Fourier Transforms for Circuit and LTI Systems Analysis

In this section we will apply what we have learned about Fourier transforms to some typical circuit problems. After a short introduction, the body of this chapter will form the basis of an examples class.

Agenda

- The system function

- Examples

The System Function

System response from system impulse response

Recall that the convolution integral of a system with impulse response $h(t)$ and input $u(t)$ is

$$h(t) * u(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)\,d\tau.$$

We let

$$g(t) = h(t) * u(t)$$

Then by the time convolution property

$$h(t) * u(t) = g(t) \Leftrightarrow G(\omega) = H(\omega).\,U(\omega)$$

The System Function

We call $H(\omega)$ the *system function*.

We note that the system function $H(\omega)$ and the impulse response $h(t)$ form the Fourier transform pair

$$h(t) \Leftrightarrow H(\omega)$$

Obtaining system response

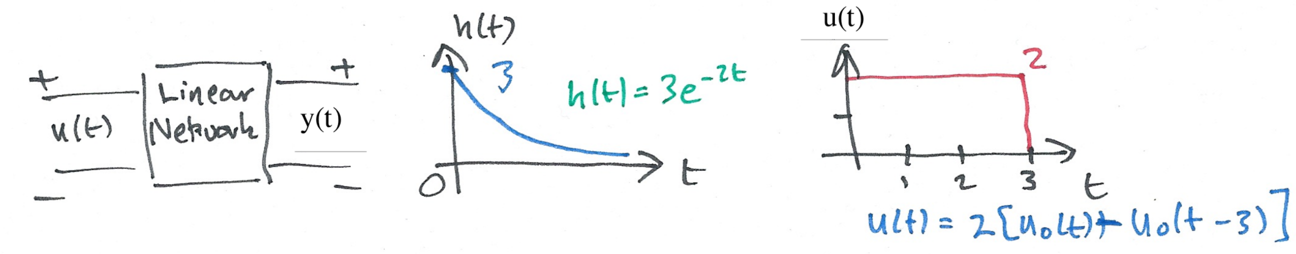
If we know the impulse resonse $h(t)$, we can compute the system response $g(t)$ of any input $u(t)$ by multiplying the Fourier transforms of $H(\omega)$ and $U(\omega)$ to obtain $G(\omega)$. Then we take the inverse Fourier transform of $G(\omega)$ to obtain the response $g(t)$.

1. Transform $h(t) \rightarrow H(\omega)$
2. Transform $u(t) \rightarrow U(\omega)$
3. Compute $G(\omega) = H(\omega).\, U(\omega)$
4. Find $\mathcal{F}^{-1}\left\{G(\omega)\right\} \rightarrow g(t)$

Examples

Example 1

Karris example 8.8: for the linear network shown below, the impulse response is $h(t) = 3e^{-2t}$. Use the Fourier transform to compute the response $y(t)$ when the input $u(t) = 2[u_0(t) - u_0(t-3)]$. Verify the result with MATLAB.



Solution

Matlab verification

In [1]:

```
syms t w
U1 = fourier(2*heaviside(t),t,w)
```

```
U1 =

2*pi*dirac(w) - 2i/w
```

In [2]:

```
H = fourier(3*exp(-2*t)*heaviside(t),t,w)
```

```
H =

3/(2 + w*1i)
```

In [3]:

```
Y1=simplify(H*U1)
```

```
Y1 =

3*pi*dirac(w) - 6i/(w*(2 + w*1i))
```

In [4]:

```
y1 = simplify(ifourier(Y1,w,t))
```

```
y1 =

(3*exp(-2*t)*(sign(t) + 1)*(exp(2*t) - 1))/2
```

Get y2

Substitute $t - 3$ into t.

In [5]:

```
y2 = subs(y1,t,t-3)
```

```
y2 =

(3*exp(6 - 2*t)*(sign(t - 3) + 1)*(exp(2*t - 6) - 1))/2
```

