Models of Discrete-Time Systems

Scope and Background Reading

This we will explore digital systems and learn more about the z-transfer function model.

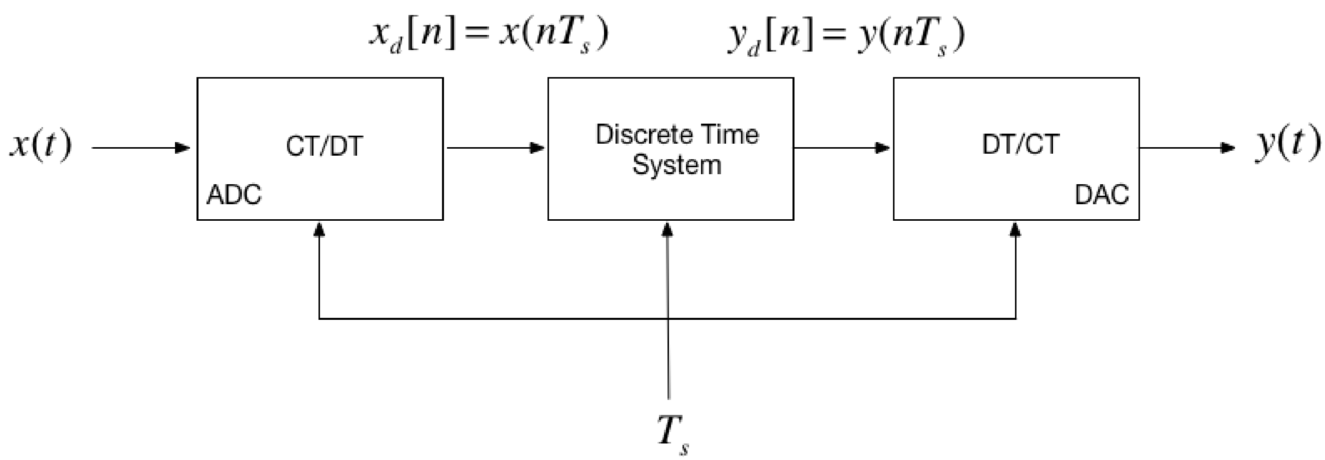
The material in this presentation and notes is based on Chapter 9 (Starting at Section 9.7) of Steven T. Karris, Signals and Systems: with Matlab Computation and Simulink Modelling, 5th Edition. (http://site.ebrary.com/lib/swansea/docDetail.action?docID=10547416) from the **Required Reading List**. I have skipped the section on digital state-space models.

Agenda

- Discrete Time Systems
- Transfer Functions in the Z-Domain
- Modelling digital systems in Matlab/Simulink
- Continuous System Equivalents
- Example: Digital Butterworth Filter

Discrete Time Systems

In the lecture that introduced the z-transform we talked about the representation of a discrete-time (DT) system by the model shown below:



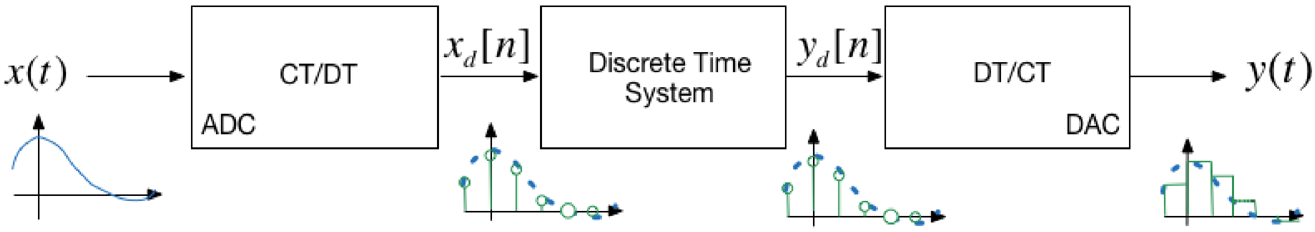
In this session, we want to explore the contents of the central block.

DT System as a Sequence Processor

- As noted in the previous slide, the discrete time system (DTS) `takes as an input the sequence $x_d[n]$[1] which in a physical signal would be obtained by sampling the continuous time signal $x(t)$ using an analogue to digital converter (ADC).

- It produces another sequence $y_d[n]$ by *processing* the input sequence in some way.

- The output sequence is converted into an analogue signal $y(t)$ by a digital to analogue converter (DAC).



What is the nature of the DTS?

- The discrete time system (DTS) is a block that converts a sequence $x_d[n]$ into another sequence $y_d[n]$

- The transformation will be a *difference equation* $h[n]$

- By analogy with CT systems, $h[n]$ is the impulse response of the DTS, and $y[n]$ can be obtained by *convolving* $h[n]$ with $x_d[n]$ so:

$$y_d[n] = h[n] * x_d[n]$$

- Taking the z-transform of $h[n]$ we get $H(z)$, and from the transform properties, convolution of the signal $x_d[n]$ by system $h[n]$ will be *multiplication* of the z-transforms:

$$Y_d(z) = H(z)X_d(z)$$

- So, what does $h[n]$ and therefore $H(z)$ look like?

