$$f(D)y = X. \quad \text{where } X = \sin ax \text{ or } \cos ax.$$

$$P.I = y_p = \frac{X}{f(D)} = \frac{\sin ax \text{ or } \cos ax}{f(D)}.$$

$$= \frac{\sin ax \text{ or } \cos ax}{f(-D^2)} = \frac{\sin ax \text{ or } \cos ax}{f(-a^2)}. \quad f(-a^2) \neq 0.$$

Prob: $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x.$

$f(D)y = (D^2 + D + 1)y = 0$

A Eqn $f(m) = 0$ $\quad m^2 + m + 1 = 0$

$$m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}.$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$y_c = e^{-x/2}\left[C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right]$$

$$y_p = \frac{\sin 2x}{D^2 + D + 1} = \frac{\sin 2x}{-2^2 + D + 1} = \frac{\sin 2x}{D - 3} \times \frac{D+3}{D+3}$$

$$= (D+3) \cdot \frac{\sin 2x}{D^2 - 9} = (D+3) \cdot \frac{\sin 2x}{-4-9} = \frac{1}{-13}(D \sin 2x + 3 \sin 2x)$$

$$= -\frac{1}{13}(2\cos 2x + 3\sin 2x).$$

$y = y_c + y_p$

$$= e^{-x/2}\left[C_1 \cos\sqrt{3}/2\, x + C_2 \sin\frac{\sqrt{3}}{2}x\right] + -\frac{1}{13}(2\cos 2x + 3\sin 2x).$$

$-\sin x$

~~$(D^3 + 4D)y = \sin^2 x.$~~

$$\frac{d^3y}{dx^3} + y = \cancel{\sin 3x}\cos^2 x/2$$

$$f(D) = (D^3+1)\,y = \cancel{\sin 3x}\cos^2 x/2 .$$

$$(m+1)\,(m^2+m+1) = 0$$

$$m = -1, \quad m = \frac{1 \pm \sqrt{3}i}{2}.$$

$$y_c = c_1 e^{-x} + e^{x/2}\left[c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \sqrt{3}/2\, x\right].$$

$$y_p = \frac{1}{2}\left[\frac{1+\cos x}{D^3+1}\right].$$

$$= \frac{1}{2}\left[\frac{e^{0.x}}{D^3+1} + \frac{\cos x}{D^2 \cdot D + 1}\right]$$

$$= \frac{1}{2}\left[1 + \frac{\cos x}{-D+1}\right] = \frac{1}{2}\left[1 + \frac{(1+D)}{1-D^2}\cos x\right]$$

$$= \frac{1}{2}\left[1 + \frac{1+\sin x}{1-(-1)}\right] = \frac{1}{2}\left[1 + \frac{\cos x - \sin x}{2}\right].$$

$$y_p = \frac{1}{2} + \frac{\cos x - \sin x}{4}.$$

G.S. $y = y_c + y_p$

$$= c_1 e^{-x} + e^{x/2}\left[c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \sqrt{3}/2\, x\right] + \frac{1}{2} + \frac{\cos x - \sin x}{4}.$$

g) $(D^2+4)y = e^x + \sin 2x.$

A.Eqn. $m^2+4=0$

$m = \pm 2i$

$y_c = C_1\cos 2x + C_2\sin 2x.$

$$y_p = \frac{e^x+\sin 2x}{D^2+4} = \frac{e^x}{D^2+4} + \frac{\sin 2x}{D^2+4}$$

$$= \frac{e^x}{5} + \frac{\sin 2x}{D^2+4}$$

$$\frac{\sin ax}{D^2+a^2} = \frac{x}{2\times a}\int \sin ax\,dx$$

$$= -\frac{x}{2a}\cdot\cos ax.$$

$$= \frac{e^x}{5} + \frac{x}{2\times 2}(-\cos 2x).$$

$$y_p = \frac{e^x}{5} - \frac{x}{4}\cos 2x$$

$$y = y_c + y_p = C_1\cos 2x + C_2\sin 2x + \frac{e^x}{5} - \frac{x}{4}\cos 2x$$

1. $(D^2+D+1)\,y = 2\sin x\cos x$
$= \sin 2x$.

1. when $\frac{X}{f(D)} = \frac{\sin ax \text{ or } \cos ax}{f(D^2)} = \frac{\sin ax \text{ or } \cos ax}{f(-a^2)}$ $\quad f(-a^2)=0$.

$$\boxed{\begin{aligned} \frac{\cos ax}{D^2+a^2} &= \frac{x}{2}\int \cos ax\,dx = \frac{x}{2a}\cdot\sin ax \\ \frac{\sin ax}{D^2+a^2} &= \frac{x}{2}\int \sin ax = -\frac{x}{2a}\cdot\cos ax \end{aligned}}$$

Prob: $(D^2+4)(D^2+1)\,y = \cos 2x + \sin x$.

