MA(2) Model

$$r_t = \mu + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}, \quad Var(a_t) = \sigma_a^2$$

1. Stationarity: always stationary

2. Mean: $E(r_t) = \mu$

$$E(r_t) = E(\mu + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2})$$

$$= E(\mu) + \underbrace{E(\cancel{a_t})}_{=0} - \theta_1 \underbrace{E(\cancel{a_{t-1}})}_{=0} - \theta_2 \underbrace{E(\cancel{a_{t-2}})}_{=0}$$

$$= \mu$$

3. Variance:

$$Var(r_t) = Var(\cancel{\mu} + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2})$$

$$= Var(a_t) + \theta_1^2 Var(a_{t-1}) + \theta_2^2 Var(a_{t-2})$$

$$- 2\theta_1 \cancel{cov(a_t, a_{t-1})} = 0$$

$$- 2\theta_2 \cancel{cov(a_t, a_{t-2})} = 0$$

$$+ 2\theta_1\theta_2 \cancel{cov(a_{t-1}, a_{t-2})} = 0$$

$$= \sigma_a^2 + \theta_1^2 \sigma_a^2 + \theta_2^2 \sigma_a^2$$

$$= (1 + \theta_1^2 + \theta_2^2)\sigma_a^2$$

4. Autocovariance: $cov(a_k, a_j) = \begin{cases} 0, & \text{if } k \neq j \\ \sigma_a^2, & \text{if } k = j \end{cases}$

lag 1: $\operatorname{cov}(r_t, r_{t-1})$

$$= \operatorname{cov}(\cancel{\mu} + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2},\ \cancel{\mu} + a_{t-1} - \theta_1 a_{t-2} - \theta_2 a_{t-3})$$

$$= -\theta_1 \operatorname{cov}(a_{t-1}, a_{t-1}) + \theta_2\theta_1 \operatorname{cov}(a_{t-2}, a_{t-2})$$

$$= -\theta_1 \sigma_a^2 + \theta_2\theta_1 \sigma_a^2$$

$$= \theta_1(\theta_2 - 1)\sigma_a^2$$

lag 2: $\operatorname{cov}(r_t, r_{t-2})$

$$= \operatorname{cov}(\mu + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2},\ \mu + a_{t-2} - \theta_1 a_{t-3} - \theta_2 a_{t-4})$$

$$= -\theta_2 \operatorname{cov}(a_{t-2}, a_{t-2})$$

$$= -\theta_2 \sigma_a^2$$

lag l ($l \geq 3$): $\operatorname{cov}(r_t, r_{t-l})$

$$= \operatorname{cov}(\cancel{\mu} + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2},\ \cancel{\mu} + a_{t-l} - \theta_1 a_{t-l-1} - \theta_2 a_{t-l-2})$$

$$= 0$$

$l=3$	a_{t-3}	a_{t-4}	a_{t-5}
$l=4$	a_{t-4}	a_{t-5}	a_{t-6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$

5. Autocorrelation:

$$\rho_1 = \frac{\operatorname{cov}(r_t, r_{t-1})}{\operatorname{Var}(r_t)} = \frac{\theta_1(\theta_2 - 1)\cancel{\sigma_a^2}}{(1+\theta_1^2+\theta_2^2)\cancel{\sigma_a^2}}$$

$$\rho_2 = \frac{\operatorname{cov}(r_t, r_{t-2})}{\operatorname{Var}(r_t)} = \frac{-\theta_2 \cancel{\sigma_a^2}}{\ldots}$$

$Var(r_t) \quad (1+\theta_1^2+\theta_2^2)\sigma_a^2$

$$l \geq 3: \rho_l = \frac{cov(r_t, r_{t-l})}{Var(r_t)} = 0$$

MA(2) model has 2-period memory!