Transfer Functions in the Z-Domain

Let us assume that the sequence transformation is a *difference equation* of the form[2]:

$$\begin{aligned} y[n] + a_1 y[n-1] + a_2 y[n-2] + \cdots + a_k y[n-k] \\ = b_0 x[n] + b_1 u[n-1] + b_2 u[n-2] + \cdots + b_k u[n-k] \end{aligned}$$

Take Z-Transform of both sides

From the z-transform properties

$$f[n-m] \Leftrightarrow z^{-m}F(z)$$

so....

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \cdots + a_k z^{-k} Y(z) = \ldots$$

$$b_0 U(z) + b_1 z^{-1} U(z) + b_2 z^{-2} U(z) + \cdots + b_k z^{-k} U(z)$$

Gather terms

$$\left(1 + a_1 z^{-1} + a_2 z^{-2} + \cdots a_k z^{-k}\right) Y(z) =$$
$$\left(b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_k z^{-k}\right) U(z)$$

from which ...

$$Y(z) = \left(\frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_k z^{-k}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots a_k z^{-k}}\right) U(z)$$

Define transfer function

We define the *discrete time transfer function* $H(z) := Y(z)/U(z)$ so...

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_k z^{-k}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots a_k z^{-k}}$$

... or more conventionally[3]:

$$H(z) = \frac{b_0 z^k + b_1 z^{k-1} + b_2 z^{k-2} + \cdots b_{k-1} z + b_k}{z^k + a_1 z^{k-1} + a_2 z^{k-2} + \cdots a_{k-1} z + a_k}$$

DT impulse response

The *discrete-time impulse reponse* $h[n]$ is the response of the DT system to the input $x[n] = \delta[n]$

Last week we showed that

$$\mathcal{Z}\left\{\delta[n]\right\}$$

was defined by the transform pair

$$\delta[n] \Leftrightarrow 1$$

so

$$h[n] = \mathcal{Z}^{-1}\left\{H(z).1\right\} = \mathcal{Z}^{-1}\left\{H(z)\right\}$$

Example 5

Karris Example 9.10:

The difference equation describing the input-output relationship of a DT system with zero initial conditions, is:

$$y[n] - 0.5y[n-1] + 0.125y[n-2] = x[n] + x[n-1]$$

Compute:

1. The transfer function $H(z)$
2. The DT impulse response $h[n]$
3. The response $y[n]$ when the input $x[n]$ is the DT unit step $u_0[n]$

5.1. The transfer function

$$H(z) = \frac{Y(z)}{U(z)} = \ldots ?$$

5.2. The DT impulse response

Start with:

$$\frac{H(z)}{z} = \frac{z-1}{z^2 + 0.5z + 0.125}$$

Matlab Solution

In [1]:

```
clear all
cd matlab
pwd
format compact
```

```
ans =

    '/Users/eechris/dev/eg-247-textbook/content/dt_systems/4/matlab'
```

See dtm_ex1_2.mlx (matlab/dtm_ex1_2.mlx). (Also available as dtm_ex1_2.m (matlab/dtm_ex1_2.m).)

The difference equation describing the input-output relationship of the DT system with zero initial conditions, is:

$$y[n] - 0.5y[n-1] + 0.125y[n-2] = x[n] + x[n-1]$$

Transfer function

Numerator $z + 1$

In [2]:

```
Nz = [0 1 1];
```

Denominator $z^2 - 0.5z + 0.125$

In [3]:

```
Dz = [1 -0.5 0.125];
```

Poles and residues

In [4]:

```
[r,p,k] = residue(Nz,Dz)
```

```
r =
   0.5000 - 2.5000i
   0.5000 + 2.5000i
p =
   0.2500 + 0.2500i
   0.2500 - 0.2500i
k =
     []
```

Impulse Response

In [5]:

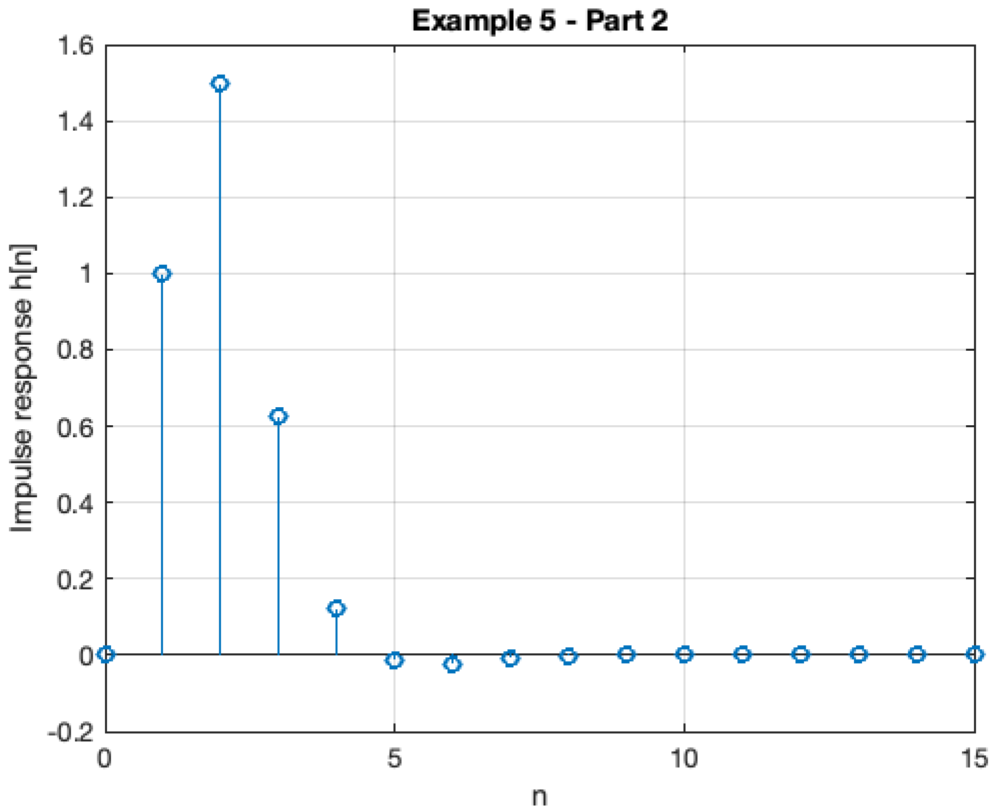
```
Hz = tf(Nz,Dz,1)
hn = impulse(Hz, 15);
```

```
Hz =

         z + 1
  -------------------
  z^2 - 0.5 z + 0.125

Sample time: 1 seconds
Discrete-time transfer function.
```

