# Worksheet 8

## To accompany Chapter 3.5 Convolution and the Impulse Response

We will step through this worksheet in class.

You are expected to have at least watched the video presentation of Chapter 3.5 (https://cpjobling.github.io/eg-247-textbook/laplace_transform/5/convolution) of the notes (https://cpjobling.github.io/eg-247-textbook) before coming to class. If you haven't watch it afterwards!

## Agenda

The material to be presented is:

### First Hour

- Even and Odd Functions of Time
- Time Convolution

## Even and Odd Functions of Time

Fill in the Blanks Quiz.

### Even Functions of Time

A function $f(t)$ is said to be an *even function* of time if the following relation holds

Polynomials with even exponents only, and with or without constants, are even functions. For example:

Write down the Taylor-series polynomial expansion of $\cos t$. Is $\cos t$ even or odd?

Odd/Even?

## Odd Functions of Time

A function $f(t)$ is said to be an *odd function* of time if the following relation holds

Polynomials with even exponents only, and with or without constants, are even functions.

Write down the Taylor-series polynomial expansion of $\cos t$. Is $\cos t$ even or odd?

Odd/Even?

## Observations

- For odd functions $f(0) =$

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- The product of *two even* or *two odd* functions is an [Even/Odd] function.
- The product of an even and an odd function, is an [Even/Odd] function.

In the following $f_e(t)$ will denote an even function and $f_o(t)$ an odd function.

## Time integrals of even and odd functions

For an even function $f_e(t)$

$$\int_{-T}^{T} f_e(t)dt = 2\int_{0}^{T} f_e(t)dt$$

For an odd function $f_o(t)$

$$\int_{-T}^{T} f_o(t)dt = 0$$

## Even/Odd Representation of an Arbitrary Function

A function $f(t)$ that is neither even nor odd can be represented as an even function by use of:

$$f_e(t) = \frac{1}{2}[f(t) + f(-t)]$$

or as an odd function by use of:

$$f_o(t) = \frac{1}{2}[f(t) - f(-t)]$$

Adding these together, an abitrary signal can be represented as

$$f(t) = f_e(t) + f_o(t)$$

That is, any function of time can be expressed as the sum of an even and an odd function.

## Example 1

Is the Dirac delta $\delta(t)$ an even or an odd function of time?

## Solution

Let $f(t)$ be an arbitrary function of time that is continous at $t = t_0$. Then by the sifting property of the delta function

and for $t_0 = 0$

Also for an even function $f_e(t)$

and for an odd function $f_o(t)$

### Even or odd?

An odd function $f_o(t)$ evaluated at $t = 0$ is zero, that is $f_o(0) = 0$.

Hence

$$\int_{-\infty}^{\infty} f_o(t)\delta(t)dt = f_o(0) = 0$$

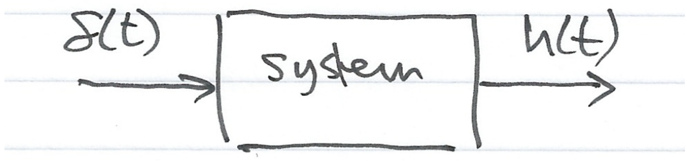
Hence the product $f_o(t)\delta(t)$ is odd function of $t$.

Since $f_o(t)$ is odd, $\delta(t)$ must be even because only an *even* function multiplied by an *odd* function can result in an *odd* function.

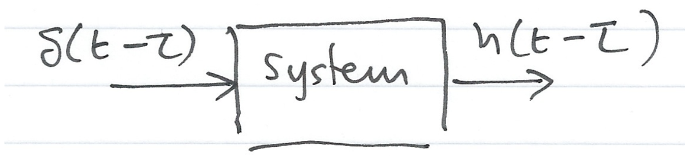
(Even times even or odd times odd produces an even function. See earlier slide)

## Time Convolution

Consider a system whose input is the Dirac delta ($\delta(t)$), and its output is the ***impulse response*** $h(t)$. We can represent the input-output relationship as a block diagram

