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Are There Any Julia Sets for the Laguerre Iteration Function?

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Abstract—For polynomials some of whose zeros are complex, little is known about the overall convergence properties of the Laguerre's method. The existence of free critical points of the Laguerre function as applied to one-parameter families of quadratic and cubic polynomials is examined. With the help of microcomputer plots, we investigate the Julia sets of the Laguerre function in the special case wherein this function is constructed to converge to the n^{th} roots of unity.

Keywords—Critical point, Fatou set, Julia set, Laguerre's method.

1. INTRODUCTION

Laguerre's method is by far the most straightforward iterative method for finding roots of polynomials all of whose zeros are real and simple. If all the zeros are real but some are not simple, the method still converges but is first order in the neighborhood of a multiple zero. Newton-Raphson-like methods will work, but large round-off errors can occur due to the large number of iterations required (see, for example, [1,2]).

For polynomials with real roots, the real line is divided into as many abutting intervals as there are distinct roots, and from any initial point in such an interval, the successive Laguerre iterates converge monotonically to the root therein. (See [3] and the references therein.) In this work, we are interested in Laguerre iterations in the complex plane. The previous property does not extend to the complex case as it stands, i.e., generally, the complex plane is not covered by abutting regions such that from an initial value in a region successive iterates converge to the zero contained therein.

The following question therefore arises naturally: up to what degree of complex polynomials does this property remain valid and what happens further in the regions of convergence. For complex polynomial equations, this question involves the iteration of functions, which is described by the classical theory of Julia [4] and Fatou [5] and its subsequent developments (see, for example, [6]), also of paramount importance in the context of numerical analysis. In what follows, we

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abbreviate as f^k the k-fold composition $f \circ f \circ \cdots \circ f$, and by *region* we mean a connected open set on the extended complex plane $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

It appears that, for cases of practical interest, convergence of the sequence of iterates $z_0, f(z_0), f^2(z_0), \ldots$ is assured for every choice of starting point z_0 in the complex plane, except when z_0 is a point of the Julia set $J(f)$. How "close" a starting point must be to the desired root depends on certain convergence conditions and how "fast" the method converges depends on the order of convergence of our iterative method.

It may happen, however, that when choosing a starting point z_0 in a certain domain, convergence takes place not to a zero of our polynomial, but to a *periodic orbit* or *cycle*, that is a set of $p \geq 2$ distinct points $\{a_1, \ldots, a_p\}$ such that

$$f(a_1) = a_2, \ldots, f(a_{p-1}) = a_p, \qquad f(a_p) = a_1,$$

so that, in fact, for each $k = 1, 2, \ldots, p$, $z = a_k$ is a solution of $f^p(z) = z$. If $p = 1$, z is called a *fixed point* of f. Hence, a point a is *periodic* if $f^p(a) = a$ for some $p > 0$; it is *repelling*, *indifferent*, or *attracting* depending on whether $|(f^p)'(a)|$ is greater than, equal to, or less than one.

The *Julia set* $J(f)$ of a function f is

$$\begin{aligned} J(f) &= \left\{z \in \mathbb{C} : \left\{f^k\right\} \text{ is not normal at } z\right\} \\ &= \left\{z \in \mathbb{C} : \left\{f^k\right\} \text{ is normal on no open set containing } z\right\}, \end{aligned}$$

and the *Fatou set* is its complement on $\mathbb{C}$. If a is an attracting fixed point of f, then

$$A(a) = \left\{z \in \mathbb{C} : \lim_{k \to \infty} f^k(z) = a\right\}$$

is the *basin of attraction* of a. If a basin of attraction is not connected, we often wish to consider the *immediate basin of attraction* $A^*(a)$ of a, namely, the connected component of $A(a)$ which contains a itself. We also have the following important property: the boundary of $A(a)$ is $J(f)$.