Plot the response

In [6]:

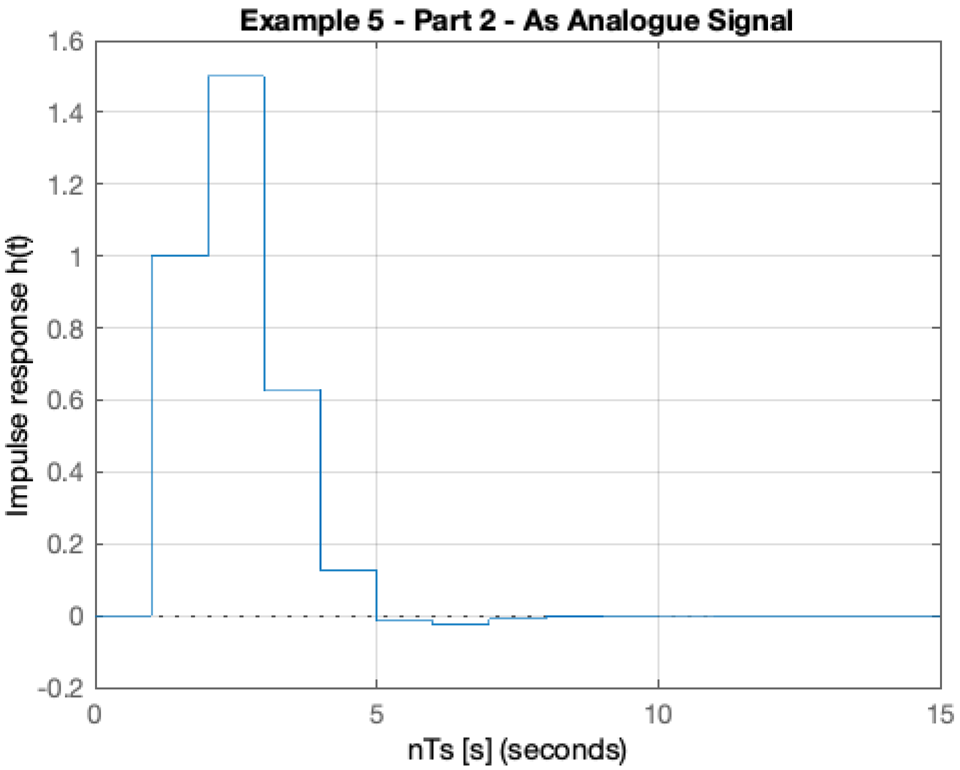
```
stem([0:15], hn)
grid
title('Example 5 - Part 2')
xlabel('n')
ylabel('Impulse response h[n]')
```



Response as stepwise continuous y(t)

In [7]:

```
impulse(Hz,15)
grid
title('Example 5 - Part 2 - As Analogue Signal')
xlabel('nTs [s]')
ylabel('Impulse response h(t)')
```



5.3. The DT step response

$$Y(z) = H(z)X(z)$$

$$u_0[n] \Leftrightarrow \frac{z}{z-1}$$

$$\begin{aligned} Y(z) = H(z)U_0(z) &= \frac{z^2+z}{z^2+0.5z+0.125} \cdot \frac{z}{z-1} \\ &= \frac{z(z^2+z)}{(z^2+0.5z+0.125)(z-1)} \end{aligned}$$

$$\frac{Y(z)}{z} = \frac{z^2+z}{(z^2+0.5z+0.125)(z-1)}$$

Solved by inverse Z-transform.

