Worksheet 12

To accompany Chapter 5.1 Defining the Fourier Transform

We will step through this worksheet in class.

You are expected to have at least watched the video presentation of Chapter 5.1 (https://cpjobling.github.io/eg-247-textbook/fourier_transform/1/ft1) of the notes (https://cpjobling.github.io/eg-247-textbook) before coming to class. If you haven't watch it afterwards!

Fourier Transform as the Limit of a Fourier Series

We start by considering the pulse train that we used in the last lecture and demonstrate that the discrete line spectra for the Fourier Series becomes a continuous spectrum as the signal becomes aperiodic.

This analysis is from Boulet pp $142-144$ and $176-180$.

Let $\tilde{x}(t)$ be the Fourier series of the rectangular pulse train shown below:
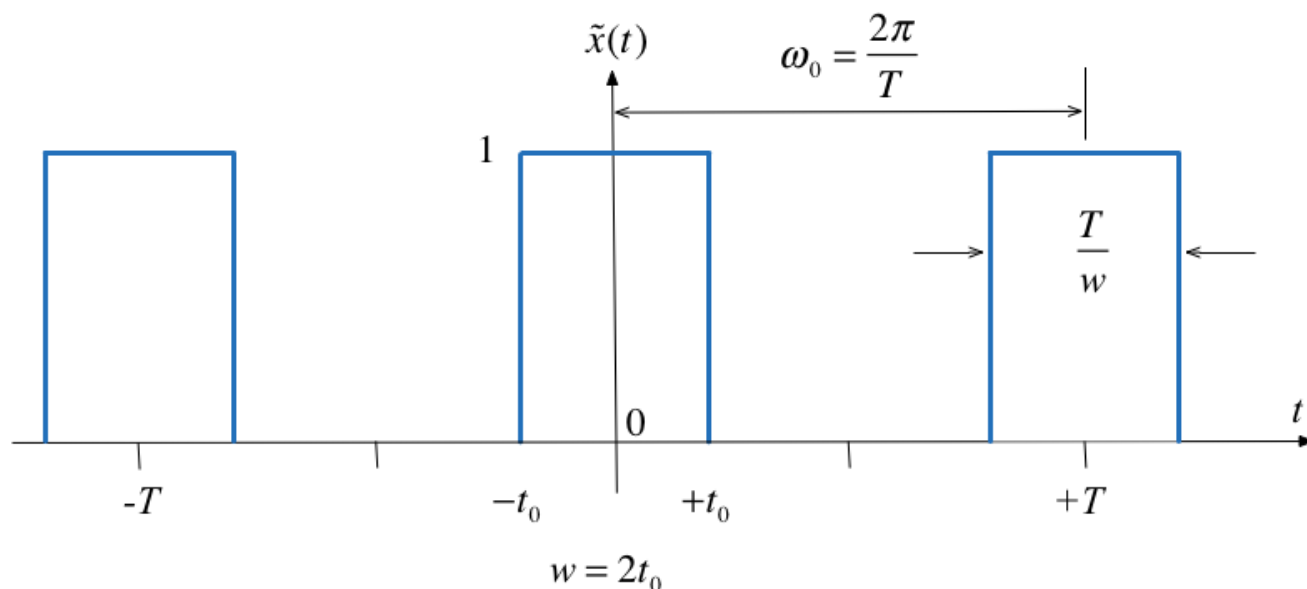
Fourier Series

In the previous section (../../fourier_series/3/exp_fs2) we used

$$C_k = \frac{1}{2\pi} \int_{-\pi/w}^{\pi/w} A e^{-jk(\Omega_0 t)} \, d(\Omega_0 t) = \frac{A}{2\pi} \int_{-\pi/w}^{\pi/w} e^{-jk(\Omega_0 t)} \, d(\Omega_0 t)$$

to compute the line spectra.

From the Time Point of View

If we instead take a time point-of-view and let $A = 1$

$$C_k = \frac{1}{T} \int_{-t_0}^{t_0} e^{-jk\Omega_0 t}\, dt.$$

Let's complete the analysis in the whiteboard.

The Sinc Function

The function, $\sin(\pi x)/\pi x$ crops up again and again in Fourier analysis. The Fourier coefficients C_k are scaled *samples* of the real continuous *normalized sinc* function defined as follows:

$$\operatorname{sinc} u := \frac{\sin \pi u}{\pi u},\ u \in \mathbb{R}.$$

The function is equal to 1 at $u = 0^1$ and has zero crossings at $u = \pm n, \ n = 1, 2, 3, \ldots$ as shown below.

Plot the sinc function

Plots:

$$sinc(u) = \frac{\sin \pi u}{\pi u}, \ u \in \mathbb{R}$$

In [ ]:

```
x = linspace(-5,5,1000);
plot(x,sin(pi.*x)./(pi.*x))
grid
title('Graph of sinc function')
ylabel('sinc(u)')
xlabel('u')
```

Duty cycle

- We define the duty cycle $\eta = 2t_0/T$ of the rectangular pulse train as the fraction of the time the signal is "on" (equal to 1) over one period.
- The duty cycle is often given as a percentage.