6. Forecast $\Rightarrow$ HW Problem Set 2

ARMA(1, 1)

$$r_t = \phi_0 + \phi_1 r_{t-1} + a_t - \theta_1 a_{t-1}, \quad Var(a_t) = \sigma_a^2$$

ARMA(1, 1): $\phi_1 \neq 0$, $\theta_1 \neq 0$

AR(1): $\phi_1 \neq 0$, $\theta_1 = 0$

MA(1): $\phi_1 = 0$, $\theta_1 \neq 0$

1. Stationarity: same as AR(1): $|\phi_1| < 1$

$$(1-\phi_1 B) r_t = \phi_0 + a_t - \theta_1 a_{t-1}$$

$$1 - \phi_1 x = 0 \Rightarrow x = \frac{1}{\phi_1}$$

$$|x| > 1 \Rightarrow |\phi_1| < 1$$

2. Invertibility: same as MA(1): $|\theta_1| < 1$

$$\text{ARMA}(1,1) \Rightarrow \text{AR}(\infty)$$

3. Mean : $\quad E(r_t) = \dfrac{\phi_0}{1-\phi_1}$

$$E(r_t) = E(\phi_0 + \phi_1 r_{t-1} + a_t - \theta_1 a_{t-1})$$

$$= E(\phi_0) + \phi_1 E(r_{t-1}) + \underbrace{\cancel{E(a_t)}}_{=0} - \theta_1 \underbrace{\cancel{E(a_{t-1})}}_{=0}$$

$$= \phi_0 + \phi_1 E(r_t)$$

$$\Rightarrow E(r_t) = \frac{\phi_0}{1-\phi_1}$$

4. Variance :

$$Var(r_t) = Var(\cancel{\phi_0} + \phi_1 r_{t-1} + a_t - \theta_1 a_{t-1})$$

$$= \phi_1^2 Var(r_{t-1}) + Var(a_t) + \theta_1^2 Var(a_{t-1})$$

$$+ 2\phi_1 \cancel{cov(r_{t-1}, a_t)} = 0$$

$$- 2\phi_1\theta_1 cov(r_{t-1}, a_{t-1}) = \sigma_a^2$$

$$- 2\theta_1 \cancel{cov(a_t, a_{t-1})} = 0$$

$$= \phi_1^2 Var(r_t) + \sigma_a^2 + \theta_1^2 \sigma_a^2 - 2\phi_1\theta_1\sigma_a^2$$

$$= \phi_1^2 Var(r_t) + (1+\theta_1^2 - 2\phi_1\theta_1)\sigma_a^2$$

$$\Rightarrow \boxed{Var(r_t) = \frac{(1+\theta_1^2 - 2\phi_1\theta_1)\sigma_a^2}{1-\phi_1^2}}$$

AR(1) : $\theta_1 = 0$

$Var(r_t) = \dfrac{\sigma_a^2}{1-\phi_1^2}$

MA(1) : $\phi_1 = 0$

$Var(r_t) = (1+\theta_1^2)\sigma_a^2$

• $Cov(r_{t-1}, a_{t-1}) = cov(\cancel{\phi_0} + \phi_1 r_{t-2} + a_{t-1} - \theta_1 a_{t-2}, a_{t-1})$

$\cdot\ \mathrm{cov}(r_{t-1}, a_{t-1}) = \mathrm{cov}(\phi_0 + \phi_1 r_{t-2} + a_{t-1} - \theta_1 a_{t-2},\ a_{t-1})$

$$= \phi_1 \underbrace{\mathrm{cov}(r_{t-2}, a_{t-1})}_{=0}$$

$$+ \underbrace{\mathrm{cov}(a_{t-1},\ a_{t-1})}_{=\sigma_a^2}$$

$$- \theta_1 \underbrace{\mathrm{cov}(a_{t-2},\ a_{t-1})}_{=0}$$

$$= \sigma_a^2$$

5. Autocovariance:

Lag 1: $\mathrm{cov}(r_t, r_{t-1})$

$$= \mathrm{cov}(\cancel{\phi_0} + \phi_1 r_{t-1} + a_t - \theta_1 a_{t-1},\ r_{t-1})$$

$$= \phi_1 \mathrm{Var}(r_{t-1}) + \underbrace{\cancel{\mathrm{cov}(a_t, r_{t-1})}}_{=0} - \theta_1 \underbrace{\mathrm{cov}(a_{t-1}, r_{t-1})}_{=\sigma_a^2}$$

$$= \phi_1 \underbrace{\mathrm{Var}(r_t)} - \theta_1 \sigma_a^2$$

Lag l ($l \geq 2$): $\mathrm{cov}(r_t, r_{t-l})$

$$= \mathrm{cov}(\phi_0 + \phi_1 r_{t-1} + a_t - \theta_1 a_{t-1},\ r_{t-l})$$

$$= \phi_1 \mathrm{cov}(r_{t-1}, r_{t-l}) + \underbrace{\cancel{\mathrm{cov}(a_t, r_{t-l})}}_{=0} - \theta_1 \underbrace{\cancel{\mathrm{cov}(a_{t-1}, r_{t-l})}}_{=0}$$