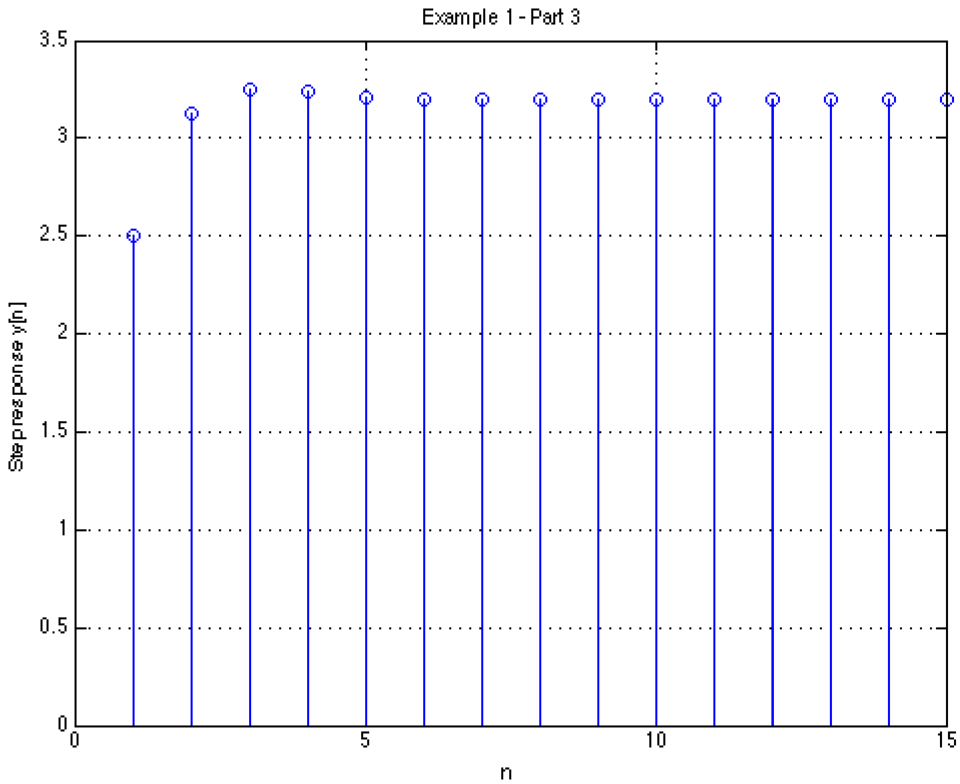
Matlab Solution

See dtm_ex1_3.mlx (matlab/dtm_ex1_3.mlx). (Also available as dtm_ex1_3.m (matlab/dtm_ex1_3.m).)

In [9]:

```
open dtm_ex1_3
```

Results



Modelling DT systems in Matlab and Simulink

Matlab

Code extracted from dtm_ex1_3.m (matlab/dtm_ex1_3.m):

In [10]:

```
Ts = 1;
z = tf('z', Ts)
```

```
z =

  z

Sample time: 1 seconds
Discrete-time transfer function.
```

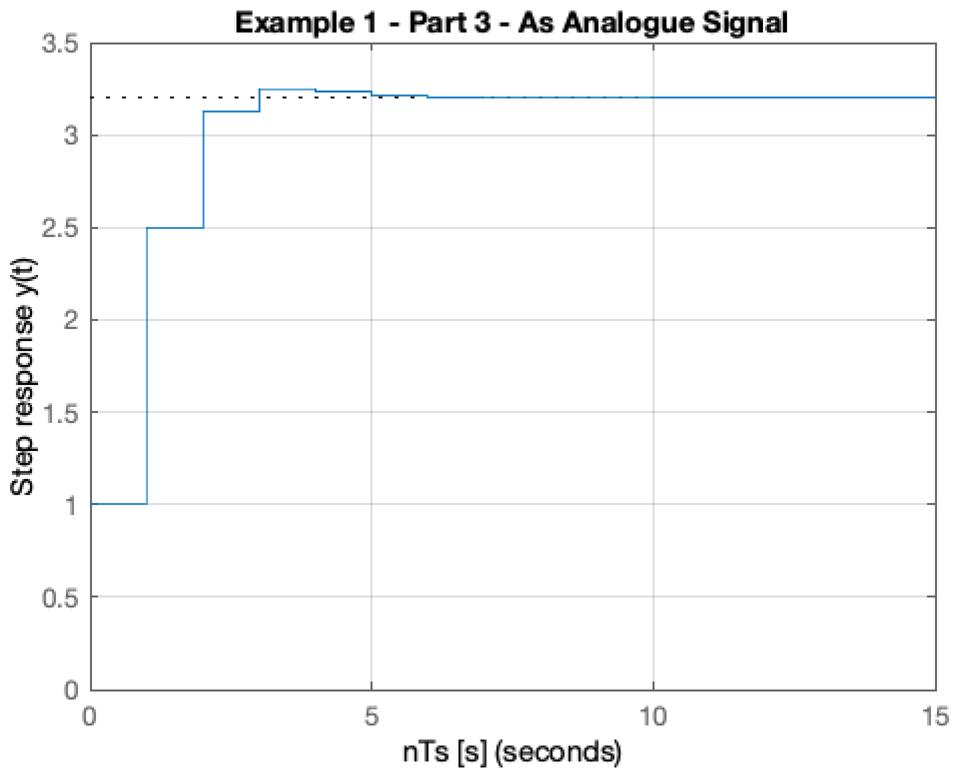
In [11]:

```
Hz = (z^2 + z)/(z^2 - 0.5 * z + 0.125)
```

```
Hz =

        z^2 + z
  -------------------
  z^2 - 0.5 z + 0.125

Sample time: 1 seconds
Discrete-time transfer function.
```

