PG4200: Algorithms And Data Structures

Lesson 05:

Tree Maps

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Overview

- Together with lists, *maps* are the most common data structures used in programs
- No “index” of position, in contrast to arrays/lists
- Association between “**keys**” and “**values**”
  - Ie keys are *mapped* to values

Insert/Find

- When inserting a new *value V*, also insert a *key K* for it
- A K key is an object (eg, String, Integer, etc.)
- When we want to retrieve V, we will query for its K
- The idea is that retrieving data for given K should be *fast*

Examples

- Collection/map of songs: the Song class is the V, where the K is the String title

- Collection/map of users: the User class is the V, where the K is the Integer id of the user

Search

- How can we efficiently search for a K?
- Let’s assume that Ks are orderable, eg a *compareTo()* is defined for them
  - This is the case for String and Integer

Search Value in Array

7	2	4	9	1	3	0	5	8	6	10

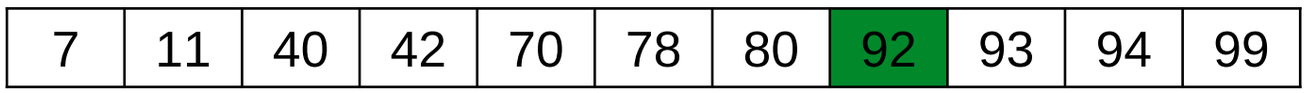
- Assume content of array is numeric keys
- And we want to search for key 8
- Scan from left-to-right, and check if == 8
- Complexity O(n)

Search In Sorted Array

7	11	40	42	70	78	80	92	93	94	99

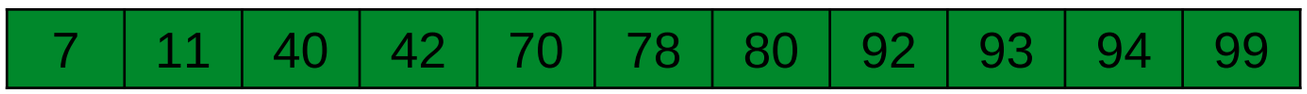
- Assume we search for 92
- Is there any clever way to check if 92 by exploiting knowledge of array being sorted?

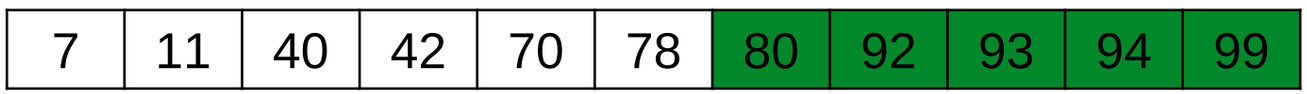
Binary Search

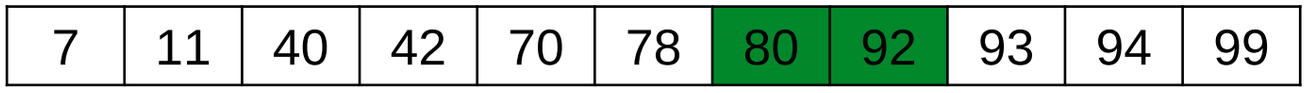


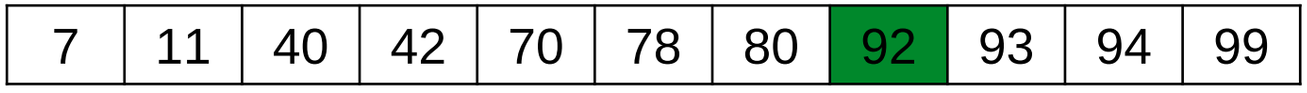
- Check middle of array, ie 78
- As 78 < 92, check middle of subarray between 80 and 99, ie 93
- As 93 > 92, ignore the rightmost values
- Look at middle of 80-92
- Finally find 92

