Trigonometric Fourier Series

Any periodic waveform can be approximated by a DC component (which may be 0) and the sum of a fundamental and harmonic sinusoidal waveforms. This has important applications in many applications of electronics but is particularly important for signal processing and communications.

Revision?

I believe that this subject has been covered in EG-150 Signals and Systems and so we present the notes as background for the Fourier transform.

Agenda

- Motivating examples
- Wave analysis and the Trig. Fourier Series
- Symmetry in Trigonometric Fourier Series
- Computing coefficients of Trig. Fourier Series in MATLAB
- Gibbs Phenomenon

Motivating Examples

This Fourier Series demo (http://dspfirst.gatech.edu/matlab/#FourierSeries), developed by Members of the Center for Signal and Image Processing (CSIP) at the School of Electrical and Computer Engineering (http://www.ece.gatech.edu/) at the Georgia Institute of Technology (http://www.gatech.edu/), shows how periodic signals can be synthesised by a sum of sinusoidal signals.

It is here used as a motivational example in our introduction to Fourier Series (http://en.wikipedia.org/wiki/Fourier_series). (See also Fourier Series (http://mathworld.wolfram.com/FourierSeries.html) from Wolfram MathWorld referenced in the **Quick Reference** on Blackboard.)

To install this example, download the zip file (http://dspfirst.gatech.edu/matlab/ZipFiles/fseriesdemo-v144.zip) and unpack it somewhere on your MATLAB path.

Wave Analysis

- Jean Baptiste Joseph Fourier (http://en.wikipedia.org/wiki/Joseph_Fourier) (21 March 1768 – 16 May 1830) discovered that any ***periodic*** signal could be represented as a series of *harmonically related* sinusoids.
- An *harmonic* is a frequency whose value is an integer multiple of some *fundamental frequency*
- For example, the frequencies 2 MHz, 3 Mhz, 4 MHz are the second, third and fourth harmonics of a sinusoid with fundamental frequency 1 Mhz.

The Trigonometric Fourier Series

Any periodic waveform $f(t)$ can be represented as

$$\begin{gathered} f(t) = \frac{1}{2}{a_0} + {a_1}\cos \Omega_0 t + {a_2}\cos 2\Omega_0 t + {a_3}\cos 3\Omega_0 t + \cdots + {a_n}\cos n\Omega_0 t + \cdots \

- {b_1}\sin \Omega_0 t + {b_2}\sin 2\Omega_0 t + {b_3}\sin 3\Omega_0 t + \cdots + {b_n}\sin n\Omega_0 t + \cdots \ \end{gathered} $$

or equivalently (if more confusingly)

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\Omega_0 t + b_n \sin n\Omega_0 t)$$

where Ω_0 rad/s is the *fundamental frequency*.

Notation

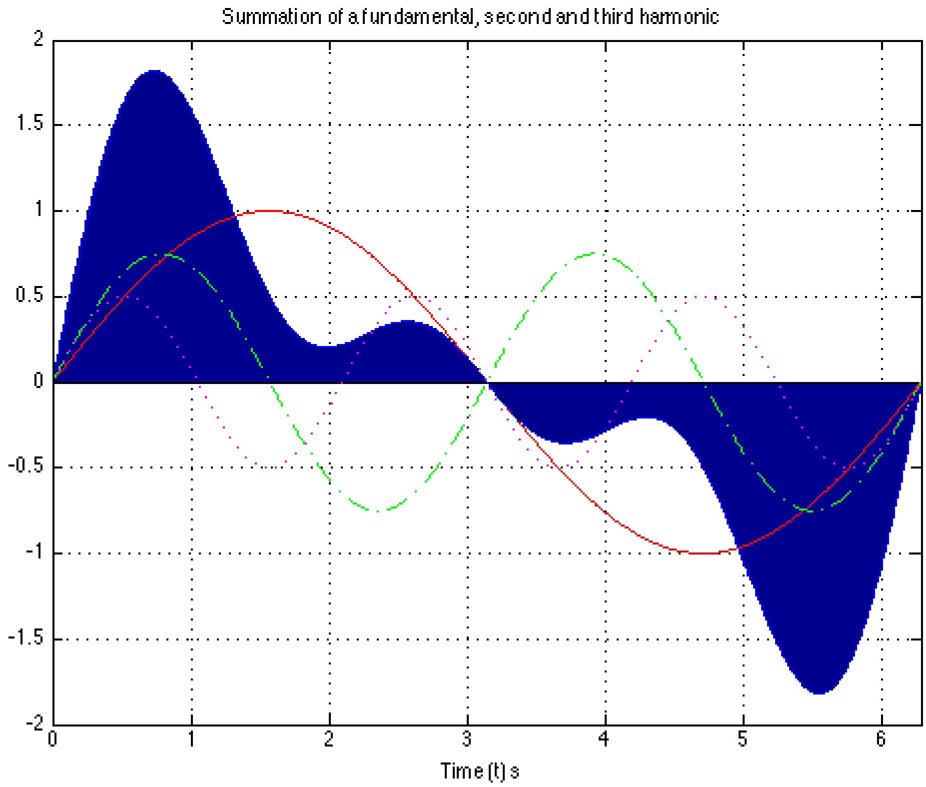
- The first term $a_o/2$ is a constant and represents the DC (average) component of the signal $f(t)$
- The terms with coefficients a_1 and b_1 together represent the fundamental frequency component of $f(t)$ at frequency Ω_0.
- The terms with coefficients a_2 and b_2 together represent the second harmonic frequency component of $f(t)$ at frequency $2\Omega_0$.

And so on.

Since any periodic function $f(t)$ can be expressed as a Fourier series, it follows that the sum of the DC, fundamental, second harmonic and so on must produce the waveform $f(t)$.

Sums of sinusoids

In general, the sum of two or more sinusoids does not produce a sinusoid as shown below.



To generate this picture use fourier_series1.m (fourier_series1.m).

