

# Introduction to Filters

## Scope and Background Reading

This section is Based on the section **Filtering** from Chapter 5 of Benoit Boulet, Fundamentals of Signals and Systems (<https://ebookcentral.proquest.com/lib/swansea-ebooks/reader.action?ppg=221&docID=3135971&tm=1518715953782>) from the **Recommended Reading List**.

This material is an introduction to analogue filters. You will find much more in-depth coverage on Pages 11-1 – 11-48 of Karris (<https://ebookcentral.proquest.com/lib/swansea-ebooks/reader.action?ppg=429&docID=3384197&tm=1518716026573>).

## Agenda

- Frequency Selective Filters
- Ideal low-pass filter
- Butterworth low-pass filter
- High-pass filter
- Bandpass filter

## Introduction

- Filter design is an important application of the Fourier transform
- Filtering is a rich topic often taught in graduate courses so we give only an introduction.
- Our introduction *will* illustrate the usefulness of the frequency domain viewpoint.
- We will explore how filters can shape the spectrum of a signal.

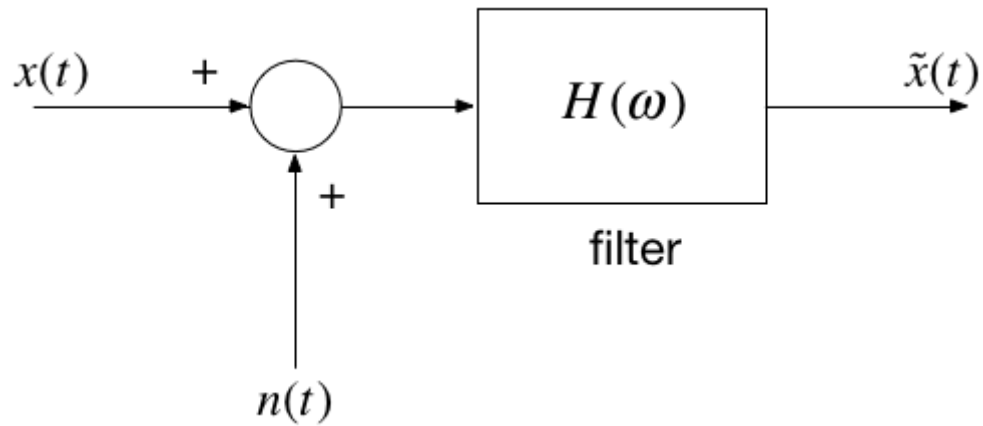
Other applications of the Fourier transform are sampling theory (introduced next week) and modulation.

## Frequency Selective Filters

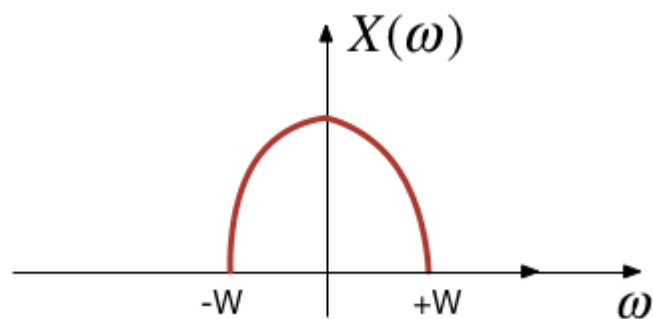
An ideal frequency-selective filter is a system that let's the frequency components of a signal through undistorted while frequency components at other components are completely cut off.

- The range of frequencies which are let through belong to the **pass Band**
- The range of frequencies which are cut-off by the filter are called the **stopband**
- A typical scenario where filtering is needed is when noise  $n(t)$  is added to a signal  $x(t)$  but that signal has most of its energy outside the bandwidth of a signal.

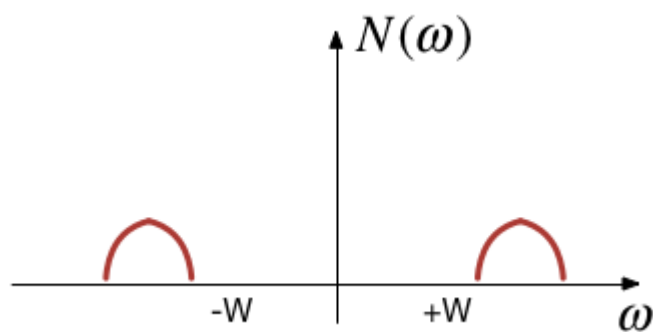
## Typical filtering problem



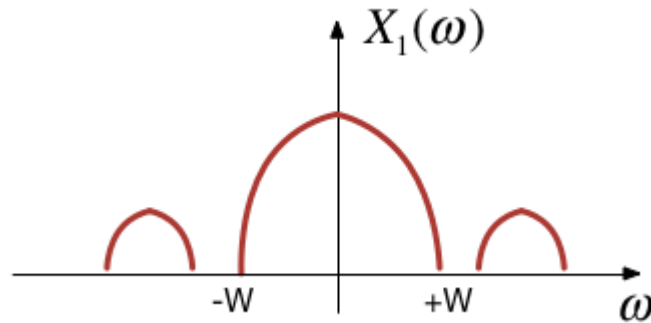
## Signal



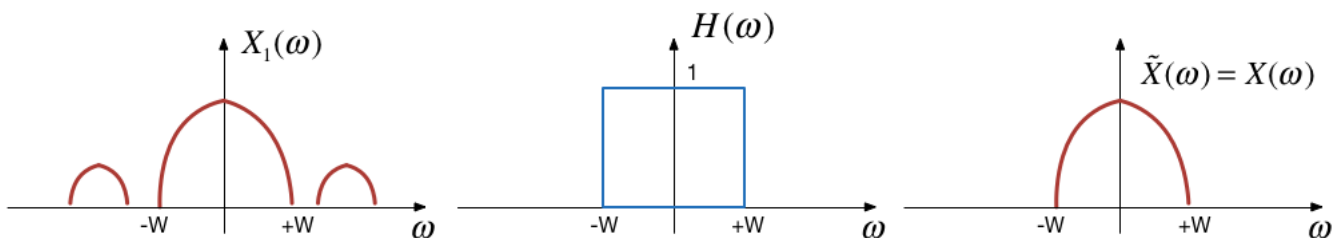
## Out-of Bandwidth Noise



## Signal plus Noise



## Filtering



## Motivating example

See the video and script in the [OneNote Class Room notebook](#)