$y_c = (c_1\cos 2x + c_2\sin 2x) + (c_3\cos x + c_4\sin x)$.

$$y_p = \frac{\cos 2x + \sin x}{(D^2+4)(D^2+1)} = \frac{\cos 2x}{(D^2+1)(D^2+4)} + \frac{\sin x}{(D^2+4)(D^2+1)}.$$

$$= \frac{\cos 2x}{(-4+1)(D^2+4)} + \frac{\sin x}{3(D^2+1)}.$$

$$= -\frac{1}{3}\cdot\frac{x}{2\times 2}\cdot\sin 2x + \frac{1}{3}\cdot\frac{x}{2\times 1}\cdot(-\cos x).$$

$$y_p = -\frac{1}{12}x\sin 2x - \frac{x}{6}\cdot\cos x$$

G.S. $y = y_c + y_p$.

$$= (c_1\cos 2x + c_2\sin 2x) + (c_3\cos x + c_4\sin x) - \frac{1}{12}x\sin 2x - \frac{x}{6}\cos x.$$

f(n) take out the lowest degree ... remaining ... $\phi(n)$.

Q $(D^2+4)y = e^x + \sin 2x$.

AEqn. $m^2+4=0$

$m = \pm 2i$

$y_c = C_1\cos 2x + C_2 \sin 2x$.

$$y_p = \frac{e^x + \sin 2x}{D^2+4} = \frac{e^x}{D^2+4} + \frac{\sin 2x}{D^2+4}$$

$$= \frac{e^x}{5} + \frac{\sin 2x}{D^2+4}$$

$$\frac{\sin ax}{D^2+a^2} = \frac{x}{2\times a}\int \sin ax\, dx$$

$$= -\frac{x}{2a}\cdot \cos ax.$$

$$= \frac{e^x}{5} + \frac{x}{2\times 2}(-\cos 2x).$$

$$y_p = \frac{e^x}{5} - \frac{x}{4}\cos 2x$$

$$y = y_c + y_p = C_1\cos 2x + C_2\sin 2x + \frac{e^x}{5} - \frac{x}{4}\cos 2x$$

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$y_p = \frac{x^m}{f(D)}$.

from $f(D)$ take out the lowest degree term and remaining factor shall be of the form $1+\phi(D)$.
We shall take this factor into the numerator with a -ve index and expand it binomially.
The expansion shall be made only up to the terms where the index of D is m.

$(1+D)^{-1} = 1 - D + D^2 - D^3 \ldots$
$(1-D)^{-1} = 1 + D + D^2 + D^3$.
$(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \ldots$
$(1-D)^{-1} = 1 + 3D + 6D^2 + 10D^3 \ldots$

Prob:

$(D^3 - D^2 - 6D)y = x^2 + 1$.
$f(D)y = x^2+1$. G.S. $y = y_c + y_p$.
A.Eqn. $f(m) = 0$.
$m^3 - m^2 - 6m = 0$
$m(m^2 - m - 6) = 0$.
$m = 0,\ m^2 - 3m + 2m - 6 = 0$
$m(m-3) + 2(m-3) = 0$
$m = 0\ m = -2, 3$. $y_c = c_1 + c_2 e^{-2x} + c_3 e^{3x}$.

$y_p = \frac{x^2+1}{D^3 - D^2 - 6D} = \frac{-1}{6D}\left[\left(1 - \frac{D^3 - D^2}{6D}\right)\right]^{-1}\cdot(x^2+1)$

$= \frac{-1}{6D}\left[\left(1 - \frac{D^2 - D}{6}\right)^{-1}\right][x^2+1]$

$= \frac{-1}{6D}\left[1 + \frac{D^2 - D}{6} + \left(\frac{D^2 - D}{6}\right)^2 + \ldots\right](x^2+1)$

$= \frac{-1}{6D}\left[1 + \frac{D^2 - D}{6} + \frac{D^2}{36}\right] x^2+1$.

$= \frac{-1}{6D}\left[x^2 + 1 + \frac{1}{6}(2 - 2x) + \frac{2}{36}\right] = \frac{1}{6D}\left[x^2 + 1 + \frac{1}{3} - \frac{x}{3} + \frac{1}{18}\right]$

$1 + \frac{1}{3} + \frac{1}{18}$
$\frac{18+6+1}{18}$

$= \frac{-1}{6D}\left[x^2 - \frac{x}{3} + \frac{25}{18}\right] = \frac{-1}{6}\left[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18}x\right]$

$= \frac{-x^3}{18} + \frac{x^2}{36} \mp \frac{25}{108}x$.