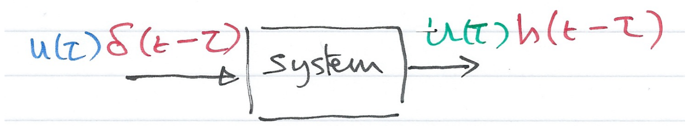



### In general




### Add an arbitrary input

Let $u(t)$ be any input whose value at $t = \tau$ is $u(\tau)$, Then because of the sampling property of the delta function



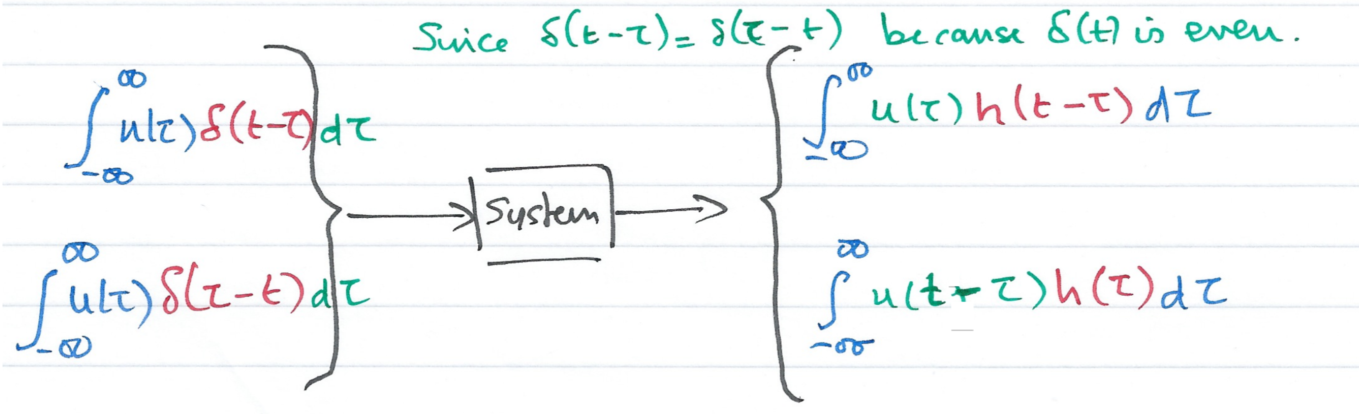

(output is $u(\tau)h(t-\tau)$)

## Integrate both sides

Integrating both sides over all values of $\tau$ ($-\infty < \tau < \infty$) and making use of the fact that the delta function is even, i.e.

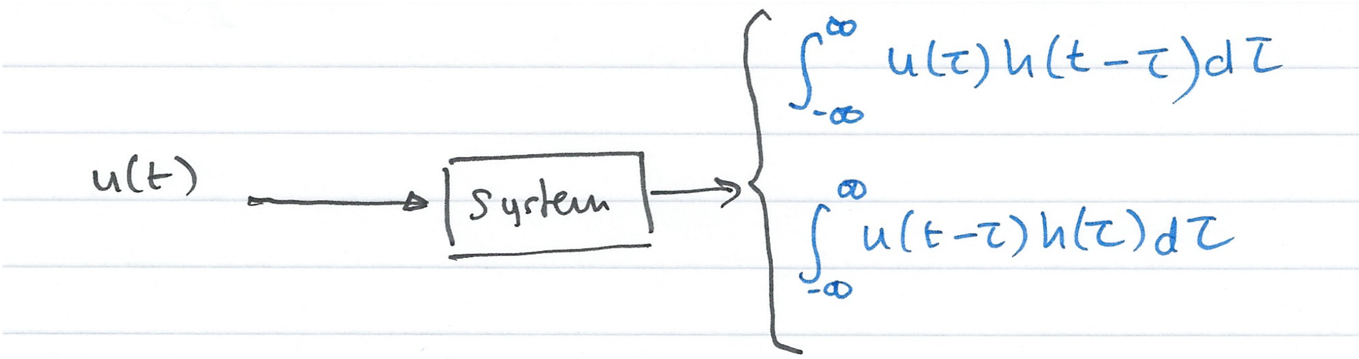
$$\delta(t-\tau) = \delta(\tau - t)$$

we have:




## Use the sifting property of delta

The second integral on the left side reduces to $u(t)$




## The Convolution Integral

The integral

$$\int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau$$

or

$$\int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau$$

is known as the *convolution integral*; it states that if we know the impulse response of a system, we can compute its time response to any input by using either of the integrals.