As far as the dynamics of the Laguerre function in the complex plane is concerned, we demonstrate the nonexistence of critical points which may be trapped by an iteration sequence associated with one-parameter families of quadratic and cubic polynomials. We then examine, with the aid of microcomputer-generated plots, the Julia sets of this method constructed to converge to the n^{th} roots of unity and, subsequently, these roots' basins of attraction. In the specific case of two roots, Cayley's classical result [7] for Newton's method, i.e., the boundary of the roots' basins of attraction is the right bisector of the line joining the two roots, is shown to hold for the Laguerre function also. In the case of three roots, the roots' basins of attraction are the connected regions separated by the lines bisecting the edges of the triangle with the roots as vertices. A similar result is also valid in the case of four roots, whereas for $n > 4$ the Julia sets of the Laguerre function are fractals. Moreover, a brief examination of a one-parameter family of cubic polynomials is included.

Although the author has found the above-mentioned results since 1998, it is only now that he discovers by accident the very interesting paper by Curry and Fiedler [8], in which the dynamics of Laguerre's iteration associated with $z^n - 1$ is examined. The reader can also find there a proof for the global convergence of the method for $n = 3$. The observed vehaviour of the Laguerre method in the dynamic space coincides in both papers.

2. LAGUERRE'S ITERATION

Section 9.5 of [9] claims that Laguerre's method, used for finding zeros of a polynomial, gives strong convergence right from any starting value. According to Ralston and Rabinowitz [10], however, this is only true if all the roots of the polynomial are real. For example, Laguerre's method runs into difficulty for the polynomial $f(x) = x^n + 1$ if the initial guess is 0, because $f'(0) = f''(0) = 0$. The method can be extended to the complex plane as follows, but refer also to [3].

Let f be analytic in some region T, let $w \in T$ be a zero of f, and let $f'(w) \neq 0$. Let ν be a real number, $\nu \neq 0, 1$. Then, there exists a neighborhood D of w such that

$$\left| \frac{\nu}{\nu - 1} \frac{f(z) f''(z)}{[f'(z)]^2} \right| < 1, \qquad z \in D.$$

Consequently, the square root

$$r(z) = \left\{ 1 - \frac{\nu}{\nu - 1} \frac{f(z) f''(z)}{[f'(z)]^2} \right\}^{1/2}$$

is analytic in D and can be defined by its principal value. For $z \in D$, we define

$$L(z) = z - \frac{f(z)}{f'(z)} \frac{\nu}{1 + (\nu - 1) r(z)} \tag{1}$$

and assert the following.

THEOREM 1. *For every $\nu \neq 0, 1$, the function L defined by equation (1) is an iteration function of order 3 for solving $f(z) = 0$.*

PROOF. See [11, p. 532]. ∎

This means that, in the neighborhood of a simple root of f, iteration of L converges at least cubically.

We define the *Laguerre iteration function* $L(z)$ as (see [11])

$$L(z) = z - \frac{\nu f(z)}{f'(z) + \left\{ (\nu - 1)^2 [f'(z)]^2 - \nu(\nu - 1) f(z) f''(z) \right\}^{1/2}}, \tag{2}$$

where the argument of the root is to be chosen to differ by less than $\pi/2$ from the argument of $(\nu - 1) f'(z)$. Equation (2) can be written in equivalent form as

$$L(z) = z - \frac{\nu [f(z)/f'(z)]}{1 + \left\{ (\nu - 1)^2 - \nu(\nu - 1) \left[f(z) f''(z) / (f'(z))^2 \right] \right\}^{1/2}}. \tag{3}$$

We observe that $L(z) = \infty$ if and only if $f'(z) = f''(z) = 0$ and $f(z) \neq 0$. Iteration (3) for $\nu = 2$ becomes

$$L(z) = z - \frac{2 [f(z)/f'(z)]}{1 + \left\{ 1 - 2 \left[f(z) f''(z) / (f'(z))^2 \right] \right\}^{1/2}}. \tag{4}$$