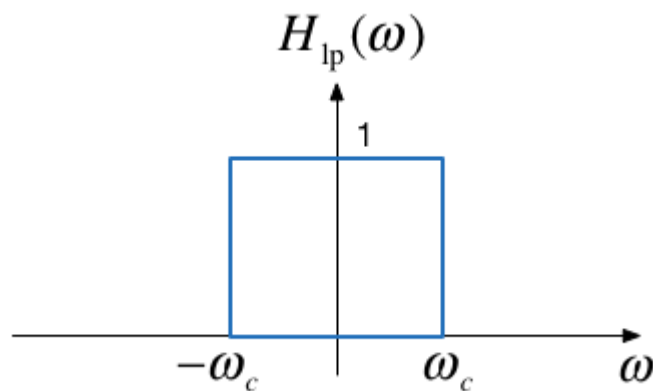
(<https://swanseauniversity.sharepoint.com/sites/EG-247SignalsandSystems2017-2108-UsrGrpcopy-UsrGrp/SiteAssets/EG-247%20Signals%20and%20Systems%202017-2108-UsrGrp%20%5bcopy%5d-UsrGrp%20Notebook/Content%20Library/Classes/Week%207.one#Motivating%20Example&section-id={681B0954-AC4E-9646-A567-FF06C3696F07}&page-id={E5AD343A-E348-0141-8096-60E0CA201E57}&end>).

## Ideal Low-Pass Filter

An ideal low pass filter cuts-off frequencies higher than its *cutoff frequency*,  $\omega_c$ .

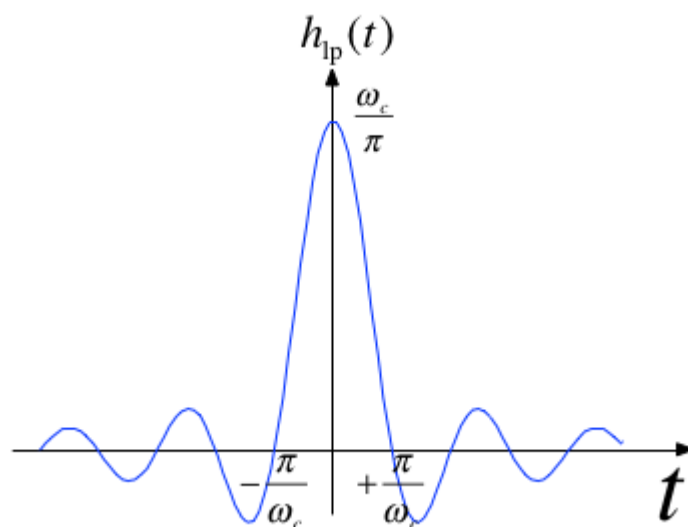
$$H_{lp}(\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| \geq \omega_c \end{cases}$$

## Frequency response



## Impulse response

$$h_{lp}(t) = \frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c}{\pi} t\right)$$



## Filtering is Convolution

The output of an LTI system with impulse response

$$h(t) \Leftrightarrow H(\omega)$$

subject to an input signal

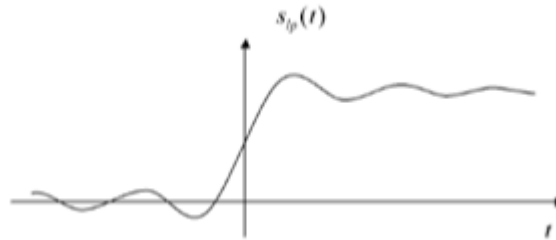
$$x(t) \Leftrightarrow X(\omega)$$

is given by

$$y(t) = h(t) * x(t) \Leftrightarrow Y(\omega) = H(\omega)X(\omega)$$

## Issues with the "ideal" filter

This is the step response:



(reproduced from Boulet Fig. 5.23 p. 205)

Ripples in the impulse response would be undesirable, and because the impulse response is non-causal it cannot actually be implemented.

## Butterworth low-pass filter

N-th Order Butterworth Filter

$$|H_B(\omega)| = \frac{1}{\left(1 + \left(\frac{\omega}{\omega_c}\right)^{2N}\right)^{\frac{1}{2}}}$$

### Remarks

- DC gain is

$$|H_B(j0)| = 1$$

- Attenuation at the cut-off frequency is

$$|H_B(j\omega_c)| = 1/\sqrt{2}$$

for any  $N$

More about the Butterworth filter: [Wikipedia Article \(http://en.wikipedia.org/wiki/Butterworth\\_filter\)](http://en.wikipedia.org/wiki/Butterworth_filter)