The convolution integral is usually written $u(t) * h(t)$ or $h(t) * u(t)$ where the asterisk ($*$) denotes convolution.

### Second Hour

- Graphical Evaluation of the Convolution Integral
- System Response by Laplace

# Graphical Evaluation of the Convolution Integral

The convolution integral is most conveniently evaluated by a graphical evaluation. The text book gives three examples (6.4-6.6) which we will demonstrate using a graphical visualization tool (http://www.mathworks.co.uk/matlabcentral/fileexchange/25199-graphical-demonstration-of-convolution) developed by Teja Muppirala of the Mathworks.

The tool: convolutiondemo.m (matlab/convolutiondemo.m) (see license.txt (matlab/license.txt)).

In [ ]:

```
clear all
cd ../matlab/convolution_demo
pwd
format compact
```

In [ ]:

```
convolutiondemo % ignore warnings
```

### Convolution by Graphical Method - Summary of Steps

For simplicity, we give the rules for $u(t)$, but the procedure is the same if we reflect and slide $h(t)$

1. Substitute $u(t)$ with $u(\tau)$ – this is a simple change of variable. It doesn't change the definition of $u(t)$.

1. Reflect $u(\tau)$ about the vertical axis to form $u(-\tau)$

1. Slide $u(-\tau)$ to the right a distance $t$ to obtain $u(t-\tau)$

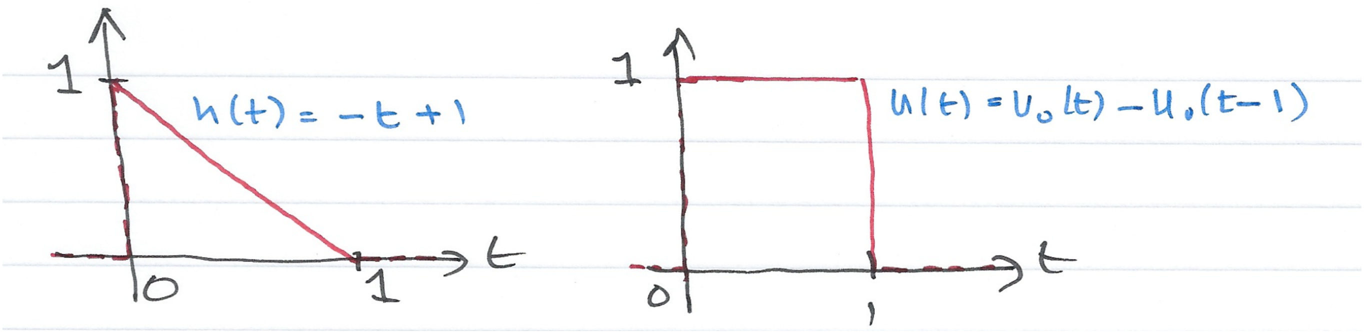
1. Multiply the two signals to obtain the product $u(t-\tau)h(\tau)$

