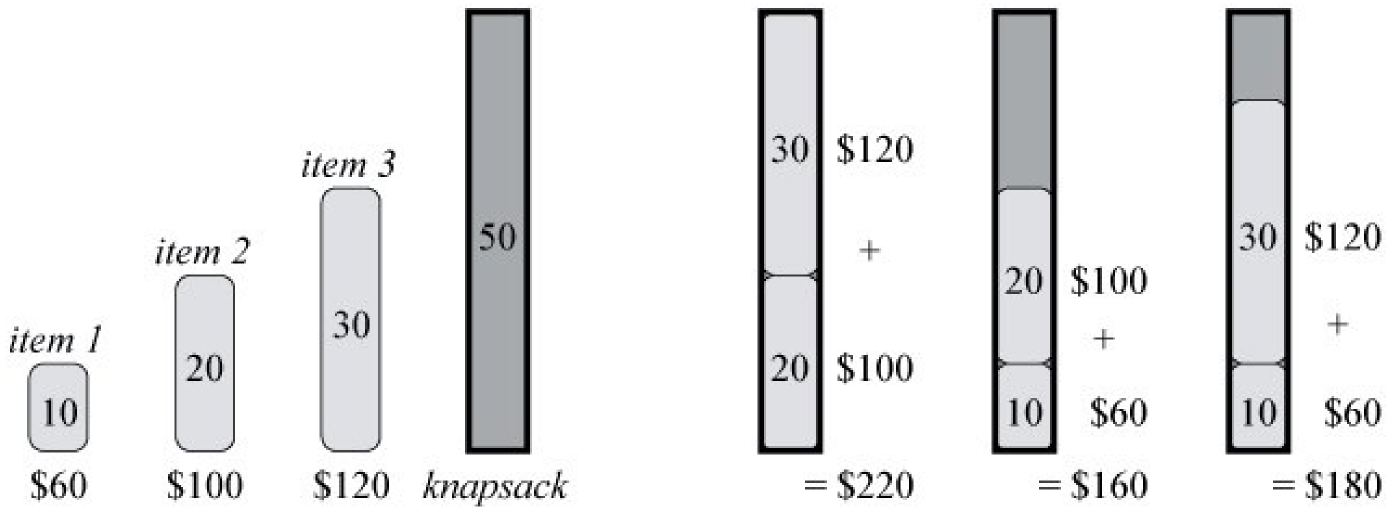
Lecture 15: 2D Dynamic Programming

2D because we use two-dimensional array to store solutions

The 0/1 Knapsack Problem



(a) (b)

Input: A set of n items,
where item i has weight w_i and value v_i,
and a knapsack with capacity W.

Goal: Find $x_1, \dots, x_n \in \{0,1\}$ satisfying $\sum_{i=1}^{n} x_i w_i \leq W$
that maximizes $\sum_{i=1}^{n} x_i v_i$.

Recall: Greedy doesn't provide optimal solution.

The Recurrence

First Attempt Definition: Let $V[w]$ be the largest obtainable value for a knapsack with capacity w.

First Attempt Recurrence:

If Optimal Solution for knapsack of size w chooses item i, remainder of optimal solution is optimal solution for subproblem of filling knapsack of size $w - w_i$ (1D solution coin denominations)

$$V[w] = \max(0,\ v_1 + V[w - w_1],\ v_2 + V[w - w_2], \dots,\ v_n + V[w - w_n])$$
$$V[j] = 0, j \leq 0$$

WRONG: This may pick the same item more than once! Non-legal Solution!

New 2D definition: Let $V[i, w]$ be the largest obtained value for a knapsack with capacity w, choosing ONLY from the first i items.

Recurrence:

Doesn't choose i

$$V[i, w] = \max(V[i-1, w],\ \ v_i + V[i-1, w - w_i])$$
$$V[i, w] = 0, i = 0 \text{ or } w = 0$$

Chooses i

So Far

Input: **A set of n items; item i has weight w_i and value v_i; a knapsack with capacity W.**
Goal: Find $x_1, \ldots, x_n \in \{0,1\}$ such that $\sum_{i=1}^{n} x_i w_i \leq W$ and $\sum_{i=1}^{n} x_i v_i$ is maximized.