Evaluation of the Fourier series coefficients

The coefficients are obtained from the following expressions (valid for any periodic waveform with fundamental frequency Ω_0 so long as we integrate over one period $0 \to T_0$ where $T_0 = 2\pi/\Omega_0$), and $\theta = \Omega_0 t$:

$$\frac{1}{2}a_0 = \frac{1}{T_0}\int_0^{T_0} f(t)dt = \frac{1}{\pi}\int_0^{2\pi} f(\theta)d\theta)$$

$$a_n = \frac{1}{T_0}\int_0^{T_0} f(t)\cos n\Omega_0 t\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta$$

$$b_n = \frac{1}{T_0}\int_0^{T_0} f(t)\sin n\Omega_0 t\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta$$

Symmetry in Trigonometric Fourier Series

There are simplifications we can make if the original periodic properties has certain properties:

- If $f(t)$ is odd, $a_0 = 0$ and there will be no cosine terms so $a_n = 0\ \forall n > 0$
- If $f(t)$ is even, there will be no sine terms and $b_n = 0\ \forall n > 0$. The DC may or may not be zero.
- If $f(t)$ has *half-wave symmetry* only the odd harmonics will be present. That is a_n and b_n is zero for all even values of n (0, 2, 4, ...)

Odd, Even and Half-wave Symmetry

Recall

- An *odd* function is one for which $f(t) = -f(-t)$. The function $\sin t$ is an *odd* function.
- An *even* function is one for which $f(t) = f(-t)$. The function $\cos t$ is an *even* function.

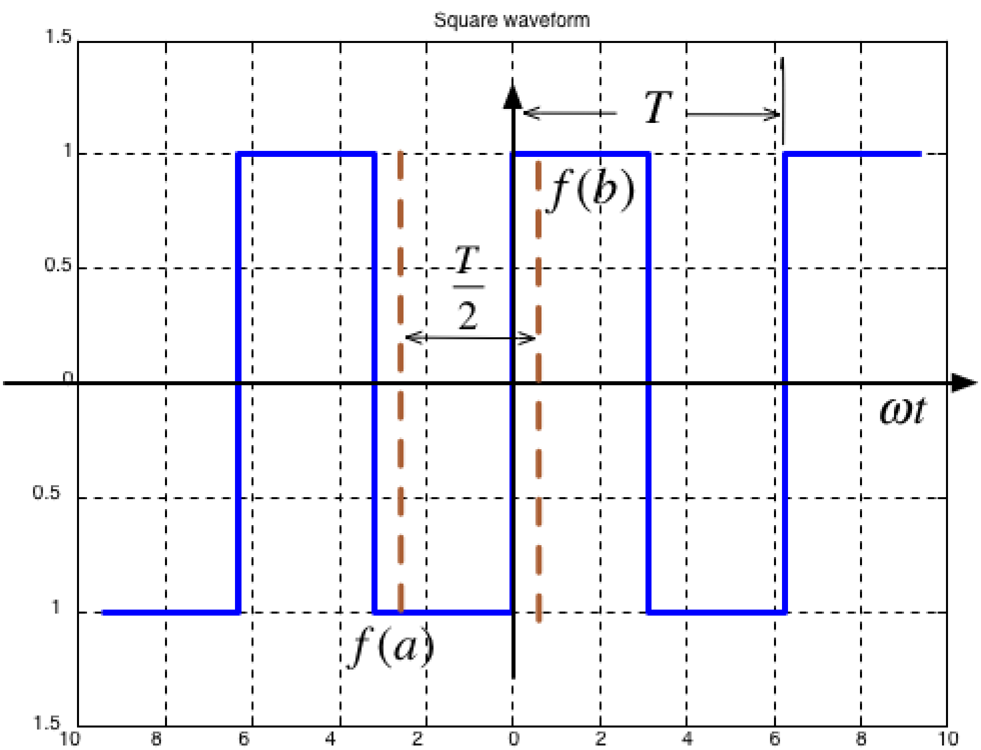
Half-wave symmetry

- A periodic function with period T is a function for which $f(t) = f(t + T)$
- A periodic function with period T, has *half-wave symmetry* if $f(t) = -f(t + T/2)$

Symmetry in Common Waveforms

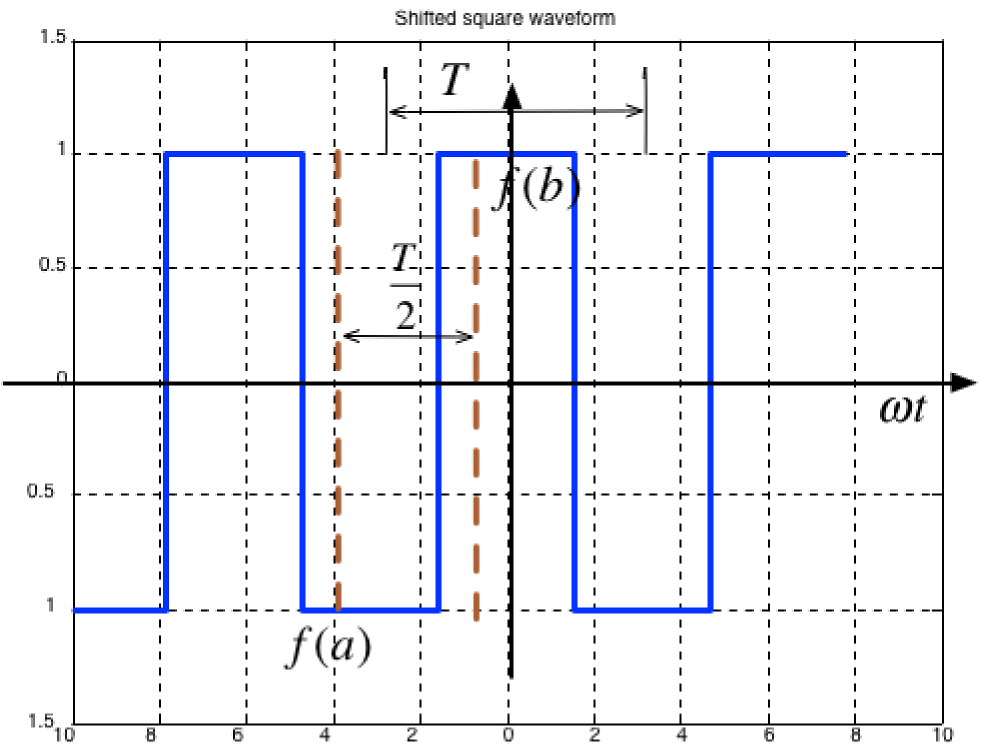
To reproduce the following waveforms (without annotation) publish the script waves.m (waves.m).