From this formula, we may derive more simple ones by introducing approximations. For example, if (for small f) we introduce the approximation

$$\sqrt{1 - 2 \frac{f(z) f''(z)}{[f'(z)]^2}} \approx 1 - \frac{f(z) f''(z)}{[f'(z)]^2}$$

in equation (4), we obtain the formula

$$H(z) = z - \frac{f(z)/f'(z)}{1 - f(z) f''(z) / \left\{ 2 [f'(z)]^2 \right\}},$$

which is also of third order, but which does not require a square-root extraction. This is the frequently rediscovered formula of *Halley*. Iterative approximation based on this formula is also sometimes called *Bailey's method* or *Lambert's method*. This formula belongs to a more general class of iterative functions called *König's functions* and is specifically $K_3(z)$. We refer the interested reader to [12,13].

If we write

$$\left[1 - \frac{f(z)f''(z)}{2\left[f'(z)\right]^2}\right]^{-1} \approx 1 + \frac{f(z)f''(z)}{2\left[f'(z)\right]^2}$$

in Halley's formula, we obtain the iteration

$$C(z) = z - \frac{f(z)}{f'(z)}\left[1 + \frac{f(z)f''(z)}{2\left[f'(z)\right]^2}\right].$$

This is a third-order iteration and sometimes called *Chebyshev's formula*. For more details on this subject, see [14]. This formula belongs also to a more general class of iterative functions called *Schröder's functions* and is specifically $S_3(z)$ (see, for example, [15]). Due to McMullen [16], there is no generally convergent purely iterative algorithm for finding the roots of a polynomial of degree 4 or more (Theorem 1.1, I). Although McMullen's theorem does not apply to Laguerre iteration since it distinctly deals with a "purely iterative algorithm" which by definition is a rational map, it does apply to all these previously mentioned methods.

The iteration sequence $z_{k+1} = L(z_k)$ converges "locally" to the roots z_i^*, $i = 1, 2, \dots, n = \deg(f)$ of $f(z) = 0$, as $O(|z_k - z_i^*|^2)$ because

$$r(z) = 1 - \frac{\nu}{2(\nu-1)} \frac{f(z)f''(z)}{\left[f'(z)\right]^2} + O\left((z-w)^2\right).$$

The Laguerre iteration function phenomenally offers no advantages over Schröder's S_3, which is also somewhat easier to compute. From a practical point of view, however, the presence of the square root has the desirable effect that if f is a real polynomial the iteration automatically branches out into the complex plane if no real roots are found. Moreover, if f is a real polynomial of degree $n \geq 2$, the choice $\nu = n$ furnishes remarkable inclusion theorems for the real zeros, such as the following.

THEOREM 2. *Let f be a polynomial of degree n and let L be the Laguerre iteration function formed with $\nu = n$. Then for each complex number z there is a zero w of f such that $|w - z| \leq \sqrt{n}\,|L(z) - z|$.*

PROOF. See [17]. ∎

3. LAGUERRE ITERATION FUNCTION REVISITED

None of the derivations of equations (2) or (3) is easy to motivate. Therefore, if $f(z) = (z-\rho_1)(z-\rho_2)\cdots(z-\rho_n)$, then

$$F(z) = \frac{f'(z)}{f(z)} = \frac{1}{z-\rho_1} + \frac{1}{z-\rho_2} + \cdots + \frac{1}{z-\rho_n} = \frac{d}{dz}\ln|f(z)|$$

and

$$G(z) = \left[F(z)\right]^2 - \frac{f''(z)}{f(z)} = \frac{1}{(z-\rho_1)^2} + \frac{1}{(z-\rho_2)^2} + \cdots + \frac{1}{(z-\rho_n)^2} = \frac{d^2}{dz^2}\ln|f(z)|.$$

Then, for z near a simple root ρ, the other roots are "far away" and so $f(z) \approx (z-\rho)(z-\xi)^{n-1}$ for some ξ. Thus,

$$F(z) \approx \frac{1}{z-\rho} + \frac{n-1}{z-\xi}, \qquad G(z) \approx \frac{1}{(z-\rho)^2} + \frac{n-1}{(z-\xi)^2}.$$