Subproblem:

$V[i, w]$ is the largest obtained value for knapsack with capacity w, choosing ONLY from items 1 … i.

Recurrence: $V[i, w] = \max(V[i-1, w], \ v_i + V[i-1, w - w_i])$

With initial condition, $\forall i, \quad V[i, 0] = 0$

Find Order for filling in table: For $i = 1$ to n

For $w = 1$ to W

Required Solution: $V[n, W]$

DP is 2-Dimensional (2 variables) and not 1-D.

The Algorithm

```
let V[0..n, 0..W] be a new 2D array of all 0
for i ← 1 to n do
    for w ← 1 to W do
        if w[i] ≤ w and v[i] + V[i − 1, w − w[i]] > V[i − 1, w] then
            V[i, w] ← v[i] + V[i − 1, w − w[i]]
        else
            V[i, w] ← V[i − 1, w]
return V[n, W]
```

Running time: $\Theta(nW)$

Space: $\Theta(nW)$, but can be improved to $\Theta(n + W)$

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

$V[i, \text{w}]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50	90	90	90	90

Reconstructing the Solution

Idea: Remember the optimal decision for each subproblem in keep[i,j]

```
let V[0..n, 0..W] and keep[0..n, 0..W] be a new array of all 0
for i ← 1 to n do
    for w ← 1 to W do
        if w[i] ≤ w and v[i] + V[i − 1, w − w[i]] > V[i − 1, w] then
            V[i, w] ← v[i] + V[i − 1, w − w[i]]
            keep[i, w] ← 1
        else
            V[i, w] ← V[i − 1, w]
            keep[i, w] ← 0
K ← W
for i ← n downto 1 do
    if keep[i, K] = 1 then
        print i
        K ← K − w[i]
```

Running time: $\Theta(nW)$

Space: $\Theta(nW)$, cannot be improved to $\Theta(n+W)$ due to the $keep$ array.

Longest Common Subsequence

Problem: Given two sequences $X = (x_1, x_2, \ldots, x_m)$ and $Y = (y_1, y_2, \ldots, y_n)$, we say that $Z = (z_1, z_2, \ldots, z_k)$ is a common subsequence of X and Y if $x_{i_p} = y_{j_p} = z_p$ for all $p = 1, 2, \ldots, k$

where $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.

The goal is to find the **longest common subsequence** of X and Y.

Example:

X: A B A C B D A B

Y: B D C A B A

Z: B C B A

The Recurrence

Observation: The problem is equivalent to finding the maximum matching between X and Y such that matched pairs don't cross.

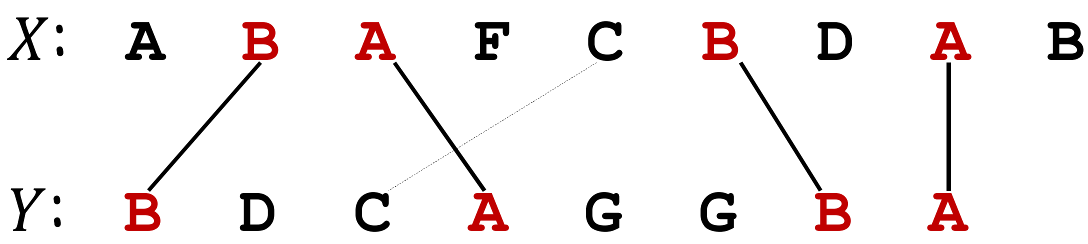
X: **A B A F C B D A B**

Y: **B D C A G G B A**

Z: **B C B A** is a solution

The Recurrence

Observation: The problem is equivalent to finding the maximum matching between X and Y such that matched pairs don't cross.