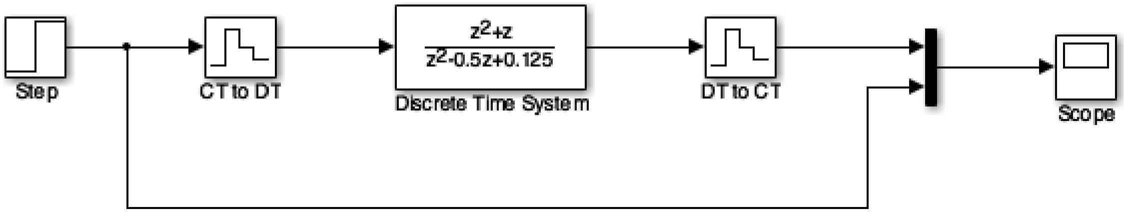
In [12]:

```
step(Hz)
grid
title('Example 1 - Part 3 - As Analogue Signal')
xlabel('nTs [s]')
ylabel('Step response y(t)')
axis([0,15,0,3.5])
```



Simulink Model

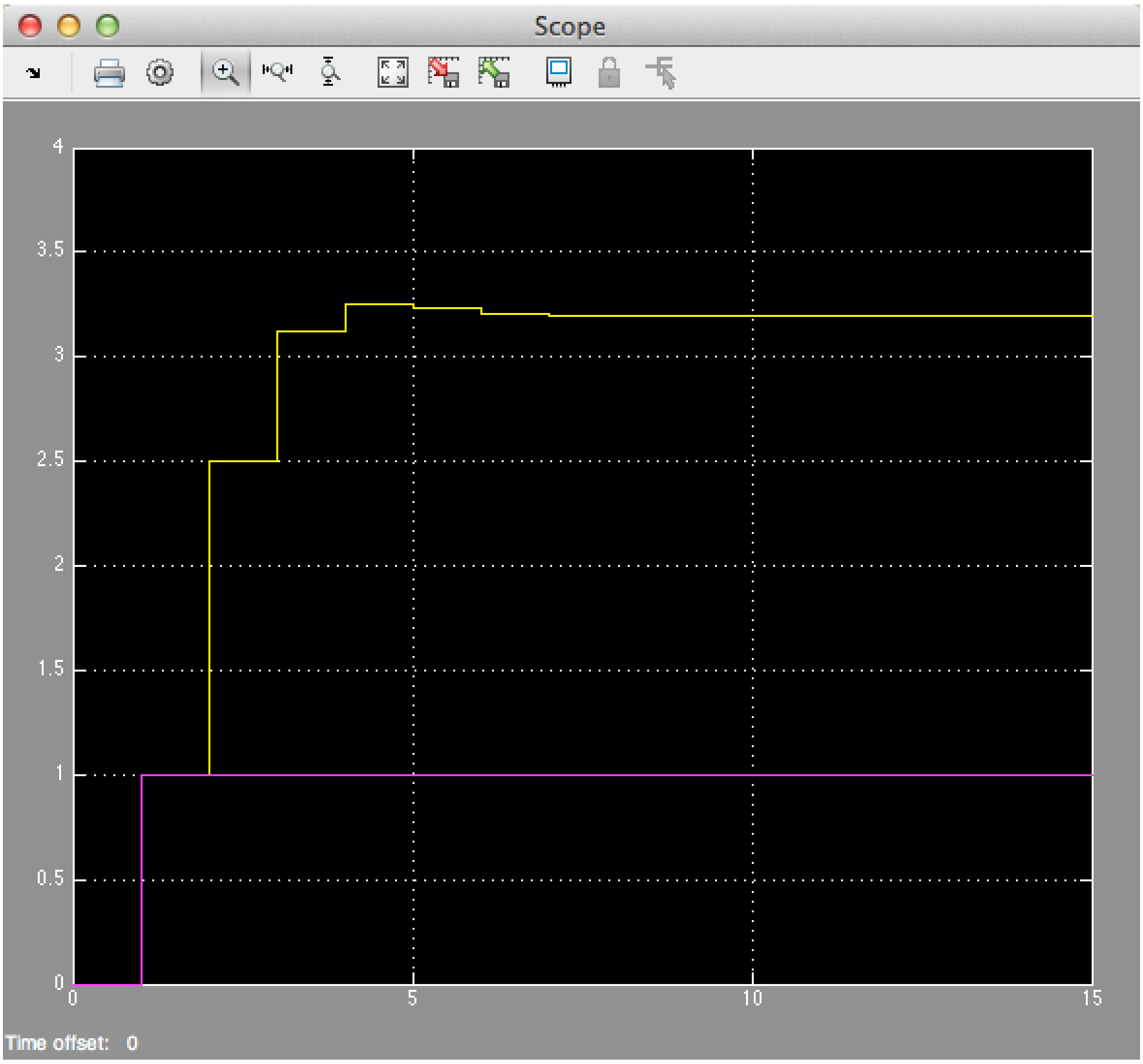
See dtm.slx (matlab/dtm.slx):



In [14]:

```
dtm
```

Results



Converting Continuous Time Systems to Discrete Time Systems

- In analogue electronics, to implement a filter we would need to resort to op-amp circuits with resistors, capacitors and inductors acting as energy dissipation, storage and release devices.
- In modern digital electronics, it is often more convenient to take the original transfer function $H(s)$ and produce an equivalent $H(z)$.
- We can then determine a *difference equation* that will respresent $h[n]$ and implement this as *computer algorithm*.
- Simple storage of past values in memory becomes the repository of past state rather than the integrators and derivative circuits that are needed in the analogue world.
- To achieve this, all we need is to be able to do is to *sample* and *process* the signals quickly enough to avoid violating Nyquist-Shannon's sampling theorem.

