AR, MA and ARMA

1. AR(1) $\Rightarrow$ MA(∞)

AR(1): $r_t = \phi_0 + \phi_1 r_{t-1} + a_t$

Mean-adjusted AR(1):

- $x_t = \phi_1 x_{t-1} + a_t$

where $x_t = r_t - u$, $u = E(r_t) = \dfrac{\phi_0}{1-\phi_1}$

$$
\begin{aligned}
x_t &= \phi_1 x_{t-1} + a_t \\
&= \phi_1(\phi_1 x_{t-2} + a_{t-1}) + a_t \\
&= \phi_1^2 x_{t-2} + \phi_1 a_{t-1} + a_t \\
&= \phi_1^2(\phi_1 x_{t-3} + a_{t-2}) + \phi_1 a_{t-1} + a_t \\
&= \phi_1^3 x_{t-3} + \phi_1^2 a_{t-2} + \phi_1 a_{t-1} + a_t \\
&\vdots \\
&= \phi_1^k x_{t-k} + \phi_1^{k-1} a_{t-(k-1)} + \phi_1^{k-2} a_{t-(k-2)} \\
&\quad + \cdots + \phi_1 a_{t-1} + a_t
\end{aligned}
$$

when $k \to \infty$, $\phi_1^k x_{t-k} \longrightarrow 0$

$(|\phi_1| < 1 \Rightarrow \phi_1^k \xrightarrow{k\to\infty} 0)$

$$x_t = \sum_{k=0}^{\infty} \phi_1^k a_{t-k}$$

$$= a_t + \phi_1 a_{t-1} + \phi_1^2 a_{t-2} + \phi_1^3 a_{t-3} + \cdots$$

$$x_t = r_t - u$$

$$\Rightarrow \boxed{r_t = u + \sum_{k=0}^{\infty} \phi_1^k a_{t-k}}$$

$$AR(1) \Rightarrow MA(\infty)$$

General Form of Linear TS Model:

$$r_t = u + \sum_{k=0}^{\infty} \psi_k a_{t-k}, \quad \psi_0 = 1$$

$$(MA(\infty))$$

$$AR(1): \psi_k = \phi_1^k$$

- $AR(p)$, $ARMA(p, q)$ $\Rightarrow MA(\infty)$

2. $MA(1) \Rightarrow AR(\infty)$

Check Week 7 In Class Notes

Invertibility of $MA(1)$: $|\theta_1| < 1$

- MA(q)
  ARMA(p, q) $\Rightarrow$ AR(∞)

Unit-Root Nonstationary Models

1. Random Walk

$$P_t = P_{t-1} + a_t, \quad Var(a_t) = \sigma_a^2, \quad P_0 \text{ given}$$

(P_t is log stock price, $r_t = P_t - P_{t-1} = a_t$)

$$P_{t-1} \xrightarrow{a_t > 0} P_t \uparrow$$

$$P_{t-1} \xrightarrow{a_t < 0} P_t \downarrow$$

① AR(1) model with $\phi_1 = 1$

$$P_t = \phi_1 P_{t-1} + a_t$$

Unit Root:

$$(1 - B) P_t = a_t \qquad B: \text{backshift operator (lag)}$$

$$1 - x = 0 \Rightarrow x = 1$$

② Mean:

$$
\begin{aligned}
P_t &= \underbrace{P_{t-1}} + a_t \\
&= P_{t-2} + a_{t-1} + a_t \\
&= P_{t-3} + a_{t-2} + a_{t-1} + a_t \\
&\vdots \\
&= P_{t-t} + a_{t-t+1} + a_{t-t+2} + \cdots + a_{t-1} + a_t \\
&= P_0 + a_1 + a_2 + \cdots + a_{t-1} + a_t
\end{aligned}
$$

$$
\begin{aligned}
E(P_t) &= E(P_0 + a_1 + a_2 + \cdots + a_{t-1} + a_t) \\
&= E(P_0) + E(a_1 + a_2 + \cdots + a_{t-1} + a_t) \\
&= P_0
\end{aligned}
$$

(3) Variance:

$$
\begin{aligned}
Var(P_t) &= Var(P_{t-1} + a_t) \\
&= Var(P_{t-1}) + Var(a_t) + \underbrace{\cancel{2cov(P_{t-1}, a_t)}}_{=0} \\
&= Var(P_{t-1}) + \sigma_a^2
\end{aligned}
$$

