Assignment 8

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1 RC Circuit

1.1 Question 1 (A) Charging of RC Circuit

In presence of external voltage source, if capacitor is initially having no charge then it gets charged. How that happens and at what rate that happens can be calculated as shown below.

For Series RC circuit we can always apply Kirchoff's Voltage Law to analyse circuit parameters. Applying KVL for circuit shown in figure(1)

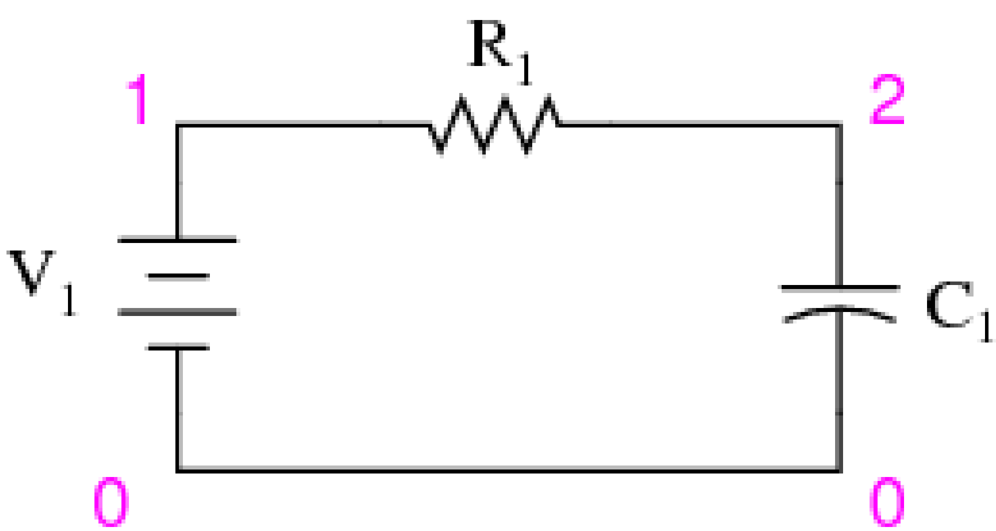

Figure 1: RC Charging Circuit

Suppose charge on Capacitor at any moment is $Q(t)$ and current through circuit is $I(t)$ then,

$$V_C + V_R = V_{ext}$$

$$V_C = \frac{Q}{C}$$

$$V_R = IR$$

$$V_R = R\frac{dQ}{dt}$$

$$R\frac{dQ}{dt} + \frac{Q}{C} = V_{ext}$$

At $t = 0$ charge on capacitor is 0 and it gradually charges to a maximum value say Q_0 then at that time, Capacitor acts as open circuit. No current passes through resistor and $V_C = V_{ext}$

$$V_{ext} = \frac{Q_0}{C}$$

$$R\frac{dQ}{dt} + \frac{Q}{C} = \frac{Q_0}{C}$$

$$\frac{dQ}{dt} = \frac{Q_0 - Q}{RC} \tag{1}$$

$$\frac{dQ}{Q - Q_0} = -\frac{dt}{RC} \tag{2}$$

1.1.1 Numerical Solution

We want to solve equation (1) numerically. For this we will use MATLAB's ode45 ODE solver. To check correctness of our numerical solution we will plot it against the analytical solution which can be obtained from solving equation (2).

1.1.2 Analytical Solution

$$\frac{dQ}{Q - Q_0} = -\frac{dt}{RC}$$

Integrating on both sides, we get,

$$\int_0^Q \frac{dQ}{Q - Q_0} = -\int_0^t \frac{dt}{RC}$$

$$ln(\frac{Q - Q_0}{-Q_0}) = -\frac{t}{RC}$$

$$Q - Q_0 = -Q_0 e^{(-\frac{t}{RC})}$$

$$Q = Q_0 \left(1 - e^{-\frac{t}{RC}}\right) \tag{3}$$

Once charge on capacitor is known, we can get the voltage across capacitor and then current in the circuit can be calculated as follows

$$V_C = \frac{Q(t)}{C}$$

$$V_C = V_{ext} \left(1 - e^{-\frac{t}{RC}}\right) \tag{4}$$

$$V_R = V_{ext} - V_C$$

$$I(t)R = V_{ext} - V_C$$

$$I(t) = \frac{V_{ext} - V_C}{R}$$

$$I(t) = \frac{dQ}{dt} = \frac{Q_0}{RC} e^{-\frac{t}{RC}} \tag{5}$$

1.1.3 Results and Graphs

Suppose for simulation purpose, Capacitance $C = 0.01F$, Resistance $R = 50\Omega$ and $V_{ext} = 10V$ Then the plots of (1) charge on capacitor, (2) Voltage across capacitor, (3) Current through circuit, (4) Analytical solution and Numerical solution

