

Trigonometric Fourier Series

Any periodic waveform can be approximated by a DC component (which may be 0) and the sum of a fundamental and harmonic sinusoidal waveforms. This has important applications in many applications of electronics but is particularly important for signal processing and communications.

Revision?

I believe that this subject has been covered in EG-150 Signals and Systems and so we present the notes as background for the Fourier transform.

Agenda

- Motivating examples
- Wave analysis and the Trig. Fourier Series
- Symmetry in Trigonometric Fourier Series
- Computing coefficients of Trig. Fourier Series in Matlab
- Gibbs Phenomenon

Motivating Examples

This [Fourier Series demo](http://users.ece.gatech.edu/mcclella/matlabGUIs#FourierSeries) (<http://users.ece.gatech.edu/mcclella/matlabGUIs#FourierSeries>), developed by Members of the [Center for Signal and Image Processing \(CSIP\)](http://www.ece.gatech.edu/CSIP) (<http://www.ece.gatech.edu/CSIP>) at the [School of Electrical and Computer Engineering](http://www.ece.gatech.edu/) (<http://www.ece.gatech.edu/>), at the [Georgia Institute of Technology](http://www.gatech.edu/) (<http://www.gatech.edu/>), shows how periodic signals can be synthesised by a sum of sinusoidal signals.

It is here used as a motivational example in our introduction to [Fourier Series](http://en.wikipedia.org/wiki/Fourier_series) (http://en.wikipedia.org/wiki/Fourier_series). (See also [Fourier Series](http://mathworld.wolfram.com/FourierSeries.html) (<http://mathworld.wolfram.com/FourierSeries.html>), from Wolfram MathWorld referenced in the **Quick Reference** on Blackboard.)

To install this example, download the [zip file](http://users.ece.gatech.edu/mcclella/matlabGUIs/Zipfseriesdemo-v130.zip) (<http://users.ece.gatech.edu/mcclella/matlabGUIs/Zipfseriesdemo-v130.zip>) and unpack it somewhere on your MATLAB path.

Wave Analysis

- Jean Baptiste Joseph Fourier (http://en.wikipedia.org/wiki/Joseph_Fourier) (21 March 1768 – 16 May 1830) discovered that any **periodic** signal could be represented as a series of *harmonically related* sinusoids.
- An *harmonic* is a frequency whose value is an integer multiple of some *fundamental frequency*
- For example, the frequencies 2 MHz, 3 Mhz, 4 MHz are the second, third and fourth harmonics of a sinusoid with fundamental frequency 1 Mhz.

The Trigonometric Fourier Series

Any periodic waveform $f(t)$ can be represented as

$$f(t) = \frac{1}{2}a_0 + \{a_1\}\cos \Omega_0 t + \{a_2\}\cos 2\Omega_0 t + \{a_3\}\cos 3\Omega_0 t + \cdots + \{a_n\}\cos n\Omega_0 t + \cdots$$

$$+ \{b_1\}\sin \Omega_0 t + \{b_2\}\sin 2\Omega_0 t + \{b_3\}\sin 3\Omega_0 t + \cdots + \{b_n\}\sin n\Omega_0 t + \cdots$$

or equivalently (if more confusingly)

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\Omega_0 t + b_n \sin n\Omega_0 t)$$

where Ω_0 rad/s is the *fundamental frequency*.

Notation

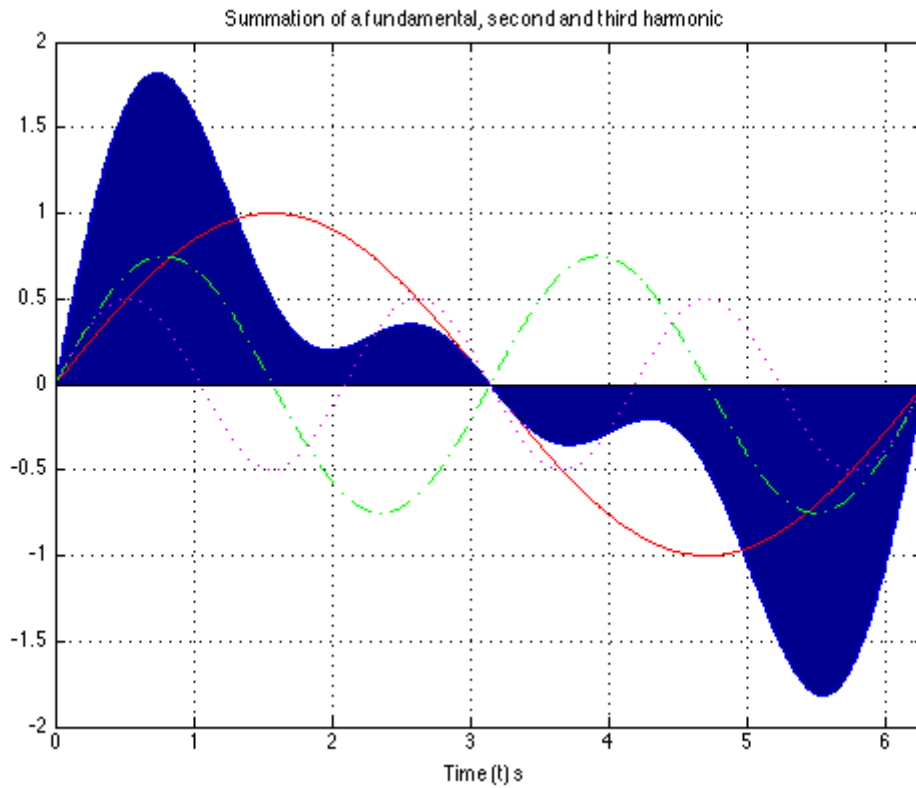
- The first term $a_0/2$ is a constant and represents the DC (average) component of the signal $f(t)$
- The terms with coefficients a_1 and b_1 together represent the fundamental frequency component of $f(t)$ at frequency Ω_0 .
- The terms with coefficients a_2 and b_2 together represent the second harmonic frequency component of $f(t)$ at frequency $2\Omega_0$.