1. Integrate the product over all $\tau$ from $-\infty$ to $\infty$.

### Example 2

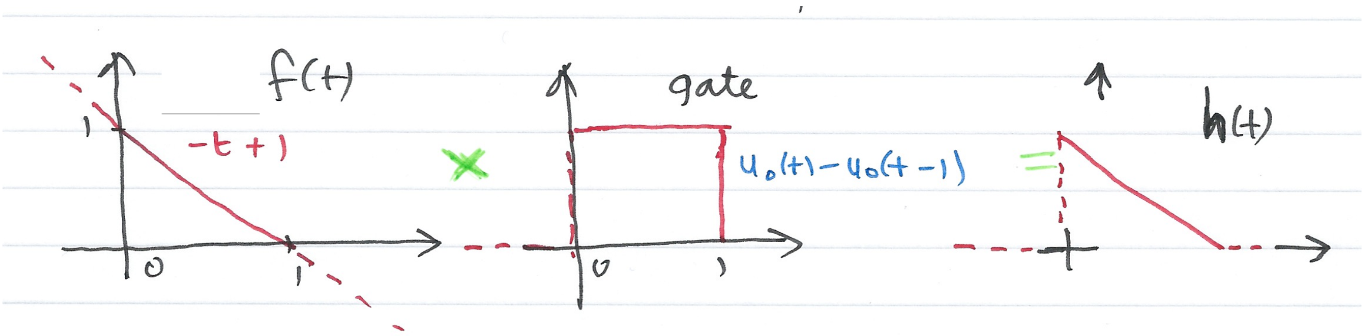
(This is example 6.4 in the Karris)

The signals $h(t)$ and $u(t)$ are shown below. Compute $h(t) * u(t)$ using the graphical technique.




## h(t)

The signal $h(t)$ is the straight line $f(t) = -t + 1$ but this is defined only between $t = 0$ and $t = 1$. We thus need to gate the function by multiplying it by $u_0(t) - u_0(t-1)$ as illustrated below:




Thus

$$h(t) \Leftrightarrow H(s)$$

$$h(t) = (-t+1)(u_0(t) - u_0(t-1)) = (-t+1)u_0(t) - (-(t-1)u_0(t-1)) = -tu_0(t) + u_0(t) + (t-1)u_0($$

$$-tu_0(t) + u_0(t) + (t-1)u_0(t-1) \Leftrightarrow -\frac{1}{s^2} + \frac{1}{s} + \frac{e^{-s}}{s^2}$$

$$H(s) = \frac{s + e^{-s} - 1}{s^2}$$

## u(t)

The input $u(t)$ is the gating function:

$$u(t) = u_0(t) - u_0(t-1)$$

so

$$U(s) = \frac{1}{s} - \frac{e^{-s}}{s} = \frac{1 - e^{-s}}{s}$$

## Prepare for convolutiondemo

To prepare this problem for evaluation in the `convolutiondemo` tool, we need to determine the Laplace Transforms of $h(t)$ and $u(t)$.

## convolutiondemo settings

- Let `g = (1 - exp(-s))/s`
- Let `h = (s + exp(-s) - 1)/s^2`
- Set range $-2 < \tau < -2$

## Summary of result

1. For $t < 0$: $u(t-\tau)h(\tau) = 0$
2. For $t = 0$: $u(t-\tau) = u(-\tau)$ and $u(-\tau)h(\tau) = 0$
3. For $0 < t \leq 1$: $h * u = \int_0^t (1)(-\tau + 1)d\tau = \tau - \tau^2/2\big|_0^t = t - t^2/2$
4. For $1 < t \leq 2$: $h * u = \int_{t-1}^1 (-\tau + 1)d\tau = \tau - \tau^2/2\big|_{t-1}^1 = t^2/2 - 2t + 2$
5. For $2 \leq t$: $u(t-\tau)h(\tau) = 0$

## Example 3

This is example 6.5 from Karris.

$$h(t) = e^{-t}$$

$$u(t) = u_0(t) - u_0(t-1)$$

## Answer 3

$$y(t) = \begin{cases} 0 : t \leq 0 \\ 1 - e^{-t} : \; 0 < t \leq 1 \\ e^{-t}\,(e-1) : \; 1 < t \leq 2 \\ 0 : \; 2 \leq t \end{cases}$$

### Check with MATLAB

In [ ]:

```
syms t tau
x1=int(exp(-tau),tau,0,t)
```

In [ ]:

```
x2=int(exp(-tau),tau,t-1,t)
```

## Example 4

This is example 6.6 from the text book.

$$h(t) = 2(u_0(t) - u_0(t-1))$$

$$u(t) = u_0(t) - u_0(t-2)$$

### Answer 4

$$y(t) = \begin{cases} 0 : t \leq 0 \\ 2t : \ 0 < t \leq 1 \\ 2 : \ 1 < t \leq 2 \\ -2t + 6 : \ 2 < t \leq 3 \\ 0 : \ 3 \leq t \end{cases}$$