In [6]:

```
y = y1 - y2
```

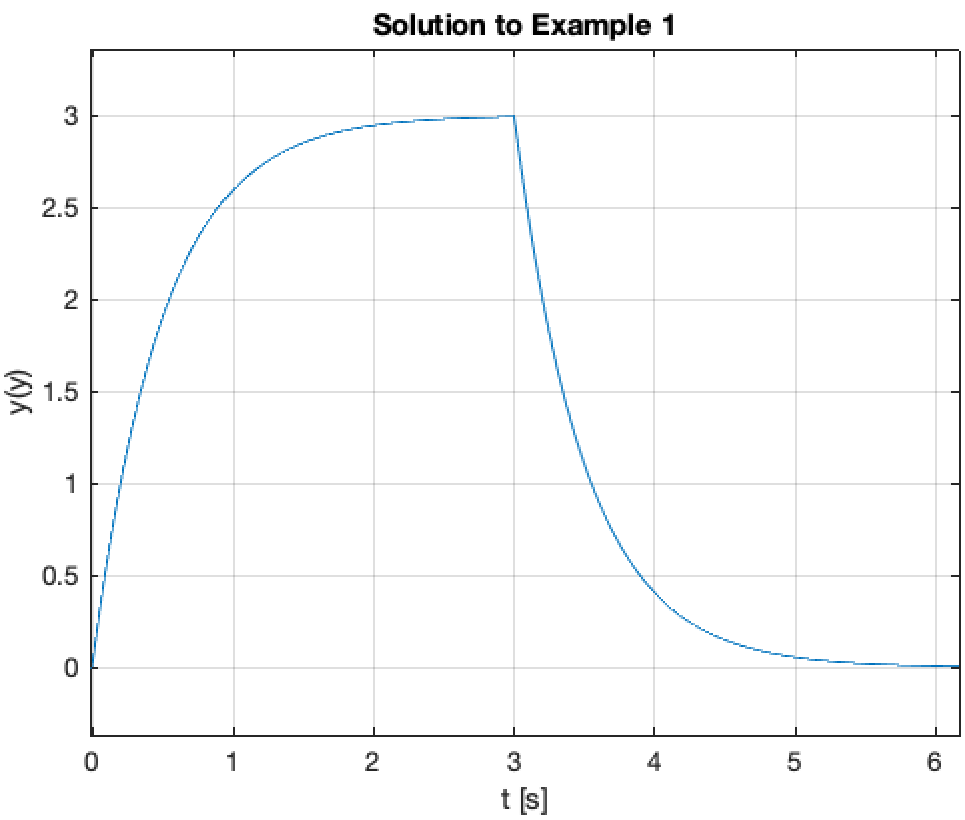
```
y =

(3*exp(-2*t)*(sign(t) + 1)*(exp(2*t) - 1))/2 - (3*exp(6 - 2*t)*(sign
(t - 3) + 1)*(exp(2*t - 6) - 1))/2
```

Plot result

In [7]:

```
ezplot(y)
title('Solution to Example 1')
ylabel('y(y)')
xlabel('t [s]')
grid
```



See [ft3_ex1.m (ft3_ex1.m)](ft3_ex1.m)

Result is equivalent to:

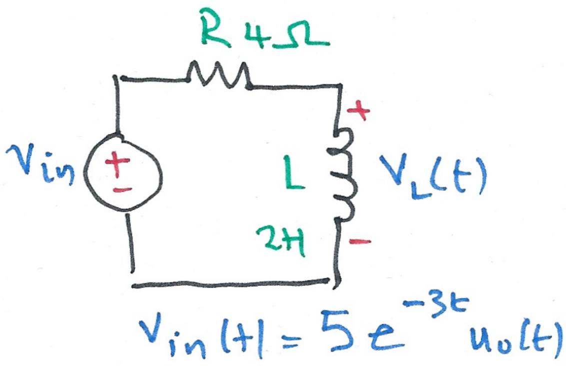
```
y = 3*heaviside(t) - 3*heaviside(t - 3) + 3*heaviside(t - 3)*exp(6 - 2*t)
 - 3*exp(-2*t)*heaviside(t)
```

Which after gathering terms gives

$$y(t) = 3(1 - 3e^{-2t})u_0(t) - 3(1 - 3e^{-2(t-3)})u_0(t-3)$$

Example 2

Karris example 8.9: for the circuit shown below, use the Fourier transfrom method, and the system function $H(\omega)$ to compute $V_L(t)$. Assume $i_L(0^-) = 0$. Verify the result with Matlab.



Solution

Matlab verification

In [8]:

```
syms t w
H = j*w/(j*w + 2)
```

```
H =

(w*1i)/(2 + w*1i)
```

In [9]:

```
Vin = fourier(5*exp(-3*t)*heaviside(t),t,w)
```

```
Vin =

5/(3 + w*1i)
```

In [10]:

```
Vout=simplify(H*Vin)
```

```
Vout =

(w*5i)/((2 + w*1i)*(3 + w*1i))
```