And so on.

Since any periodic function $f(t)$ can be expressed as a Fourier series, it follows that the sum of the DC, fundamental, second harmonic and so on must produce the waveform $f(t)$.

Sums of sinusoids

In general, the sum of two or more sinusoids does not produce a sinusoid as shown below.



To generate this picture use `fourier_series1.m` (`fourier_series1.m`).

Evaluation of the Fourier series coefficients

The coefficients are obtained from the following expressions (valid for any periodic waveform with fundamental frequency Ω_0 so long as we integrate over one period $0 \rightarrow T_0$ where $T_0 = 2\pi/\Omega_0$), and $\theta = \Omega_0 t$:

$$\begin{aligned} \frac{1}{2}a_0 &= \frac{1}{T_0} \int_0^{T_0} f(t) dt = \frac{1}{\pi} \int_0^{2\pi} f(\theta) d\theta \\ a_n &= \frac{1}{T_0} \int_0^{T_0} f(t) \cos n\Omega_0 t dt = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \cos n\theta d\theta \\ b_n &= \frac{1}{T_0} \int_0^{T_0} f(t) \sin n\Omega_0 t dt = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \sin n\theta d\theta \end{aligned}$$

Symmetry in Trigonometric Fourier Series

There are simplifications we can make if the original periodic properties has certain properties:

- If $f(t)$ is odd, $a_0 = 0$ and there will be no cosine terms so $a_n = 0 \forall n > 0$
- If $f(t)$ is even, there will be no sine terms and $b_n = 0 \forall n > 0$. The DC may or may not be zero.
- If $f(t)$ has *half-wave symmetry* only the odd harmonics will be present. That is a_n and b_n is zero for all even values of n (0, 2, 4, ...)

Odd, Even and Half-wave Symmetry

Recall

- An *odd* function is one for which $f(t) = -f(-t)$. The function $\sin t$ is an *odd* function.
- An *even* function is one for which $f(t) = f(-t)$. The function $\cos t$ is an *even* function.

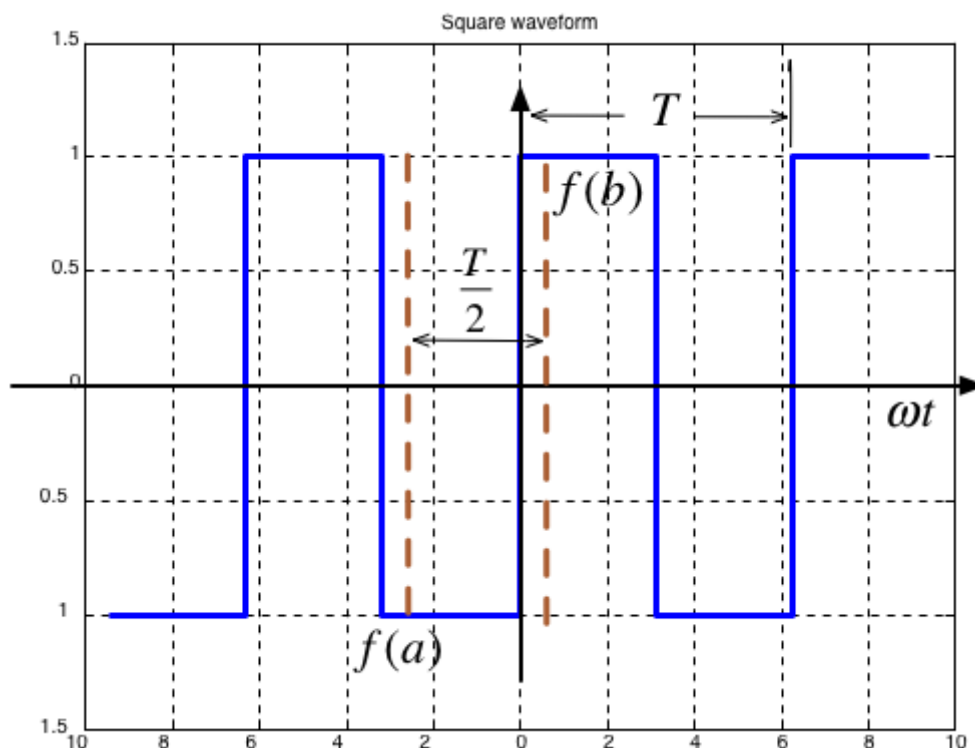
Half-wave symmetry

- A periodic function with period T is a function for which $f(t) = f(t + T)$
- A periodic function with period T , has *half-wave symmetry* if $f(t) = -f(t + T/2)$