Assuming the approximations are exact yields a solution

$$z - \rho = \frac{n}{\max\left[F(z) \pm \sqrt{(n-1)(nG(z) - [F(z)]^2)}\right]},$$

which leads immediately to equation (2). So, if

$$\Lambda(z) = \frac{F(z) + \delta(z)\sqrt{\nu - 1}}{\nu},$$

where $\delta(z) = \pm\sqrt{\nu G(z) - [F(z)]^2}$, with the sign chosen to maximize $|\Lambda|$, equation (2) becomes

$$L(z) = z - \frac{1}{\Lambda(z)},$$

where the sign should be taken to yield the largest magnitude for the denominator.

It is well known that the dynamics of polynomials and rational maps is determined to a large extent by the fate of the orbits of critical values. *Critical values* of a function f are defined as those values $v \in \mathbb{C}$ for which $f(z) = v$ has a multiple root. The multiple root $z = c$ is called the *critical point* of f. This is equivalent to the condition $f'(c) = 0$. In some cases, such as in $E_\lambda(z) = \lambda e^z$, $\lambda \in \mathbb{R}$, that has no critical points, the role of the critical value is played by the *asymptotic value* 0, which is an omitted value for E_λ. In this article, we intend to exclude the case in which some critical point will converge to an attracting cycle, should such a cycle exist.

Among the critical points of the Laguerre function L, determined by the condition $L'(z) = 0$, are the zeros z_i^* which are also attracting fixed points of L. These points are obviously not free to converge to any other attracting cycles. In the next section, we seek the existence of other roots, which we shall call the *free critical points.*

Following the notation of this section, the condition $L'(z) = 0$ implies that

$$[\Lambda(z)]^2 + \Lambda'(z) = 0,$$

which yields

$$\begin{aligned} &3(\nu-1)(\nu-2)^2\left[f'(z)\right]^2\left[f''(z)\right]^2 - 4(\nu-1)^2(\nu-2)\left[f'(z)\right]^3 f'''(z) \\ &- 4\nu(\nu-2)^2 f(z)\left[f''(z)\right]^3 + 6\nu(\nu-1)(\nu-2)f(z)f'(z)f''(z)f'''(z) \\ &\qquad - \nu^2(\nu-1)[f(z)]^2\left[f'''(z)\right]^2 = 0. \end{aligned} \tag{5}$$

The following proposition is a great constraint for the study of the Laguerre function in both the parameter and dynamic spaces.

PROPOSITION 1. *The Laguerre iteration function remains invariant under every Möbius transformation.*

PROOF. See [3]. ∎

4. EXAMPLES

We now focus attention on the Laguerre iteration method associated with the quadratic family

$$p_c(z) = z^2 + c, \qquad c \in \mathbb{C},$$

and with the particular one-parameter family of cubic polynomials,

$$p_\lambda(z) = z^3 + (\lambda - 1)z - \lambda = (z-1)\left(z^2 + z + \lambda\right), \qquad \lambda \in \mathbb{C},$$

the zeros of which are $z_1^* = 1$, $z_2^* = (-1 + \sqrt{1-4\lambda})/2$, and $z_3^* = (-1 - \sqrt{1-4\lambda})/2$.

Note the λ-dependence of z_2^* and z_3^*. The polynomials p_λ are exactly the monic cubics whose roots sum to zero and which have 1 as a root. Since any quadratic can be transformed into a p_c and any cubic can be transformed into a p_λ or into z^3 by an affine change of the variable and multiplication by a constant, thus analyzing Laguerre's method for a general quadratic or cubic reduces essentially to analyzing it for the p_cs or p_λs, respectively.