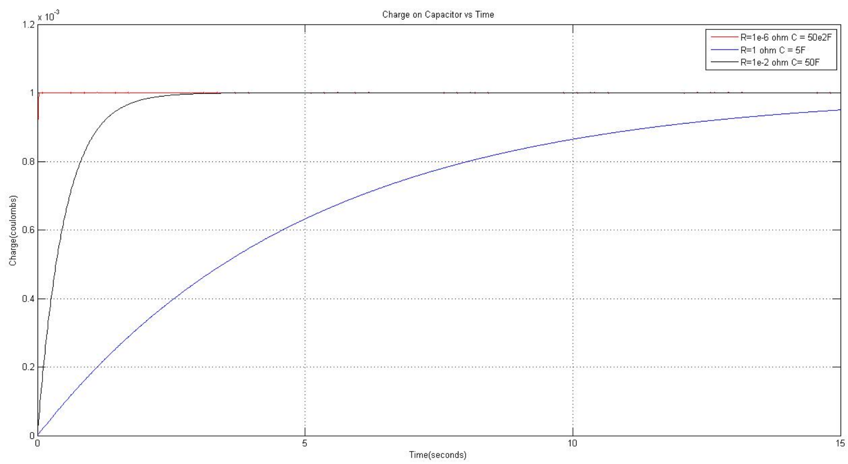

Figure 2: Charge on capacitor

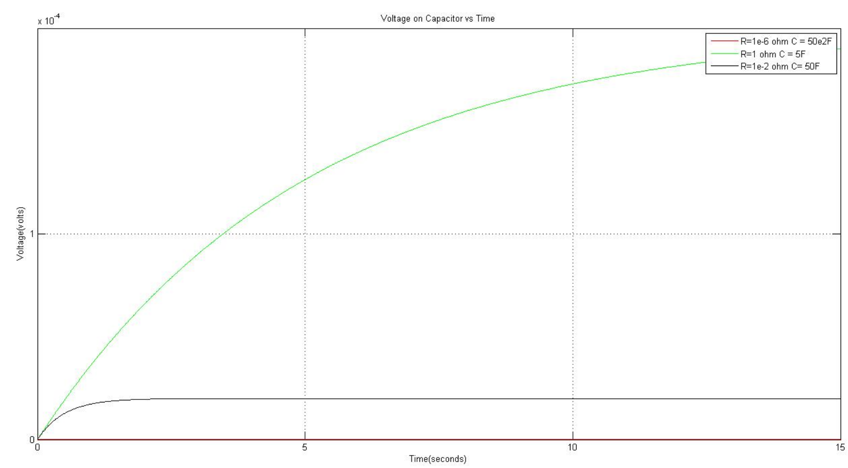

Figure 3: Voltage across capacitor

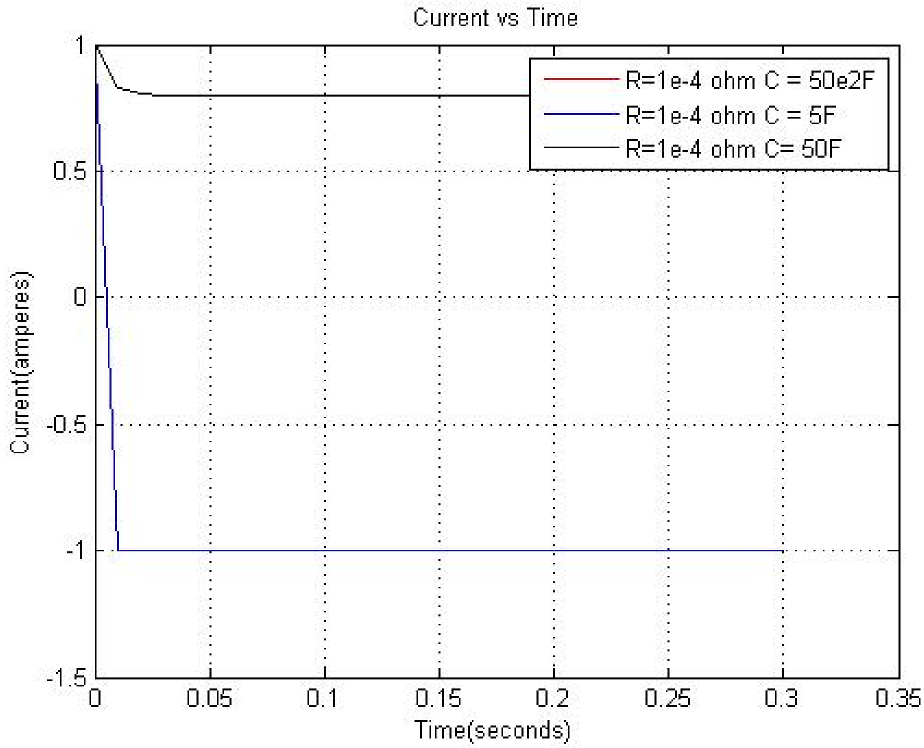

Figure 4: Current in circuit

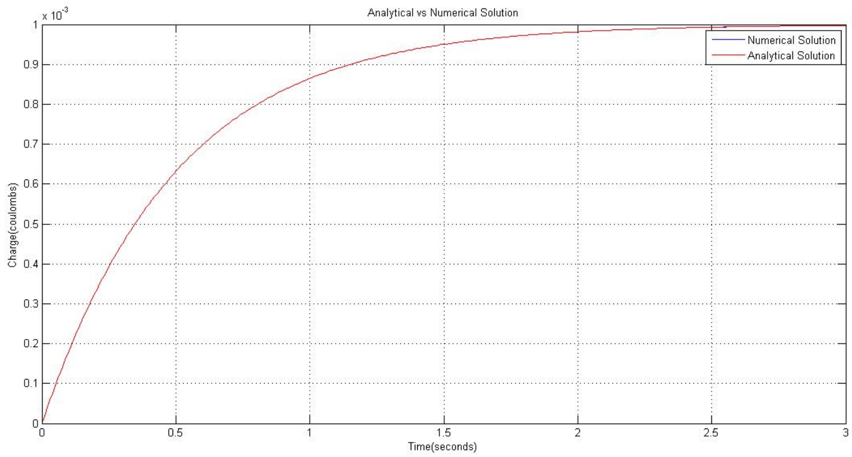

Figure 5: Analytical Solution vs Numerical Solution (Error)

From figure(5), it can be seen that for step size $dt = 0.01$ the numerical solution accurately approximates the analytical solution.

1.1.4 Dependence on values of R,C

As seen from the equation(5) (equation of $\frac{dQ}{dt}$), we can see that, the rate of accumulation of charge inversely depends on the product of resistance and capacitance i.e.,

$$\frac{dQ}{dt} \propto \frac{1}{RC}$$

So if the value of RC is less, then $\frac{dQ}{dt}$ is more and that RC charging circuit reaches steady state faster than the RC charging circuit having more RC value. Due to this fact, product RC is also known as time-constant τ of RC circuit.

This can be observed from the figure(2), in which the RC circuit with least RC product reaches steady state before than other circuits. Also approximately linear behaviour is observed in the RC circuits which has less time-constants.

1.2 Question 1 (B) Discharging RC Circuit

We will assume that there are no external voltage sources present and at $t = 0$ capacitor is having charge $Q = Q_0$. Applying KVL for the circuit shown in figure(6), we get

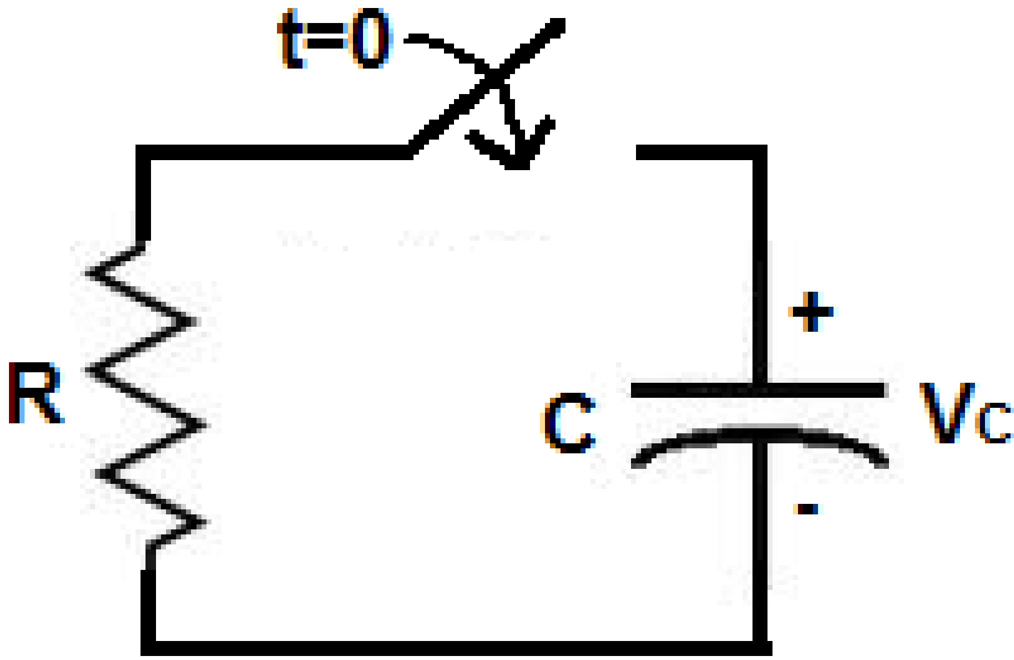

Figure 6: Discharging of capacitor circuit

$$V_C - V_R = 0$$

$$V_C = \frac{Q}{C}$$

$$V_R = IR$$

$$\frac{Q}{C} - IR = 0$$

But, $I = -\frac{dQ}{dt}$ as capacitor charge is decreasing.

$$\frac{Q}{C} + R\frac{dQ}{dt} = 0 \tag{6}$$

$$\frac{dQ}{dt} = -\frac{Q}{RC} \tag{7}$$

1.2.1 Numerical Solution

Equation(7) can be solved numerically using MATLAB's ode45 ODE solver. We will solve it analytically also and plot both solution to see how close they are.