The spectral coefficients expressed using the normalized sinc function and the duty cycle can be written as

$$C_k = \frac{2t_0}{T} \frac{\sin\left(\frac{\pi k 2 t_0}{T}\right)}{\frac{\pi k 2 t_0}{T}} = \frac{2t_0}{T} \mathrm{sinc}\left(\frac{k 2 t_0}{T}\right)$$

$$C_k = \eta \, \mathrm{sinc}(k\eta)$$

Normalize the spectral coefficients

Let us normalize the spectral coefficients of $\tilde{x}(t)$ by multiplying them by T, and assume t_0 is fixed so that the duty cycle $\eta = 2t_0/T$ will decrease as we increase T:

$$TC_k = T\eta \, \mathrm{sinc}(k\eta) = 2t_0 \, \mathrm{sinc}\left(k\frac{2t_0}{T}\right)$$

Then the normalized coefficients TC_k of the rectangular wave is a sinc envelope with constant amplitude at the origin equal to $2t_0$, and a zero crossing at fixed frequency π/t_0 rad/s, both independent of T.

In [ ]:

```
open duty_cycle
```

Demo

Run duty_cycle with values of:

- 50% ($\eta = 1/2$)

- 25% ($\eta =?$)

- 12.5% ($\eta =?$)

- 5% ($\eta =?$)

Comments

- As the fundamental period increases, we get more spectral lines packed into the lobes of the sinc envelope.
- These normalized spectral coefficients turn out to be samples of the continuous sinc function on the spectrum of $\tilde{x}(t)$
- The two spectra are plotted against the frequency variable $k\omega_0$ with units of rad/s rather than index of harmonic component
- The first zeros of each side of the main lobe are at frequencies $\omega = \pm\pi/t_0$ rad/s
- The zero-crossing points of sinc envelope are independent of the period T. They only depend on t_0.

Intuition leading to the Fourier Transform

- An aperiodic signal that has been made periodic by "repeating" its graph every T seconds will have a line spectrum that becomes more and more dense as the fundamental period is made longer and longer.
- The line spectrum has the same continuous envelope.
- As T goes to infinity, the line spectrum will become a continuous function of ω.
- The *envelope* is this function.

Doing the Maths

See the notes (ft1)

Inverse Fourier Transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} \, d\omega := \mathcal{F}^{-1}\left\{X(j\omega)\right\}$$

Similarly, given the expression we have already seen for an arbitrary $x(t)$:

Fourier Transform:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt := \mathcal{F}\left\{x(t)\right\}.$$

Fourier Transform Pair

- The two equations on the previous slide are called the *Fourier transform pair*.

Properties of the Fourier Transform

Again, we will provide any properties that you might need in the examination.

You will find a number of these in the accompanying notes.

Table of Properites of the Fourier Transform

As was the case of the Laplace Transform, properties of Fourier transforms are usually summarized in Tables of Fourier Transform properties. For example this one: <u>Properties of the Fourier Transform (Wikpedia)</u> <u>(http://en.wikipedia.org/wiki/Fourier_transform#Properties_of_the_Fourier_transform)</u> and Table 8.8 in Karris (page 8-17).

More detail and some commentry is given in the printable version of these notes.

	Name	$f(t)$	$F(j\omega)$	Re
1	Linearity	$a_1 f_1(t) + a_2 f_2(t) + \cdots + a_n f_n(t)$	$a_1 F_1(j\omega) + a_2 F_2(j\omega) + \cdots + a_n F_n(j\omega)$	Fou tran a lir ope
2	Symmetry	$2\pi f(-j\omega)$	$F(t)$	
3.	Time and frequency scaling	$f(\alpha t)$	$\dfrac{1}{|\alpha|} F\left(j\dfrac{\omega}{\alpha}\right)$	time con is fr exp and vers
4.	Time shifting	$f(t - t_0)$	$e^{-j\omega t_0} F(j\omega)$	A ti cor to a shif freq don
5.	Frequency shifting	$e^{j\omega_0 t} f(t)$	$F(j\omega - j\omega_0)$	Mul sigr con exp resu freq shif
6.	Time differentiation	$\dfrac{d^n}{dt^n} f(t)$	$(j\omega)^n F(j\omega)$	
7.	Frequency differentiation	$(-jt)^n f(t)$	$\dfrac{d^n}{d\omega^n} F(j\omega)$	
8.	Time integration	$\displaystyle\int_{-\infty}^{t} f(\tau)d\tau$	$\dfrac{F(j\omega)}{j\omega} + \pi F(0)\delta(\omega)$	
9.	Conjugation	$f^*(t)$	$F^*(-j\omega)$	
10.	Time convolution	$f_1(t) * f_2(t)$	$F_1(j\omega) F_2(j\omega)$	Cor with Trar

