# AR(1) Model

$$r_t = \phi_0 + \phi_1 r_{t-1} + a_t, \qquad Var(a_t) = \sigma_a^2$$

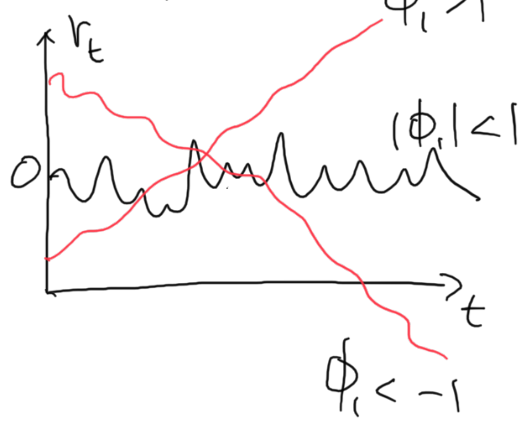
(Weak)

1. Stationarity condition: $|\phi_1| < 1$

$$(1-\phi_1 B) r_t = \phi_0 + a_t$$

$$1 - \phi_1 x = 0 \Rightarrow x = \frac{1}{\phi_1}$$

$$|x| > 1 \Rightarrow |\phi_1| < 1$$




2. $E(r_t) = \dfrac{\phi_0}{1-\phi_1}$

$$E(r_t) = E(\phi_0 + \phi_1 r_{t-1} + a_t)$$

$$= E(\phi_0) + \phi_1 E(r_{t-1}) + \cancel{E(a_t)} \quad {=0}$$

$$= \phi_0 + \phi_1 E(r_t)$$

$$\Rightarrow E(r_t) = \frac{\phi_0}{1-\phi_1}$$

3. $Var(r_t) = \dfrac{\sigma_a^2}{1-\phi_1^2}$

$$Var(r_t) = Var(\cancel{\phi_0} + \phi_1 r_{t-1} + a_t)$$

$$= \phi_1^2 Var(r_{t-1}) + Var(a_t)$$

$$+2\phi_1 \underbrace{\cancel{cov(r_{t-1}, a_t)}}_{} = 0$$

$$= \phi_1^2 Var(r_t) + \sigma_a^2$$

$$\Rightarrow Var(r_t) = \frac{\sigma_a^2}{1-\phi_1^2}$$

4. $\rho_1 = \phi_1, \quad \rho_2 = \phi_1^2, \cdots, \rho_k = \phi_1^k$

$$\rho_1 = \frac{cov(r_t, r_{t-1})}{Var(r_t)} = \frac{cov(\phi_0 + \phi_1 r_{t-1} + a_t, r_{t-1})}{Var(r_t)}$$

$$= \frac{\overset{=0}{\cancel{cov(\phi_0, r_{t-1})}} + \phi_1 cov(r_{t-1}, r_{t-1}) + \overset{=0}{\cancel{cov(a_t, r_{t-1})}}}{Var(r_t)}$$

$$= \frac{\phi_1 Var(r_{t-1})}{Var(r_t)} = \phi_1 \qquad Var(r_{t-1}) = Var(r_t) \text{ weak stationarity}$$

$$\rho_2 = \frac{cov(r_t, r_{t-2})}{Var(r_t)} = \frac{cov(\phi_0 + \phi_1 r_{t-1} + a_t, r_{t-2})}{Var(r_t)}$$

$$= \frac{\overset{=0}{\cancel{cov(\phi_0, r_{t-2})}} + \phi_1 cov(r_{t-1}, r_{t-2}) + \overset{=0}{\cancel{cov(a_t, r_{t-2})}}}{Var(r_t)}$$

$$= \frac{\phi_1 cov(r_{t-1}, r_{t-2})}{Var(r_t)} \qquad cov(r_{t-1}, r_{t-2}) = cov(r_t, r_{t-1}) \text{ weak stationarity}$$

$$= \frac{\phi_1 cov(r_t, r_{t-1})}{} = \phi \rho = \phi^2$$

$$\mathrm{Var}(r_t)$$

$$\rho_k = \frac{\mathrm{cov}(r_t, r_{t-k})}{\mathrm{Var}(r_t)} = \frac{\mathrm{cov}(\phi_0 + \phi_1 r_{t-1} + a_t, r_{t-k})}{\mathrm{Var}(r_t)}$$