1.2.2 Analytical Solution

Equation(7) can be re-written as follows after seperating variables

$$\frac{dQ}{Q} = -\frac{dt}{RC}$$

Integrating over $t = 0$ to $t = t$, we get limits of Q as $Q = Q_0$ at $t = 0$ and $Q = Q$ at $t = t$

$$\int_{Q_0}^{Q} \frac{dQ}{Q} = -\int_0^t \frac{dt}{RC}$$

$$ln(\frac{Q}{Q_0}) = -\frac{t}{RC}$$

$$Q = Q_0 e^{(-\frac{t}{RC})} \tag{8}$$

Voltage across Capacitor as a function of time can be calculated as

$$V_C = \frac{Q(t)}{C}$$

$$V_C = \frac{Q_0 e^{(-\frac{t}{RC})}}{C} \tag{9}$$

Voltage across Resistor as a function of time can be calculated as

$$V_R = V_C$$

$$V_R = \frac{Q_0 e^{(-\frac{t}{RC})}}{C} \tag{10}$$

Current through Resistor as a function of time can be calculated from voltage across resistor,

$$IR = -\frac{Q_0 e^{(-\frac{t}{RC})}}{C}$$

$$I = \frac{1}{RC} Q_0 e^{(-\frac{t}{RC})} \tag{11}$$

1.2.3 Results and Graphs

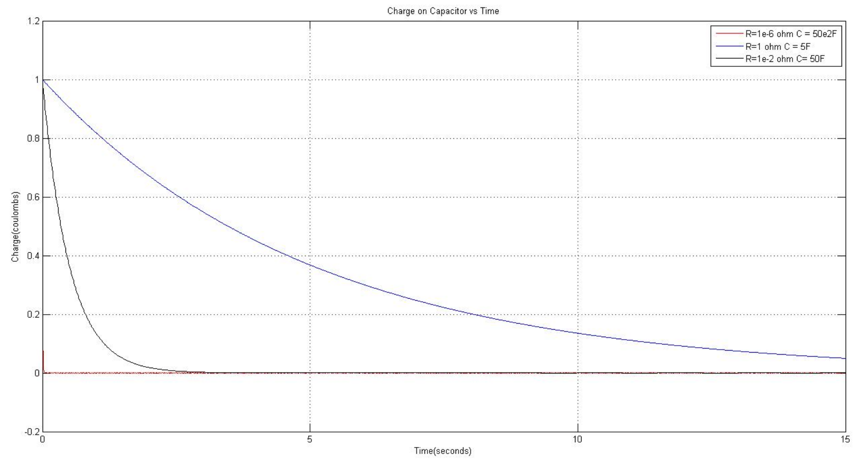

Figure 7: Charge on capacitor

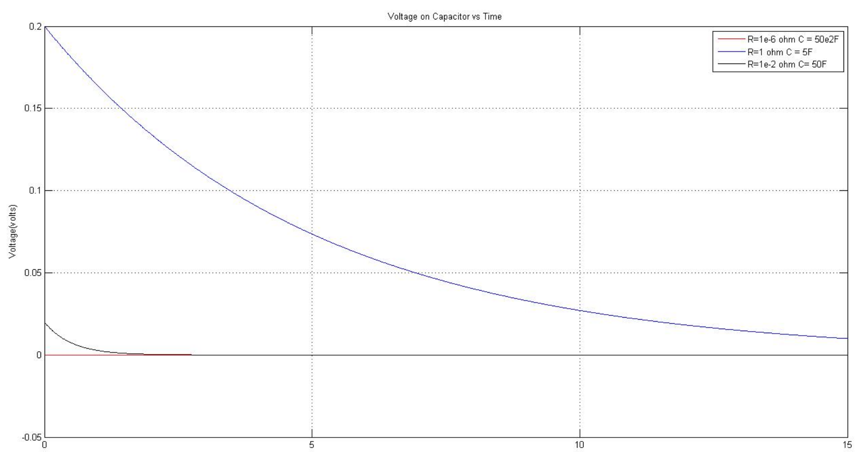

Figure 8: Voltage across capacitor

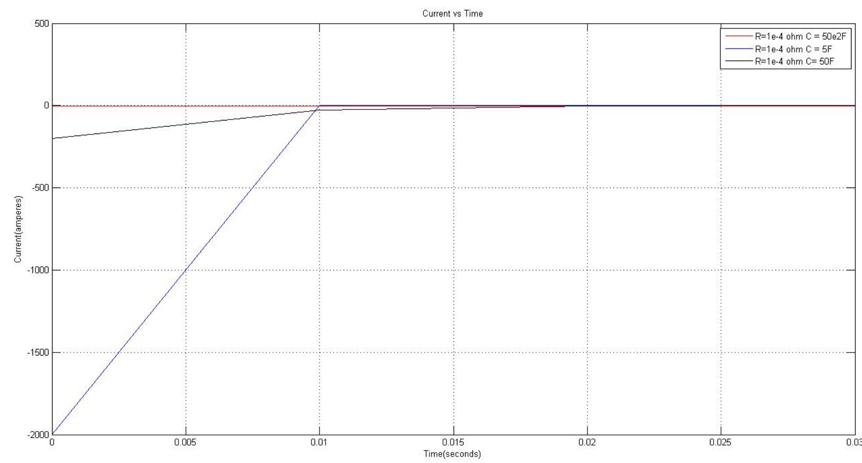

Figure 9: Current in circuit

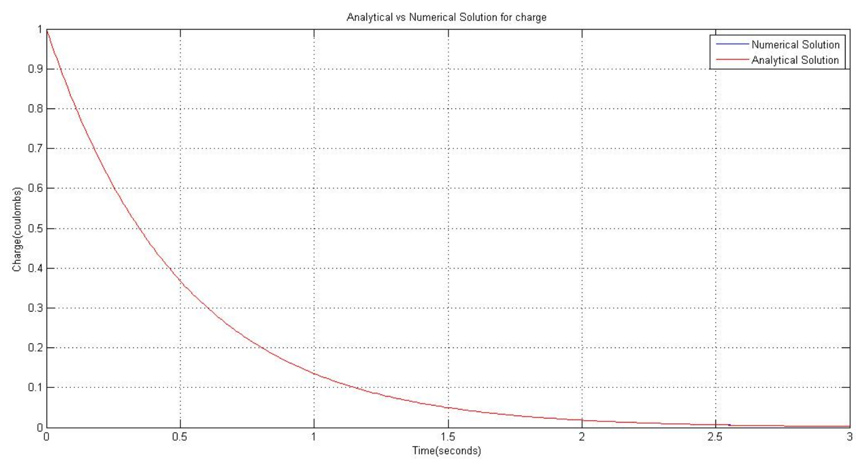

Figure 10: Analytical Solution vs Numerical Solution (Error)