One main question is, are there any regions in the parameter spaces where attracting periodic cycles exist in addition to the (attracting) fixed points associated with the zeros of p_c or p_λ? To detect the existence of attracting cycles which could interfere with the Laguerre search for the z_i^*, we examine the existence of the free critical points of the L function.

From equation (5) and setting $\nu = \deg(p_c) = 2$, the free critical points (should there be any) for the L function associated with p_c can be derived from

$$[p_c(z)]^2 \left[p_c'''(z)\right]^2 = 0, \tag{6}$$

which holds for every $z \in \mathbb{C}$.

From equation (5) the free critical points for the L function associated with p_λ can be derived from

$$\begin{aligned} 6(n-3)^2(n-4)z^6 - 6(n-3)^2(n+2)(\lambda-1)z^4 + 12n(n-3)(2n-5)\lambda z^3 \\ + 6n(n-1)(n-3)(\lambda-1)^2 z^2 - 12n(n-1)(n-3)\lambda(\lambda-1)z \\ - 2(n-1)^2(n-2)(\lambda-1)^3 - 3n^2(n-1)\lambda^2 = 0. \end{aligned}$$

Setting $n = \deg(p_\lambda) = 3$ in the above equation, we deduce that for the three specific values of λ, $\lambda = -2, -2, 1/4$, this also holds for every $z \in \mathbb{C}$, as in equation (6). We conclude that there are not any free critical points, something that suggests that the dynamics of Laguerre's method for all the complex quadratic and cubic polynomials is unaffected by critical points.

Now we examine, by means of computer-generated plots, the basins of attraction and Julia sets of the Laguerre function constructed to converge to the n^{th} roots of unity as well as the roots' basins of attraction of $L(p_\lambda)$. Substituting f_n in the Laguerre transformation (3), we get

$$L(z) = z\frac{z^{-n/2} + (n-1)}{z^{n/2} + (n-1)} = \frac{1 \pm (n-1)z^{n/2}}{z^{(n-2)/2}\left[z^{n/2} \pm (n-1)\right]}. \tag{7}$$

It is readily apparent that, when n is an even integer, equation (7) is a rational map and, when n is odd, it is algebraic.

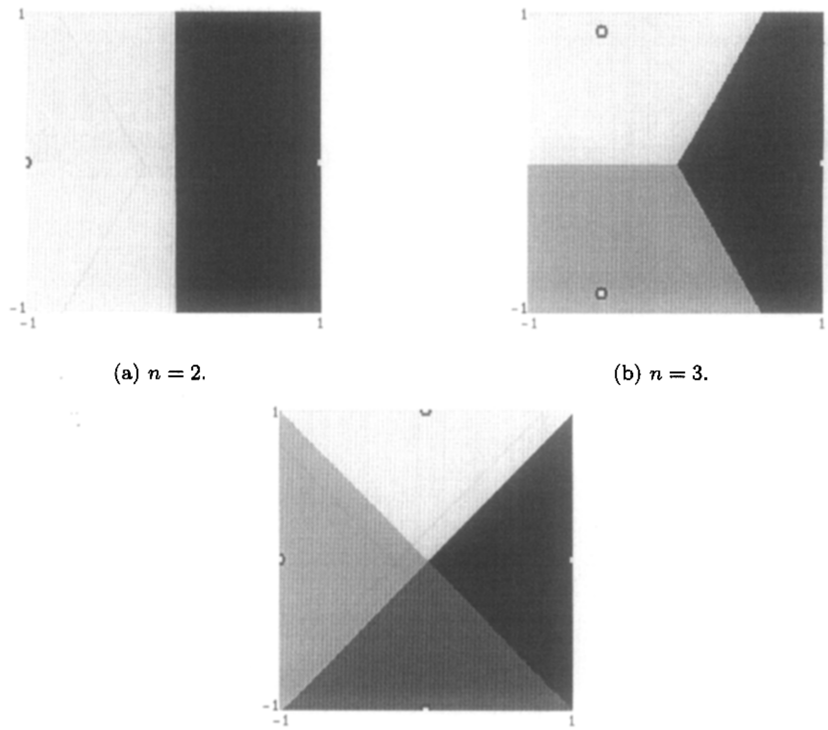
The fixed-point condition $L(z) = z$ implies that

(i) $f_n(z) = z^n - 1 = 0$, or

(ii) $z = 0$ for $n = 3, 4 \dots$.