Squarewave



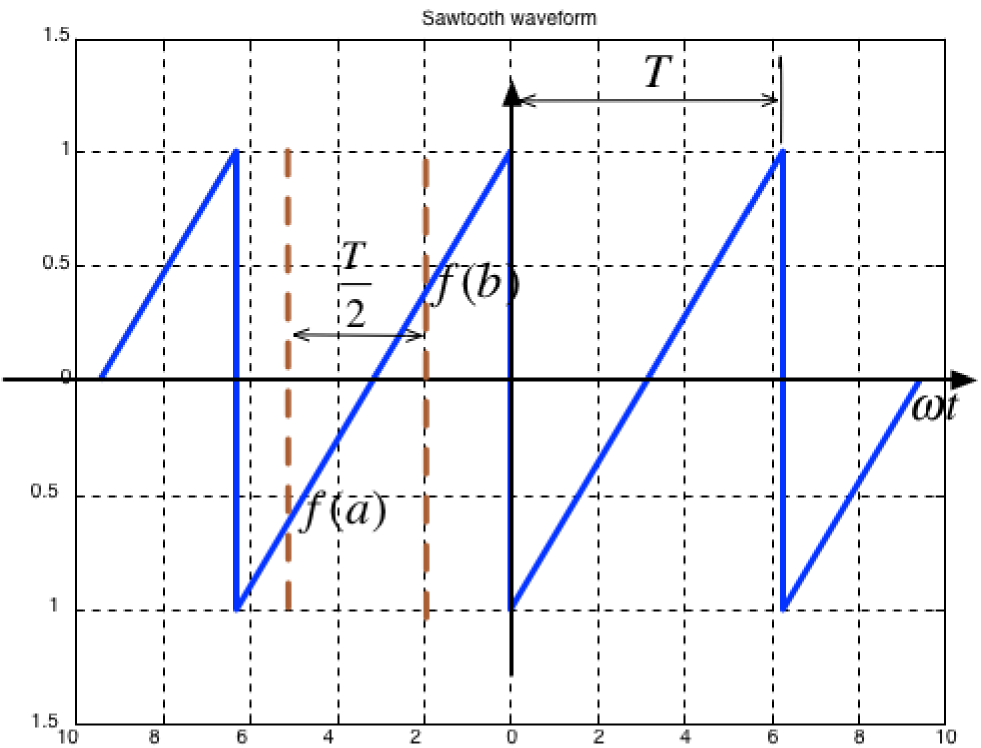
- Average value over period T is ...?
- It is an **odd/even**function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Shifted Squarewave



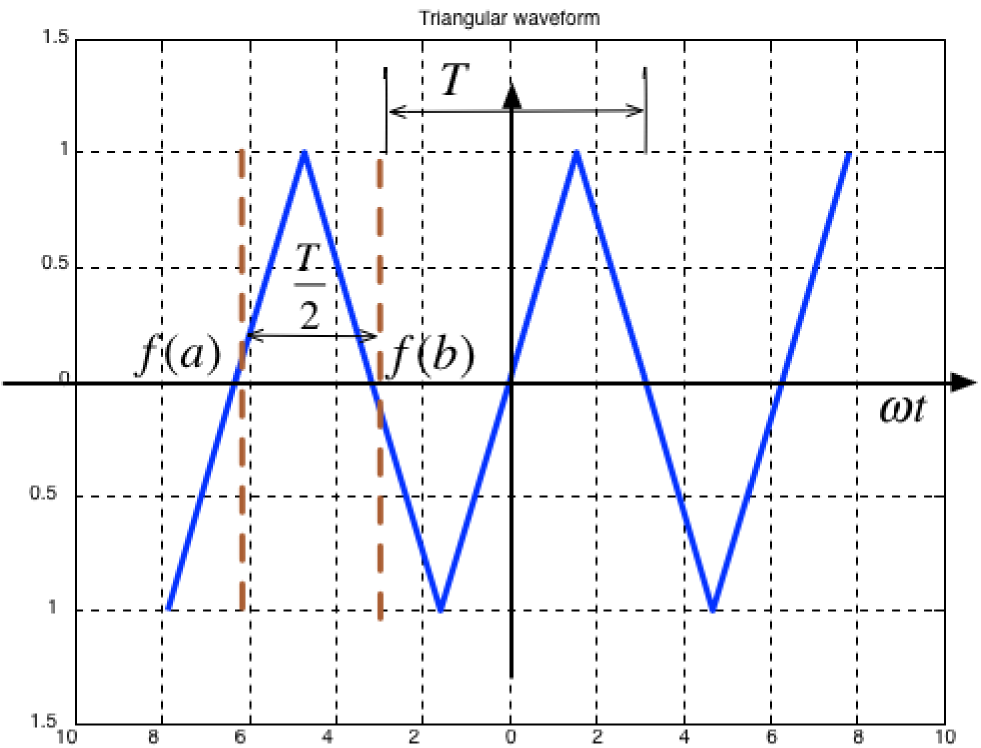
- Average value over period T is ...?
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Sawtooth



- Average value over period T is ...?
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Triangle