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Prob: $\left(\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y\right) = \sin 3x + x^2 + x + e^{4x}$.

$\frac{8}{49}$
$\frac{}{130}$

$f(D)y = 0$

A. Eqn $f(m) = 0$ $\quad m^2 - 3m + 2 = 0$. $\quad m = 2, 1$

$y_c = c_1 e^{x} + c_2 e^{2x}$.

$$y_{p_1} = \frac{\sin 3x}{D^2 - 3D + 2} = \frac{\sin 3x}{-3^2 - 3D + 2} = \frac{\sin 3x}{-3D - 7} = -\frac{(3D - 7)\sin 3x}{(9D^2 - 49)}.$$

$$= \frac{-[3D\sin 3x - 7\sin 3x]}{9(-3^2) - 49} = \frac{-[3\cdot 3\cos 3x - 7\sin 3x]}{-130}$$

$$= \frac{9\cos 3x - 7\sin 3x}{130}.$$

$$y_{p_2} = \frac{x^2 + x}{D^2 - 3D + 2} = \frac{1}{2\left(1 + \frac{D^2 - 3D}{2}\right)} \cdot x^2 + x.$$

$$= \frac{1}{2}\left(1 + \frac{D^2 - 3D}{2}\right)^{-1} (x^2 + x)$$

$$= \frac{1}{2}\left(1 - \left(\frac{D^2 - 3D}{2}\right) + \left(\frac{D^2 - 3D}{2}\right)^2 + \cdots\right) x^2 + x.$$

$$= \frac{1}{2}\left(x^2 + x - 1 + \frac{3}{2}(2x + 1) + \frac{9}{4}\cdot(2)\right].$$

$$= \frac{1}{2}\left[x^2 + x - 1 + 3x + 3/2 + 9/2\right] = \frac{1}{2}\left[x^2 + 4x + 5\right]$$

$$y_{p_3} = \frac{e^{4x}}{D^2 - 3D + 2} = \frac{e^{4x}}{16 - 3\cdot 4 + 2} = \frac{e^{4x}}{6}.$$

$$y_p = y_{p_1} + y_{p_2} + y_{p_3} = \frac{9\cos 3x - 7\sin 3x}{130} + \frac{1}{2}[x^2 + 4x + 5] + \frac{e^{4x}}{6}$$

G.S. $y = y_c + y_p$

$$= c_1 e^{x} + c_2 e^{2x} + \frac{1}{130}[9\cos 3x - 7\sin 3x] + \frac{1}{2}(x^2 + 4x + 5) + \frac{e^{4x}}{6}$$

When X is of the form of $e^{ax}V$. where $V = x^m$ or $\sin ax$ or $\cos ax$.

P.I for $\dfrac{e^{ax}.V}{f(D)} = e^{ax}.\left\{\dfrac{1}{f(D+a)}\right\} V$.

Prob: $(D^2 - 4D + 3) y = e^{2x} \sin 3x$.

$f(D)y = X$.

Homo eqn. $f(D)y = 0$

A Eqn. $f(m) = 0$.

$m^2 - 4m + 3 = 0$

$m^2 - 3m - m + 3 = 0$

$m(m-3) - 1(m-3) = 0$

$m = 1, 3$.

$y_c = c_1 e^{x} + c_2 e^{3x}$:

$y_p = \dfrac{e^{2x} \sin 3x}{D^2 - 4D + 3}$

PI: $\dfrac{e^{ax}.V}{f(D)} = e^{ax}\left\{\dfrac{1}{f(D+a)}\right\}V$

$= e^{2x} \dfrac{1}{(D+2)^2 - 4(D+2) + 3}. \sin 3x.$

$= e^{2x}\left\{\dfrac{1}{D^2 + \cancel{4D} + 4 - \cancel{4D} - 5}\right\} \sin 3x.$

$= e^{2x}\left\{\dfrac{\sin 3x}{D^2 - 1}\right\}. = e^{2x}. \dfrac{\sin 3x}{-10}$

$y_p = -\dfrac{1}{10} e^{2x} \sin 3x$

G.S $= y = y_c + y_p$

$= c_1 e^{x} + c_2 e^{3x} - \dfrac{1}{10} e^{2x} \sin 3x.$

§ $(D^2-2D+1)y = e^{3x}x^2$.

$f(D)y = 0$.