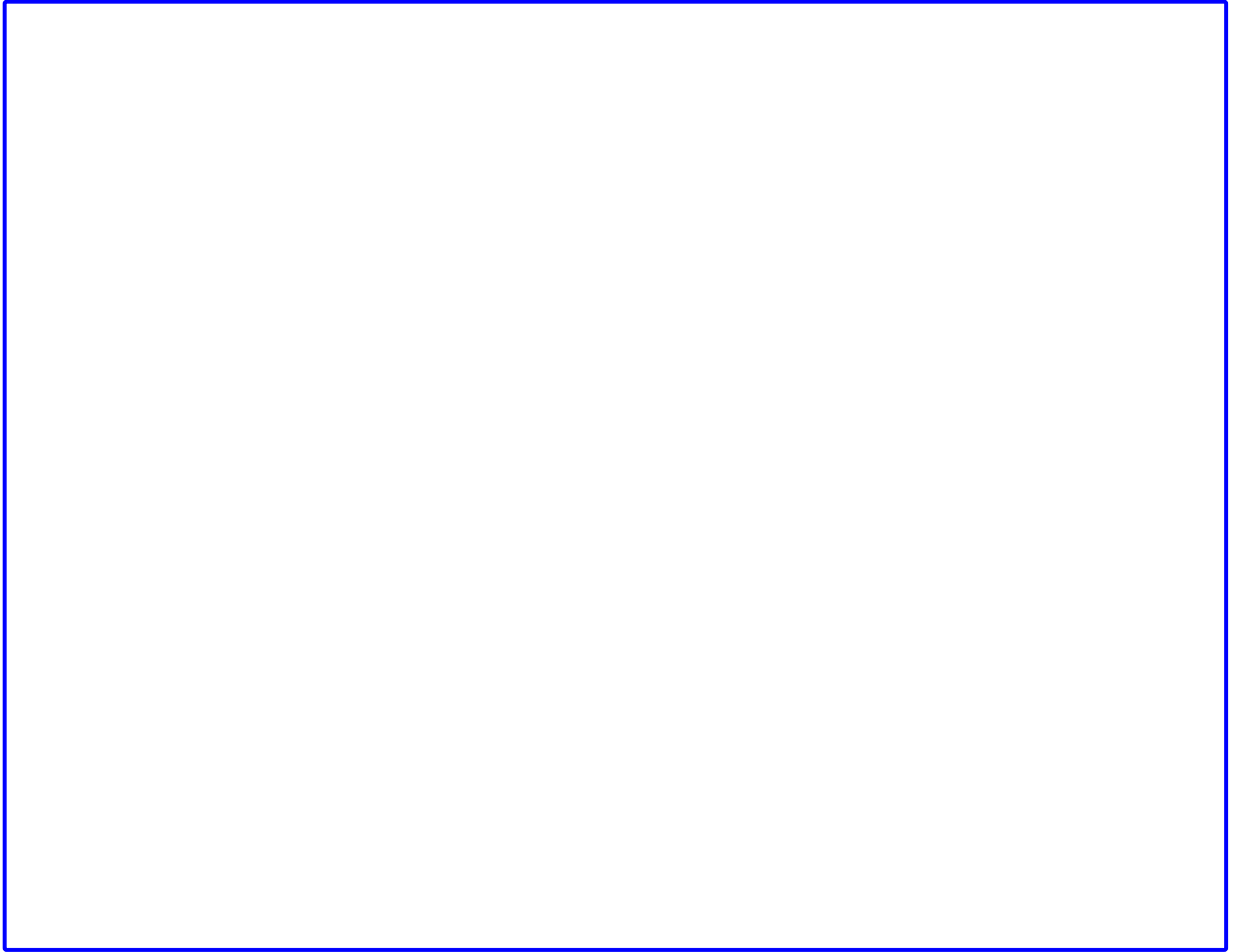
## Example 5: Second-order BW Filter

The second-order butterworth Filter is defined by its Characteristic Equation (CE):

$$p(s) = s^2 + \omega_c\sqrt{2}s + \omega_c^2 = 0^*$$

Calculate the roots of  $p(s)$  (the poles of the filter transfer function) in both Cartesian and polar form.

**Note:** This has the same characteristic as a control system with damping ratio  $\zeta = 1/\sqrt{2}$  and  $\omega_n = \omega_c$ !

**Solution****Example 6**

Derive the differential equation relating the input  $x(t)$  to output  $y(t)$  of the 2nd-Order Butterworth Low-Pass Filter with cutoff frequency  $\omega_c$ .

**Solution****Example 7**

Determine the frequency response  $H_B(\omega) = Y(\omega)/X(\omega)$

## Solution



### Magnitude of frequency response of a 2nd-order Butterworth Filter

In [1]:

```
wc = 100;
```

Transfer function



In [2]:

```
H = tf(wc^2,[1, wc*sqrt(2), wc^2])
```

H =

```

      10000
-----
s^2 + 141.4 s + 10000

```

Continuous-time transfer function.