r_{t-2}
r_{t-3}
$\vdots$

$$= \phi_1 \mathrm{cov}(r_t, r_{t-(l-1)})$$

6. Autocorrelations:

Lag 1: $\rho_1 = \dfrac{\mathrm{cov}(r_t, r_{t-1})}{\ldots}$

$$\frac{Var(r_t)}{}$$

$$= \frac{\phi_1 Var(r_t) - \theta_1 \sigma_a^2}{Var(r_t)}$$

$$= \phi_1 - \frac{\theta_1 \sigma_a^2}{Var(r_t)} \neq \phi_1$$

AR(1): $\theta_1 = 0$, $\rho_1 = \phi_1$

lag l ($l \geq 2$): $\rho_l = \frac{cov(r_t, r_{t-l})}{Var(r_t)}$

$$= \frac{\phi_1 cov(r_t, r_{t-(l-1)})}{Var(r_t)}$$

$$= \phi_1 \rho_{l-1}$$

$$\rho_2 = \phi_1 \rho_1$$

$$\rho_3 = \phi_1 \rho_2 = \phi_1 (\phi_1 \rho_1) = \phi_1^2 \rho_1$$

$$\rho_4 = \phi_1 \rho_3 = \phi_1 (\phi_1^2 \rho_1) = \phi_1^3 \rho_1$$

$$\vdots \qquad \vdots$$

$$\rho_k = \phi_1^{k-1} \rho_1$$

AR(1): $\theta_1 = 0$, $\rho_l = \phi_1 \rho_{l-1}$

7. Forecast: at time origin n

① 1-step ahead forecast:

$$\hat{r}_n(1) = E(r_{n+1} \mid F_n)$$
$$= E(\phi_0 + \phi_1 r_n + a_{n+1} - \theta_1 a_n \mid F_n)$$
$$= E(\phi_0 \mid F_n) + \phi_1 E(r_n \mid F_n) + \underbrace{E(a_{n+1} \mid F_n)}_{=0} - \theta_1 E(a_n \mid F_n)$$
$$= \phi_0 + \phi_1 r_n - \theta_1 a_n$$

② 1-step ahead forecast error:

$$\hat{e}_n(1) = r_{n+1} - \hat{r}_n(1)$$
$$= \phi_0 + \phi_1 r_n + a_{n+1} - \theta_1 a_n - (\phi_0 + \phi_1 r_n - \theta_1 a_n)$$
$$= a_{n+1}$$

$$Var(\hat{e}_n(1)) = Var(a_{n+1}) = \sigma_a^2$$

③ Multi-step ahead forecast : $l \geq 2$

$$\hat{r}_n(l) = E(r_{n+l} \mid F_n)$$
$$= E(\phi_0 + \phi_1 r_{n+l-1} + a_{n+l} - \theta_1 a_{n+l-1} \mid F_n)$$
$$= \phi_0 + \phi_1 E(r_{n+l-1} \mid F_n) + \underbrace{\cancel{E(a_{n+l} \mid F_n)}}_{=0} - \theta_1 \underbrace{E(a_{n+l-1} \mid F_n)}_{=}$$

a_{n+1}
a_{n+2}
$\vdots$

$$= \phi_0 + \phi_1 \hat{r}_n(l-1)$$

$$\hat{r}_n(2) = \phi_0 + \phi_1 \hat{r}_n(1)$$
$$\hat{r}_n(3) = \phi_0 + \phi_1 \hat{r}_n(2)$$
$$\hat{r}_n(4) = \phi_0 + \phi_1 \hat{r}_n(3)$$

$$r_n(4) = \phi_0 + \phi_1 r_n(3)$$

$$\vdots \qquad\qquad \vdots$$

④ Multi-step ahead forecast error:

$$\hat{e}_n(l) = r_{n+l} - \hat{r}_n(l)$$

$$= \phi_0 + \phi_1 r_{n+l-1} + a_{n+l} - \theta_1 a_{n+l-1}$$

$$- (\phi_0 + \phi_1 \hat{r}_n(l-1))$$

$$= a_{n+l} - \theta_1 a_{n+l-1} + \phi_1 (r_{n+l-1} - \hat{r}_n(l-1))$$

$$= a_{n+l} - \theta_1 a_{n+l-1} + \phi_1 \hat{e}_n(l-1)$$