A Eqn $f(m)=0$ $\quad m^2-2m+1=0$

$(m-1)^2=0 \quad m=1,1$.

$$\frac{e^{ax}V}{f(D)} = e^{ax}\frac{1}{f(D+a)}V.$$

$y_c = (c_1+c_2x)e^x$.

$$y_p = \frac{e^{3x}.x^2}{D^2-2D+1} = e^{3x}.\frac{1}{(D+3)^2-2(D-}$$

$$= \frac{e^{3x}.x^2}{(D-1)^2} = e^{3x}\left\{\frac{x^2}{(D+3-1)^2}\right\}$$

$$= e^{3x}\left\{\frac{x^2}{(D+2)^2}\right\} = \frac{e^{3x}}{4}\left(\left(1+\frac{D}{2}\right)^{-2}\right)x^2$$

$$= \frac{e^{3x}}{4}\left(1-2(D/2)+3(D/2)^2+\dots\right)x^2$$

$$= \frac{e^{3x}}{4}\left(x^2-2x+\frac{3}{4}.2\right)$$

$$= \frac{e^{3x}}{4}(x^2-2x+3/2)$$

$$y = y_c+y_p = (c_1+c_2x)e^x + \frac{e^{3x}}{4}(x^2-2x+3/2).$$

§. $(D^2+2D-3)y = e^{-3x}.x^2$.

$m=1,-3 \quad y_c = c_1e^x+c_2e^{-3x}$.

$$y_p = \frac{e^{-3x}x^2}{D^2+2D-3} = e^{-3x}\left\{\frac{1}{(D-3)^2+2(D-3)-3}\right\}x^2$$

$$= e^{-3x}\left\{\frac{x^2}{D^2-4D}\right\} = e^{-3x}\frac{1}{-4D}\{(1-D/4)^{-1}.x^2.$$

$$= e^{-3x}\frac{1}{-4D}\left(1+D/4+D^2/16\right)x^2$$

$$= e^{-3x}\cdot\frac{-1}{4D}\left(x^2-\frac{x}{2}+\frac{1}{8}\right)$$

$$= -\frac{e^{-3x}}{4}\left\{\frac{x^3}{3}-x^2/4+x/8\right\}.$$

$$y = y_c+y_p = c_1e^x+c_2e^{-3x}-\frac{e^{-3x}}{4}(x^3/3+x^2/4+x/8)$$

P.I $\quad f(D)y = xV$ form $\quad$ where $V = \cos x$ or $\sin x$.

<u>Prob:</u> $\quad \frac{d^2y}{dx^2} - y = x^2 \cos x.$

$e^{ix} = \cos x + i \sin x$

$y_c = c_1 e^{x} + c_2 e^{-x}.$

$\frac{e^{ax} V}{f(D)} = \frac{e^{ax} V}{f(D+a)}$

$y_p = R.P \text{ of } e^{ix} \cdot \frac{x^2}{D^2 - 1} = R.P \text{ of } e^{ix} \cdot \frac{1}{(D+i)^2 - 1} x^2.$

$= R.P \text{ of } e^{ix} \cdot \frac{1}{D^2 - 1 + 2iD \cancel{-1}} x^2$

$= \text{''} \quad \text{''} \quad \frac{1}{D^2 + 2Di - 2} x^2$

$= -\frac{1}{2} \text{''} \quad \text{''} \quad \left[1 - \frac{D^2 + 2Di}{2}\right]^{-1} \cdot x^2$

$= -\frac{1}{2} \text{''} \quad \text{''} \quad \left[1 + \frac{D^2 + 2Di}{2} + \left(\frac{D^2 + 2Di}{2}\right)^2 \cdots\right] x^2$

$= \text{''} \quad \text{''} \quad \left[1 + \frac{D^2}{2} + \frac{\cancel{2}Di}{\cancel{2}} + D^2(-1)\right] x^2$

$= \text{''} \quad \text{''} \quad \left[x^2 + 1 + i2x - 2\right].$

$= \text{''} \quad \text{''} \quad \left[(x^2 - 1) + i2x\right]$

$= -\frac{1}{2} \text{ RP of } (\cos x + i \sin x)\left[(x^2 - 1) + i2x\right]$

$= -\frac{1}{2}\left[(x^2 - 1)\cos x - 2x \sin x\right] = \frac{1}{2}\left[(1 - x^2)\cos x + 2x \sin x\right]$

G.S $\quad y = y_c + y_p$

$= c_1 e^{x} + c_2 e^{-x} + \frac{1}{2}\left[(1 - x^2)\cos x + 2x \sin x\right]$

$(D^2-4D+4)y = 8x^2 e^{2x}\sin 2x.$

$y_c = (c_1+c_2x)e^{2x}.$

$$y_p = \frac{8x^2 e^{2x}\sin 2x}{(D-2)^2} = 8e^{2x}.\frac{1}{(D-2)^2}.x^2\sin 2x.$$

$$y_p = 8.e^{2x}\underbrace{\left[\frac{1}{D^2}\right]x^2\sin 2x}_{I}, \quad = 8e^{2x}.I.$$