Figure(7) shows how the charge on capacitor varies with time. Figure(8) shows how the voltage across capacitor vaires with time. Figure(9) shows the variation of current through the circuit and figure(10) compares accuracy of numeical and analytical solution. The nature of Discharging circuit can be explained as follows:

1) Resistor provides a dissipative medium for capacitor through which the electrical energy is dissipated in form of heat energy and thus the charge on capacitor decreases.
2) Current in the circuit also decreases because the voltage which drives current, i.e., capacitor voltage is also decreasing. Eventually leading to zero current in circuit.

From figure(10), it can be seen that for step size $dt = 0.01$ the numerical solution accurately approximates the analytical solution.

1.2.4 Dependence on values of R,C

As seen from the equation(11) (equation of I or $\frac{dQ}{dt}$), we can see that, the rate of discharging of capacitor charge inversely depends on the product of resistance and capacitance i.e.,

$$\frac{dQ}{dt} \propto \frac{1}{RC}$$

So if the value of RC is less, then $\frac{dQ}{dt}$ is more and that RC discharging circuit zero potential faster than the RC discharging circuit having more RC value.

This can be observed from the figure(7), in which the RC circuit with least RC product reaches zero potential before than other circuits. Also approximately linear behaviour is observed in the RC circuits which has less RC value.

2 RL Circuits

Circuit diagram of RL Circuit is shown in figure(11). As shown in circuit, a DC voltage source of potential V_s is applied to Resistor and Inductor connected in series with each other.

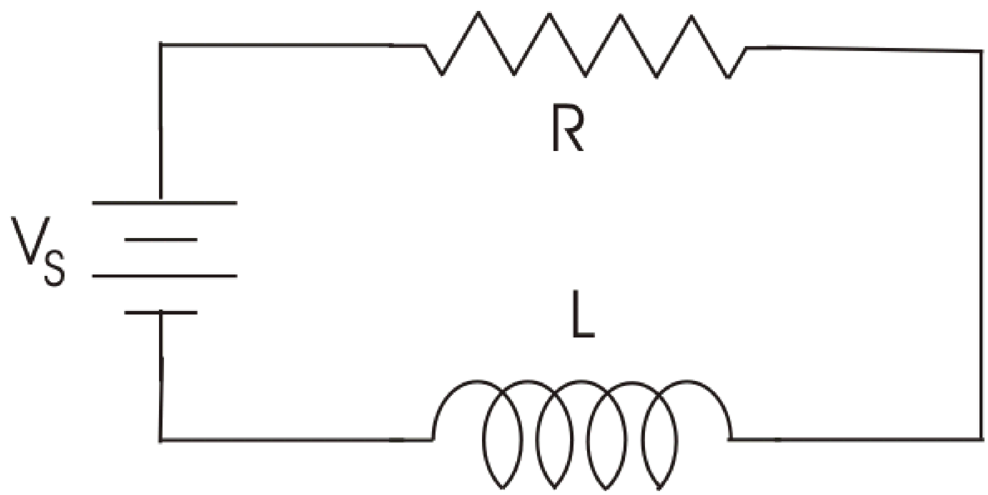

Figure 11: RL Circuit

At $t = 0$ when external voltage is applied, Inductor will oppose flow of current in circuit and acts as infinite resistor which leads to zero current in the circuit at $t = 0$. Gradually current increases and when steady state is obtained, Inductor behaves as zero resistance and maximum current $I = I_0$ flows through the circuit.

$$I_0 = \frac{V_s}{R}$$

Or we can write,

$$V_s = I_0 R \tag{12}$$

Applying KVL for circuit in figure(11)

$$V_s = V_R + V_L$$

$$V_R = IR$$

$$V_L = L\frac{dI}{dt}$$

$$V_s = IR + L\frac{dI}{dt}$$

$$V_s - IR = L\frac{dI}{dt}$$

$$\frac{V_s - IR}{L} = \frac{dI}{dt}$$

Using equation(12), we can write

$$\frac{I_0R - IR}{L} = \frac{dI}{dt}$$

$$\frac{dI}{dt} = (I_0 - I)\frac{R}{L} \tag{13}$$

2.1 Numerical Solution

Once we have ODE for I we can use MATLAB's ode45 ODE solver to get the current I in the circuit as a function of time. Also we will try to find analytic solution for this ODE and plot it against numerical solution to get idea of how close both solutions are.

2.2 Analytical Solution

Equation (13) can be re-writren as

$$\frac{dI}{I - I_0} = -\frac{R}{L}dt \tag{14}$$

Integrating on both sides of equation (14) we can get $I(t)$

$$\int_0^I \frac{dI}{I - I_0} = -\frac{R}{L}\int_0^t dt$$

$$ln(\frac{I - I_0}{-I_0}) = -\frac{R}{L}t$$

$$I = I_0(1 - e^{(-\frac{Rt}{L})}) \tag{15}$$

Equation(15) represents the variation of current with time. Voltage across resistor V_R can be calculated as follows

$$V_R = IR$$

$$V_R = I_0R(1 - e^{(-\frac{Rt}{L})})$$

Using equation(12), we can write

$$V_R = V_s(1 - e^{(-\frac{Rt}{L})}) \tag{16}$$

Voltage across inductor V_L can be simply calculated from KVL as follows

$$V_L = V_s - V_R$$

From equation(16),

$$V_L = V_s - V_s(1 - e^{(-\frac{Rt}{L}))})$$

$$V_L = V_s e^{(-\frac{Rt}{L})} \tag{17}$$

Once Voltage across Inductor is known then one can simply find rate of change of current i.e., $\frac{dI}{dt}$ as follow

$$L\frac{dI}{dt} = V_L$$

$$\frac{dI}{dt} = \frac{V_L}{L}$$

$$\frac{dI}{dt} = I_0\frac{R}{L}e^{(-\frac{Rt}{L})} \tag{18}$$

2.3 Results and Graphs

Using KVL we can easily find everything once we are able to measure $I(t)$. So our computational cost is reduced and we only need to solve equation(13) computationally.