Complexity









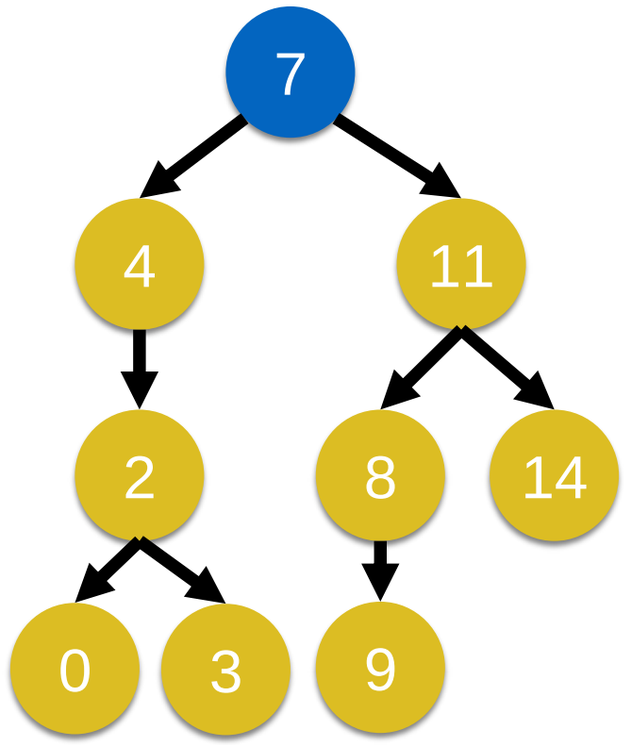
- At each step, look at area that is half in size
- How many times *x* we can divide N by 2 till we arrive to value 1?
- $N = 2^x \rightarrow x = \log(N)$
- Binary Search has O(log N) complexity, which is much better than O(N)

List/Array Based Map

- Each element is an Entry object, containing K and V
- **Insertion: O(N)**, can keep list/array sorted
- **Deletion: O(N)**
- **Search** for K: **O(log N)** if sorted, otherwise O(N)
- *Can we do better?* Eg, **O(log N)** in all operations

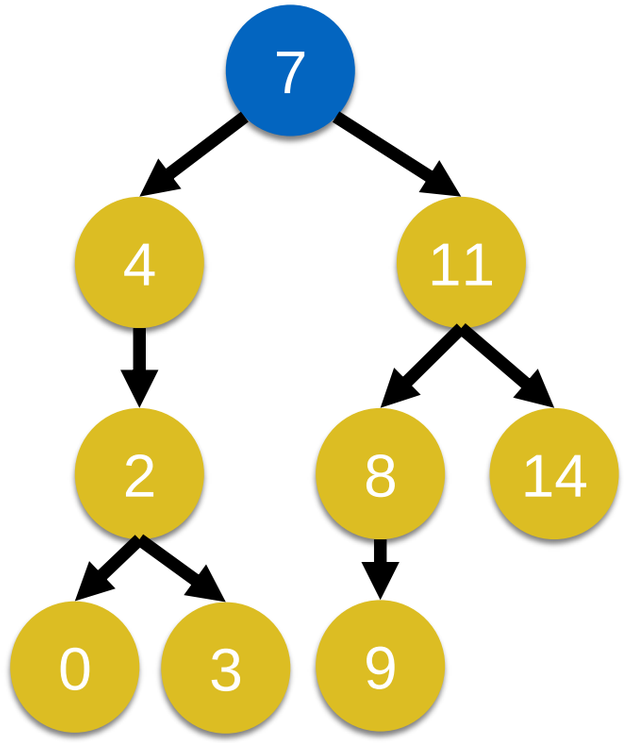
Tree-based Maps

- Data structures resembling a tree
- Nodes contain values, and links to child nodes
- Starting point is the root of the tree
- Two main / most famous versions
  - Binary Trees (“simple”)
  - Red-Black Trees (“very complex”, but high performance)



