Worksheet 15

To accompany Chapter 5.4 Introduction to Filters

We will step through this worksheet in class.

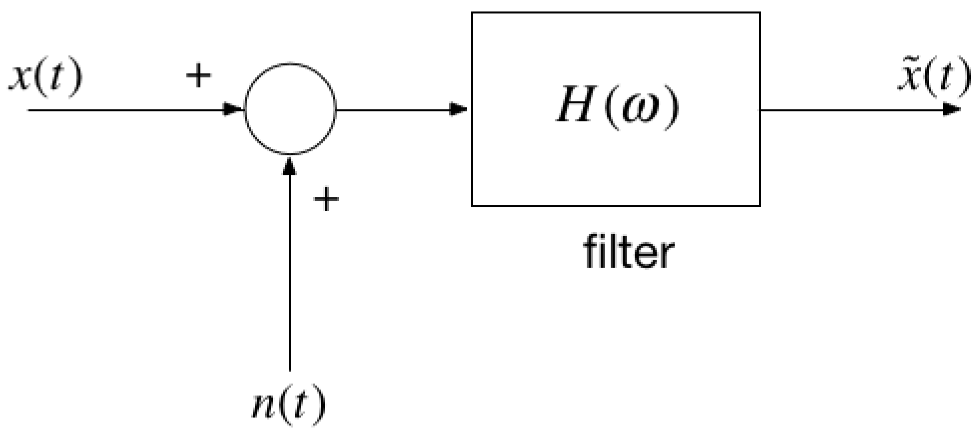
You are expected to have at least watched the video presentation of Chapter 5.4 (https://cpjobling.github.io/eg-247-textbook/fourier_transform/4/ft4) of the notes (https://cpjobling.github.io/eg-247-textbook) before coming to class. If you haven't watch it afterwards!

Frequency Selective Filters

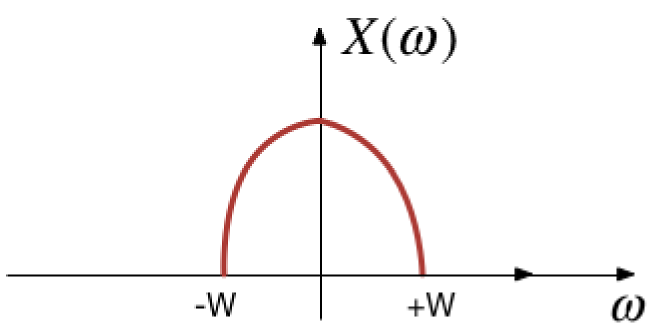
An ideal frequency-selective filter is a system that let's the frequency components of a signal through undistorted while frequency components at other components are completely cut off.

- The range of frequencies which are let through belong to the **pass Band**
- The range of frequencies which are cut-off by the filter are called the **stopband**
- A typical scenario where filtering is needed is when noise $n(t)$ is added to a signal $x(t)$ but that signal has most of its energy outside the bandwidth of a signal.