Magnitude frequency response

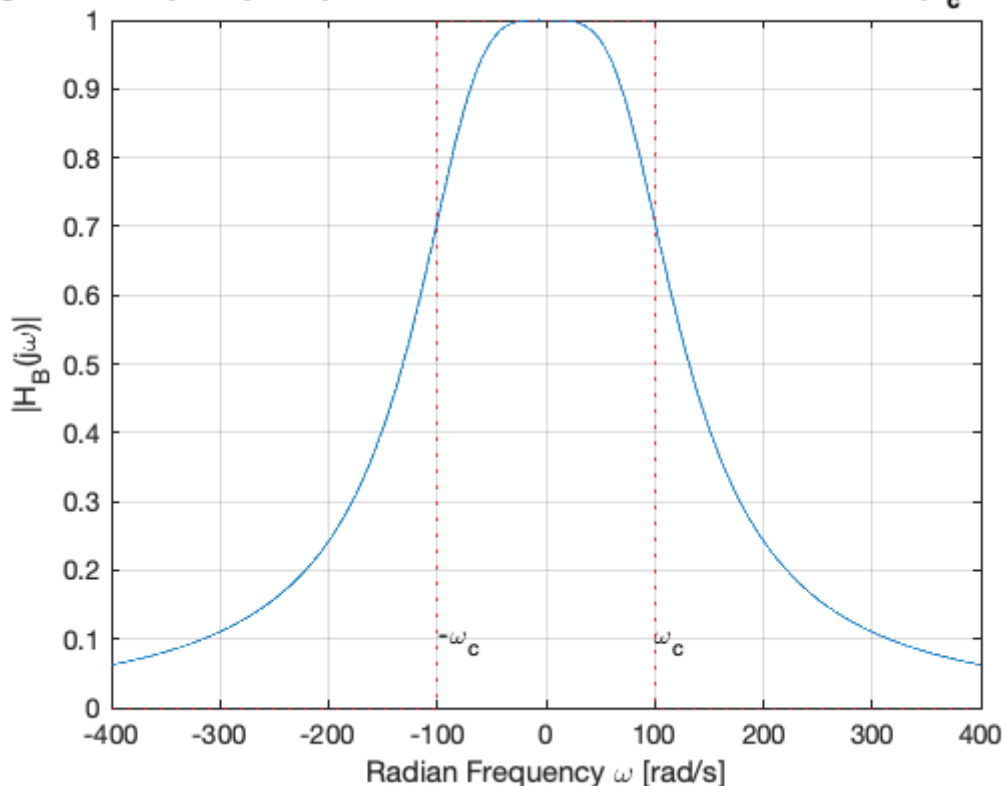
In [3]:

```

w = -400:400;
mHlp = 1./(sqrt(1 + (w./wc).^4));
plot(w,mHlp)
grid
ylabel('|H_B(j\omega)|')
title('Magnitude Frequency Response for 2nd-Order LP Butterworth Filter (\omega_c = 100 rad/s)')
xlabel('Radian Frequency \omega [rad/s]')
text(100,0.1,'\omega_c')
text(-100,0.1,'-\omega_c')
hold on
plot([-400,-100,-100,100,100,400],[0,0,1,1,0,0],'r:')
hold off

```

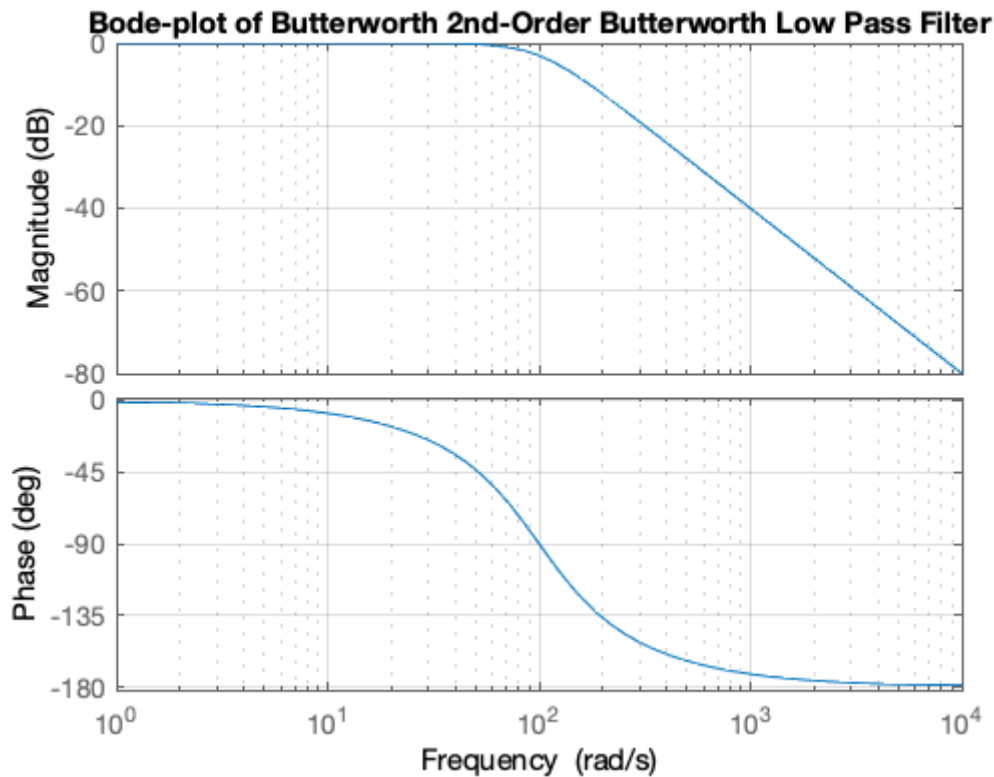
**Magnitude Frequency Response for 2nd-Order LP Butterworth Filter ( $\omega_c = 100$  rad/s)**



Bode plot

In [4]:

```
bode(H)
grid
title('Bode-plot of Butterworth 2nd-Order Butterworth Low Pass Filter')
```