In [11]:

```
vout = simplify(ifourier(Vout,w,t))
```

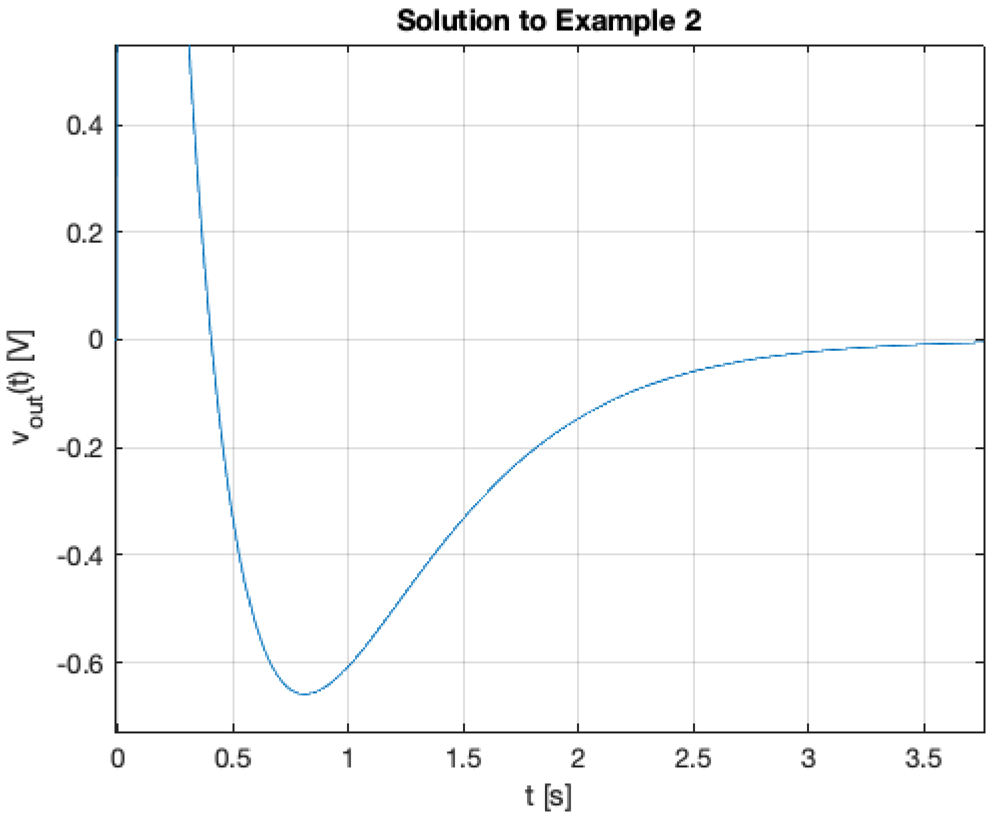
```
vout =

-(5*exp(-3*t)*(sign(t) + 1)*(2*exp(t) - 3))/2
```

Plot result

In [12]:

```
ezplot(vout)
title('Solution to Example 2')
ylabel('v_{out}(t) [V]')
xlabel('t [s]')
grid
```



See ft3_ex2.m (matlab/ft3_ex2.m)

Result is equivalent to:

```
vout = -5*exp(-3*t)*heaviside(t)*(2*exp(t) - 3)
```

Which after gathering terms gives

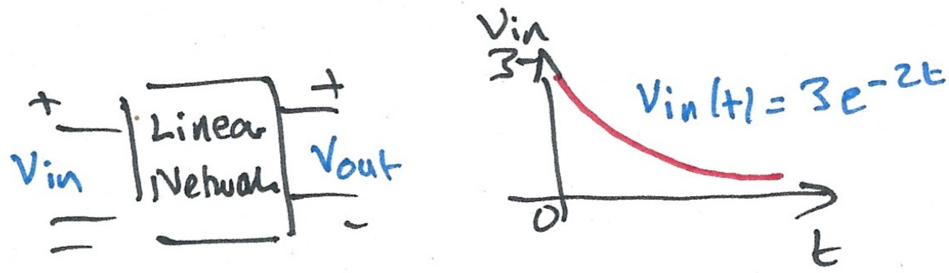
$$v_{\text{out}} = 5\left(3e^{-3t} - 2e^{-2t}\right) u_0(t)$$

Example 3

Karris example 8.10: for the linear network shown below, the input-output relationship is:

$$\frac{d}{dt} v_{\text{out}} + 4v_{\text{out}} = 10v_{\text{in}}$$

where $v_{\text{in}} = 3e^{-2t}$. Use the Fourier transform method, and the system function $H(\omega)$ to compute the output v_{out}. Verify the result with Matlab.