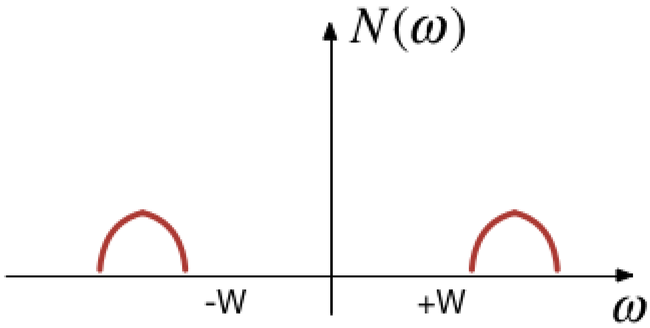
Typical filtering problem



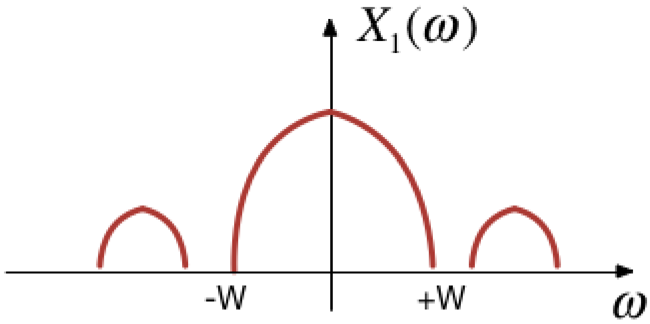
Signal



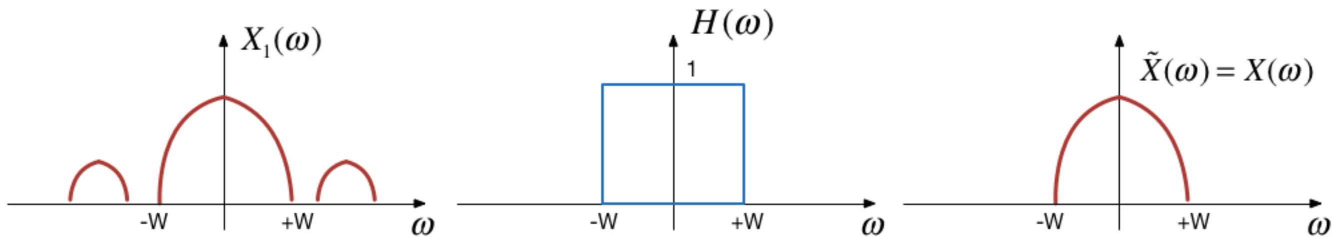
Out-of Bandwidth Noise



Signal plus Noise



Filtering



Motivating example

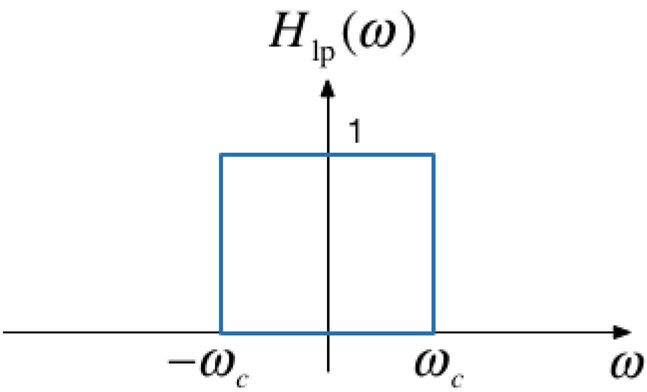
See the video and script in the OneNote Class Room notebook (https://swanseauniversity.sharepoint.com/sites/EG-247SignalsandSystems2017-2108-UsrGrpcopy-UsrGrp/SiteAssets/EG-247%20Signals%20and%20Systems%202017-2108-UsrGrp%20%5bcopy%5d-UsrGrp%20Notebook/_Content%20Library/Classes/Week%207.one#Motivating%20Example&section-id={681B0954-AC4E-9646-A567-FF06C3696F07}&page-id={E5AD343A-E348-0141-8096-60E0CA201E57}&end).

Ideal Low-Pass Filter

An ideal low pass filter cuts-off frequencies higher than its *cutoff frequency*, ω_c.

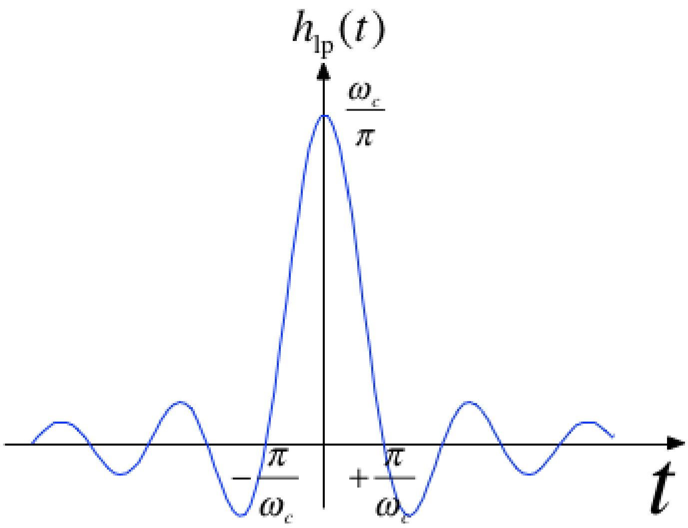
$$H_{\mathrm{lp}}(\omega) = \begin{cases} 1, & |\omega| \ < \omega_c \\ 0, & |\omega| \ \geq \omega_c \end{cases}$$

Frequency response



Impulse response

$$h_{\mathrm{lp}}(t) = \frac{\omega_c}{\pi} \mathrm{sinc}\left(\frac{\omega_c}{\pi} t\right)$$