$$= \frac{\cancel{\underbrace{\mathrm{cov}(\phi_0, r_{t-k})}}^{=0} + \phi_1 \mathrm{cov}(r_{t-1}, r_{t-k}) + \cancel{\underbrace{\mathrm{cov}(a_t, r_{t-k})}}^{=0}}{\mathrm{Var}(r_t)}$$

$$= \frac{\phi_1 \mathrm{cov}(r_{t-1}, r_{t-k})}{\mathrm{Var}(r_t)}$$

$$= \frac{\phi_1 \mathrm{cov}(r_t, r_{t-(k-1)})}{\mathrm{Var}(r_t)} = \phi_1 \rho_{k-1}$$

$$\begin{aligned}
k=1, \quad & \rho_1 = \phi_1 \rho_0 = \phi_1 \\
k=2, \quad & \rho_2 = \phi_1 \rho_1 = \phi_1^2 \\
k=3, \quad & \rho_3 = \phi_1 \rho_2 = \phi_1^3 \\
\vdots \quad & \quad \vdots \qquad \vdots \qquad \vdots \\
& \rho_k = \phi_1 \rho_{k-1} = \phi_1^k
\end{aligned}$$

5. An alternative representation for AR(1):

$$r_t = \phi_0 + \phi_1 r_{t-1} + a_t$$

$$\Rightarrow (r_t - \mu) = \phi_1 (r_{t-1} - \mu) + a_t$$

where $\mu = E(r_t) = \dfrac{\phi_0}{1-\phi_1}$

6. Forecast: $x_t = r_t - \mu$

- $x_t = \phi_1 x_{t-1} + a_t$

Suppose we are at time $n$

(a) 1-step ahead forecast : $F_n = \{x_1, x_2, \cdots, x_n\}$

$$\hat{x}_n(1) = E(x_{n+1} \mid F_n)$$

$$= E(\phi_1 x_n + a_{n+1} \mid F_n)$$

$$= \phi_1 E(x_n \mid F_n) + \underbrace{E(a_{n+1} \mid F_n)}_{= E(a_{n+1}) = 0}$$

$$= \phi_1 x_n$$

(b) 1-step ahead forecast error :

$$\hat{e}_n(1) = x_{n+1} - \hat{x}_n(1)$$

$$= \phi_1 x_n + a_{n+1} - \phi_1 x_n$$

$$= a_{n+1}$$

(c) Variance of 1-step ahead forecast error :

$$Var(\hat{e}_n(1)) = Var(a_{n+1}) = \sigma_a^2$$

(d) 2-step ahead forecast :

$$\hat{x}_n(2) = E(x_{n+2} \mid F_n)$$

$$= E(\phi_1 x_{n+1} + a_{n+2} \mid F_n)$$

$= \phi_1 E(x_{n+1} | F_n) + \underbrace{E(a_{n+2} | F_n)}_{= E(a_{n+2}) = 0}$

$= \phi_1 \cdot \phi_1 x_n$

$= \phi_1^2 x_n$

(e) 2-step ahead forecast error:

$\hat{e}_n(2) = x_{n+2} - \hat{x}_n(2)$

$= \phi_1 x_{n+1} + a_{n+2} - \phi_1^2 x_n$

$= \phi_1 (\cancel{\phi_1 x_n} + a_{n+1}) + a_{n+2} - \cancel{\phi_1^2 x_n}$

$= \phi_1 a_{n+1} + a_{n+2}$

(f) Variance of 2-step ahead forecast error:

$Var(\hat{e}_n(2)) = Var(\phi_1 a_{n+1} + a_{n+2})$

$= \phi_1^2 Var(a_{n+1}) + Var(a_{n+2})$

$+ 2 \phi_1 \underbrace{\cancel{cov(a_{n+1}, a_{n+2})}}_{} = 0$

$= \phi_1^2 \sigma_a^2 + \sigma_a^2$

$= (1 + \phi_1^2) \sigma_a^2$