$e^{2ix} = \cos 2x + i\sin 2x$

$$I = \frac{1}{D^2}x^2\sin 2x$$

$$= I.P.\text{ of }\frac{1}{D^2}x^2 e^{2ix}.$$

$$= IP\text{ of } e^{2ix}.\frac{1}{D^2}x^2$$

$$= \text{"}\quad e^{2ix}\frac{1}{(D+2i)^2}x^2$$

$(1-x)^{-2} = 1+2x+3x^2$

$$= \text{" "}\quad \frac{e^{2ix}}{4i^2}.(1+D/2i)^{-2}.x^2$$

$$= \text{" "}\quad \frac{e^{2ix}}{-4}\left(1-\frac{Di}{2}\right)^{-2}x^2$$

$$= \text{" "}\quad \frac{e^{2ix}}{-4}\left(1+\frac{2Di}{2}+3\left(\frac{Di}{2}\right)^2+\ldots\right)x^2$$

$$= \text{" "}\quad \frac{e^{2ix}}{-4}\left(x^2+i\frac{2\cdot 2x}{2}+\frac{3}{4}.(-1).2\right)$$

$$= \text{" "}\quad \frac{e^{2ix}}{-4}\left[(x^2-3/2)+i\,2x\right].$$

$$= \text{" "}\quad -\frac{1}{4}(\cos 2x + i\sin 2x)\left[(x^2-3/2)+i\,2x\right]$$

$$= -\frac{1}{4}\left(2x\cos 2x + (x^2-3/2).\sin 2x\right].$$

$$y_p = -\frac{8}{4}e^{2x}.\left(2x\cos 2x + \frac{(2x^2-3)}{2}\sin 2x\right].$$

$$y_p = -e^{2x}\left(4x\cos 2x + (2x^2-3)\sin 2x\right)$$

$$G.S.\ y = y_c + y_p = (c_1+c_2x)e^{2x} - e^{2x}\left(4x\cos 2x + (2x^2-3)\sin 2x\right)$$

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The Second order Constant coeff ODEqn.

$$y'' + Py' + Qy = R. \qquad P, Q \text{ are consts}$$

$y'' + Py' + Qy = 0$ is a homo. d.e.

It has two solutions $y_c = c_1 u_1 + c_2 v_1$

$u_1\ v_1$ are two solutions are said to be 'Linearly Independent in the interval $[a, b]$ their ratio on the interval $\frac{u_1}{v_1} \neq$ Constant

Otherwise the solutions are Linearly Dependent (L.D)

In otherwords two solutions u_1, v_1 are called 'L.D. on the interval $[a, b]$ then there exists constant number λ.

Such that. $\frac{u}{v} = \lambda$.

Wronskian: Wronskian is a solution determinant

If u, v are two solutions (functions of x) determinent

$$W(u, v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - u'v$$

If $W(u, v) \neq 0$ Solutions are L-I.

$W(u, v) = 0$ Solutions are L-D.

S.t. e^{2x}, e^{3x} are L.I solution of $y'' - 5y' + 6y = 0$ — (1)

find the solution (1) $y(x)$ w.r.t. $y(0) = 0$, $y'(0) = 1$

$u = e^{2x} \quad v = e^{3x}$ — (2)

if u and v are two solutions then u, v satisfy eqn (1).

$$4e^{2x} - 5 \cdot 2e^{2x} + 6 \cdot e^{2x} = 0.$$

$$9e^{3x} - 5 \cdot 3e^{3x} + 6e^{3x} = 0.$$

Now, the wronskian of u, v are $W(u, v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$.

$$= \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{5x} - 2e^{5x} = e^{5x} \neq 0.$$

$\therefore$ e^{2x}, e^{3x} are L.I. soln of (1).

G.S.

$y = c_1 e^{2x} + c_2 e^{3x}$. $y(0) = 0$.

at $x = 0$ $y = 1$

$0 = c_1 + c_2$

$y'(0) = 1$

$y' = 2c_1 e^{2x} + 3c_2 e^{3x}$.

at $x = 0$ $y' = 1$

$1 = 2c_1 + 3c_2$

$c_2 = -c_1$.

$2c_1 - 3c_1 = 1$.

$c_1 = -1$.

$c_2 = 1$.

$\cancel{2c_1 + 2c_2 = 0}$

$\cancel{2c_1 + 3c_2 = 0}$

$-c_2 = 1$

$\cancel{c_2 = 1 \Rightarrow c_1 = 0\ 2}$.