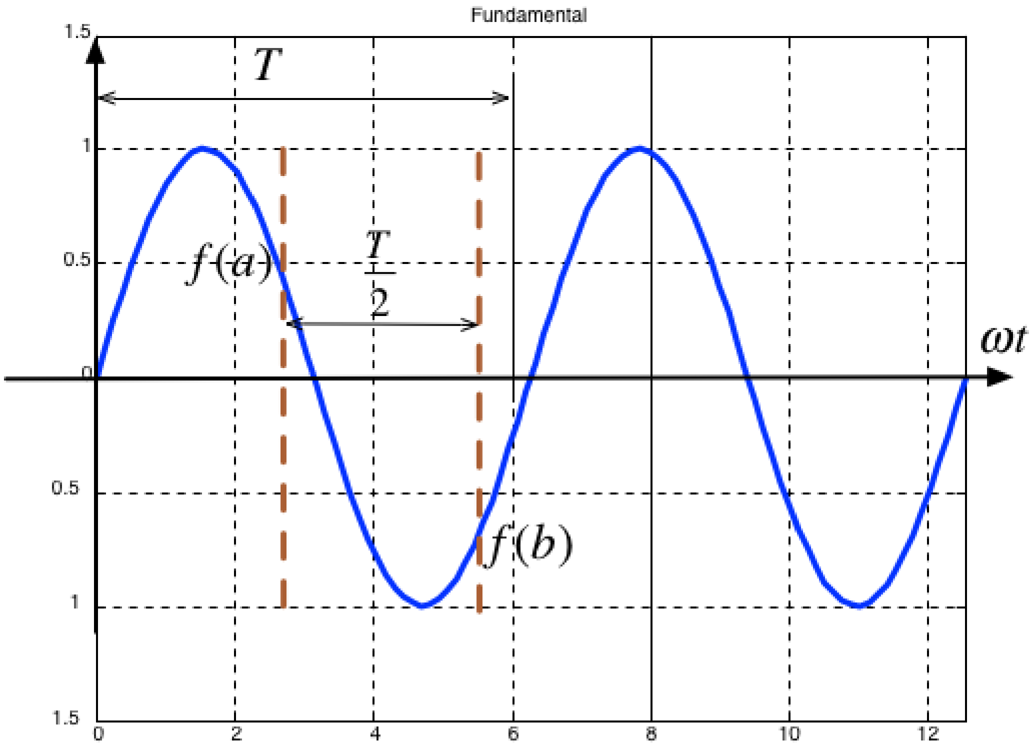


- Average value over period T is ...?
- It is an **odd/even**function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Symmetry in fundamental, Second and Third Harmonics

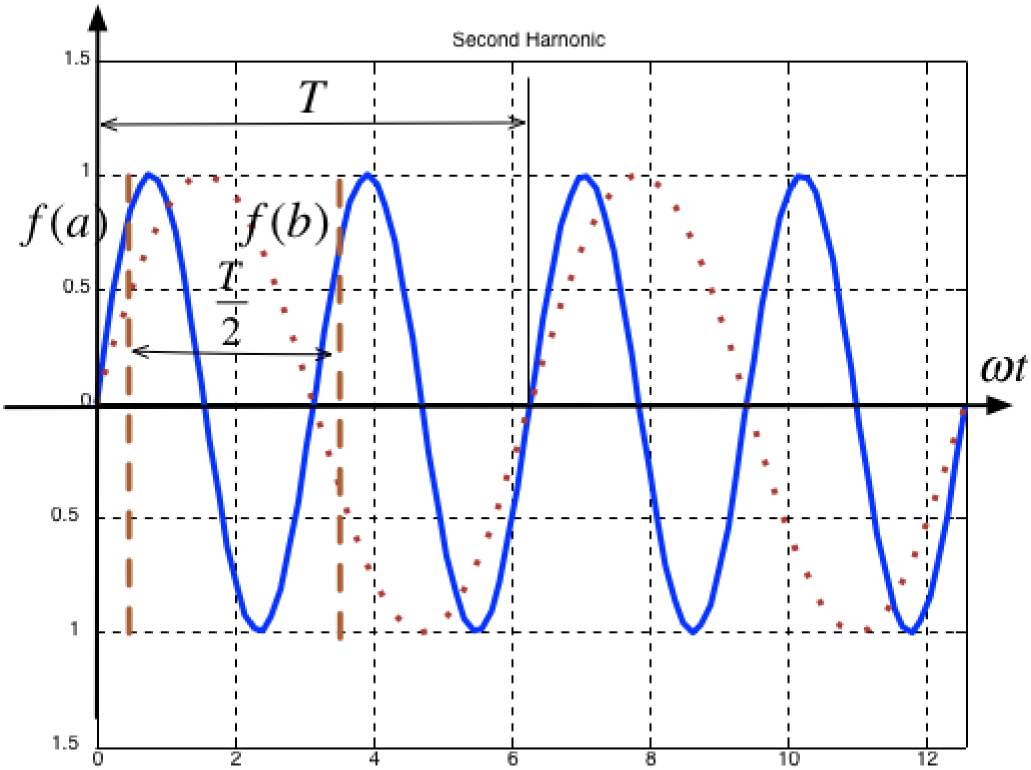
In the following, $T/2$ is taken to be the half-period of the fundamental sinewave.

Fundamental



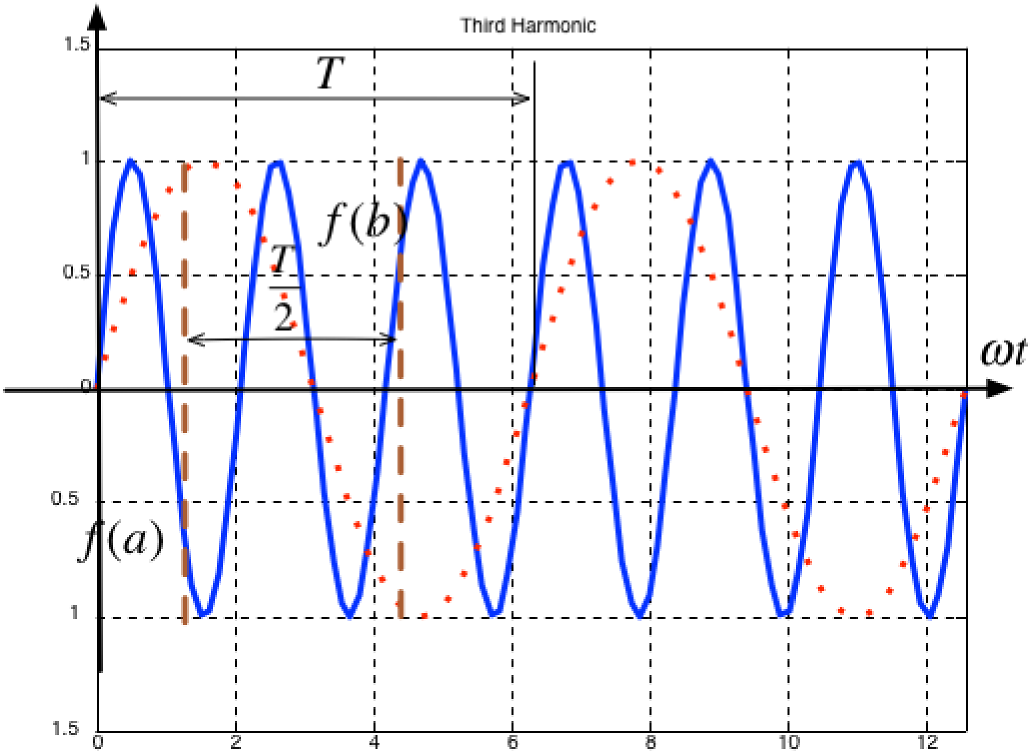
- Average value over period T is ...?
- It is an **odd/even**function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Second Harmonic