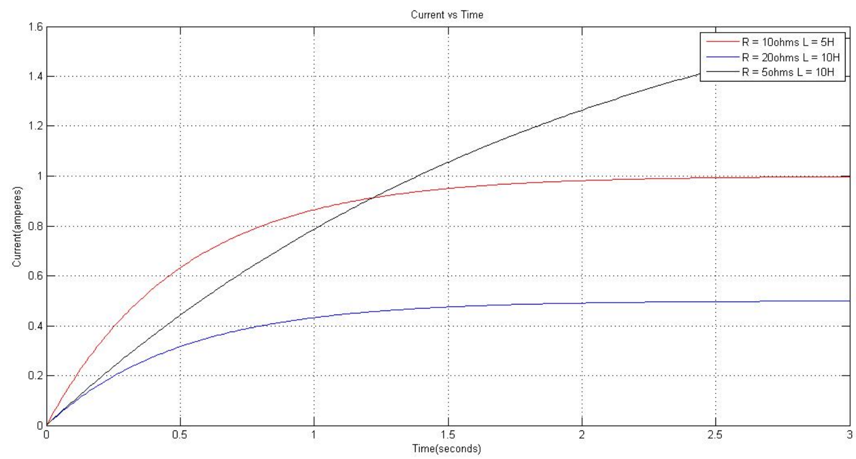

Figure 12: Current in circuit

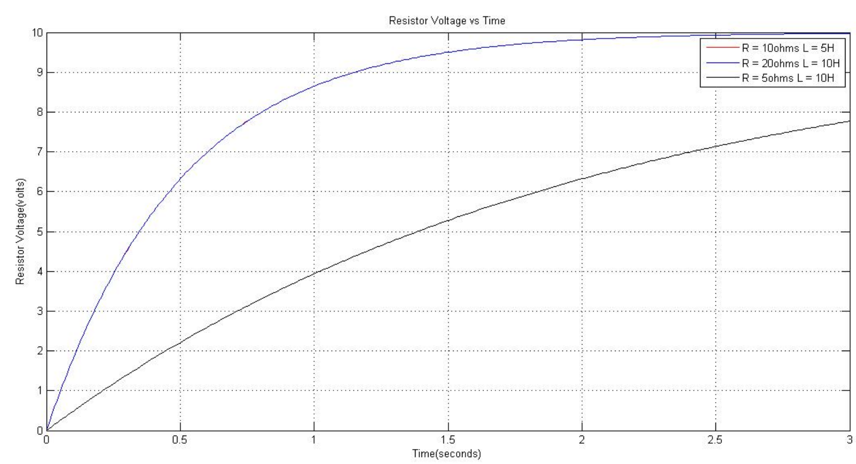

Figure 13: Voltage across Resistor

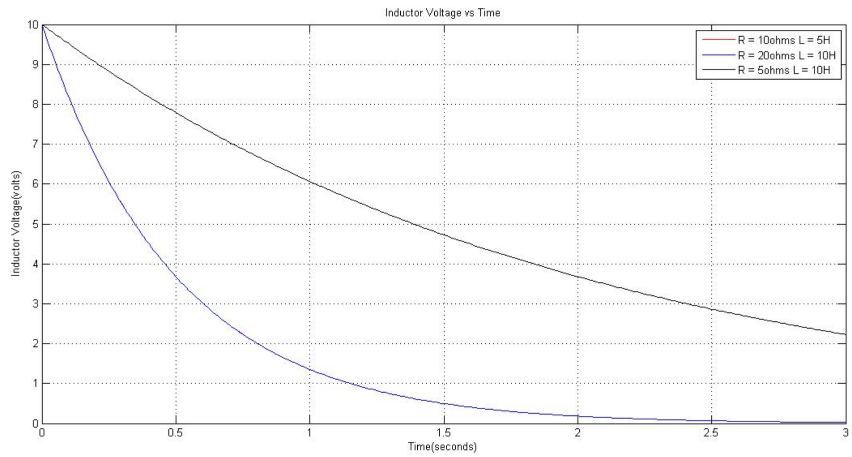

Figure 14: Voltage across Inductor

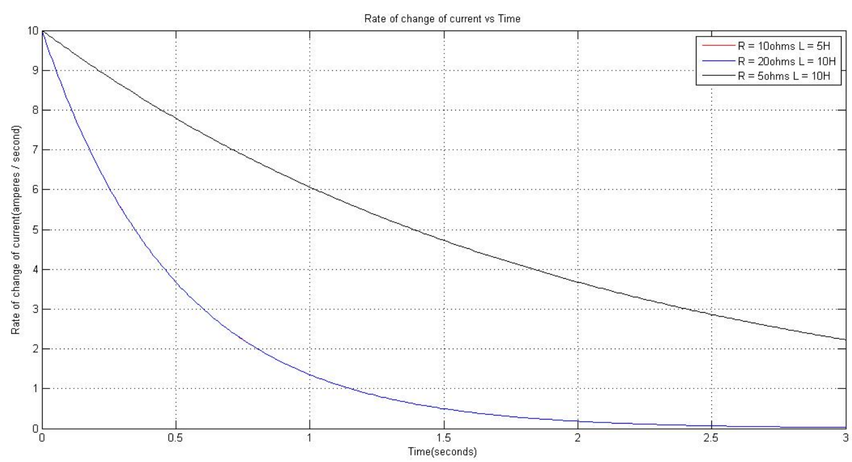

Figure 15: Rate of change of current in circuit

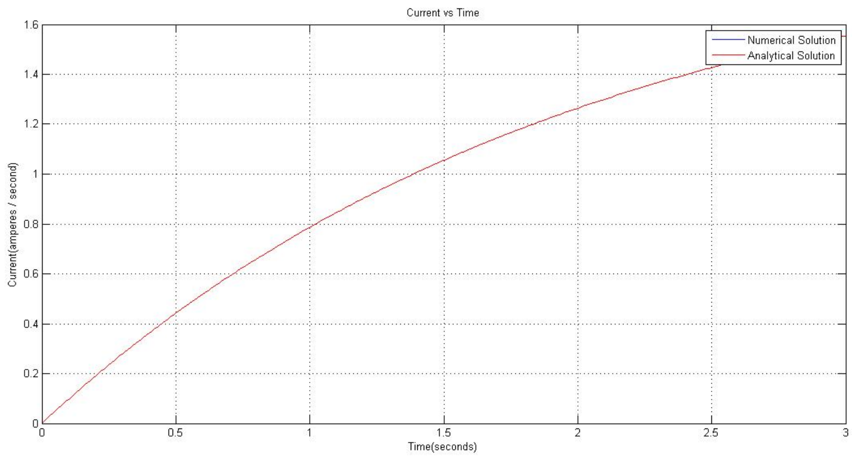

Figure 16: Numerical Solution vs Analytical Solution

2.4 Comparison with falling body problem

For a body, falling freely under the presence of earth's gravitational force in resistive medium, having small velocity, The equations of motion are

$$F_{net} = F_{gravity} - F_{drag}$$

Due to small velocity drag force can be approximated as, $F_{drag} = bv$ and equation of motion can be re-written as,

$$F_{net} = F_{gravity} - bv$$

$$m\frac{dv}{dt} = mg - bv$$

$$\frac{dv}{dt} = g - \frac{b}{m}v \tag{19}$$

If we analyse equation(19), we know that after sometime the falling body will attain steady state and move with constant velocity or zero acceleration state, (also known as terminal velocity) Also equation(19) is similar to the equation(1) and equation(13).