$y = -e^{2x} + e^{3x} \Rightarrow \underline{e^{3x} - e^{2x}}$. is complete sol.

Method of Variation of Parameters

let the Second order Differential Eqn.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R.$$

$$f(D)y = R.$$

A Eqn $f(m) = 0$

$$(m^2 + Pm + Q)y = 0.$$

let $y_c = C_1 u + C_2 v$ is a complementary soln.

$$y_p = Au + Bv.$$

where $A = -\int \frac{vR}{W(u,v)}\,dx.$

$$B = \int \frac{uR}{W(u,v)}\,dx.$$

$$W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - vu' \neq 0.$$

$$y_p = Au + Bv.$$

G.S $y = y_c + y_p.$

Prob: $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x.$ $\quad R = \operatorname{cosec} x.$

$$(D^2+1)y = \operatorname{cosec} x.$$

$$f(D)y = \operatorname{cosec} x.$$

A Eqn. $f(m) = 0.$

$$m^2 + 1 = 0$$

$$y_c = C_1\cos x + C_2 \sin x, \quad \text{where } u = \cos x \quad v = \sin x.$$

$$W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \neq 0 \text{ LI}.$$

$$y_p = Au + Bv$$

$$= A\cos x + B\sin x.$$

$$A = -\int \frac{vR}{W(u,v)}\,dx$$

$$= -\int \frac{\operatorname{cosec} x \cdot \sin x}{1}\,dx$$

$$= -x.$$

$$B = \int \frac{uR}{W(u,v)} dx$$

$$= \int \frac{\operatorname{cosec} x \cdot \cos x}{1} dx$$

$$= \int \cot x = \ln \sin x .$$

$$y_p = Au + Bv$$

$$= -x \cos x + \sin x \ln \sin x$$

$$G.S = \quad y = y_c + y_p .$$

$$= C_1 \cos x + C_2 \sin x - x \cos x + \sin x \ln \sin x .$$

§ $(D^2+1) y = \sec x .$ $\qquad R = \sec x .$

$y_c = C_1 \cos x + C_2 \sin x .$ $\qquad u = \cos x \quad v = \sin x$

$y_p = Au + Bv$

where $A = -\int \frac{vR}{W(u,v)} dx$ $\qquad W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$

$= \int \frac{\sec x \cdot \sin x}{1} dx$ $\qquad = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 .$

$$= \int \tan x \, dx = \ln (\sec x)$$

$$B = \int \frac{uR}{W(u,v)} dx = \int \frac{\cos x \cdot \sec x}{1} dx = \int 1 \, dx = x .$$

$$y_p = \cos x \ln (\sec x) + x \sin x .$$

$G.S. \quad y = y_c + y_p = C_1 \cos x + C_2 \sin x + \cos x \ln (\sec x) + x \sin x$

$(D^2-3D+2)y = e^{4x}$.

A.Equ $m^2-3m+2=0$. $\qquad R = e^{4x}$.

$m = 1, 2$.

$y_c = c_1 e^{x} + c_2 e^{2x}$ $\qquad u = e^{x} \quad v = e^{2x}$.

$y_p = Au + Bv$

$W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x} \neq 0$

$A = \int \frac{-vR}{W(u,v)}\,dx$.

$= -\int \frac{e^{2x}\, e^{4x}}{e^{3x}}\,dx$

$= -\int \frac{e^{6x}}{e^{3x}}\,dx = -\int e^{3x}\,dx = -\frac{e^{3x}}{3}$

$B = \int \frac{uR}{W(u,v)}\,dx = \int \frac{e^{x}.e^{4x}}{e^{3x}}\,dx = \int \frac{e^{5x}}{e^{3x}}\,dx$.

$= \frac{e^{2x}}{2}$

$y_p = Au + Bv = -\frac{e^{3x}}{3}.e^{x} + \frac{e^{2x}}{2}.e^{2x}$.

$= -\frac{e^{4x}}{3} + \frac{e^{4x}}{2}$

G.S $y = y_c + y_p = c_1 e^{x} + c_2 e^{2x} - \frac{e^{4x}}{3} + \frac{e^{4x}}{2}$.