So, $z = 0$ is an unwanted repelling additional periodic point of period 2, because $L(0) = \infty$ and $L(\infty) = 0$. The basins of attraction for $n = 2, 3, \dots, 7$ are presented in Figures 1a–1c and 2a–2c, respectively, whereas for $\lambda = -2, 1/4, \imath$ are presented in Figure 5a–5c, respectively, where the small (semi) circles indicate the position of the corresponding roots. Figures 3 and 4 show the corresponding Julia sets $J(L(f_n))$ for $n = 2, 3, \dots, 7$ and were computed using the BSM, whereas Figures 1, 2, and 5 were computed using the LSM, both methods described in [18].

A square grid corresponding to 200×200 pixels of a monitor represent a region in the complex plane. After each iteration, the Euclidean distances between the iterate z_k and the zeros z_i^* of $f_n(z)$ (or of $p_\lambda(z)$) were computed. If any of the distances were less than 0.0001, it was assumed that the sequence would converge to that particular root. The basin of attraction $A(z_i^*)$ for each root of unity z_i^* would be assigned a characteristic color. For $n = 2$, blue regions constitute $A(1)$; yellow regions constitute $A(-1)$. For $n = 3$, blue regions constitute $A(1)$; yellow regions constitute $A(-0.5 + \imath\, 0.5\sqrt{3})$; green regions constitute $A(-0.5 - \imath\, 0.5\sqrt{3})$. For $n = 4$, blue regions constitute $A(1)$; yellow regions constitute $A(\imath)$; green regions constitute $A(-1)$; red regions

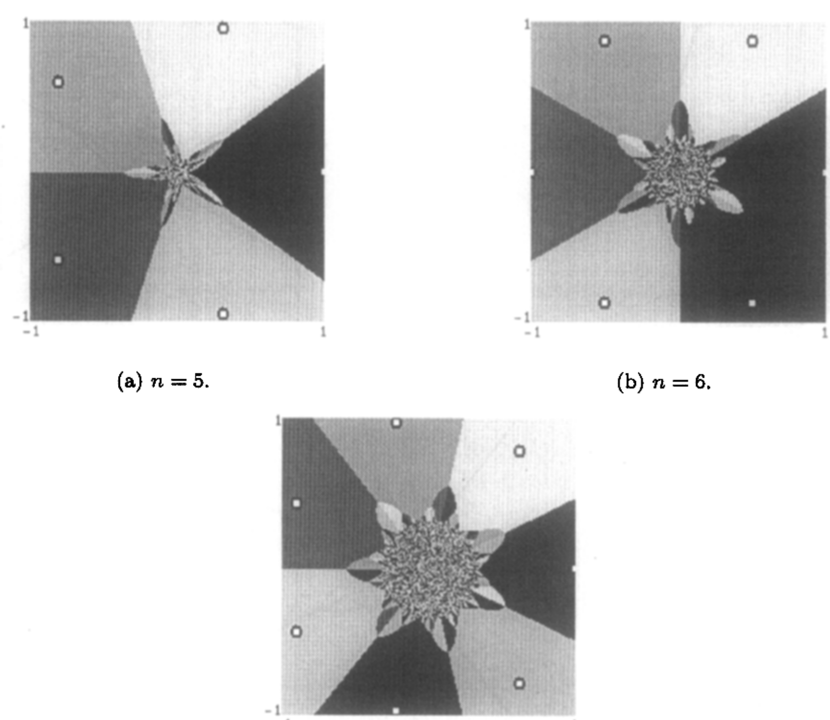


(a) $n = 2$.

(b) $n = 3$.