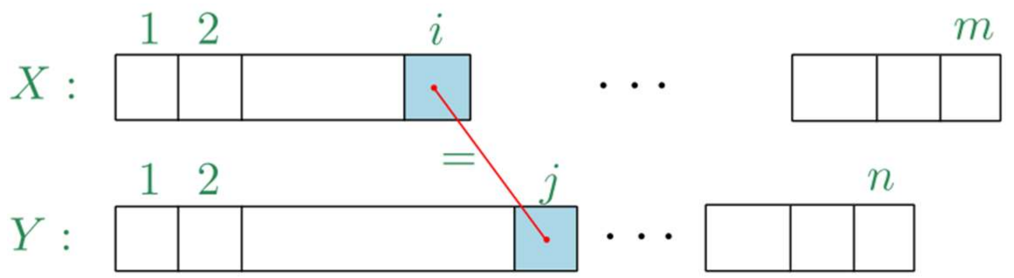
Z: **B C B A** is a solution

Z': **B A B A** is another legal solution

The Recurrence

Def: $c[i, j]$ is length of the longest common subsequence of $X[1..i]$ and $Y[1..j]$.

Observations: The problem is equivalent to finding the maximum matching between X and Y such that matched pairs don't cross.



The recurrence:

- Case 1: If $x_i = y_j$, then we match x_i and y_j.
- Case 2: If $x_i \neq y_j$, then either x_i or y_j is not matched.

  Optimal solution reduces to either $c[i-1, j]$ or $c[i, j-1]$.

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j \\ \max\{c[i, j-1], c[i-1, j]\} & \text{if } i, j > 0 \text{ and } x_i \neq y_j \end{cases}$$

The Recurrence and Algorithm

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1,j-1]+1 & \text{if } i,j > 0 \text{ and } x_i = y_j \\ \max\{c[i,j-1], c[i-1,j]\} & \text{if } i,j > 0 \text{ and } x_i \neq y_j \end{cases}$$

```
let c[0..m,0..n] and b[0..m,0..n] be new arrays of all 0
for i ← 1 to m
    for j ← 1 to n
        if x_i = y_j then
            c[i,j] ← c[i-1,j-1] + 1
            b[i,j] ← "↖"                         MATCH x_i, y_j
        else if c[i-1,j] ≥ c[i,j-1] then
            c[i,j] ← c[i-1,j]                    x_i not matched
            b[i,j] ← "↑"
        else
            c[i,j] ← c[i,j-1]
            b[i,j] ← "←"                         y_j not matched
Print-LCS(b,m,n)
```

Running time: $\Theta(mn)$

Space: $\Theta(mn)$, can be improved to $\Theta(m+n)$ if we only need to return the optimal length.

Reconstruct the Optimal Solution

```
Print-LCS(b,i,j):
    if i = 0 or j = 0 then return
    if b[i,j] = "↖" then
        Print-LCS(b,i − 1,j − 1)
        print x_i
    else if b[i,j] = "↑"
        Print-LCS(b,i − 1,j)
    else Print-LCS(b,i,j − 1)
```

i \ j		0	1	2	3	4	5	6
	y_j		B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2	B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3	C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4	B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	← 3
5	D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4

Value of b[i,j] indicates whether

↖: x_i, y_j matched:
then write x_i and return LCS (i-1,j-1)

↑: x_i not matched
skip x_i and return LCS (i-1,j)

←: y_j not matched
skip y_j and return LCS (i,j-1)

Longest Common Substring

Problem: Given two strings $X = x_1 x_2 \ldots x_m$ and $Y = y_1 y_2 \ldots y_n$, we wish to find their longest common substring Z, that is, the largest k for which there are indices i and j with $x_i x_{i+1} \ldots x_{i+k-1} = y_j y_{j+1} \ldots y_{j+k-1}$.

Ex:

X : **DEADBEEF**

Y : **EATBEEF**

Z : **BEEF** //pick the longest contiguous substring

Note: Brute-force algorithm takes $O(n^4)$ time.

Different from LCS problem because, in this problem, letters have to be together.

The Recurrence

Def: $d[i,j]$ = the length of the longest common substring of $X[1..i]$ and $Y[1..j]$. (Does this work?)