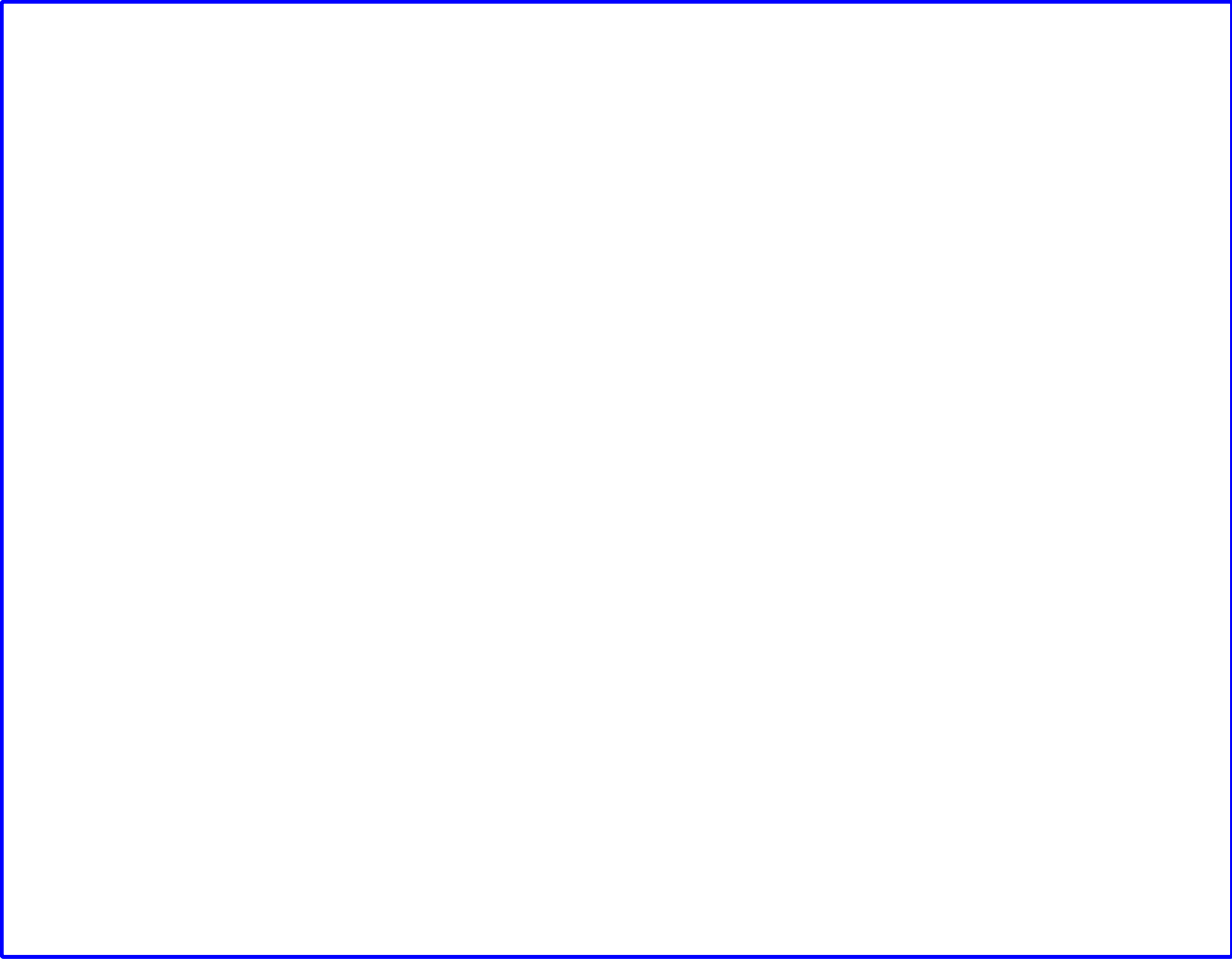
## Example 8

Determine the impulse and step response of a butterworth low-pass filter.

You will find this Fourier transform pair useful:

$$e^{-at} \sin \omega_0 t u_0(t) \Leftrightarrow \frac{\omega_0}{(j\omega + a)^2 + \omega_0^2}$$

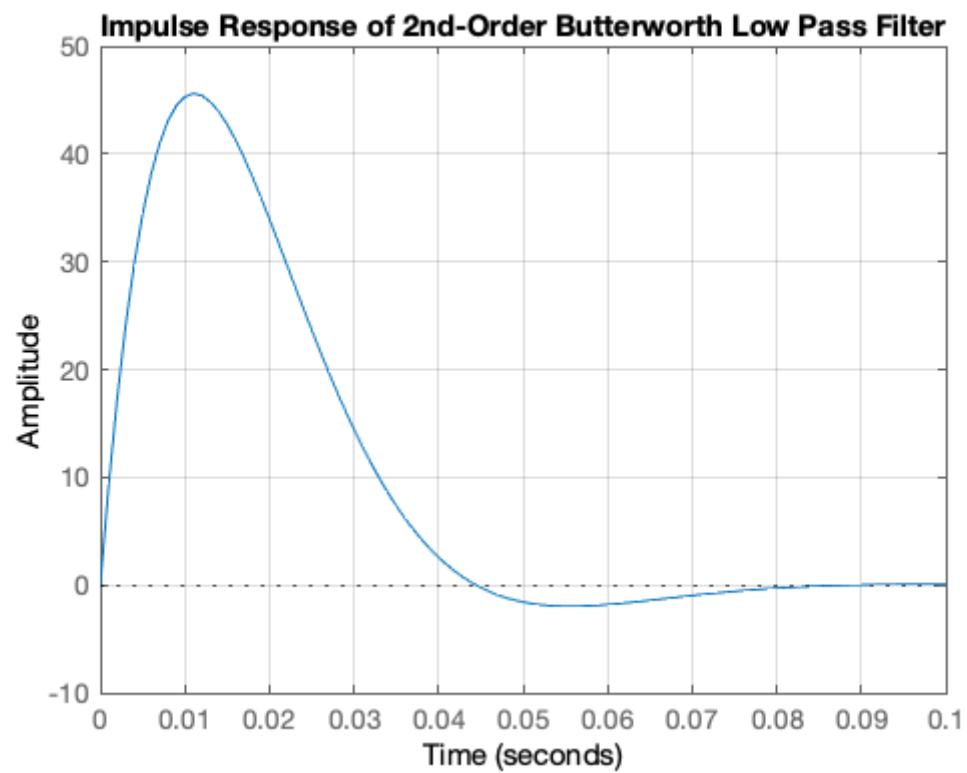
**Solution**



Impulse response

In [5]:

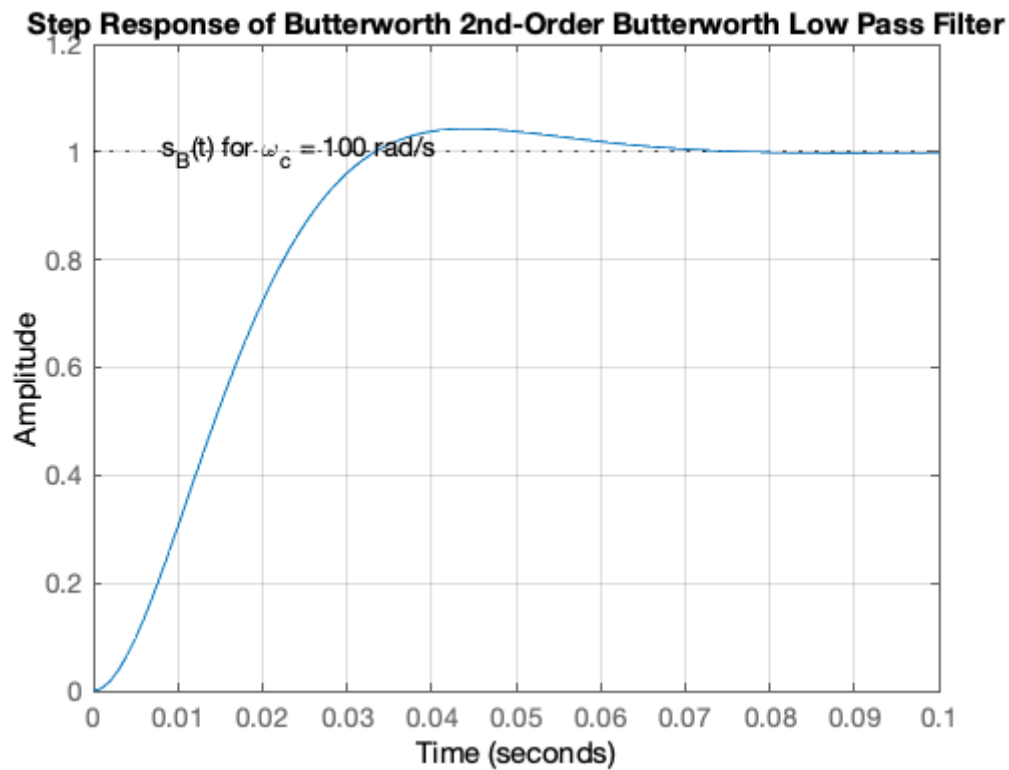
```
impulse(H,0.1)
grid
title('Impulse Response of 2nd-Order Butterworth Low Pass Filter')
```



Step response

In [6]:

```
step(H,0.1)
title('Step Response of Butterworth 2nd-Order Butterworth Low Pass Filter')
grid
text(0.008,1,'s_B(t) for \omega_c = 100 rad/s')
```

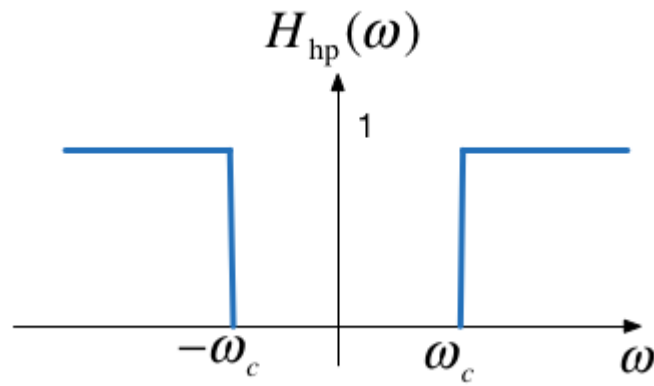


## High-pass filter

An ideal highpass filter cuts-off frequencies lower than its *cutoff frequency*,  $\omega_c$ .

$$H_{\text{hp}}(\omega) = \begin{cases} 0, & |\omega| \leq \omega_c \\ 1, & |\omega| > \omega_c \end{cases}$$