Solution

Matlab verification

In [13]:

```
syms t w
H = 10/(j*w + 4)
```

```
H =

10/(4 + w*1i)
```

In [14]:

```
Vin = fourier(3*exp(-2*t)*heaviside(t),t,w)
```

```
Vin =

3/(2 + w*1i)
```

In [15]:

```
Vout=simplify(H*Vin)
```

```
Vout =

30/((2 + w*1i)*(4 + w*1i))
```

In [16]:

```
vout = simplify(ifourier(Vout,w,t))
```

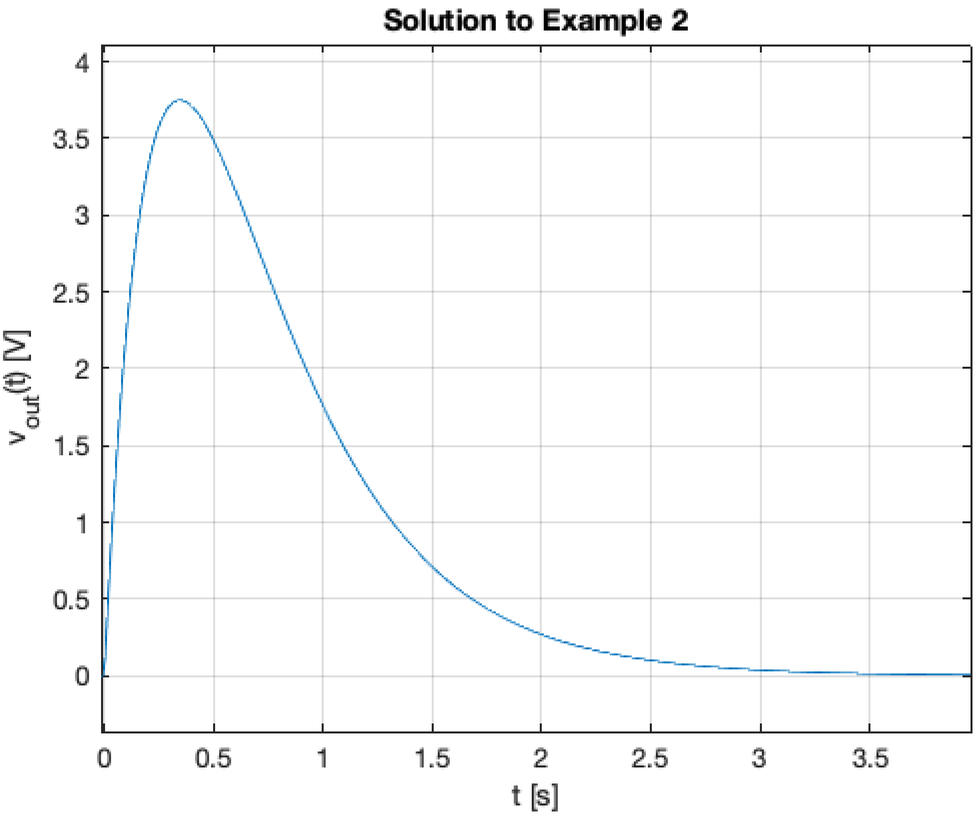
```
vout =

(15*exp(-4*t)*(sign(t) + 1)*(exp(2*t) - 1))/2
```

Plot result

In [17]:

```
ezplot(vout)
title('Solution to Example 2')
ylabel('v_{out}(t) [V]')
xlabel('t [s]')
grid
```



See ft3_ex3.m (ft3_ex3.m)

Result is equiavlent to:

```
15*exp(-4*t)*heaviside(t)*(exp(2*t) - 1)
```

Which after gathering terms gives

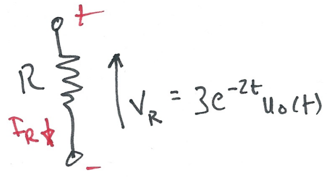
$$v_{\mathrm{out}}(t) = 15\left(e^{-2t}) - e^{-4t}\right) u_0(t)$$

Example 4

Karris example 8.11: the voltage across a 1 Ω resistor is known to be $V_R(t) = 3e^{-2t}u_0(t)$. Compute the energy dissipated in the resistor for $0 < t < \infty$, and verify the result using Parseval's theorem. Verify the result with Matlab.

Note from tables of integrals (http://en.wikipedia.org/wiki/Lists_of_integrals)

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\arctan\frac{x}{a} + C.$$



Solution

Matlab verification

In [18]:

```
syms t w
```

Calcuate energy from time function

In [19]:

```
Vr = 3*exp(-2*t)*heaviside(t);
R = 1;
Pr = Vr^2/R
Wr = int(Pr,t,0,inf)
```

```
Pr =

9*exp(-4*t)*heaviside(t)^2


Wr =

9/4
```

Calculate using Parseval's theorem

In [20]:

```
Fw = fourier(Vr,t,w)
```

```
Fw =

3/(2 + w*1i)
```

In [21]:

```
Fw2 = simplify(abs(Fw)^2)
```

```
Fw2 =

9/abs(2 + w*1i)^2
```

In [22]:

```
Wr=2/(2*pi)*int(Fw2,w,0,inf)
```

```
Wr =

(51607450253003931*pi)/72057594037927936
```

See [ft3_ex4.m (ft3_ex4.m)](ft3_ex4.m)

Solutions

See worked solutions in OneNote Week 7.