Symmetry in Common Waveforms

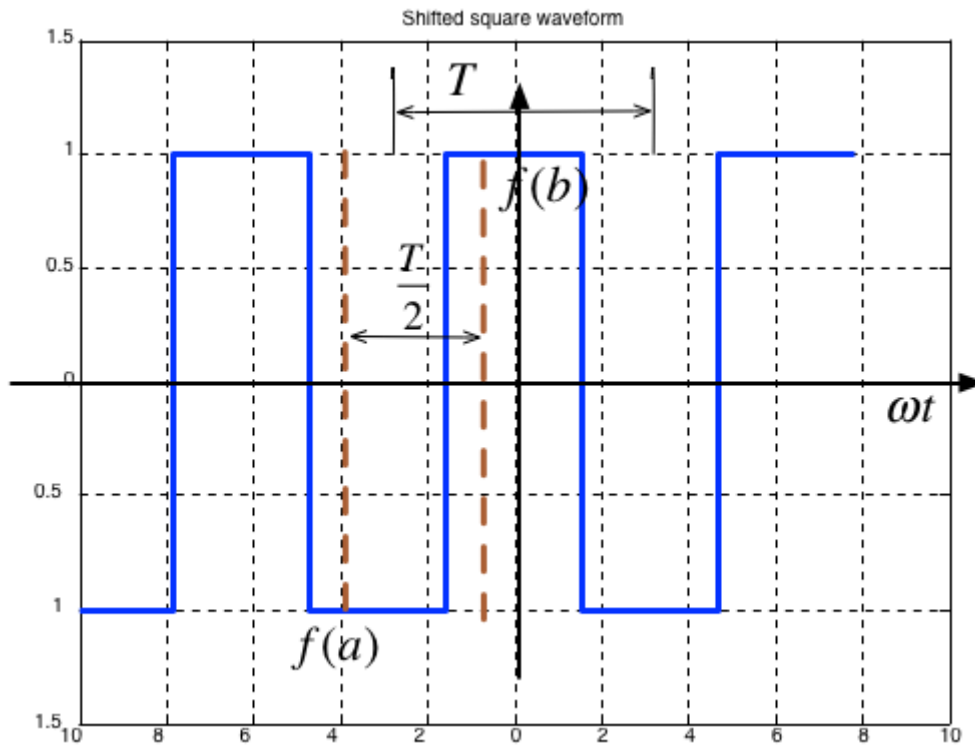
To reproduce the following waveforms (without annotation) publish the script [waves.m \(waves.m\)](#).

Squarewave



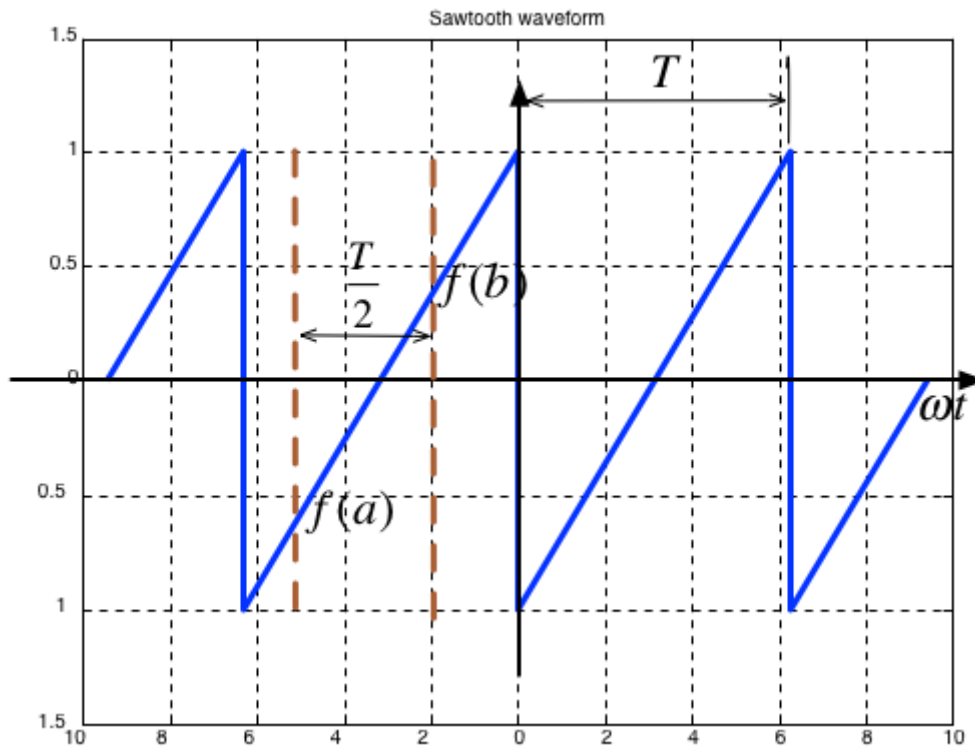
- Average value over period T is ...?
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Shifted Squarewave