- Average value over period T is ...?
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Third Harmonic



- Average value over period T is ...?
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

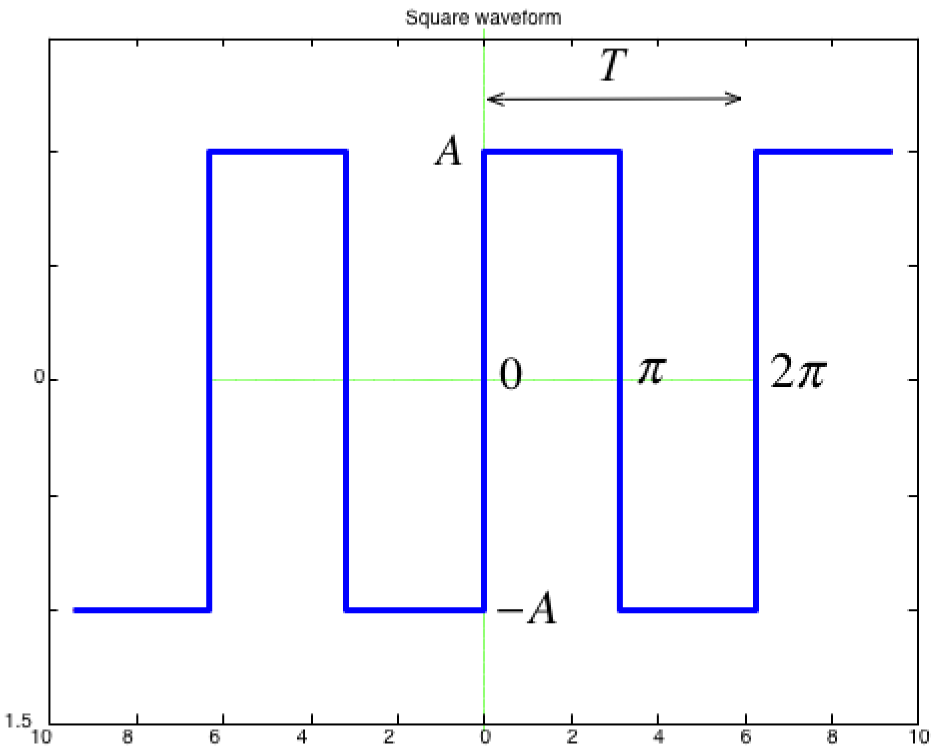
Some simplifications that result from symmetry

- The limits of the integrals used to compute the coefficents a_n and b_n of the Fourier series are given as $0 \rightarrow 2\pi$ which is one period T
- We could also choose to integrate from $-\pi \rightarrow \pi$
- If the function is *odd*, or *even* or has *half-wave symmetry* we can compute a_n and b_n by integrating from $0 \rightarrow \pi$ and multiplying by 2.
- If we have *half-wave symmetry* we can compute a_n and b_n by integrating from $0 \rightarrow \pi/2$ and multiplying by 4.

(For more details see page 7-10 of Karris)

Computing coefficients of Trig. Fourier Series in Matlab

As an example let's take a square wave with amplitude $\pm A$ and period T.



Solution

In [1]:

```
format compact
clear all
```

In [4]:

```
syms t n A pi
n = [1:11];
```

DC component

In [5]:

```
half_a0 = 1/(2*pi)*(int(A,t,0,pi)+int(-A,t,pi,2*pi))
```

```
half_a0 =
0
```

Compute harmonics

In [6]:

```
ai = 1/pi*(int(A*cos(n*t),t,0,pi)+int(-A*cos(n*t),t,pi,2*pi))
bi = 1/pi*(int(A*sin(n*t),t,0,pi)+int(-A*sin(n*t),t,pi,2*pi))
```

```
ai =
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
bi =
[ (4*A)/pi, 0, (4*A)/(3*pi), 0, (4*A)/(5*pi), 0, (4*A)/(7*pi), 0, (4
*A)/(9*pi), 0, (4*A)/(11*pi)]
```

Reconstruct $f(t)$ from harmonic sine functions

In [9]:

```
ft = half_a0;
for k=1:length(n)
    ft = ft + ai(k)*cos(k*t) + bi(k)*sin(k*t);
end;
ft
```

```
ft =
(4*A*sin(t))/pi + (4*A*sin(3*t))/(3*pi) + (4*A*sin(5*t))/(5*pi) + (4
*A*sin(7*t))/(7*pi) + (4*A*sin(9*t))/(9*pi) + (4*A*sin(11*t))/(11*p
i)
```

Make numeric

In [10]:

```
ft_num = subs(ft,A,1.0)
```

```
ft_num =
(4*sin(3*t))/(3*pi) + (4*sin(5*t))/(5*pi) + (4*sin(7*t))/(7*pi) + (4
*sin(9*t))/(9*pi) + (4*sin(11*t))/(11*pi) + (4*sin(t))/pi
```

Print using 4 sig digits

In [11]:

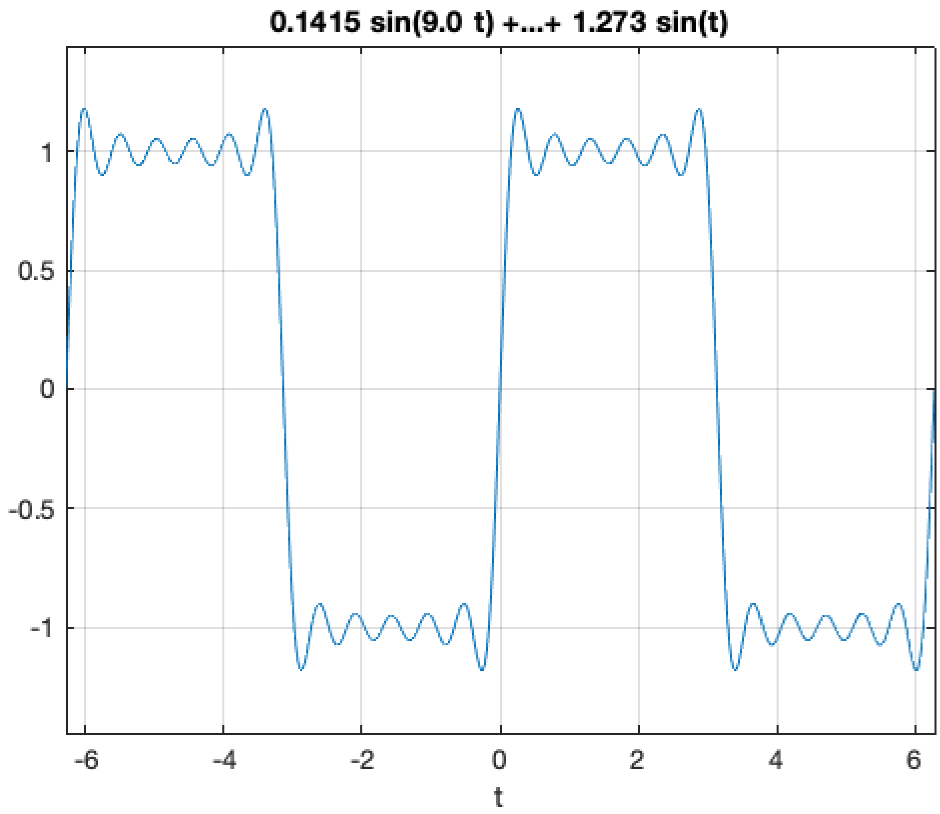
```
ft_num = vpa(ft_num, 4)
```

```
ft_num =
0.1415*sin(9.0*t) + 0.2546*sin(5.0*t) + 0.1157*sin(11.0*t) + 0.4244*
sin(3.0*t) + 0.1819*sin(7.0*t) + 1.273*sin(t)
```

Plot result

In [15]:

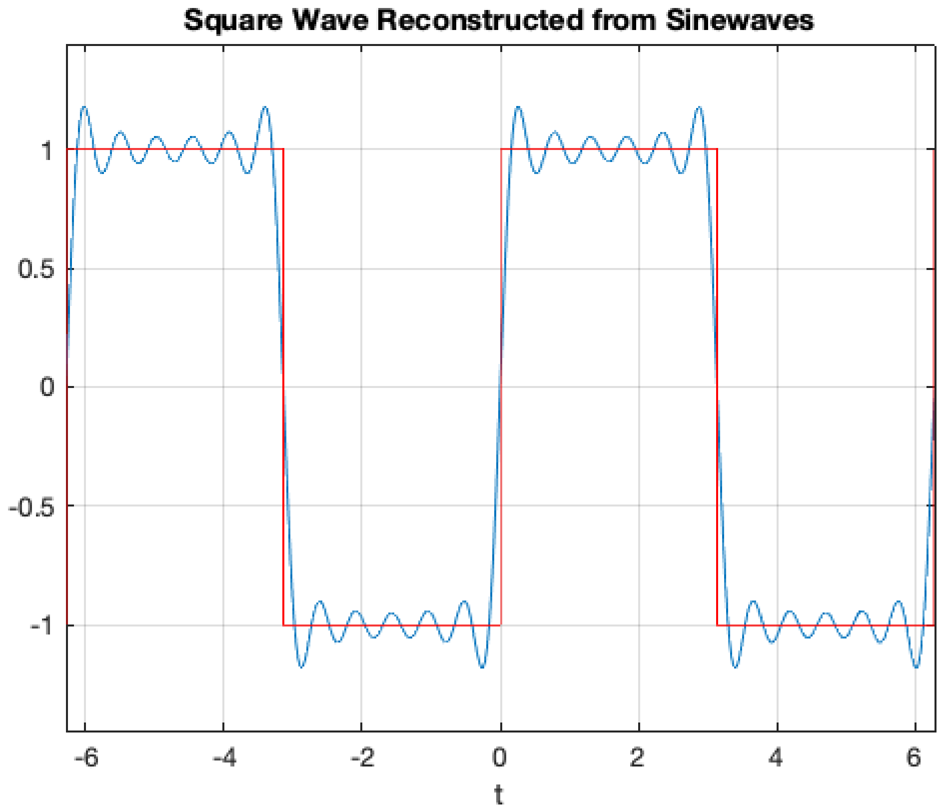
```
ezplot(ft_num),grid
```



Plot original signal (we could use `heaviside` for this as well)

In [14]:

```
ezplot(ft_num)
hold on
t = [-3,-2,-2,-2,-1,-1,-1,0,0,0,1,1,1,2,2,2,3]*pi;
f = [-1,-1,0,1,1,0,-1,-1,0,1,1,0,-1,-1,0,1,1];
plot(t,f,'r-')
grid
title('Square Wave Reconstructed from Sinewaves')
hold off
```



To run the full solution yourself download and run [square_ftrig.mlx (square_ftrig.mlx)](square_ftrig.mlx).

The Result confirms that:

- $a_0 = 0$
- $a_i = 0$: function is odd
- $b_i = 0$: for i even - half-wave symmetry

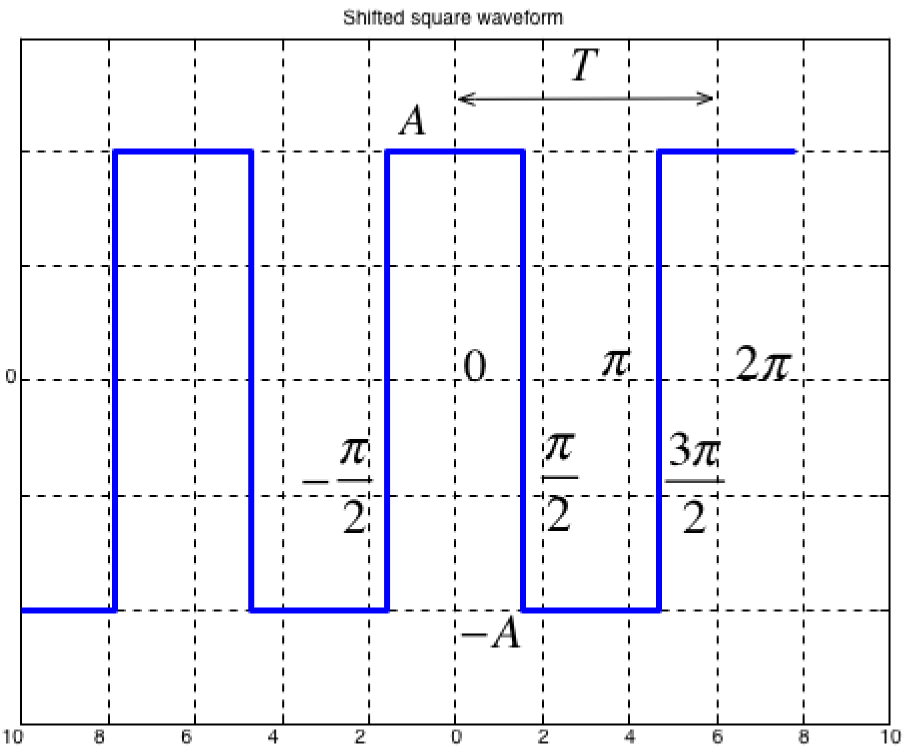
```
ft =

(4*A*sin(t))/pi + (4*A*sin(3*t))/(3*pi) + (4*A*sin(5*t))/(5*pi) + (4*A*sin
(7*t))/(7*pi) + (4*A*sin(9*t))/(9*pi) + (4*A*sin(11*t))/(11*pi)
```