(c) $n = 4$.

Figure 1. Basins of attraction for the roots of $f_n(z) = z^n - 1$ in the complex region $[-1, 1] \times [-1, 1]$ using Laguerre function.

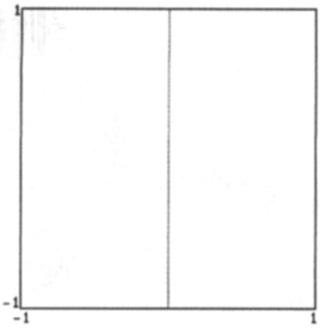


(a) $n = 5$.

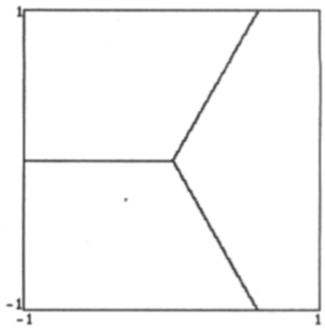
(b) $n = 6$.

(c) $n = 7$.

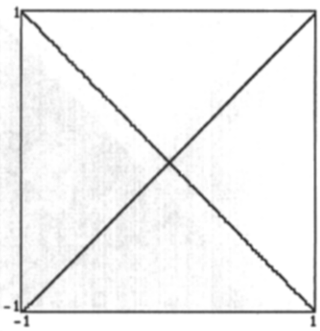
Figure 2. Basins of attraction for the roots of $f_n(z) = z^n - 1$ in the complex region $[-1, 1] \times [-1, 1]$ using Laguerre function.



(a) $n = 2$.

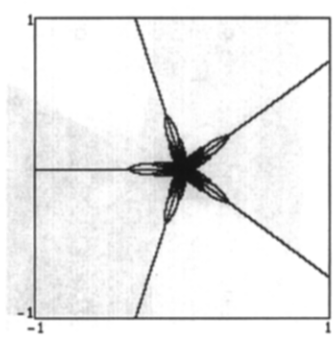


(b) $n = 3$.

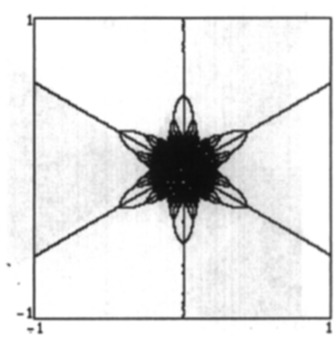


(c) $n = 4$.

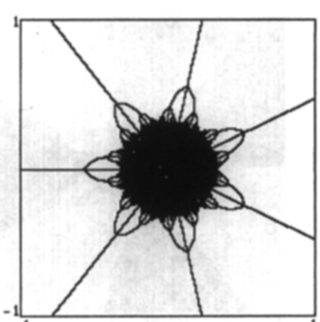
Figure 3. Laguerre Julia sets for the functions $f_n(z) = z^n - 1$ in the complex region $[-1, 1] \times [-1, 1]$.



(a) $n = 5$.