Continuous System Equivalents

- There is no digital system that uniquely represents a continuous system
- This is because as we are sampling, we only have knowledge of signals being processed at the sampling instants, and need to *reconstruct* the inter-sample behaviour.
- In practice, only a small number of transformations are used.
- The derivation of these is beyond the scope of this module, but we'll mention the ones that Matlab provides in a function called `c2d`

Matlab c2d function

Let's see what the help function says:

In [15]:

```
help c2d
```

```
 C2D  Converts continuous-time dynamic system to discrete time.

    SYSD = C2D(SYSC,TS,METHOD) computes a discrete-time model SYSD w
ith
    sample time TS that approximates the continuous-time model SYSC.
    The string METHOD selects the discretization method among the fo
llowing:
       'zoh'           Zero-order hold on the inputs
       'foh'           Linear interpolation of inputs
       'impulse'       Impulse-invariant discretization
       'tustin'        Bilinear (Tustin) approximation.
       'matched'       Matched pole-zero method (for SISO systems on
ly).
       'least-squares' Least-squares minimization of the error betwe
en
                       frequency responses of the continuous and dis
crete
                       systems (for SISO systems only).
    The default is 'zoh' when METHOD is omitted. The sample time TS
should
    be specified in the time units of SYSC (see "TimeUnit" propert
y).

    C2D(SYSC,TS,OPTIONS) gives access to additional discretization o
ptions.
    Use C2DOPTIONS to create and configure the option set OPTIONS. F
or
    example, you can specify a prewarping frequency for the Tustin m
ethod by:
       opt = c2dOptions('Method','tustin','PrewarpFrequency',.5);
       sysd = c2d(sysc,.1,opt);

    For state-space models,
       [SYSD,G] = C2D(SYSC,Ts,METHOD)
    also returns the matrix G mapping the states xc(t) of SYSC to th
e states
    xd[k] of SYSD:
       xd[k] = G * [xc(k*Ts) ; u[k]]
    Given an initial condition x0 for SYSC and an initial input valu
e u0=u(0),
    the equivalent initial condition for SYSD is (assuming u(t)=0 fo
r t<0):
       xd[0] = G * [x0;u0] .

    See also C2DOPTIONS, D2C, D2D, DYNAMICSYSTEM.

    Reference page in Doc Center
       doc c2d

    Other functions named c2d

       DynamicSystem/c2d    ltipack.tfdata/c2d
```

In [16]:

```
doc c2d
```

Example 6

- Design a 2nd-order butterworth low-pass anti-aliasing filter with transfer function $H(s)$ for use in sampling music.
- The cut-off frequency $\omega_c = 20$ kHz and the filter should have an attenuation of at least -80 dB in the stop band.
- Choose a suitable sampling frequency for the audio signal and give the transfer function $H(z)$ and an algorithm to implement $h[n]$

Solution

See digit_butter.m (matlab/digit_butter.m).

First determine the cut-off frequency ω_c

$$\omega_c = 2\pi f_c = 2 \times \pi \times 20 \times 10^3 \text{ rad/s}$$

In [17]:

```
wc = 2*pi*20e3
```

```
wc =
   1.2566e+05
```

$$\omega_c = 125.66 \times 10^3 \text{ rad/s}$$

From the lecture on filters, we know the 2nd-order butterworth filter has transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{\omega_c^2}{s^2 + \omega_c\sqrt{2}\,s + \omega_c^2}$$

Substituting for $\omega_c = 125.6637 \times 10^3$ this is ...?

In [18]:

```
Hs = tf(wc^2,[1 wc*sqrt(2), wc^2])
```

```
Hs =
 
            1.579e10
  ---------------------------
  s^2 + 1.777e05 s + 1.579e10
 
Continuous-time transfer function.
```

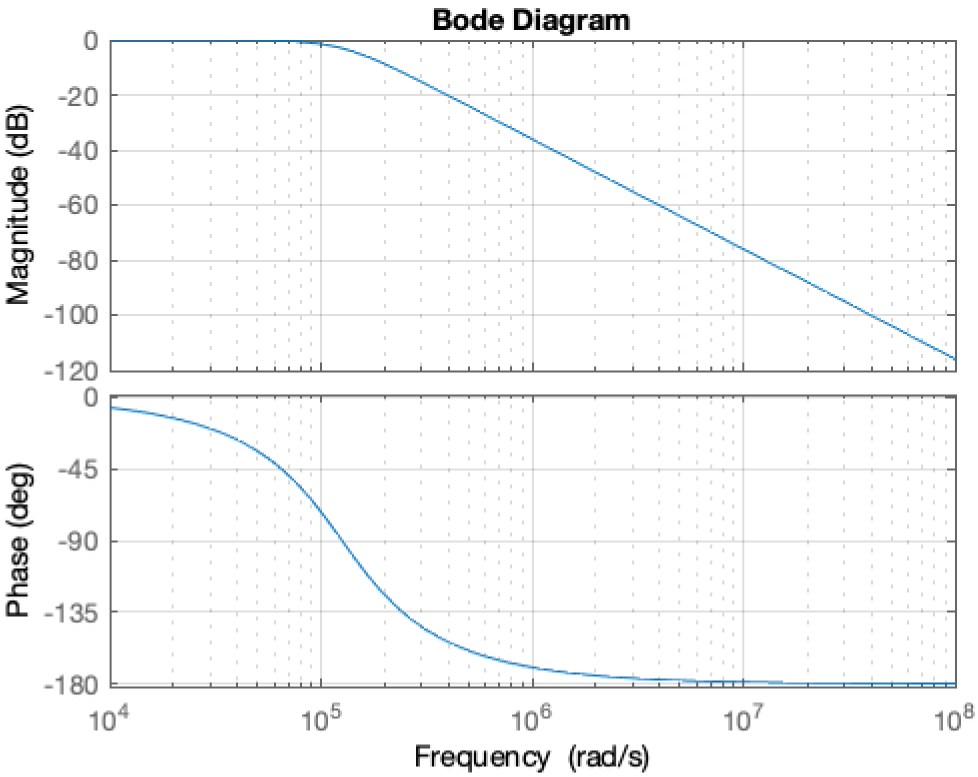
$$H(s) = \frac{15.79 \times 10^9}{s^2 + 177.7 \times 10^3 s + 15.79 \times 10^9}$$