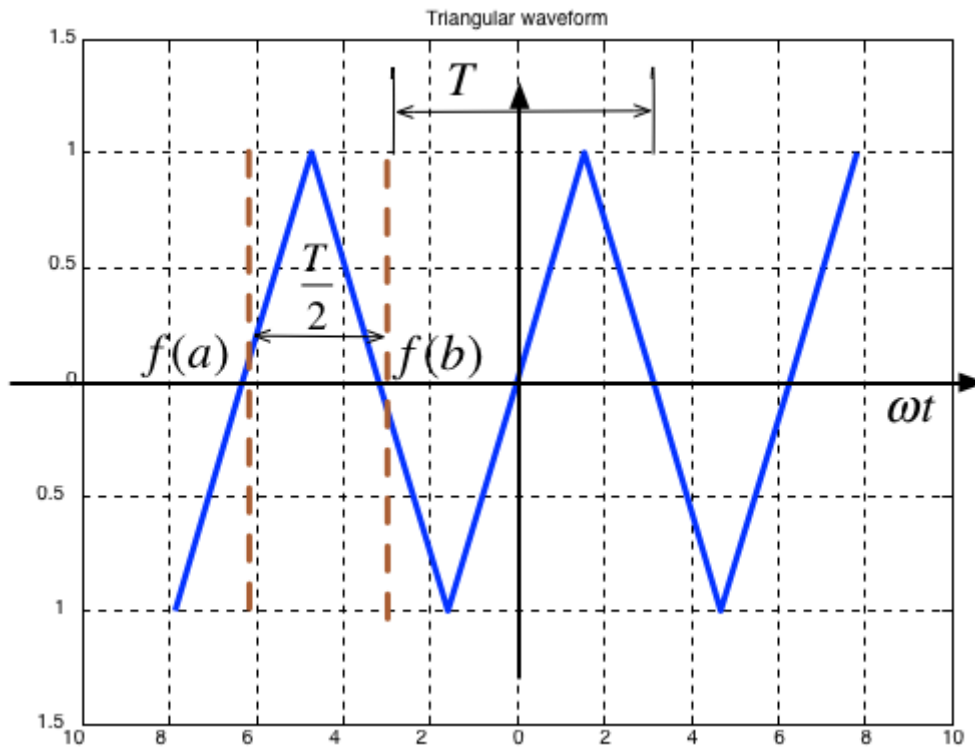
- Average value over period T is ...?
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Sawtooth



- Average value over period T is ...?
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Triangle

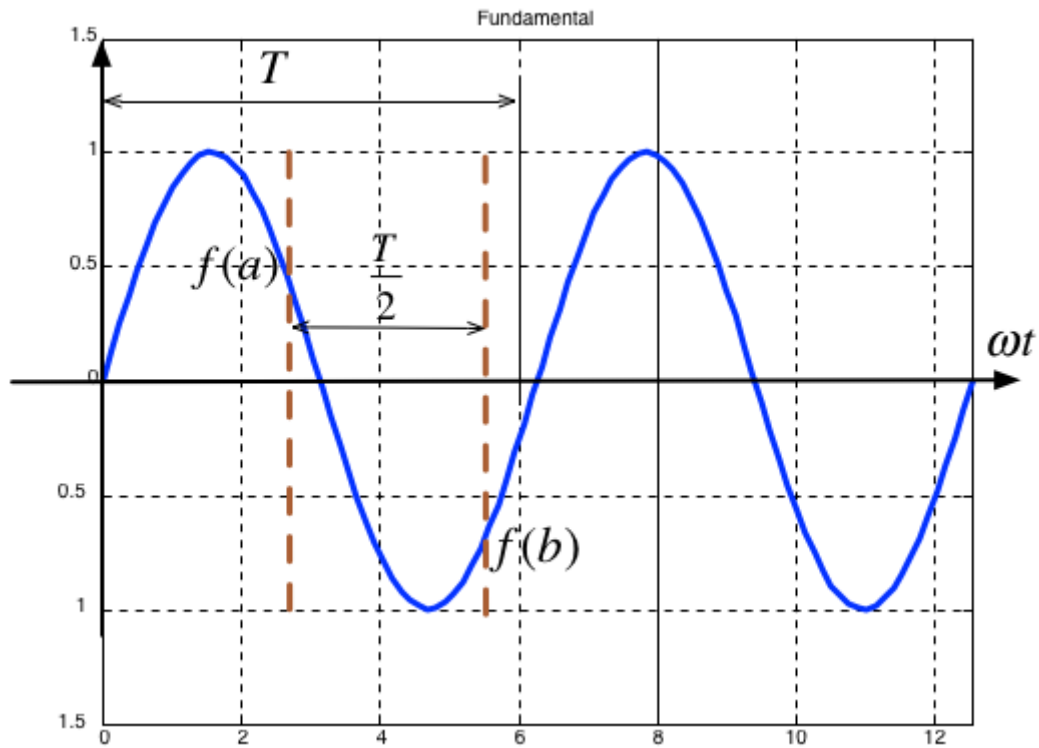


- Average value over period T is ...?
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Symmetry in fundamental, Second and Third Harmonics