Note that the coefficients match those given in the textbook (Section 7.4.1).

$$f(t) = \frac{4A}{\pi}\left(\sin \Omega_0 t + \frac{1}{3}\sin 3\Omega_0 t + \frac{1}{5}\sin 5\Omega_0 t + \cdots\right) = \frac{4A}{\pi}\sum_{n=\text{odd}} \frac{1}{n}\sin n\Omega_0 t$$

Using symmetry - computing the Fourier series coefficients of the shifted square wave



Calculation of Fourier coefficients for Shifted Square Wave Exploiting half-wave symmetry. This is almost the same procedure as before. You can confirm the results by downloading and executing this file: shifted_sq_ftrig.mlx (shifted_sq_ftrig.mlx).

In [18]:

```
syms t n A pi
```

Define harmonics

In [19]:

```
n = [1:11];
```

DC component

In [21]:

```
half_a0 = 0
```

```
half_a0 =
     0
```

Compute harmonics - use half-wave symmetry

In [23]:

```
ai = 4/pi*int(A*cos(n*t),t,0,(pi/2))
```

```
ai =
[ (4*A)/pi, 0, -(4*A)/(3*pi), 0, (4*A)/(5*pi), 0, -(4*A)/(7*pi), 0,
(4*A)/(9*pi), 0, -(4*A)/(11*pi)]
```

In [24]:

```
bi = zeros(size(n))
```

```
bi =
     0     0     0     0     0     0     0     0     0     0     0
```

Reconstruct f(t) from harmonic sine functions

In [25]:

```
ft = half_a0;
for k=1:length(n)
    ft = ft + ai(k)*cos(k*t) + bi(k)*sin(k*t);
end
ft
```

```
ft =
(4*A*cos(t))/pi - (4*A*cos(3*t))/(3*pi) + (4*A*cos(5*t))/(5*pi) - (4
*A*cos(7*t))/(7*pi) + (4*A*cos(9*t))/(9*pi) - (4*A*cos(11*t))/(11*p
i)
```

Make numeric and print to 4 sig. figs.

In [28]:

```
ft_num = subs(ft,A,1.0);
ft_num = vpa(ft_num, 4)
```

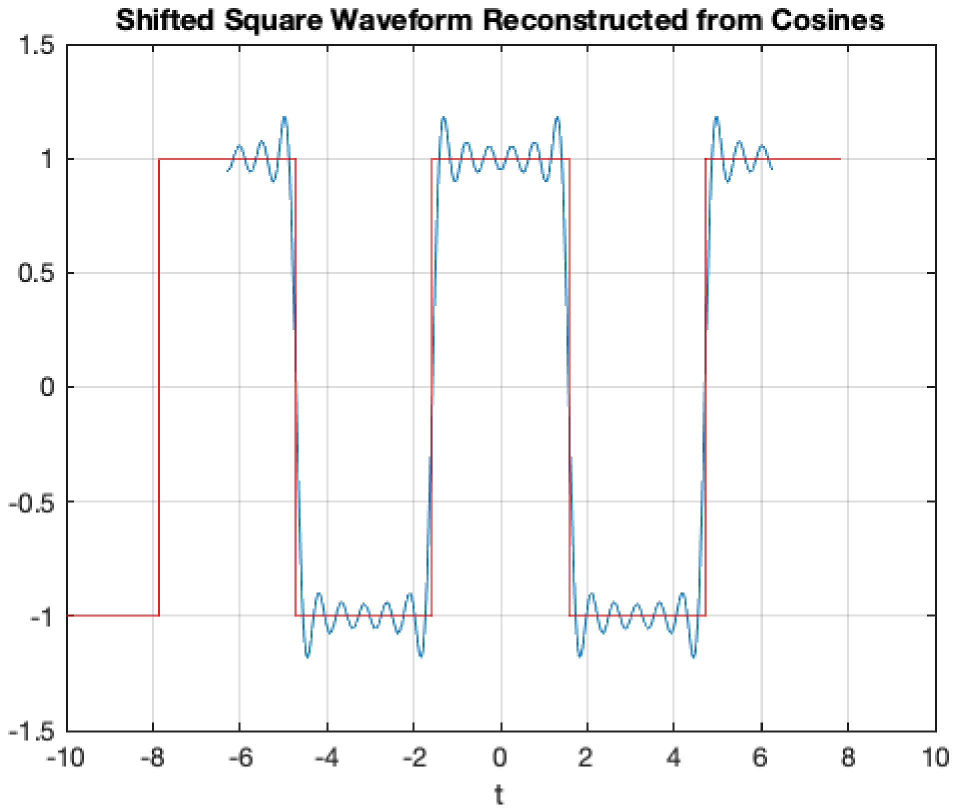
```
ft_num =
0.1415*cos(9.0*t) + 0.2546*cos(5.0*t) - 0.1157*cos(11.0*t) - 0.4244*
cos(3.0*t) - 0.1819*cos(7.0*t) + 1.273*cos(t)
```

plot result and overlay original signal (we could use `heaviside` for this as well.