## Frequency response



## Responses

### Frequency response

$$H_{hp}(\omega) = 1 - H_{lp}(\omega)$$

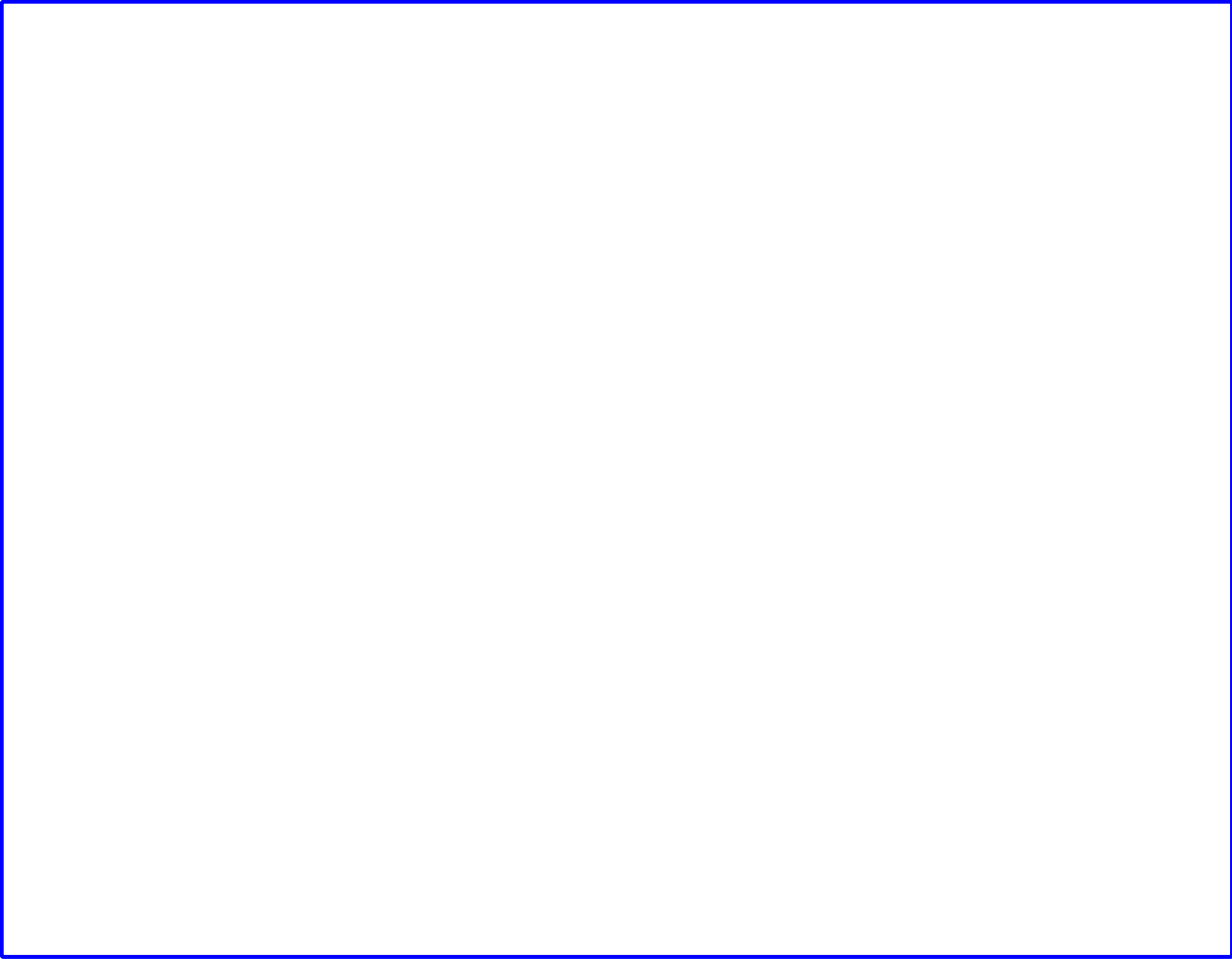
### Impulse response

$$h_{hp}(t) = \delta(t) - h_{lp}(t)$$

## Example 9

Determine the frequency response of a 2nd-order butterworth highpass filter

**Solution**

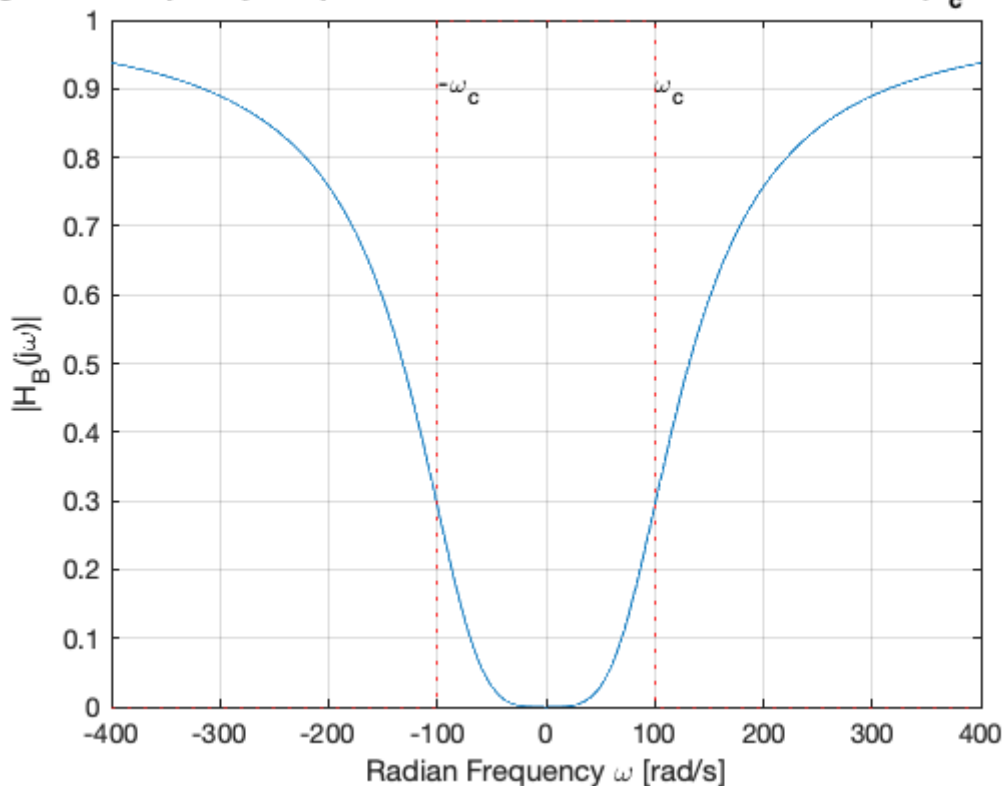


Magnitude frequency response

In [7]:

```
w = -400:400;
plot(w,1-mHlp)
grid
ylabel('|H_B(j\omega)|')
title('Magnitude Frequency Response for 2nd-Order HP Butterworth Filter (\omega_c = 100 rad/s)')
xlabel('Radian Frequency \omega [rad/s]')
text(100,0.9,'\omega_c')
text(-100,0.9,'-\omega_c')
hold on
plot([-400,-100,-100,100,100,400],[0,0,1,1,0,0],'r:')
hold off
```