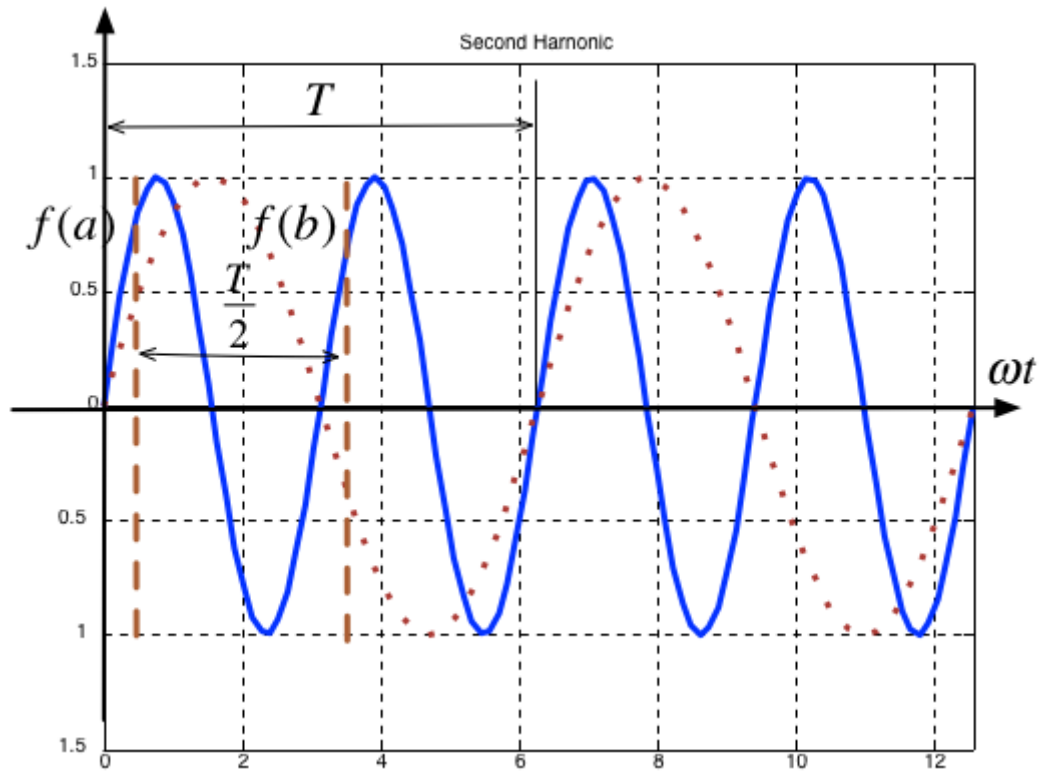
In the following, $T/2$ is taken to be the half-period of the fundamental sinewave.

Fundamental



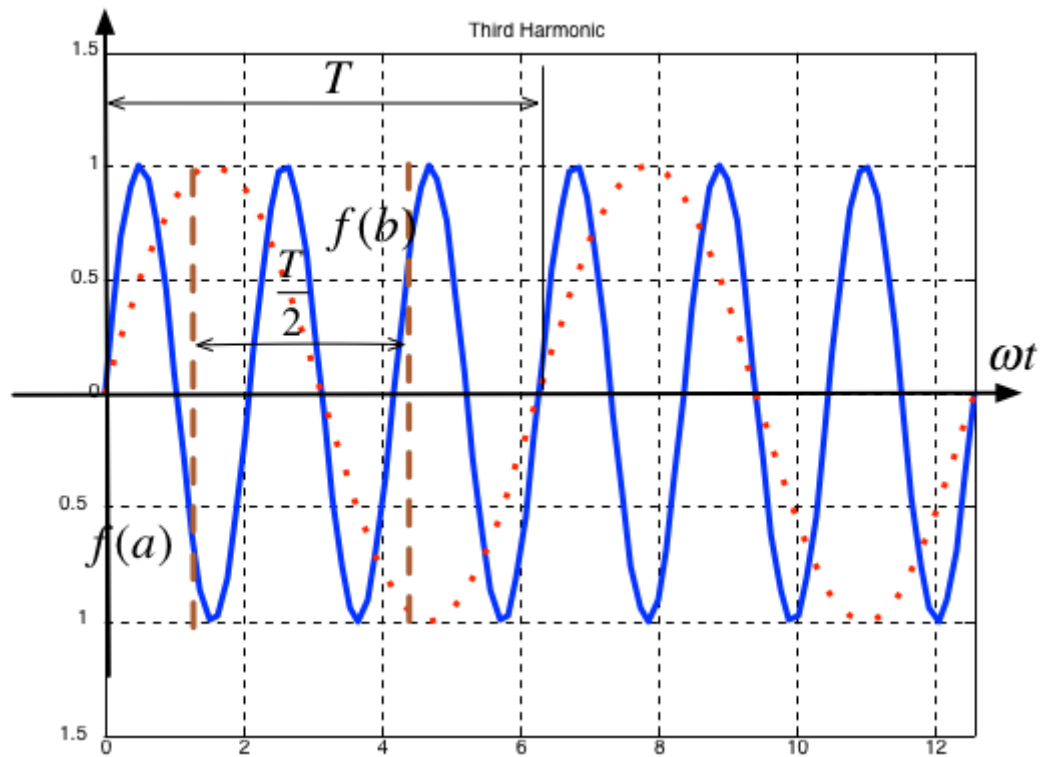
- Average value over period T is ...?
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Second Harmonic