$Var(P_t) \neq Var(P_{t-1}) \Rightarrow$ P_t is Nonstationary

$$
\begin{aligned}
Var(P_t) &= Var(\cancel{P_0} + a_1 + a_2 + \cdots + a_{t-1} + a_t) \\
&= Var(a_1) + Var(a_2) + \cdots + Var(a_{t-1}) + Var(a_t) \\
&= t \cdot \sigma_a^2 \quad \text{depends on time } t
\end{aligned}
$$

when $t \to \infty$, $Var(P_t) \to \infty$

④ Autocovariance:

$$cov(P_t, P_s) \qquad s<t$$

$$= cov(P_{t-1} + a_t, P_s)$$

$$= cov(P_{t-2} + a_{t-1} + a_t, P_s)$$

$\vdots$

$$= cov(P_s + a_{s+1} + a_{s+2} + \cdots + a_{t-1} + a_t, P_s)$$

$$= cov(P_s, P_s)$$

$$= Var(P_s) = s \cdot \sigma_a^2$$

- $cov(P_t, P_s) = \min\{t, s\} \cdot \sigma_a^2$

⑤ Autocorrelations:

$$\rho_{t,s} = \frac{cov(P_t, P_s)}{\sqrt{Var(P_t) \cdot Var(P_s)}}$$

$$= \frac{s\, \cancel{\sigma_a^2}}{\sqrt{t\, \cancel{\sigma_a^2} \cdot s\, \cancel{\sigma_a^2}}} = \sqrt{\frac{s}{t}} < 1$$

- $\rho_{t,s}$ is close to 1

if s is not far away from t

- Strong memory

(6) Forecast: at time origin n

- $\hat{P}_n(1) = E(P_{n+1} | F_n)$

$$P_t = P_{t-1} + a_t$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$n+1 \quad n+1 \quad n+1$$

$$= E(P_n + a_{n+1} | F_n)$$

$$= E(P_n | F_n) + \underbrace{E(a_{n+1} | F_n)}_{=0}$$

$$= P_n$$

$$\hat{e}_n(1) = P_{n+1} - \hat{P}_n(1)$$

$$= P_n + a_{n+1} - P_n$$

$$= a_{n+1}$$

$$Var(\hat{e}_n(1)) = Var(a_{n+1}) = \sigma_a^2$$

- $\hat{P}_n(2) = E(P_{n+2} | F_n)$

$$= E(P_{n+1} + a_{n+2} | F_n)$$

$$= E(P_n + a_{n+1} + a_{n+2} | F_n)$$

$$= E(P_n | F_n) + E(a_{n+1} + a_{n+2} | F_n)$$

$$= P_n$$

$$\hat{e}_n(2) = P_{n+2} - \hat{P}_n(2)$$
$$= P_n + a_{n+1} + a_{n+2} - P_n$$
$$= a_{n+1} + a_{n+2}$$

$$Var(\hat{e}_n(2)) = Var(a_{n+1} + a_{n+2})$$
$$= Var(a_{n+1}) + Var(a_{n+2}) + \underbrace{2cov(a_{n+1}, a_{n+2})}_{=0}$$
$$= 2\sigma_a^2$$

- $$\hat{P}_n(l) = E(P_{n+l} \mid F_n)$$
$$= E(P_n + a_{n+1} + a_{n+2} + \cdots + a_{n+l} \mid F_n)$$
$$= E(P_n \mid F_n) + E(a_{n+1} + a_{n+2} + \cdots + a_{n+l} \mid F_n)$$
$$= P_n$$

$$\hat{e}_n(l) = P_{n+l} - \hat{P}_n(l)$$
$$= a_{n+1} + a_{n+2} + \cdots + a_{n+l}$$

$$Var(\hat{e}_n(l)) = Var(a_{n+1} + a_{n+2} + \cdots + a_{n+l})$$
$$= l \cdot \sigma_a^2$$

when $l \to \infty$, $Var(\hat{e}_n(l)) \to \infty$

2. Random Walk with Drift

$$P_t = \mu + P_{t-1} + a_t, \quad \mu \neq 0, \quad P_0 \text{ given}$$

$$Var(a_t) = \sigma_a^2$$

① has an unit root

② Nonstationary

③ strong memory

④ has a time trend with slope μ

$$
\begin{aligned}
P_t &= \mu + P_{t-1} + a_t \\
&= \mu + (\mu + P_{t-2} + a_{t-1}) + a_t \\
&= 2\mu + P_{t-2} + a_{t-1} + a_t \\
&= 2\mu + (\mu + P_{t-3} + a_{t-2}) + a_{t-1} + a_t \\
&= 3\mu + P_{t-3} + a_{t-2} + a_{t-1} + a_t \\
&\vdots \\
&= t \cdot \mu + P_0 + a_1 + a_2 + \cdots + a_{t-1} + a_t
\end{aligned}
$$