So if we are to draw comparison between falling body system, RC Charging circuit sysetm and RL charging circuit system, we can make following observation

In case of RC charging system, velocity v of falling body and charge q of capacitor, both play same role. Also the role of mass m in falling body system is analogous to that of resistance R. The role of drag coefficient b is also analogous to the role that inverse of capacitance C^{-1} in latter system.

In case of RL charging system, velocity v of falling body and current I in circuit, both play same role. Also the role of mass m in falling body system is analogous to that of inductor L. The role of drag coefficient b is also analogous to the role that resistance R plays in latter system.

3 LC Circuit

Applying Kirchoff's voltage law for the following circuit we get,

$$\frac{Q}{C} + L\frac{d^2Q}{dt^2} = 0$$

$$\frac{d^2Q}{dt^2} = -\frac{Q}{LC} \tag{20}$$

3.1 Numerical Solution

Equation(20) is a second order ordinary differential equation.This second order ode can be decomposed into two first order odes and then we can combine both to get $Q(t)$. This can be achieved as follow,

$$\frac{dQ}{dt} = I$$

$$\frac{dI}{dt} = -\frac{Q}{LC}$$

Initially current in the circuit at $t = 0$ is $I = 0$ and charge $Q = Q_0$. Once initial conditions are known and odes are known we can use MATLAB's ode45 ode solver to get the variations of charge and current i.e., $Q(t)$ and $I(t)$ simultaneously.

3.2 Analytical Solution

Equation(20) is analogous to differential equation of "Simple Harmonic Motion" and general solution to this would be

$$Q = A\sin(\omega t) + B\cos(\omega t)$$

From given initial conditions, charge at $t = 0$ is $Q = Q_0$ So solution would be of the form of,

$$Q = Q_0 \cos(\omega t) \tag{21}$$

$$I = -\frac{Q_0}{\sqrt{LC}} \sin(\omega t) \tag{22}$$

Here $\omega = \frac{1}{\sqrt{LC}}$ which is natural frequency of LC Circuit, the frequency with which charge oscillates in circuit.