- Average value over period T is ...?
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Third Harmonic



- Average value over period T is ...?
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

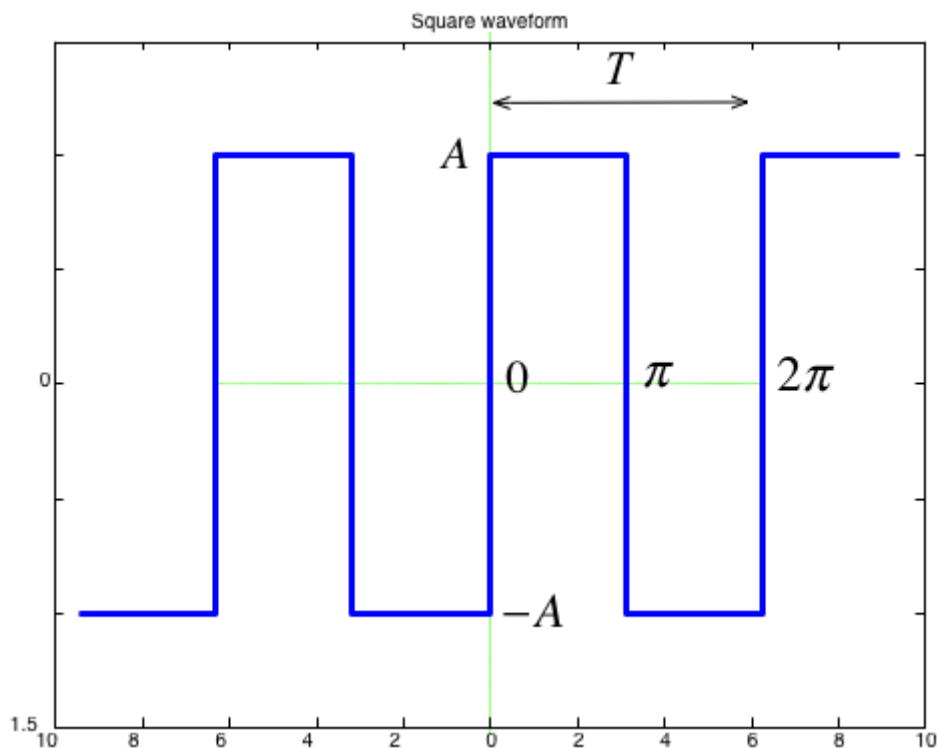
Some simplifications that result from symmetry

- The limits of the integrals used to compute the coefficients a_n and b_n of the Fourier series are given as $0 \rightarrow 2\pi$ which is one period T
- We could also choose to integrate from $-\pi \rightarrow \pi$
- If the function is *odd*, or *even* or has *half-wave symmetry* we can compute a_n and b_n by integrating from $0 \rightarrow \pi$ and multiplying by 2.
- If we have *half-wave symmetry* we can compute a_n and b_n by integrating from $0 \rightarrow \pi/2$ and multiplying by 4.

(For more details see page 7-10 of the textbook)

Computing coefficients of Trig. Fourier Series in Matlab

As an example let's take a square wave with amplitude $\pm A$ and period T .



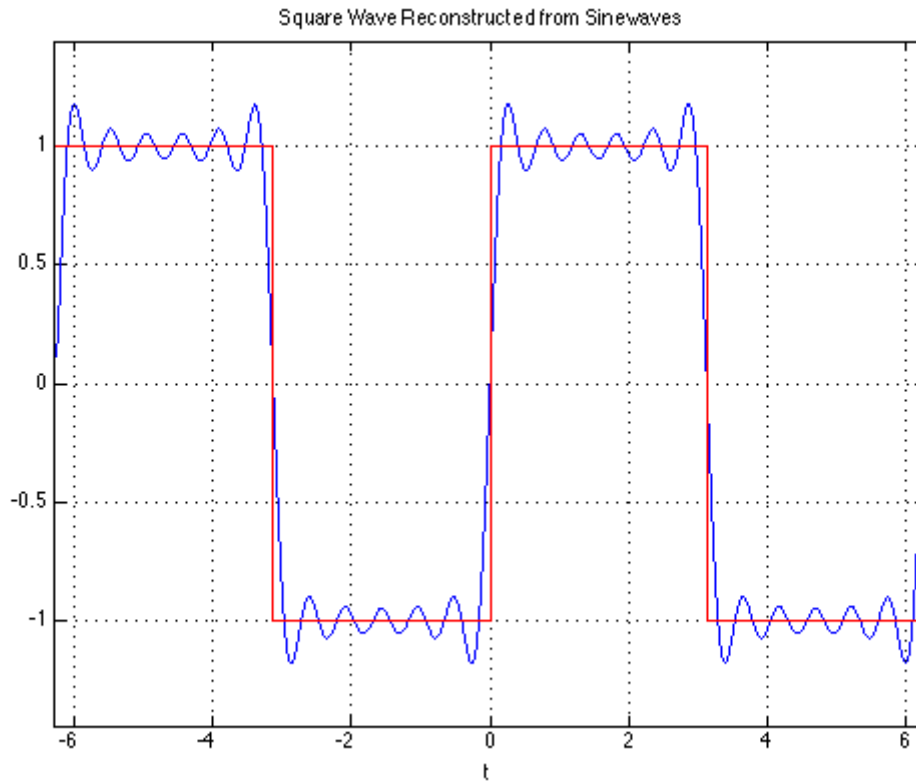
Solution

Solution: See [square_ftsig.m](#) ([square_ftsig.m](#)). Script confirms that:

- $a_0 = 0$
- $a_i = 0$: function is odd
- $b_i = 0$: for i even - half-wave symmetry

ft =

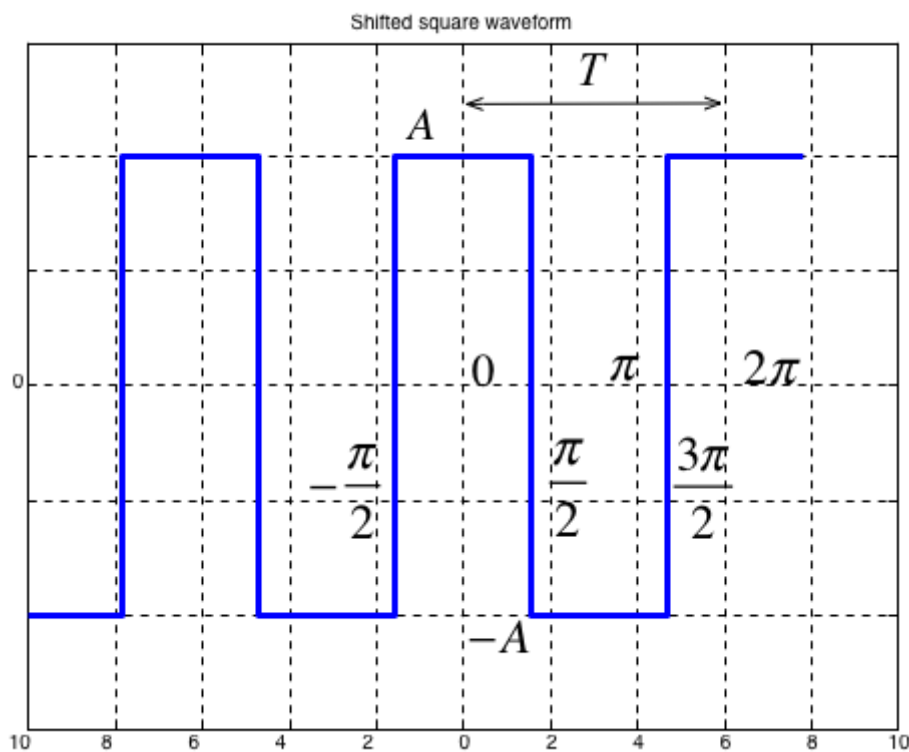
$$(4*A*\sin(t))/\pi + (4*A*\sin(3*t))/(3*\pi) + (4*A*\sin(5*t))/(5*\pi) + (4*A*\sin(7*t))/(7*\pi) + (4*A*\sin(9*t))/(9*\pi) + (4*A*\sin(11*t))/(11*\pi)$$



Note that the coefficients match those given in the textbook (Section 7.4.1).

$$f(t) = \frac{4A}{\pi} \left(\sin \Omega_0 t + \frac{1}{3} \sin 3\Omega_0 t + \frac{1}{5} \sin 5\Omega_0 t + \dots \right) = \frac{4A}{\pi} \sum_{n=\text{odd}} \frac{1}{n} \sin n\Omega_0 t$$

Using symmetry - computing the Fourier series coefficients of the shifted square wave