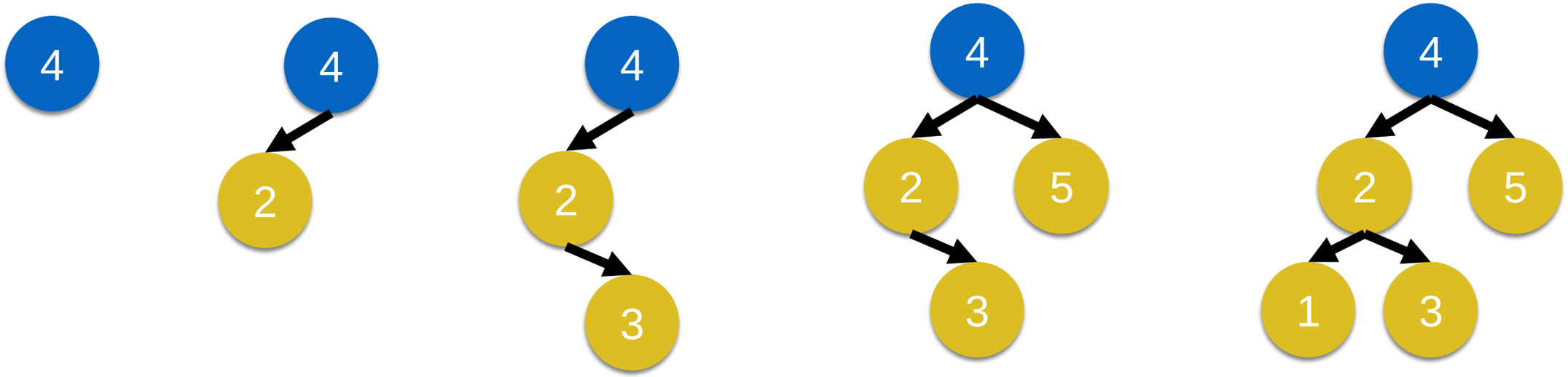
Binary Tree

- Each Node has 1 value and up to 2 children: *left* and *right*
- Values in *left* subtree are all *smaller*
- Values in *right* subtree are all *larger*
- Insertion/deletion do keep these invariants
- Search starts from root