Bode plot

Matlab:

In [19]:

```
bode(Hs,{1e4,1e8})
grid
```



Sampling Frequency

From the bode diagram, the frequency at which $|H(j\omega)|$ is -80 dB is approx 12.6×10^6 rad/s.

To avoid aliasing, we should choose a sampling frequency twice this = ?

$\omega_s = 2 \times 12.6 \times 10^6$ rad/s.

In [20]:

```
ws = 2* 12.6e6
```

```
ws =
    25200000
```

So

$\omega_s = 25.2 \times 10^6$ rad/s.

Sampling frequency (f_s) in Hz = ?

$$f_s = \omega_s/(2\pi) \text{ Mhz}$$

In [21]:

```
fs = ws/(2*pi)
```

```
fs =
   4.0107e+06
```

$$f_s = 40.11 \text{ Mhz}$$

Sampling time $T_s = ?$

$T_s = 1/fs$ s

In [22]:

```
Ts = 1/fs
```

```
Ts =
   2.4933e-07
```

$$T_s = 1/f_s \approx 0.25\ \mu\text{s}$$

Digital Butterworth

zero-order-hold equivalent

In [23]:

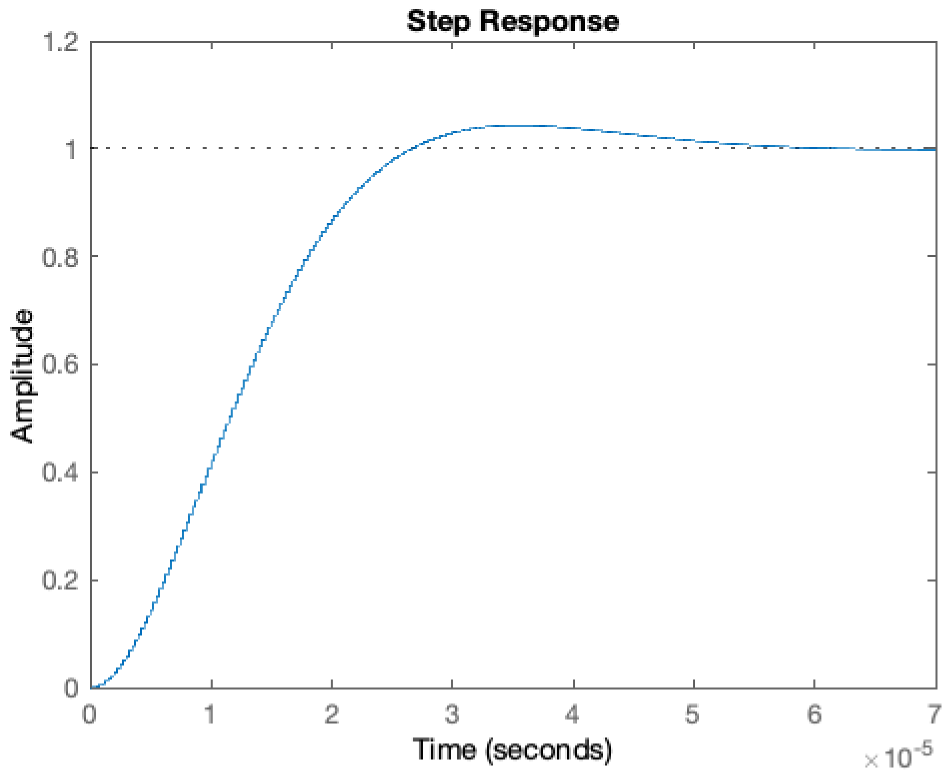
```
Hz = c2d(Hs, Ts)
```

```
Hz =

  0.0004836 z + 0.0004765
  -----------------------
  z^2 - 1.956 z + 0.9567

Sample time: 2.4933e-07 seconds
Discrete-time transfer function.
```

Step response

In [24]:

```
step(Hz)
```



Algorithm

From previous result:

$$H(z) = \frac{Y(z)}{U(z)} = \frac{486.6 \times 10^{-6} z + 476.5 \times 10^{-6}}{z^2 - 1.956z + 0.9567}$$