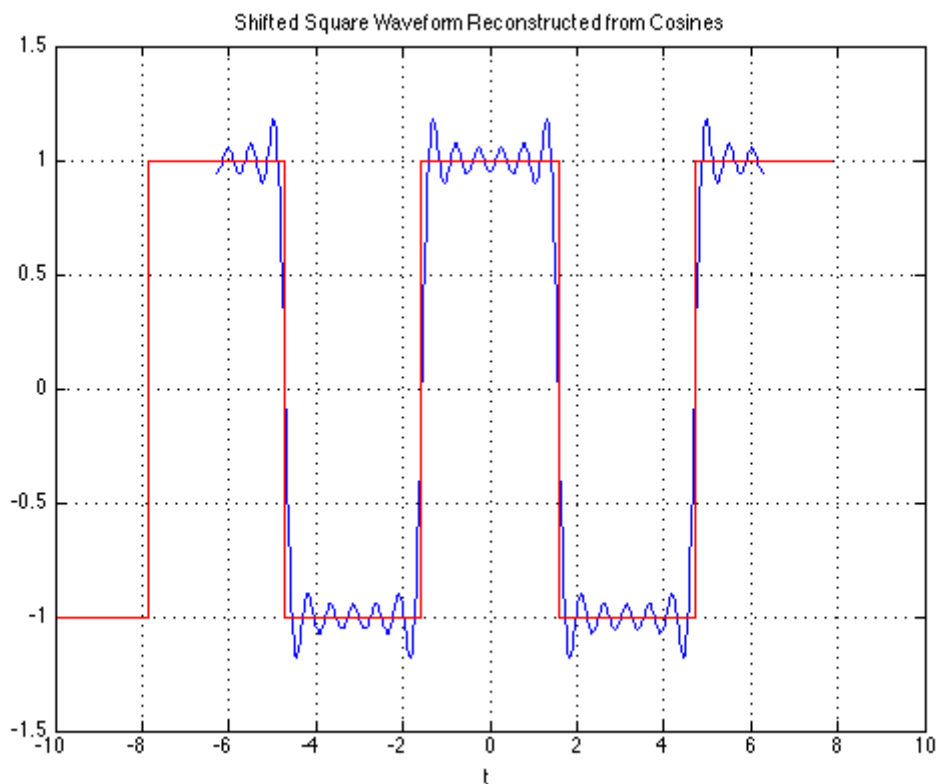


- As before $a_0 = 0$
- We observe that this function is even, so all b_k coefficients will be zero
- The waveform has half-wave symmetry, so only odd indexed coefficients will be present.
- Further more, because it has half-wave symmetry we can just integrate from $0 \rightarrow \pi/2$ and multiply the result by 4.

See `shifted_sq_ftrig.m` (`shifted_sq_ftrig.m`).

`ft =`

```
(4*A*cos(t))/pi - (4*A*cos(3*t))/(3*pi) + (4*A*cos(5*t))/(5*pi) - (4*A*cos(
7*t))/(7*pi) + (4*A*cos(9*t))/(9*pi) - (4*A*cos(11*t))/(11*pi)
```



Note that the coefficients match those given in the textbook (Section 7.4.2).

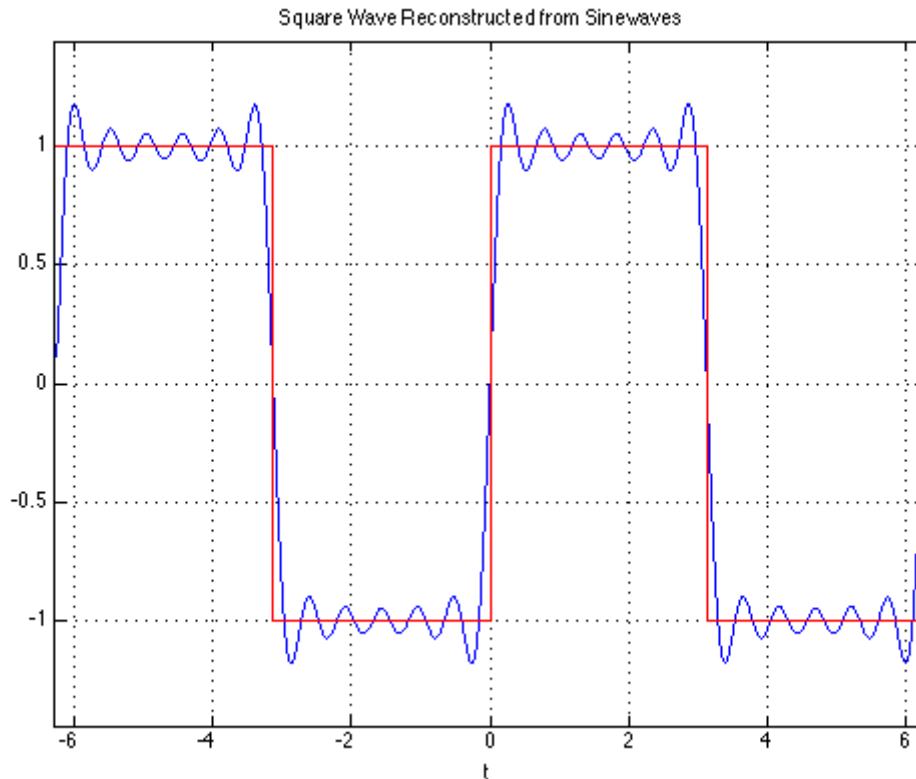
$$f(t) = \frac{4A}{\pi} \left(\cos \Omega_0 t - \frac{1}{3} \cos 3\Omega_0 t + \frac{1}{5} \cos 5\Omega_0 t - \dots \right) = \frac{4A}{\pi} \sum_{n=\text{odd}} (-1)^{\frac{n-1}{2}} \frac{1}{n} \cos n\Omega_0 t$$

Gibbs Phenomenon

In an earlier slide we found that the trigonometric for of the Fourier series of the square waveform is

$$f(t) = \frac{4A}{\pi} \left(\sin \Omega_0 t + \frac{1}{3} \sin 3\Omega_0 t + \frac{1}{5} \sin 5\Omega_0 t + \dots \right) = \frac{4A}{\pi} \sum_{n=\text{odd}} \frac{1}{n} \sin n\Omega_0 t$$

This figure shows the approximation for the first 11 harmonics:



As we add more harmonics, the sum looks more and more like a square wave. However the crests do not become flattened; this is known as *Gibbs Phenomenon* and it occurs because of the discontinuity of the perfect square waveform as it changes from $+A$ to $-A$ and *vice versa*.