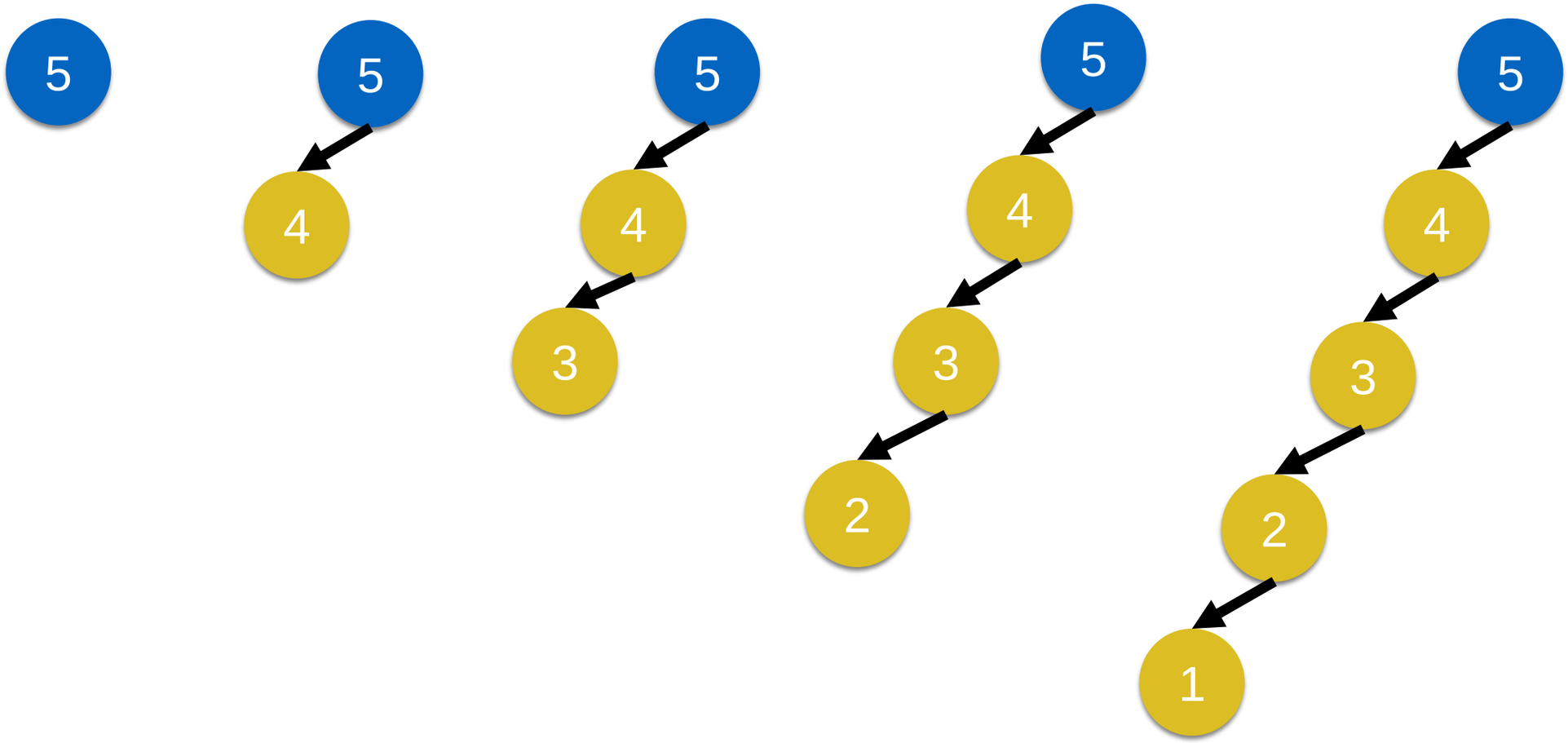
Insertion

- Assume inserting entries with keys 4, 2, 3, 5 and 1
- Recall: each node has a key and an associated value
- Here in the circles I am just showing the keys, not the values



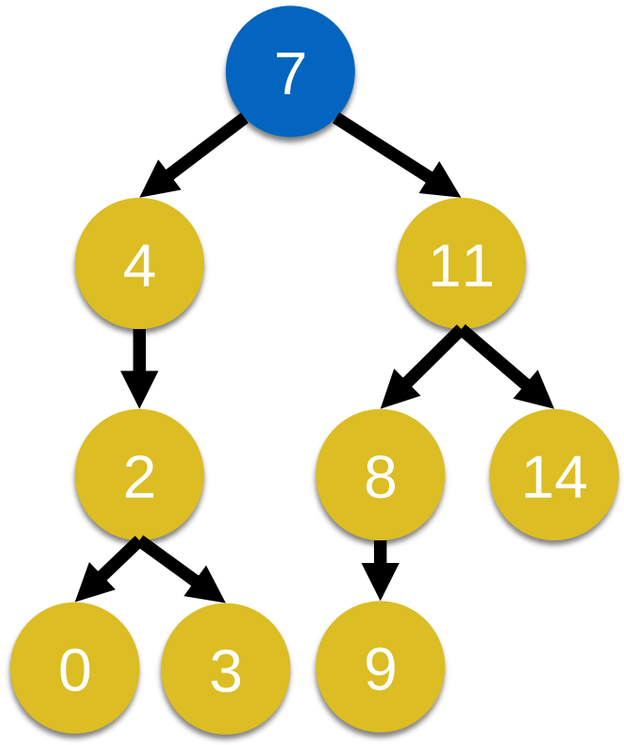
Insertion Order

- Assume inserting same entries but in this order: 5, 4, 3, 2, 1
- Shape of tree depends on order in which elements are inserted
- Worst case the tree degenerates into a list