3. $4y'' - y = e^x + e^{3x}$.
4. $3y'' + 2y' - y = e^{-2x} + x$.
5. $y'' + 6y' + 8y = e^{-3x} + e^x$.
6. $y'' + 4y' + 3y = 6e^{-x}$.
7. $2y'' + 3y' - 2y = 5e^{-2x} + e^x$.
8. $y'' - y' - 6y = 5e^{-2x} + 10e^{3x}$.
9. $3y'' + 5y' - 2y = 14e^{x/3}$.
10. $y'' + 3y' + 2y = \cos x + \sin x$.
11. $y'' + y' - 6y = 39 \cos 3x$.
12. $y'' + 4y' - 5y = 34 \cos 2x - 2 \sin 2x$.
13. $y'' + 25y = 50 \cos 5x + 30 \sin 5x$.
14. $y'' + 16y = 16 \sin 4x$.
15. $y'' - 4y' + 4y = 8e^{2x} + e^{3x}$.
16. $4y'' - 4y' + y = 6e^{x/2}$.
17. $y'' + 6y' + 9y = 26e^{-3x} + 5e^{2x}$.
18. $y'' + y = e^x \sin x$.
19. $y'' + 2y' + 10y = e^{-x} \sin 3x$.
20. $y'' - 4y' + 5y = 16e^{2x} \cos x$.
21. $y'' - 6y' + 13y = 6e^{3x} \sin x \cos x$.
22. $y'' + 4y' + 4y = 6e^{-2x} \cos^2 x$.
23. $y'' + 3y' + 2y = 12e^{-x} \sin^3 x$.
24. $y'' - 4y' + 3y = 4 \cosh 3x$.
25. $y''' + 4y'' - y' - 4y = 18e^{-x}$.
26. $y''' + 3y'' - 4y = 12e^{-2x} + 9e^x$.
27. $y''' - 9y'' + 27y' - 27y = 36e^{3x}$.
28. $y''' - y'' + y' - y = 6 \cos 2x$.
29. $y''' - 2y'' + 4y' - 8y = 8(x^2 + \cos 2x)$.
30. $y^{iv} - 256y = 128 \cos 4x$.
31. $y^{iv} - y = x^4 + 1$.
32. $y^{iv} + 3y''' + 3y'' + y' = 2x + 4$.
33. $y^{iv} - 3y'' - 4y = 60e^{2x}$.
34. $y^{iv} + 6y''' + 12y'' + 8y' = 60e^{-2x}$.
35. $y^{iv} - 16y'' = 8x + 16$.

5.4.3 Solution of Euler-Cauchy Equation

or

$$y(x) = c_1^* x + \frac{1}{x} c_2 + \frac{x}{2} \ln|x|, \text{ where } c_1^* = c_1 - \frac{1}{4}.$$

Exercise 5.4

Find the general solution of the following differential equations, using the method of variation of parameters.

1. $y'' - 2y' - 3y = e^x$.
2. $y'' - 4y' + 4y = e^{-2x}$.
3. $y'' + 4y = \cos x$.
4. $y'' + y = \sec x$.
5. $y'' + y = \operatorname{cosec} x$.
6. $y'' + y = \tan x$.
7. $y'' - 4y' + 3y = e^x$.
8. $y'' + 4y = \sec 2x$.
9. $y'' + 4y = \cos 2x$.
10. $y'' + 4y' + 4y = e^{-2x} \sin x$.
11. $y'' + 6y' + 9y = e^{-3x}/x$.
12. $y'' + 2y' + 2y = e^{-x} \cos x$.

In the following problems, using the method of variation of parameters and the given linearly independent solutions, find a particular integral and the general solution.

13. $x^2y'' + xy' - y = x^3$, $y_1 = x$, $y_2 = 1/x$.
14. $x^2y'' + xy' - 4y = x^2 \ln|x|$, $y_1 = x^2$, $y_2 = 1/x^2$.
15. $x^2y'' - xy' + y = 1/x^4$, $y_1 = x$, $y_2 = x \ln|x|$.
16. $x^2y'' - 2xy' + 2y = x^3 + x$, $y_1 = x$, $y_2 = x^2$.
17. $y'' + 4y' + 8y = 16\, e^{-2x} \operatorname{cosec}^2 2x$, $y_1 = e^{-2x} \cos 2x$, $y_2 = e^{-2x} \sin 2x$.
18. $y''' + 4y' = \sec 2x$, $y_1 = 1$, $y_2 = \cos 2x$, $y_3 = \sin 2x$.
19. $y''' - 6y'' + 12y' - 8y = e^{2x}/x$, $y_1 = e^{2x}$, $y_2 = xe^{2x}$, $y_3 = x^2e^{2x}$.
20. Show that the general solution of the equation $y'' + k^2y = g(x)$, where $k \neq 0$ and $g(x)$ is continuous on I, can always be written as

$$\ldots \frac{1}{k}\int^{x} \sin k(x - t) g(t)\, dt.$$

Ans. Given functions are linearly independent if $m_1 \neq m_2 \neq m_3$.

