AR(2) Model

$$r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + a_t, \quad Var(a_t) = \sigma_a^2$$

1. Stationarity condition:

$$(1 - \phi_1 B - \phi_2 B^2)\, r_t = \phi_0 + a_t$$

$$1 - \phi_1 x - \phi_2 x^2 = 0 \Rightarrow x$$

requirement for stationarity : $|x| > 1$ for all root

AR(1): $\quad 1 - \phi_1 x = 0 \Rightarrow x = \frac{1}{\phi_1}$

$$|x| > 1 \Rightarrow |\phi_1| < 1$$

2. $E(r_t) = \dfrac{\phi_0}{1 - \phi_1 - \phi_2}$

$$E(r_t) = E(\phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + a_t)$$

$$= E(\phi_0) + \phi_1 E(r_{t-1}) + \phi_2 E(r_{t-2}) + \cancel{E(a_t)}^{=0}$$

$$= \phi_0 + \phi_1 E(r_t) + \phi_2 E(r_t)$$

$$\Rightarrow E(r_t) = \frac{\phi_0}{1 - \phi_1 - \phi_2}$$

3. $Var(r_t) = \dfrac{\sigma_a^2}{1 - \phi_1^2 - \phi_2^2 - 2\phi_1\phi_2\rho_1}$

$$Var(r_t) = Var(\cancel{\phi_0} + \phi_1 r_{t-1} + \phi_2 r_{t-2} + a_t)$$

$$Var(r_t) = Var(\cancel{\phi_0} + \phi_1 r_{t-1} + \phi_2 r_{t-2} + a_t)$$

$$= \phi_1^2 Var(r_{t-1}) + \phi_2^2 Var(r_{t-2}) + Var(a_t)$$

$$+ 2\phi_1 \phi_2 cov(r_{t-1}, r_{t-2})$$

$$+ 2\phi_1 \cancel{cov(r_{t-1}, a_t)} = 0$$

$$+ 2\phi_2 \cancel{cov(r_{t-2}, a_t)} = 0$$

$$= \phi_1^2 Var(r_t) + \phi_2^2 Var(r_t) + \sigma_a^2$$

$$+ 2\phi_1 \phi_2 \rho_1 Var(r_t)$$

$$\Rightarrow Var(r_t) = \frac{\sigma_a^2}{1 - \phi_1^2 - \phi_2^2 - 2\phi_1\phi_2\rho_1}$$

4. Autocorrelations:

$$\rho_0 = 1, \quad \rho_1 = \frac{\phi_1}{1-\phi_2}$$

$$\rho_l = \phi_1 \rho_{l-1} + \phi_2 \rho_{l-2}, \quad l \geq 2$$

$l \geq 1$:

$$\rho_l = \frac{cov(r_t, r_{t-l})}{Var(r_t)}$$

$$= \frac{cov(\cancel{\phi_0} + \phi_1 r_{t-1} + \phi_2 r_{t-2} + a_t,\ r_{t-l})}{Var(r_t)}$$

$$= \frac{\phi_1 cov(r_{t-1}, r_{t-l}) + \phi_2 cov(r_{t-2}, r_{t-l}) + \overbrace{\cancel{cov(a_t, r_{t-l})}}^{=0}}{Var(r_t)}$$

$$= \phi_1 \frac{cov(r_t, r_{t-(l-1)})}{Var(r_t)} + \phi_2 \frac{cov(r_t, r_{t-(l-2)})}{Var(r_t)}$$