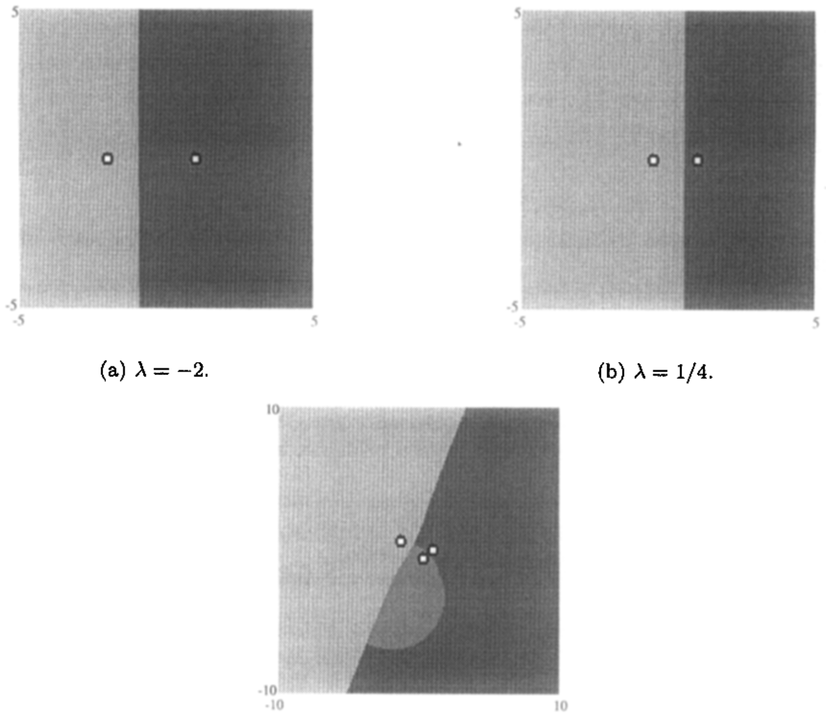


(b) $n = 6$.



(c) $n = 7$.

Figure 4. Laguerre Julia sets for the functions $f_n(z) = z^n - 1$ in the complex region $[-1, 1] \times [-1, 1]$.



(a) $\lambda = -2$.

(b) $\lambda = 1/4$.

(c) $\lambda = \imath$.

Figure 5. Basins of attraction for the roots of $p_\lambda(z) = z^3 + (\lambda - 1)z - \lambda$ using Laguerre function.

constitute $A(-\imath)$. For $\lambda = -2$, blue regions constitute $A(1)$; green regions constitute $A(-2)$. For $\lambda = 1/4$, blue regions constitute $A(1)$; green regions constitute $A(-1/2)$. For $\lambda = \imath$, blue regions constitute $A(1)$; green regions constitute $A(-0.5 + 0.5\sqrt{1 - 4\imath})$; red regions constitute $A(-0.5 - 0.5\sqrt{1 - 4\imath})$. The common boundaries of these basins of attraction constitute the Julia set $J(L)$. If after 200 iterations no such convergence was observed, the routine would skip to the next grid point.

5. CONCLUSIONS

In practical problems there is often enough *a priori* knowledge of the desired root of the equation to ensure that convergence of the iterations is not a problem. When *a priori* knowledge is poor, it is well known that it is often advisable to use a method which converges independently of the starting values, such as the Laguerre function, until a good approximation is obtained and then to switch over to a more rapidly converging method such as Schröder's or König's method.

We observe that for $n = 2$, 3, and 4 the Julia set of the Laguerre method as applied to $f_n(z) = z^n - 1$ is identical to the Voronoi diagram of the corresponding roots; in the case $n = 2$, the boundary of the roots' basins of attraction is the right bisector of the line joining the two roots (Figure 3a); in the case $n = 3$, the roots' basins of attraction are the connected regions separated by the lines bisecting the edges of the triangle with the roots as vertices (Figure 3b); in the case $n = 4$, the roots' basins of attraction are the connected regions separated by the darkened lines of Figure 3c, which bisect the edges of the square with the roots as vertices. For $n > 4$ a slight perturbation in the roots' immediate basins of attraction exists in a small neighbourhood near the origin, the unwanted periodic point.

A similar result holds also for the Julia set of the Laguerre method as applied to $p_\lambda(z)$; in the case $\lambda = -2$, $1/4$, the boundary of the roots' basins of attraction is perpendicular to the line joining the two roots (Figure 5a,b) whereas for all the other values of λ the boundary of the roots' basins of attraction is a regular Euclidean figure (see, for example, Figure 5c).

We conclude that the Laguerre function has excellent results when applied to polynomials up to four order, while for greater degree polynomials it has very good results only if we extend its radius of convergence (see [8]), because it exhibits fractal Julia sets in a small neighborhood around the origin, which grows as the degree of the polynomial increases.

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