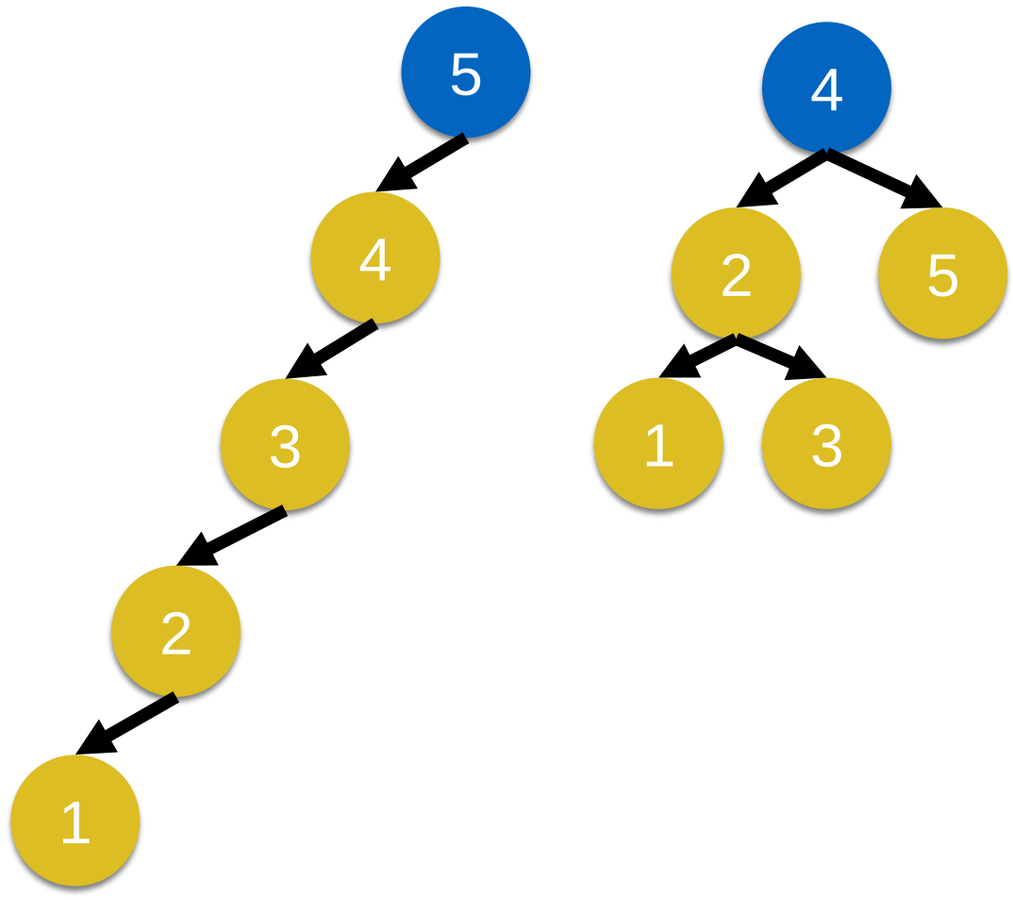
Search

- Start from root
- Recursively navigate left or right subtree based on when searched key is smaller or bigger than the one in the node
- Worst case: searched key is in a leaf node
- Max number of comparisons bound by *depth* of tree
  - Eg 4 in the example