$$= \phi_1 \rho_{l-1} + \phi_2 \rho_{l-2}$$

when $l=1$, $\rho_1 = \phi_1 \rho_0 + \phi_2 \rho_{-1}$

$$= \phi_1 + \phi_2 \rho_1$$

$$\Rightarrow \rho_1 = \frac{\phi_1}{1-\phi_2}$$

$l=2$, $\rho_2 = \phi_1 \rho_1 + \phi_2 \rho_0$

$l=3$, $\rho_3 = \phi_1 \rho_2 + \phi_2 \rho_1$

$\vdots \qquad \vdots$

5. Mean-adjusted format:

$$(r_t - \mu) = \phi_1 (r_{t-1} - \mu) + \phi_2 (r_{t-2} - \mu) + a_t$$

where $\mu = E(r_t) = \frac{\phi_0}{1-\phi_1-\phi_2}$

MA(1) Model

$$r_t = \mu + a_t - \theta a_{t-1}, \quad Var(a_t) = \sigma_a^2$$

1. Stationarity: always stationary

(weekly)

2. Mean: $E(r_t) = \mu$

$$E(r_t) = E(\mu + a_t - \theta a_{t-1})$$
$$= E(\mu) + \underbrace{\cancel{E(a_t)}}_{=0} - \theta \underbrace{\cancel{E(a_{t-1})}}_{=0}$$
$$= \mu$$

3. Variance: $Var(r_t) = (1+\theta^2)\sigma_a^2$

$$Var(r_t) = Var(\cancel{\mu} + a_t - \theta a_{t-1})$$
$$= Var(a_t) + \theta^2 Var(a_{t-1})$$
$$- 2\theta \underbrace{\cancel{cov(a_t, a_{t-1})}}_{} = 0$$
$$= \sigma_a^2 + \theta^2 \sigma_a^2$$
$$= (1+\theta^2)\sigma_a^2$$

- $cov(a_k, a_j) = \begin{cases} 0 & \text{if } k \neq j \\ \sigma_a^2 & \text{if } k = j \end{cases}$

4. Autocovariance:

Lag 1: $cov(r_t, r_{t-1})$

$$= cov(\cancel{\mu} + a_t - \theta a_{t-1}, \cancel{\mu} + a_{t-1} - \theta a_{t-2})$$
$$= \underbrace{cov(a_t, a_{t-1})}_{}^{=0} - \theta \underbrace{cov(a_t, a_{t-2})}_{}^{=0}$$
$$- \theta\, cov(a_{t-1}, a_{t-1}) + \theta^2 \underbrace{cov(a_{t-1}, a_{t-2})}_{}^{=0}$$
$$= -\theta \sigma_a^2$$

lag 2: $\operatorname{cov}(r_t, r_{t-2})$

$$= \operatorname{cov}(\cancel{\mu} + a_t - \theta a_{t-1}, \cancel{\mu} + a_{t-2} - \theta a_{t-3})$$

$$= \operatorname{cov}(a_t, a_{t-2}) - \theta \operatorname{cov}(a_t, a_{t-3})$$
$$- \theta \operatorname{cov}(a_{t-1}, a_{t-2}) + \theta^2 \operatorname{cov}(a_{t-1}, a_{t-3})$$

$$= 0$$

lag l ($l \geq 2$): $\operatorname{cov}(r_t, r_{t-l})$

$$= \operatorname{cov}(\cancel{\mu} + a_t - \theta a_{t-1}, \cancel{\mu} + a_{t-l} - \theta a_{t-l-1})$$

$$\begin{array}{cc} a_{t-2} & a_{t-3} \\ a_{t-3} & a_{t-4} \\ \vdots & \vdots \end{array}$$

$$= 0$$

5. Autocorrelation:

lag 1: $\rho_1 = \dfrac{\operatorname{cov}(r_t, r_{t-1})}{\operatorname{Var}(r_t)} = \dfrac{-\theta \sigma_a^2}{(1+\theta^2)\sigma_a^2} = \dfrac{-\theta}{1+\theta^2}$

lag l ($l \geq 2$): $\rho_l = \dfrac{\operatorname{cov}(r_t, r_{t-l})}{\operatorname{Var}(r_t)} = 0$