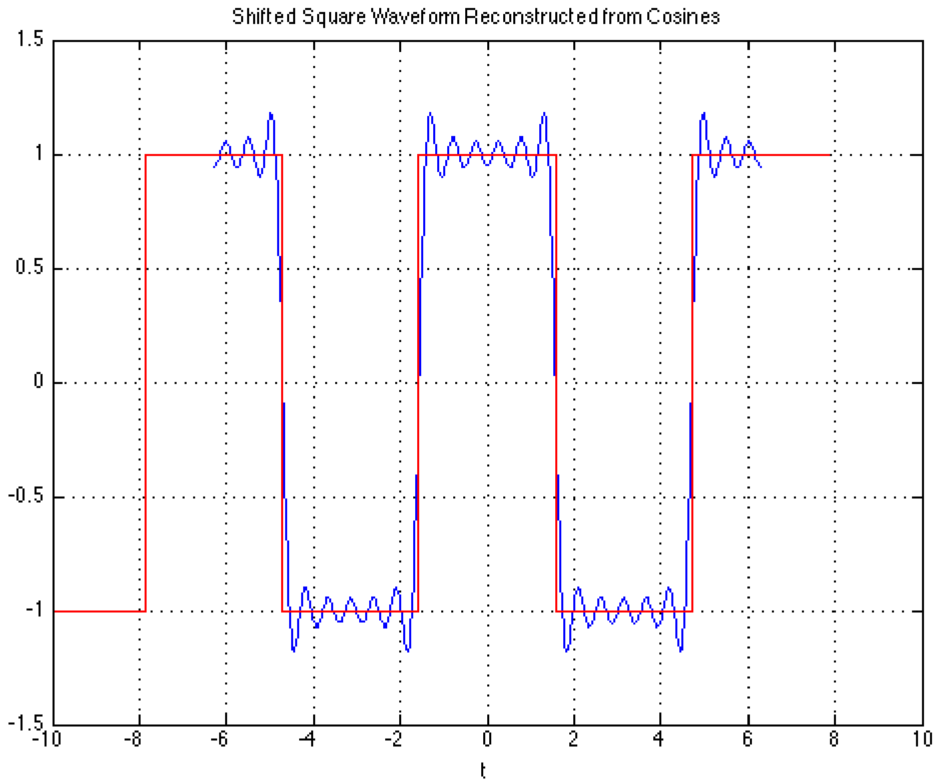
In [29]:

```
ezplot(ft_num)
hold on

t = [-3,-2,-2,-2,-1,-1,-1,0,0,0,1,1,1,2,2,2,3]*pi;
f = [-1,-1,0,1,1,0,-1,-1,0,1,1,0,-1,-1,0,1,1];
plot(t-pi/2,f,'r-')
axis([-10,10,-1.5,1.5])
grid
title('Shifted Square Waveform Reconstructed from Cosines')
hold off
```



- As before $a_0 = 0$
- We observe that this function is even, so all b_k coefficents will be zero
- The waveform has half-wave symmetry, so only odd indexed coefficents will be present.
- Further more, because it has half-wave symmetry we can just integrate from $0 \to \pi/2$ and multiply the result by 4.



Note that the coefficients match those given in the textbook (Section 7.4.2).

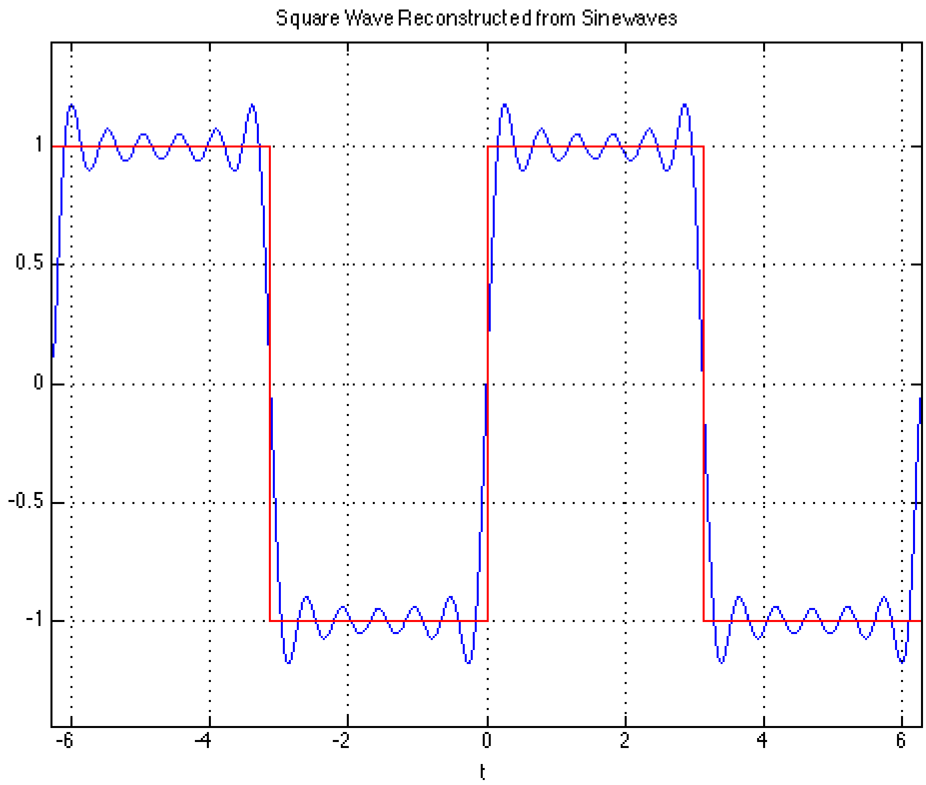
$$f(t) = \frac{4A}{\pi}\left(\cos\Omega_0 t - \frac{1}{3}\cos 3\Omega_0 t + \frac{1}{5}\cos 5\Omega_0 t - \cdots\right) = \frac{4A}{\pi}\sum_{n=\mathrm{odd}}(-1)^{\frac{n-1}{2}}\frac{1}{n}\cos n\Omega_0 t$$

Gibbs Phenomenon

In an earlier slide we found that the trigonometric for of the Fourier series of the square waveform is

$$f(t) = \frac{4A}{\pi}\left(\sin \Omega_0 t + \frac{1}{3}\sin 3\Omega_0 t + \frac{1}{5}\sin 5\Omega_0 t + \cdots\right) = \frac{4A}{\pi}\sum_{n=\mathrm{odd}} \frac{1}{n}\sin n\Omega_0 t$$

This figure shows the approximation for the first 11 harmonics:



As we add more harmonics, the sum looks more and more like a square wave. However the crests do not become flattened; this is known as *Gibbs Phenomenon* and it occurs because of the discontinuity of the perfect sqare waveform as it changes from $+A$ to $-A$ and *vice versa*.