	Name	$f(t)$	$F(j\omega)$	Rer
11.	Frequency convolution	$f_1(t)f_2(t)$	$\dfrac{1}{2\pi}F_1(j\omega) * F_2(j\omega)$	This app to a moc as s Bou 182
12.	Area under $f(t)$	$\displaystyle\int_{-\infty}^{\infty} f(t)\,dt = F(0)$		Way calc (or valu sigr
13.	Area under $F(j\omega)$	$f(0) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} F(j\omega)\,d\omega$		
14.	Energy-Density Spectrum	$E_{[\omega_1,\omega_2]} := \dfrac{1}{2\pi}\displaystyle\int_{\omega_1}^{\omega_2} |F(j\omega)|^2\,d\omega.$		
15.	Parseval's theorem	$\displaystyle\int_{-\infty}^{\infty} |f(t)|^2\,dt = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} |F(j\omega)|^2\,d\omega.$		Def RM fror

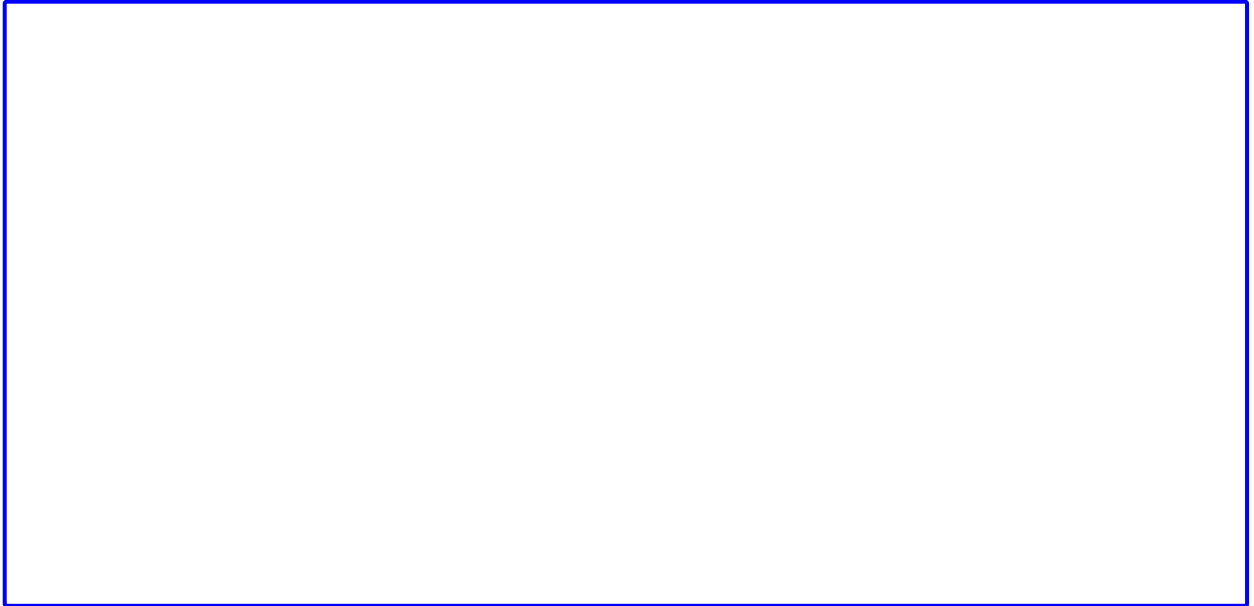See also: Wikibooks: Engineering Tables/Fourier Transform Properties

Examples

1. Amplitude Modulation
2. Impulse response
3. Energy computation

Example 1: Amplitude Modulation

Compute the result of multiplying a signal $f(t)$ by a carrier waveform $\cos \omega_c t$.

Hint use Euler's identity and the frequency shift property

Example 2: Impulse response

A system has impulse response $f(t) = e^{-t}u_0(t)$. Compute the frequency sprectrum of this system.

Example 3: Energy computation

An aperiodic real signal $f(t)$ has Fourier transform $F(j\omega)$. Compute the energy contained the signal between 5kHz and 10kHz.

Computing Fourier Transforms in Matlab

MATLAB has the built-in **fourier** and **ifourier** functions that can be used to compute the Fourier transform and its inverse. We will explore some of thes in the next lab.

For now, here's an example:

Example

Use Matlab to confirm the Fourier transform pair:

$$e^{-\frac{1}{2}t^2} \Leftrightarrow \sqrt{2\pi}e^{-\frac{1}{2}\omega^2}$$

In [ ]:

```
syms t v omega x;
ft = exp(-t^2/2);
Fw = fourier(ft,omega)
```

In [ ]:

```
pretty(Fw)
```

Check by computing the inverse using ifourier

In [ ]:

```
ft = ifourier(Fw)
```