6. Forecast: at time origin n

① 1-step ahead forecast:

$$\hat{r}_n(1) = E(r_{n+1} \mid F_n)$$
$$= E(\mu + a_{n+1} - \theta a_n \mid F_n)$$
$$= E(\mu \mid F_n) + E(a_{n+1} \mid F_n) - \theta E(a_n \mid F_n)$$

$$\underbrace{\qquad\qquad}_{=0} \qquad \underbrace{\qquad\qquad}_{a_n}$$

$$= \mu - \theta a_n$$

② 1-step ahead forecast error:

$$\hat{e}_n(1) = r_{n+1} - \hat{r}_n(1)$$

$$= \mu + a_{n+1} - \theta a_n - (\mu - \theta a_n)$$

$$= a_{n+1}$$

$$Var(\hat{e}_n(1)) = Var(a_{n+1}) = \sigma_a^2$$

③ Multi-step ahead forecast:

$$l \geq 2: \quad \hat{r}_n(l) = E(r_{n+l} \mid F_n)$$

$$= E(\mu + a_{n+l} - \theta a_{n+l-1} \mid F_n)$$

$$= E(\mu \mid F_n) + \underbrace{E(a_{n+l} \mid F_n)}_{=0} - \theta \underbrace{E(a_{n+l-1} \mid F_n)}_{=0}$$

$$= \mu \qquad \begin{matrix} a_{n+2} \\ a_{n+3} \\ \vdots \end{matrix} \qquad \begin{matrix} a_{n+1} \\ a_{n+2} \\ \vdots \end{matrix}$$

④ Multi-step ahead forecast error:

$$l \geq 2: \quad \hat{e}_n(l) = r_{n+l} - \hat{r}_n(l)$$

$$= \mu + a_{n+l} - \theta a_{n+l-1} - \mu$$

$$= a_{n+l} - \theta a_{n+l-1}$$

$$Var(\hat{e}_n(l)) = Var(a_{n+l} - \theta a_{n+l-1})$$

$$= Var(a_{n+l}) + \theta^2 Var(a_{n+l-1})$$

$$= Var(a_{n+l}) + \theta^2 Var(a_{n+l-1})$$

$$- 2\theta \, cov(a_{n+l}, a_{n+l-1}) = 0$$

$$= \sigma_a^2 + \theta^2 \sigma_a^2$$

$$= (1+\theta^2)\,\sigma_a^2 = Var(r_t)$$

7. Invertibility : MA(1) $\Rightarrow$ AR(∞)

Condition: $|\theta| < 1$

$$r_t = \mu + a_t - \theta a_{t-1}$$

$$\Rightarrow a_t = -\mu + r_t + \theta a_{t-1}$$

$$= -\mu + r_t + \theta(-\mu + r_{t-1} + \theta a_{t-2})$$

$$= -(1+\theta)\mu + r_t + \theta r_{t-1} + \theta^2 a_{t-2}$$

$$= -(1+\theta)\mu + r_t + \theta r_{t-1} + \theta^2(-\mu + r_{t-2} + \theta a_{t-3})$$

$$= -(1+\theta+\theta^2)\mu + r_t + \theta r_{t-1} + \theta^2 r_{t-2} + \theta^3 a_{t-3}$$

$$\vdots$$

$$= -(1+\theta+\theta^2+\cdots+\theta^l)\mu$$

$$+ r_t + \theta r_{t-1} + \theta^2 r_{t-2} + \cdots + \theta^l r_{t-l}$$

$$+ \boxed{\theta^{l+1} a_{t-(l+1)}}$$

when $l \to \infty$, $\theta^{l+1} \to 0$ if $|\theta| < 1$

$$a_t = -\frac{1}{1-\theta}\mu + r_t + \sum \theta^j r_{t-j}$$

$$\Rightarrow r_t = \frac{1}{1-\theta}\mu - \sum_{j=1}^{\infty} \theta^j r_{t-j} + a_t$$

$$MA(1) \xRightarrow[|\theta|<1]{} AR(\infty)\,!!$$