Filtering is Convolution

The output of an LTI system with impulse response

$$h(t) \Leftrightarrow H(\omega)$$

subject to an input signal

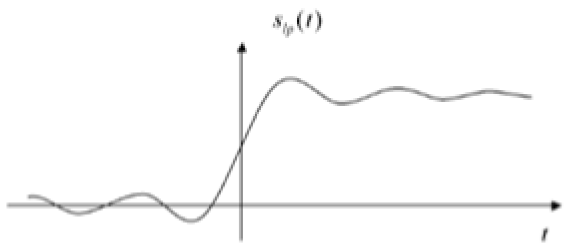
$$x(t) \Leftrightarrow X(\omega)$$

is given by

$$y(t) = h(t) * x(t) \Leftrightarrow Y(\omega) = H(\omega)X(\omega)$$

Issues with the "ideal" filter

This is the step response:



(reproduced from Boulet Fig. 5.23 p. 205)

Ripples in the impulse resonse would be undesireable, and because the impulse response is non-causal it cannot actually be implemented.

Butterworth low-pass filter

N-th Order Butterworth Filter

$$|H_B(\omega)| = \frac{1}{\left(1 + \left(\frac{\omega}{\omega_c}\right)^{2N}\right)^{\frac{1}{2}}}$$

Remarks

- DC gain is
$$|H_B(j0)| = 1$$
- Attenuation at the cut-off frequency is
$$|H_B(j\omega_c)| = 1/\sqrt{2}$$
for any N

More about the Butterworth filter: Wikipedia Article (http://en.wikipedia.org/wiki/Butterworth_filter)

Example 5: Second-order BW Filter

The second-order butterworth Filter is defined by is Characteristic Equation (CE):

$$p(s) = s^2 + \omega_c\sqrt{2}s + \omega_c^2 = 0^*$$

Calculate the roots of $p(s)$ (the poles of the filter transfer function) in both Cartesian and polar form.

Note: This has the same characteristic as a control system with damping ratio $\zeta = 1/\sqrt{2}$ and $\omega_n = \omega_c$!

Solution

Example 6

Derive the differential equation relating the input $x(t)$ to output $y(t)$ of the 2nd-Order Butterworth Low-Pass Filter with cutoff frequency ω_c.

Solution

Example 7

Determine the frequency response $H_B(\omega) = Y(\omega)/X(\omega)$

Solution

Magnitude of frequency response of a 2nd-order Butterworth Filter

In [ ]:

```
wc = 100;
```

Transfer function

In [ ]:

```
H = tf(wc^2,[1, wc*sqrt(2), wc^2])
```

Magnitude frequency response

In [ ]:

```
w = -400:400;
mHlp = 1./(sqrt(1 + (w./wc).^4));
plot(w,mHlp)
grid
ylabel('|H_B(j\omega)|')
title('Magnitude Frequency Response for 2nd-Order LP Butterworth Filter (\omega_
c = 100 rad/s)')
xlabel('Radian Frequency \omega [rad/s]')
text(100,0.1,'\omega_c')
text(-100,0.1,'-\omega_c')
hold on
plot([-400,-100,-100,100,100,400],[0,0,1,1,0,0],'r:')
hold off
```

Bode plot

In [ ]:

```
bode(H)
grid
title('Bode-plot of Butterworth 2nd-Order Butterworth Low Pass Filter')
```

Example 8

Determine the impulse and step response of a butterworth low-pass filter.

You will find this Fourier transform pair useful:

$$e^{-at}\sin\omega_0 t\; u_0(t) \Leftrightarrow \frac{\omega_0}{(j\omega+a)^2+\omega_0^2}$$

Solution

Impulse response

In [ ]:

```
impulse(H,0.1)
grid
title('Impulse Response of 2nd-Order Butterworth Low Pass Filter')
```

Step response

In [ ]:

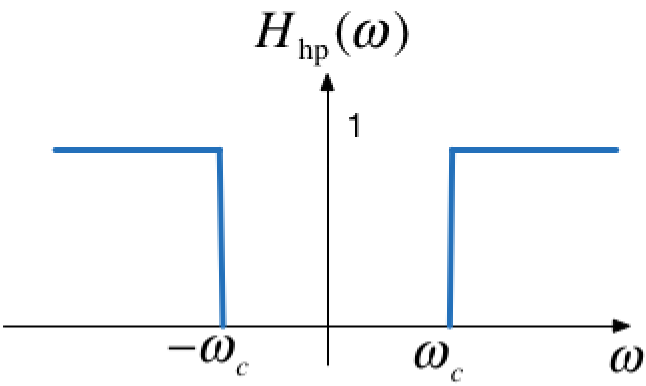
```
step(H,0.1)
title('Step Response of Butterworth 2nd-Order Butterworth Low Pass Filter')
grid
text(0.008,1,'s_B(t) for \omega_c = 100 rad/s')
```

High-pass filter

An ideal highpass filter cuts-off frequencies lower than its *cutoff frequency*, ω_c.

$$H_{\mathrm{hp}}(\omega) = \begin{cases} 0, & |\omega| \ \leq \omega_c \\ 1, & |\omega| \ > \omega_c \end{cases}$$

Frequency response



Responses

Frequency response

$$H_{\mathrm{hp}}(\omega) = 1 - H_{\mathrm{lp}}(\omega)$$

Impulse response

$$h_{\mathrm{hp}}(t) = \delta(t) - h_{\mathrm{lp}}(t)$$

Example 9

Determine the frequency response of a 2nd-order butterworth highpass filter

Solution

Magnitude frequency response

In [ ]:

```
w = -400:400;
plot(w,1-mHlp)
grid
ylabel('|H_B(j\omega)|')
title('Magnitude Frequency Response for 2nd-Order HP Butterworth Filter (\omega_
c = 100 rad/s)')
xlabel('Radian Frequency \omega [rad/s]')
text(100,0.9,'\omega_c')
text(-100,0.9,'-\omega_c')
hold on
plot([-400,-100,-100,100,100,400],[0,0,1,1,0,0],'r:')
hold off
```

High-pass filter

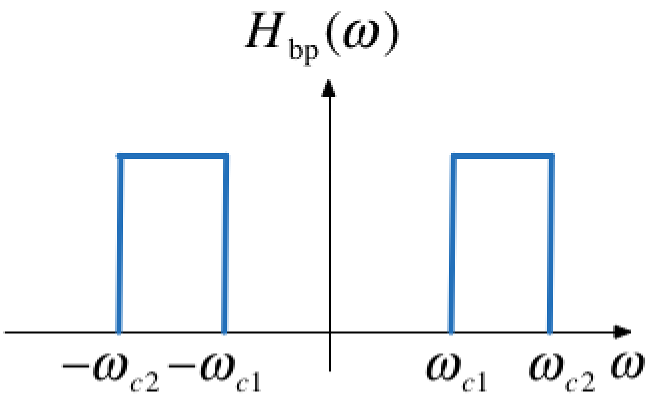
In [ ]:

```
Hhp = 1 - H
bode(Hhp)
grid
title('Bode-plot of Butterworth 2nd-Order Butterworth High Pass Filter')
```

Band-pass filter

An ideal bandpass filter cuts-off frequencies lower than its first *cutoff frequency* ω_{c1}, and higher than its second *cutoff frequency* ω_{c2}.

$$H_{\mathrm{bp}}(\omega) = \begin{cases} 1, & \omega_{c1} < |\omega| < \omega_{c2} \\ 0, & \text{otherwise} \end{cases}$$



Bandpass filter design

A bandpass filter can be obtained by multiplying the frequency responses of a lowpass filter by a highpass filter.

$$H_{\mathrm{bp}}(\omega) = H_{\mathrm{hp}}(\omega) H_{\mathrm{lp}}(\omega)$$

- The highpass filter should have cut-off frequency of ω_{c1}
- The lowpass filter should have cut-off frequency of ω_{c2}

Solutions

Solutions to Examples 5-9 are captured as a PenCast which you will find attached to the Worked Solutions section of the Week 7 Section (https://swanseauniversity.sharepoint.com/sites/EG-247SignalsandSystems2017-2108-UsrGrpcopy-UsrGrp/SiteAssets/EG-247%20Signals%20and%20Systems%202017-2108-UsrGrp%20%5bcopy%5d-UsrGrp%20Notebook/_Content%20Library/Classes/Week%207.one#Week%207%20FT%20for%20Circuit%2 id={681B0954-AC4E-9646-A567-FF06C3696F07}&page-id={4CC13EA9-40BD-7B4F-B0B6-61B392AC4943}&end) of the OneNote Class Notebook.