Dividing top and bottom by z^2 ...

$$H(z) = \frac{Y(z)}{U(z)} = \frac{486.6 \times 10^{-6} z^{-1} + 476.5 \times 10^{-6} z^{-2}}{1 - 1.956z^{-1} + 0.9567z^{-2}}$$

expanding out ...

$$\begin{aligned} Y(z) - 1.956z^{-1}Y(z) + 0.9567z^{-2}Y(z) = \\ 486.6 \times 10^{-6} z^{-1} U(z) + 476.5 \times 10^{-6} z^{-2} U(z) \end{aligned}$$

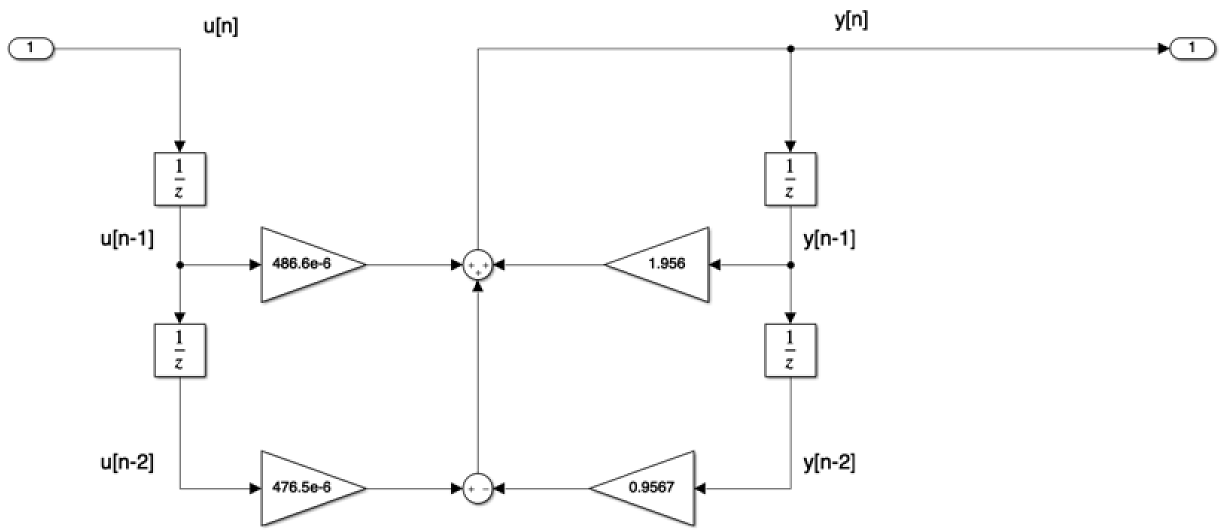
Inverse z-transform gives ...

$$y[n] - 1.956y[n-1] + 0.9567y[n-2] =$$
$$486.6 \times 10^{-6}u[n-1] + 476.5 \times 10^{-6}u[n-2]$$

in algorithmic form (compute $y[n]$ from past values of u and y) ...

$$y[n] = 1.956y[n-1] - 0.9567y[n-2] + 486.6 \times 10^{-6}u[n-1] + \ldots$$
$$476.5 \times 10^{-6}u[n-2]$$

Block Diagram of the digital BW filter



As Simulink Model

In [3]:

```
open digifilter
```

Convert to code

To implement:

$$y[n] = 1.956y[n-1] - 0.9567y[n-2] + 486.6 \times 10^{-6}u[n-1] + 476.5 \times 10^{-6}u[n-2]$$

```
/* Initialize */
Ts = 2.4933e-07; /* more probably some fraction of clock speed */
ynm1 = 0; ynm2 = 0; unm1 = 0; unm2 = 0;
while (true) {
    un = read_adc;
    yn = 1.956*ynm1 - 0.9567*ynm2 + 486.6e-6*unm1 + 476.5e-6*unm2;
    write_dac(yn);
    /* store past values */
    ynm2 = ynm1; ynm1 = yn;
    unm2 = unm1; unm1 = un;
    wait(Ts);
}
```

Comments

PC soundcards can sample audio at 44.1 kHz so this implies that the anti-aliasing filter is much sharper than this one as $f_s/2 = 22.05$ kHz.

You might wish to find out what order butterworth filter would be needed to have $f_c = 20$ kHz and f_{stop} of 22.05 kHz.

Summary

- Discrete Time Systems
- Transfer Functions in the Z-Domain
- Modelling digital systems in Matlab/Simulink
- Continuous System Equivalents
- Example: Digital Butterworth Filter

Solutions to Example 5

Solution to 5.1.

The transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2 + z}{z^2 - 0.5z + 0.125}$$

Solution to 5.2.

The DT impulse response:

$$h[n] = \left(\frac{\sqrt{2}}{4}\right)^n \left(\cos\left(\frac{n\pi}{4}\right) + 5\sin\left(\frac{n\pi}{4}\right)\right)$$

Solution to 5.3.

Step response:

$$y[n] = \left(3.2 - \left(\frac{\sqrt{2}}{4}\right)^n \left(2.2\cos\left(\frac{n\pi}{4}\right) + 0.6\sin\left(\frac{n\pi}{4}\right)\right)\right) u_0[n]$$