2. Test the linear independence of the following sets of functions:

(*i*) $\sin x, \cos x$. **Ans.** Linearly independent

(*ii*) $1 + x, 1 + 2x, x^2$. **Ans.** Linearly independent

(*iii*) $x^2 - 1, x^2 - x + 1, 3x^2 - x - 1$. **Ans.** Linearly dependent

(*iv*) $\sin x, \cos x, \sin 2x$. **[Meerut 2010]** **Ans.** Linearly independent

(*v*) $e^x, e^{-x}, \sin ax$. **Ans.** Linearly independent

(*vi*) $e^x, xe^x, \sinh x$. **Ans.** Linearly independent

(*vii*) $\sin 3x, \sin x, \sin^3 x$. **Ans.** Linearly dependent

3. Show that the functions $e^x \cos x$ and $e^x \sin x$ are linearly independent. Form the differential equation

Exercise 5(H)

1. $d^3y/dx^3 + d^2y/dx^2 + dy/dx = e^{2x} + x^2 + x$. [Meerut 2009; Lucknow 1994,

Ans. $y = c_1 + (c_2 + c_3x)\ e^{-x} + (1/18) \times e^{2x} + (1/3) \times x^3 - (3/2) \times x^2 +$

2. $(D^2 - 4D + 4)\ y = \sin 2x + x^2$. [G.N.D.U. Amritsar 20

Ans. $y = (c_1 + c_2x)\ e^{2x} + (3 \sin 2x + 8 \cos 2x)/25 + (2x^2 + 4x +$

3. $(D^2 + 4)\ y = e^x + \sin 2x$. [Allahabad 1994 ; Agra 2005 ; Rohilkhand 1

Ans. $y = c_1 \cos 2x + c_2 \sin 2x + (1/5) \times e^x - (1/4) \times x$ co

4. $(D^4 + 2D^3 - 3D^2)\ y = 3e^{2x} + 4 \sin x$. [Kanpur 1

Ans. $y = c_1 + c_2x + c_3e^x + c_4e^{-3x} + (3/20) \times e^{2x} + (2/5) \times (2 \sin x +$ c

5. $(D^2 + D - 2)\ y = x + \sin x$. [Guwahati 1998; Meerut 1998 ; Delhi 2007, 09 ; Utkal 2

Ans. $y = c_1e^x + c_2e^{-2x} - (x/2) - (1/4) - (1/10) \times (\cos x + 3$ s

6. (*i*) $D^3 - 3D^2 + 3D - 1)\ y = x\ e^{-x} + e^x$. **Ans.** $y = (c_1 + c_2x + c_3x^2)\ e^x - (1/16) \times (2x + 3)\ e^{-x} + (x^3/6)$

(*ii*) $(D^3 - 3D^2 + 3D - 1)\ y = x\ e^x + e^x$. **Ans.** $y = e^x\ [c_1 + c_2x + c_3x^2 + (1/6) \times x^3 + (1/24)$

7. $(D^2 + 5D + 6)\ y = e^{-2x} + 5 \sin 4x$. **Ans.** $y = c_1e^{-3x} + c_2e^{-2x} + xe^{-2x} - (1/10) \times (\sin 4x + 2$ c

8. $(D^2 + 1)\ y = e^{-x} + \cos x$. **Ans.** $y = c_1\cos x + c_2 \sin x + (1/2) \times e^{-x} + (1/2) \times$

9. $(D^2 + 4)\ y = \sin^2 x$. **Ans.** $y = c_1 \cos 2x + c_2 \sin 2x + (1/8) \times (9 - x$ s

10. $(D^2 + 1)\ y = e^{-x} + \cos x + x^3 + e^x \sin x$.

Ans. $y = c_1 \cos x + c_2 \sin x + (1/2) \times e^{-x} + (1/2) \times x \sin x + x^3 - 6x - (1/5) \times e^x\ (2 \cos x -$

11. $(D^2 + 1)\ y = \cos^2 (x/2)$. **Ans.** $y = c_1 \cos x + c_2 \sin x + (1/2) + (1/4) \times$

12. $(D^2 + 4)\ y = x^2 + 3 \sin x$ **Ans.** $y = c_1 \cos 2x + c_2 \sin 2x + (2x - 1)/8 +$

13. $(D^2 + 4)\ y = \sin 2x + x^2$. **Ans.** $y = c_1 \cos 2x + c_2 \sin 2x - (1/4) \times x \cos 2x + (1/8) \times (2$

14. $(2D^2 - D - 6)\ y = e^{-(3x/2)} + \sin x$ [Pune

Ans. $y = c_1\ e^{2x} + c_2\ e^{-(3x/2)} + (\cos x - 8 \sin x) / 65 - (x/7) \times$