Voltage across capacitor

$$V_C = \frac{Q}{C}$$

$$V_C = \frac{Q_0}{C} \cos(\omega t) \tag{23}$$

Voltage across Inductor, from KVL

$$V_L = -V_c$$

$$V_L = -\frac{Q_0}{C} \cos(\omega t) \tag{24}$$

$$V_L = L\frac{dI}{dt}$$

$$\frac{dI}{dt} = \frac{V_L}{L}$$

$$\frac{dI}{dt} = -\frac{Q_0}{LC} \cos(\omega t) \tag{25}$$

3.3 Results and Graphs

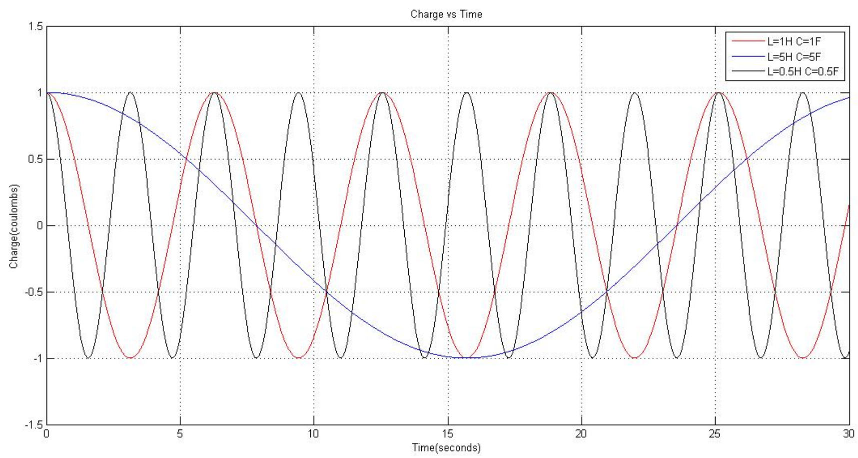

Figure 17: Charge on capacitor

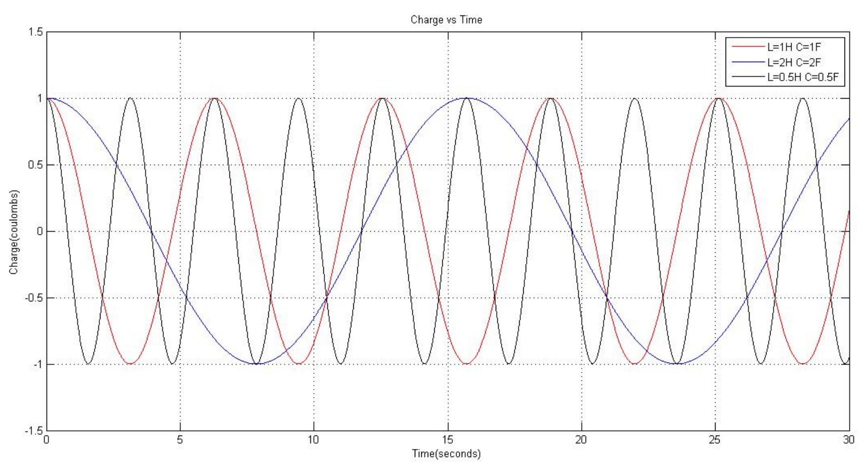

Figure 18: Voltage across Resistor

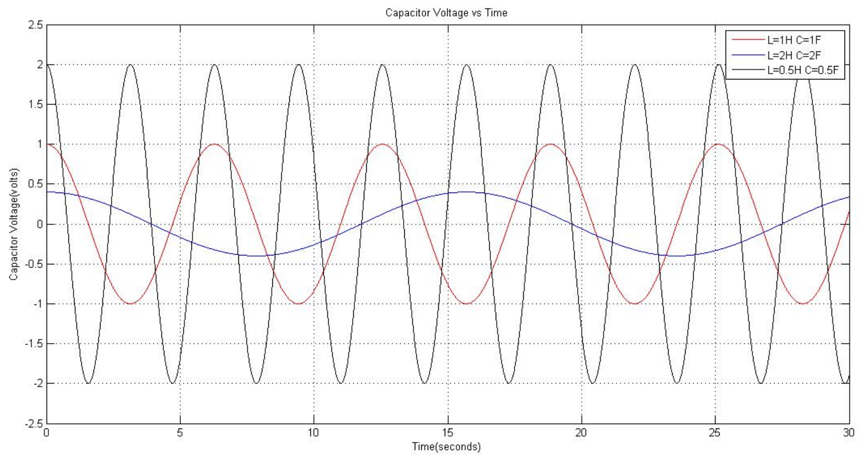

Figure 19: Voltage across Capacitor

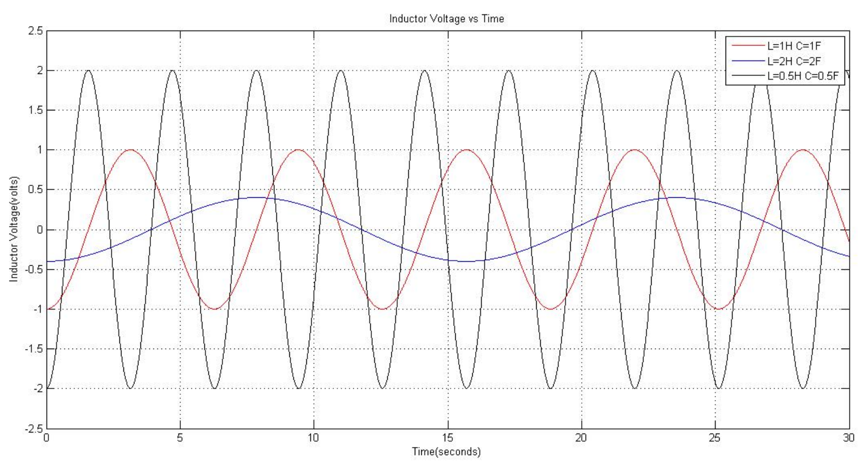

Figure 20: Voltage across Inductor

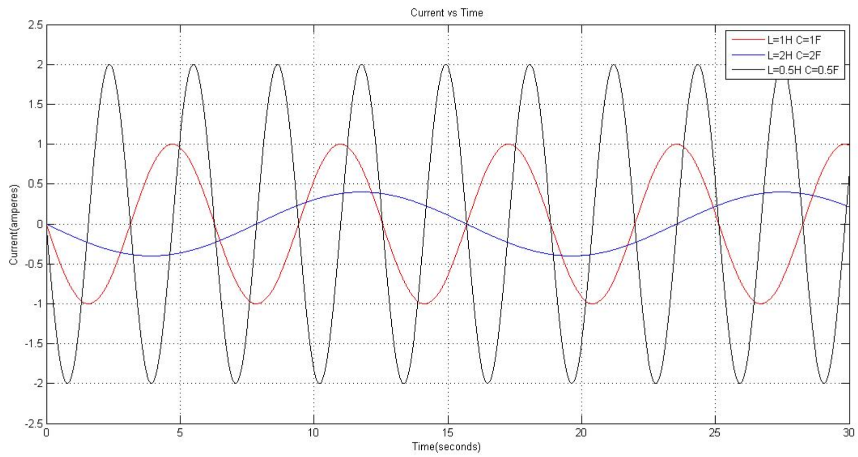

Figure 21: Rate of change of current in circuit

3.4 Effect of changing values of L and C

As seen from equation(21), the frequency with which the charge oscillates is ω and

$$\omega = \frac{1}{\sqrt{LC}}$$

So, if value of LC is more then value of ω is less and time-period is $T = \frac{2\pi}{\omega}$ is more. Also if value of LC is less than value of ω is more and time-period is less and oscillations are fast. This fact is also incorporated in figures 17 to 21.

4 RLC Series Circuit

Circuit Diagram of RLC Series Circuit is shown in figure(22). As shown in figure(22), an external AC voltage source with amplitude V_m and frequency ω_m is applied to series connection of Resistor, Inductor and Capacitor.

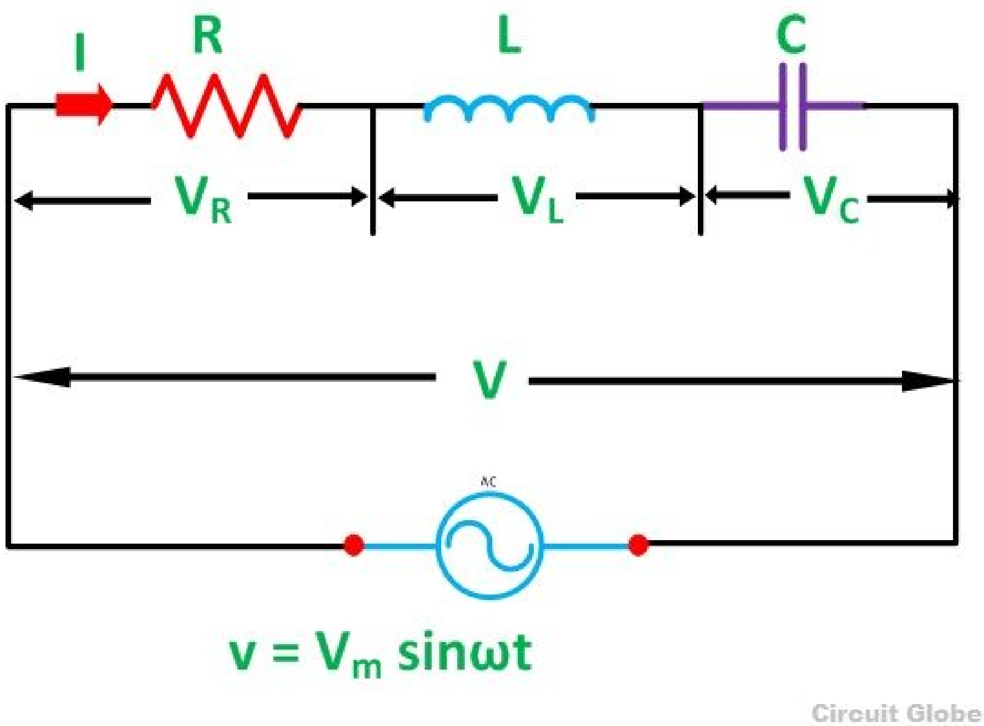

Figure 22: RLC Series Circuit with an external AC Voltage source

Assuming that value of resistor does not change with temprature, we can have following KVL Loop equation

$$V_R + V_L + V_C = V_S$$

$$IR + L\frac{dI}{dt} + \frac{Q}{C} = V_m \sin(\omega_m t)$$

Replacing I by $\frac{dQ}{dt}$ in above equation we get

$$R\frac{dQ}{dt} + L\frac{d^2Q}{dt^2} + \frac{Q}{C} = V_m \sin(\omega_m t)$$

$$L\frac{d^2Q}{dt^2} = V_m \sin(\omega_m t) - R\frac{dQ}{dt} - \frac{Q}{C}$$

$$\frac{d^2Q}{dt^2} = \frac{V_m}{L}\sin(\omega_m t) - \frac{R}{L}\frac{dQ}{dt} - \frac{Q}{LC} \tag{26}$$

4.1 Analytical Solution

Equation(24) is a second order ordinary differential equation with presence of both first order derivative and dependent variable itself. Solving such equations analytically requires a lot of calculation related to complex numbers.

4.2 Numerical Solution

Once the initial values of all dependent variables are known (Here charge and current) then solving equations of type similar to that of equation(26) numerically turns out to be very easy. So with the help of MATLAB's ode45 ODE solver we get following graphs depicting the variations of charge and current in transient and steady state. So this methodology is also applicable for completely different length scaled, time scaled systems for example, motion of an oscillator in presence of external force with damping force present.

4.3 Results and Graphs

4.3.1 Transient State Analysis

Figure(23) and figure(24) represent transition of RLC Series circuit from its natural frequency to the frequency of external AC source.