## System Response by Laplace

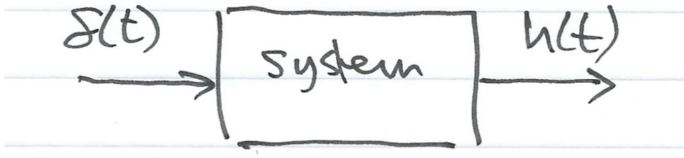
In the discussion of Laplace, we stated that

$$\mathcal{L}\left\{f(t) * g(t)\right\} = F(s)G(s)$$

We can use this property to make the solution of convolution problems even simpler.

## Impulse Response and Transfer Functions

Returning to the example we started with




Then the impulse response of the system $h(t)$ will be given by:

$$\mathcal{L}\left\{h(t) * \delta(t)\right\} = H(s)\Delta(s)$$

Where $H(s)$ be the laplace transform of the impulse response of the system $h(t)$. From properties of the Laplace transform we know that

$$\delta(t) \Leftrightarrow 1$$

so that $\Delta(s) = 1$ and

$$h(t) * \delta(t) \Leftrightarrow H(s).1 = H(s)$$

A consequence of this is that the transform of the impulse response $h(t)$ of a system with transfer function $H(s)$ is completely defined by the transfer function itself.

Previously we argued that the response of system with impulse response $h(t)$ was given by the convolution integrals:

$$h(t) * u(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau$$

Thus the Laplace transform of any system subject to an input $u(t)$ is simply

$$Y(s) = H(s)U(s)$$

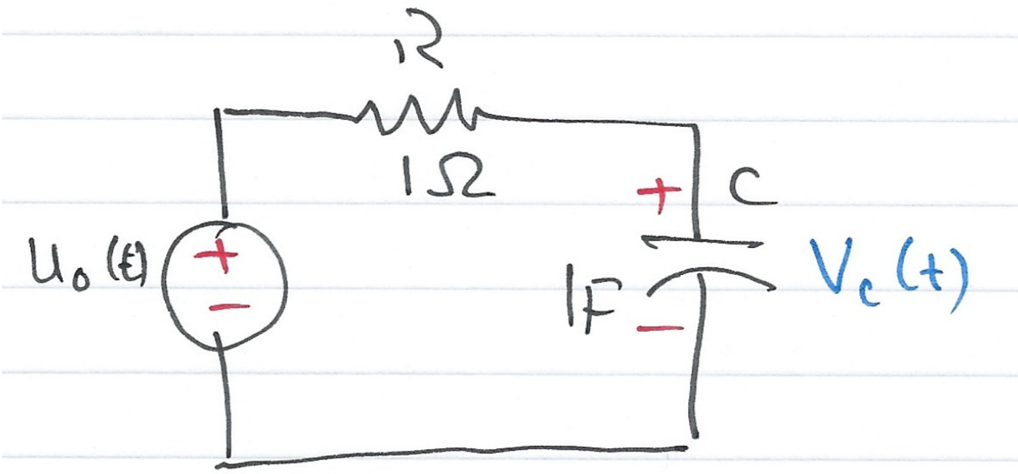
and

$$y(t) = \mathcal{L}^{-1}\left\{G(s)U(s)\right\}$$

Using tables, solution of a convolution problem by Laplace is usually simpler than using convolution directly.

## Example 5

This is example 6.7 from Karris.




For the circuit shown above, show that the transfer function of the circuit is:

$$H(s) = \frac{V_c(s)}{V_s(s)} = \frac{1/RC}{s + 1/RC}$$

Determine the impulse respone $h(t)$ of the circuit and the response of the capacitor voltage when the input is the unit step function $u_0(t)$ and $v_c(0^-) = 0$.

Assume $C = 1\ \mathrm{F}$ and $R = 1\ \Omega$.

### Solution 5a - Impulse response

### Solution 5b - Step response

### Homework

Verify this result using the convolution integral

$$h(t) * u(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau$$