb}T_{ab}^{-1} &= T_{cb} \begin{bmatrix} R_{ab}^T & -R_{ab}^T p_{ab} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} - \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 1 - \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} - \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$