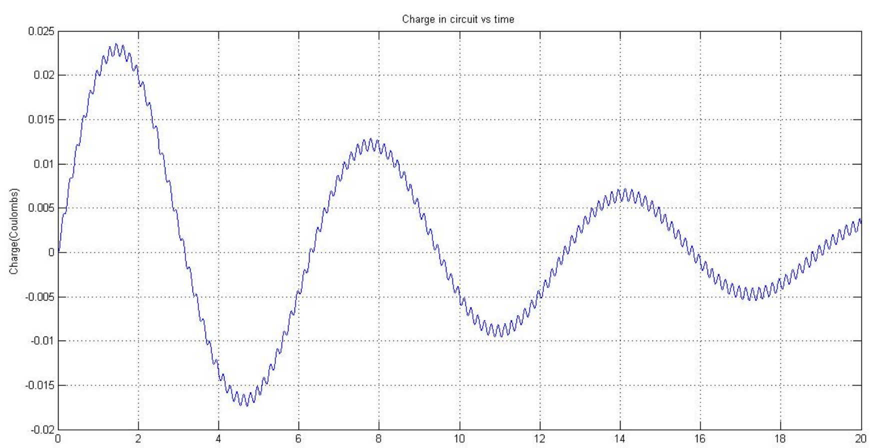

Figure 23: Charge in RLC Series circuit in presence of external AC source

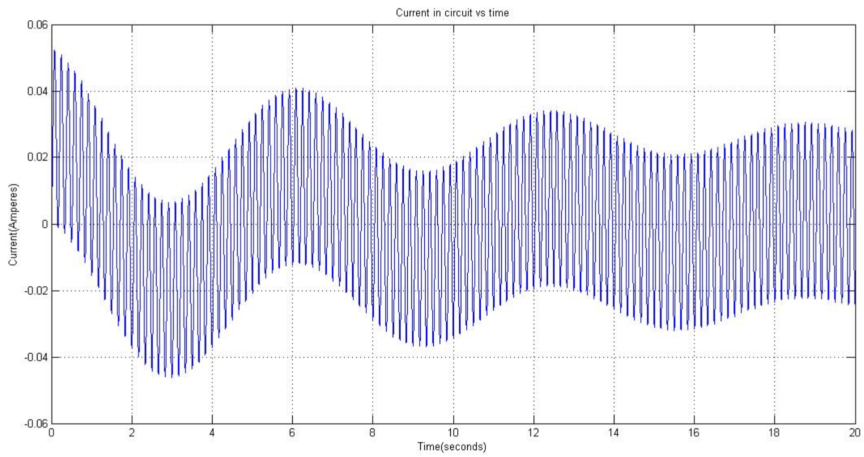

Figure 24: Current in RLC Series circuit in presence of external AC source

4.3.2 Steady State Analysis

It turns out that system requires some time to settle down and after that time, the charge and current in the system oscillates with same frequency as that of external AC source. This is also observed when an oscillator is oscillating in presence of an external sinusoidal force.

Figure(25) and figure(26) represents the above explained steady state nature of the system.

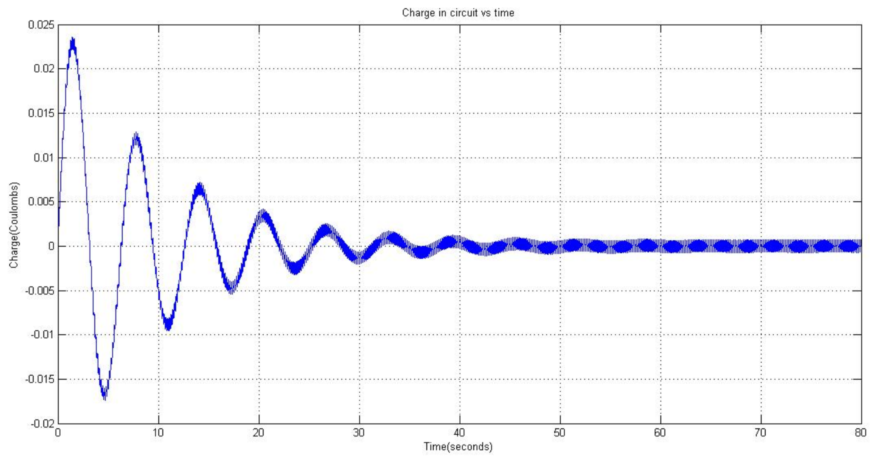

Figure 25: Steady state charge variation in RLC Series circuit in presence of external AC source

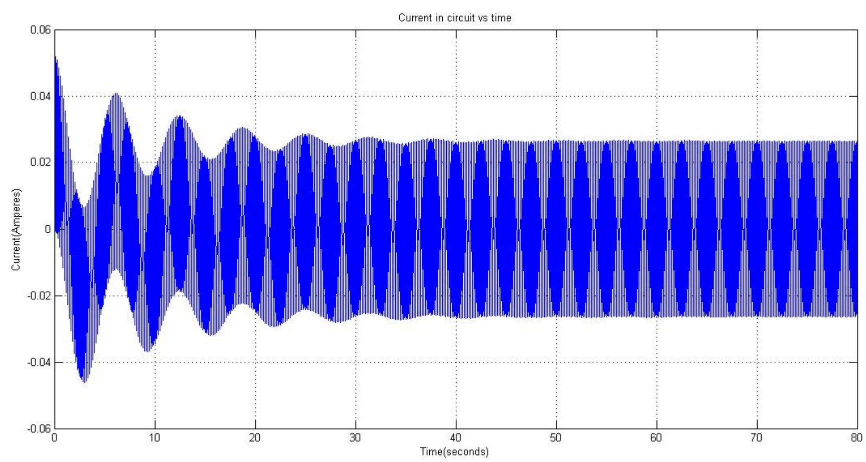

Figure 26: Steady state current variation in RLC Series circuit in presence of external AC source

4.4 Analogy to 1-D Oscillator

Equation of motion for 1-Dimensional Simple Harmonic Oscillator, On which an external sinusoidal force of frequency ω_m is aplied, in a medium of drag-coefficient b is,

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\cos(\omega_m t) \tag{27}$$

Comparing above equation(27) with equation (26), we get following observations

1) Both the equations belong to same kind.
2) Role of mass in equation(27) is played by inductor in equation(26)
3) Role of drag-coefficient in equation(27) is played by resistor in equation(26), in a way this is self-explanatory in the sense that they both tend to dissipate the energy of respective systems.
4) Role of spring-constant in equation(27) is played by inverse of capacitance in equation(26).
5) External voltage source is similar to that of external force, they both tend to give energy to system.

So, these both systems are exactly analogous and as we have seen effect of resonance in the forced-oscillator system, the same effect is observed here when the natural frequency of RLC series system $\omega_o = (LC)^{-\frac{1}{2}}$ equals to external voltage source's frequency.

Also two similar kind of regimes are observed here, one capacitor controlled (analogous to stiffness controlled) and one inductor controlled (analogous to mass controlled regime). Both are respectively called capacitive and inductive regimes.