3. Differencing

1st difference: $r_t = P_t - P_{t-1} = \Delta P_t$

- $P_t = P_{t-1} + a_t \Rightarrow r_t = a_t$

  P_t : ARIMA(0, 1, 0)

- $P_t = u + P_{t-1} + a_t \Rightarrow r_t = u + a_t$

  P_t : ARIMA(0, 1, 0)

- $P_t = u + P_{t-1} + \phi_1 \Delta P_{t-1} + a_t + \theta_2 a_{t-2}$

  P_t : ARIMA(1, 1, 2)

$\Rightarrow r_t = u + \phi_1 r_{t-1} + a_t + \theta_2 a_{t-2}$

$r_t = \Delta P_t = P_t - P_{t-1}$

- $P_t = u + \phi_1 P_{t-1} + \phi_2 \Delta P_{t-1} + a_t + \theta_2 a_{t-2}$

  P_t : ARIMA(2, 0, 2)

$\Rightarrow P_t = u + \phi_1 P_{t-1} + \phi_2 (P_{t-1} - P_{t-2}) + a_t + \theta_2 a_{t-2}$

$= u + (\phi_1 + \phi_2) P_{t-1} - \phi_2 P_{t-2} + a_t + \theta_2 a_{t-2}$

when $\phi_1 = 1$, P_t : ARIMA(1, 1, 2)

$\phi_1 \neq 1$, P_t : ARIMA(2, 0, 2)

4. Unit-Root test

- Test for AR(1) for P_t:

$$P_t = \phi_1 P_{t-1} + a_t$$

or $$P_t = \mu + \phi_1 P_{t-1} + a_t$$

$H_0: \phi_1 = 1$ vs $H_1: \phi_1 < 1$

H_0: there is an unit root for P_t

Dicky-Fuller test

p value > 5% $\Rightarrow$ P_t has an unit root

- Test for AR(p) for P_t:

$$P_t = C_t + \beta P_{t-1} + \sum_{i=1}^{p-1} \phi_i \Delta P_{t-i} + a_t$$

$$\left(\begin{array}{l} \Delta P_{t-1} = P_{t-1} - P_{t-2} \\ \Delta P_{t-2} = P_{t-2} - P_{t-3} \\ \vdots \end{array} \right)$$

- $\boxed{H_0: \beta = 1 \quad \text{vs} \quad H_1: \beta < 1}$

H_0: there is an unit root

$$H_0 \Rightarrow \Delta P_t = C_t + \sum_{i=1}^{p-1} \phi_i \Delta P_{t-i} + a_t$$

$$\Delta P_t \sim AR(p-1)$$

$$P_t \sim ARIMA(p-1, 1, 0)$$

- $H_1: \beta < 1 \Rightarrow P_t \sim ARIMA(p, 0, 0)$

$$\underline{\underline{AR(p)}}$$

$$P_t = C_t + \beta P_{t-1} + \phi_1 \Delta P_{t-1} + \phi_2 \Delta P_{t-2} + \dots \phi_{p-1} \Delta P_{t-(p-1)} + a_t$$

$$= C_t + \beta P_{t-1} + \phi_1 (P_{t-1} - P_{t-2}) + \phi_2 (P_{t-2} - P_{t-3}) + \dots + \phi_{p-1} (P_{t-(p-1)} - P_{t-p}) + a_t$$

$$= C_t + (\beta + \phi_1) P_{t-1} + (\phi_2 - \phi_1) P_{t-2} + \dots + (-\phi_{p-1}) \underline{\underline{P_{t-p}}} + a_t$$

$$P_t \sim AR(p)$$

$\boxed{\text{Augmented Dicky-Fuller test}}$

p value $> 5\% \Rightarrow P_t$ has an unit root

- Alternative form of the model

$$P_t = C_t + \beta P_{t-1} + \sum_{i=1}^{p-1} \phi_i \Delta P_{t-i} + a_t$$

$P_t - P_{t-1}$

$p-1$

$$\Rightarrow \Delta P_t = C_t + (\beta - 1) P_{t-1} + \sum_{i=1} \phi_i \Delta P_{t-i} + a_t$$

$$= C_t + \beta_r P_{t-1} + \sum_{i=1}^{p-1} \phi_i \Delta P_{t-i} + a_t$$

$$\beta_r \triangleq \beta - 1$$

$$H_0 : \underbrace{\beta_r}_{=\beta-1} = 0 \quad \text{vs} \quad H_1 : \underbrace{\beta_r}_{=\beta-1} < 0$$