Def: $d[i,j]$ = the length of the longest common substring of $X[1..i]$ and $Y[1..j]$ that ends at x_i and y_j.

Q: Wait, are we changing the problem?

A: Yes, but it's OK. Optimal solution to the original is just $\max_{i,j}\{d[i,j]\}$

Recurrence:

- If $x_i = y_j$, then the LCS of $X[1..i]$ and $Y[1..j]$ is just the LCS of $X[1..i-1]$ and $Y[1..j-1]$, plus $x_i = y_j$
- If $x_i \neq y_j$, then there can't be a common substring ending at x_i and y_j!

$$\boxed{d[i,j] = \begin{cases} d[i-1,j-1]+1 & \text{if } x_i = y_j \\ 0 & \text{if } x_i \neq y_j \end{cases}}$$

The Algorithm

```
let d[0..m, 0..n] be a new array of all 0
l_m ← 0, p_m ← 0
for i ← 1 to m
    for j ← 1 to n
        if x_i = y_j then
            d[i, j] ← d[i − 1, j − 1] + 1
            if d[i, j] > l_m then
                l_m ← d[i, j]
                p_m ← i
for i ← p_m − l_m + 1 to p_m
    print x_i
```

Note: For this problem, reconstructing the optimal solution just needs the location of the LCS.

Running time: $\Theta(mn)$

Space: $\Theta(mn)$ but can be improved to $\Theta(m+n)$.

Exercise on Edit Distance

Given two strings s and t, the edit distance edit(s,t) is the smallest number of following edit operations to turn s into t:

Insertion: add a letter

Deletion: remove a letter

Substitution: replace a character with another one.

Example: s = abode and t = blog.

Then, edit(s,t) = 4 operations

Start from abode

1 delete a ⇒ bode

2 insert l after b ⇒ blode

3 delete d ⇒ bloe

4 substitute e with g ⇒ blog

Impossible to do so with at most 3 operations.

Exercise on Edit Distance (cont)

Let s and t be two strings with lengths m and n, respectively.

1 If $m = 0$, then $edit(s,t) = n$.

2 If $n = 0$, then $edit(s,t) = m$.

3 If $m > 0$, $n > 0$, and $s[m] = t[n]$, then $edit(s,t)$ is min(

$$1 + edit(s[1..m],t[1..n-1])$$
$$1 + edit(s[1..m-1],t[1..n])$$
$$edit(s[1..m-1],t[1..n-1]))$$

Explanation of Case 3

i] Delete t[n], and use the least number of edit operations to change s[1..m] into t[1..n – 1]. The total number of edit operations is therefore 1 + edit(s[1..m],t[1..n – 1]). (example: s:abc, t=abcc)

ii] Delete s[m], and use the least number of edit operations to change s[1..m – 1] into t[1..n]. The total number of edit operations is therefore 1 + edit(s[1..m – 1],t[1..n]). (example: s:abcc, t=abc)

iii] Simply change s[1..m – 1] into t[1..n – 1]. The total number of edit operations is therefore edit(s[1..m – 1],t[1..n – 1]).

Exercise on Edit Distance (cont2)

Let s and t be two strings with lengths m and n, respectively.

4 If $m > 0$, $n > 0$, and $s[m] \neq t[n]$, then edit(s,t) is min(

1 + edit(s[1..m],t[1..n − 1])

1 + edit(s[1..m − 1],t[1..n])

1 + edit(s[1..m − 1],t[1..n − 1]))

Lets store edit(i,j) in an array E[i,j]. Then E[i,j]=min(

1 + E[i,j-1]

1 + E[i-1,j]

E[i-1,j-1] if s[i] = t[j], or E[i-1,j-1]+1 if s[i] ≠ t[j])

DP Algorithm for filling array E

1 Fill in row 0 and column 0.

2 Fill in the cells of row 1 from left to right.

3 Fill in the cells of row 2 from left to right.

4 ...

5 Fill in the cells of row m from left to right.