**Magnitude Frequency Response for 2nd-Order HP Butterworth Filter ( $\omega_c = 100$  rad/s)**



High-pass filter



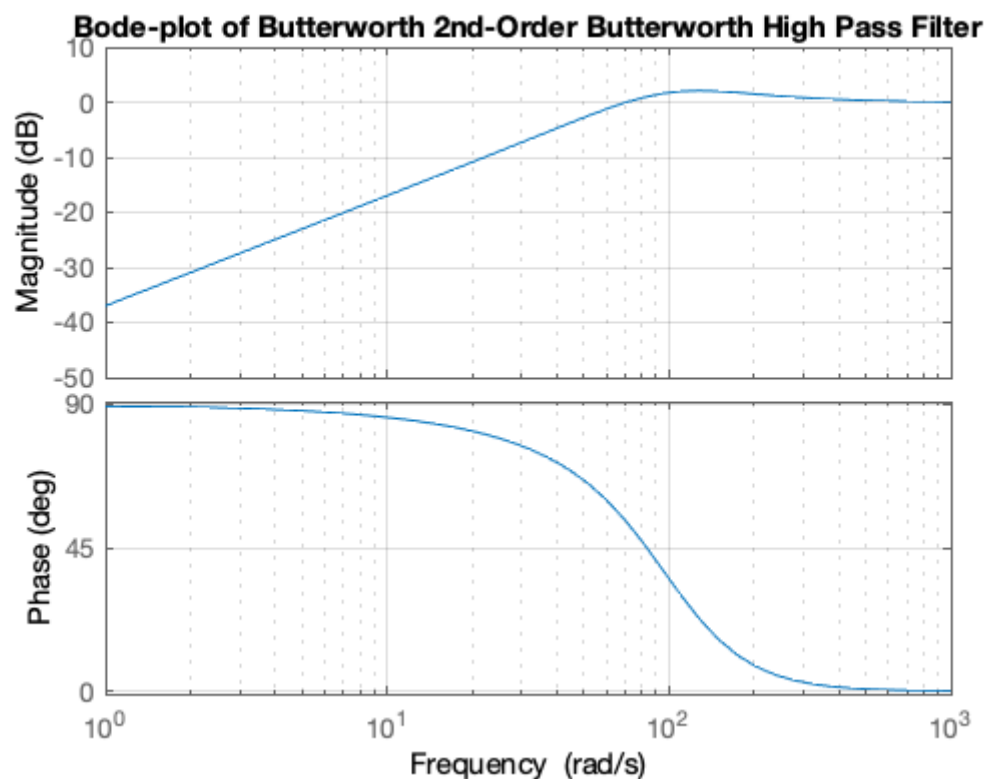
In [8]:

```
Hhp = 1 - H
bode(Hhp)
grid
title('Bode-plot of Butterworth 2nd-Order Butterworth High Pass Filter')
```

Hhp =

$$\frac{s^2 + 141.4 s}{s^2 + 141.4 s + 10000}$$

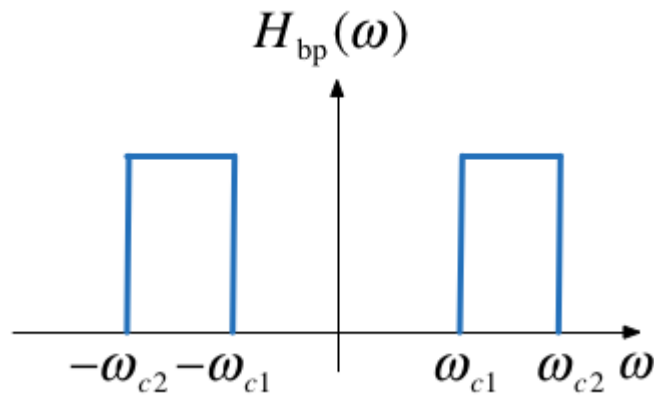
Continuous-time transfer function.



## Band-pass filter

An ideal bandpass filter cuts-off frequencies lower than its first *cutoff frequency*  $\omega_{c1}$ , and higher than its second *cutoff frequency*  $\omega_{c2}$ .

$$H_{bp}(\omega) = \begin{cases} 1, & \omega_{c1} < |\omega| < \omega_{c2} \\ 0, & \text{otherwise} \end{cases}$$



## Bandpass filter design

A bandpass filter can be obtained by multiplying the frequency responses of a lowpass filter by a highpass filter.

$$H_{bp}(\omega) = H_{hp}(\omega)H_{lp}(\omega)$$

- The highpass filter should have cut-off frequency of  $\omega_{c1}$
- The lowpass filter should have cut-off frequency of  $\omega_{c2}$

To generate all the plots shown in this presentation, you can use [butter2\\_ex.mlx](https://cpjobling.github.io/eg-247-textbook/fourier_transform/4/butter2_ex.mlx) ([https://cpjobling.github.io/eg-247-textbook/fourier\\_transform/4/butter2\\_ex.mlx](https://cpjobling.github.io/eg-247-textbook/fourier_transform/4/butter2_ex.mlx))

## Summary

- Frequency-Selective Filters
- Ideal low-pass filter
- Butterworth low-pass filter
- High-pass filter
- Bandpass filter

## Solutions

Solutions to Examples 5-9 are captured as a PenCast in [filters.pdf](https://cpjobling.github.io/eg-247-textbook/fourier_transform/solutions/filters.pdf) ([https://cpjobling.github.io/eg-247-textbook/fourier\\_transform/solutions/filters.pdf](https://cpjobling.github.io/eg-247-textbook/fourier_transform/solutions/filters.pdf)).