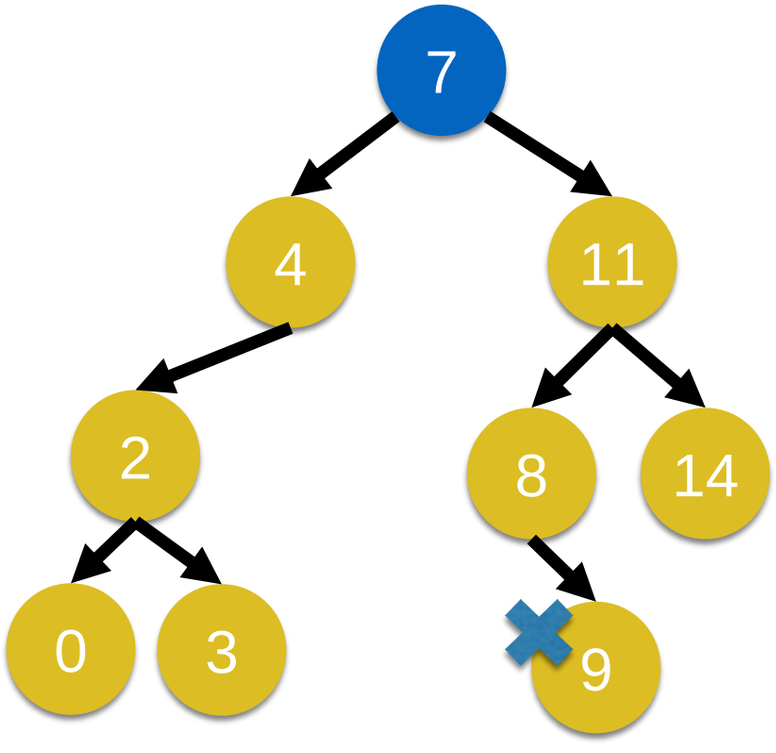
Tree Depth

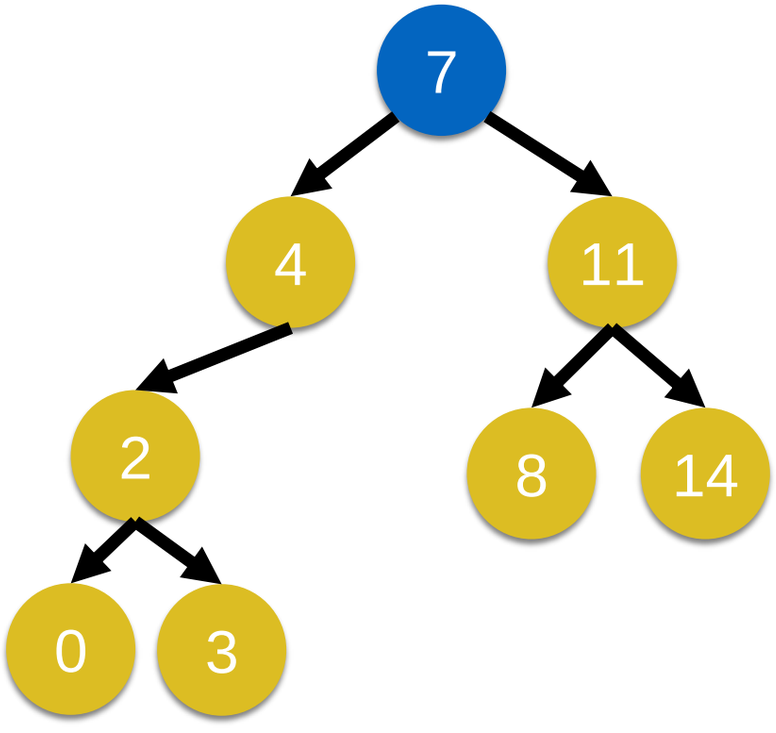
- Depth: longest distance between root and a leaf
- Given N nodes
- Depth bounded between N (worst case) and log(N) (best case)



Deletion

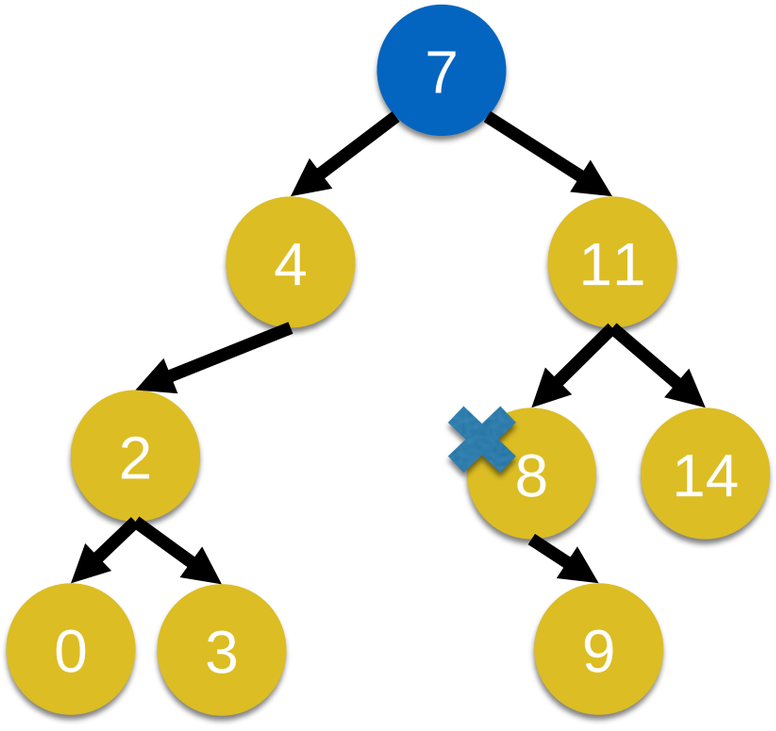
- To delete K, need to find it first
- Actual deletion depends on numbers of children of K
- Simple case: deleting with no children, eg delete key 9, just set to null the link of the parent (8 in this case)

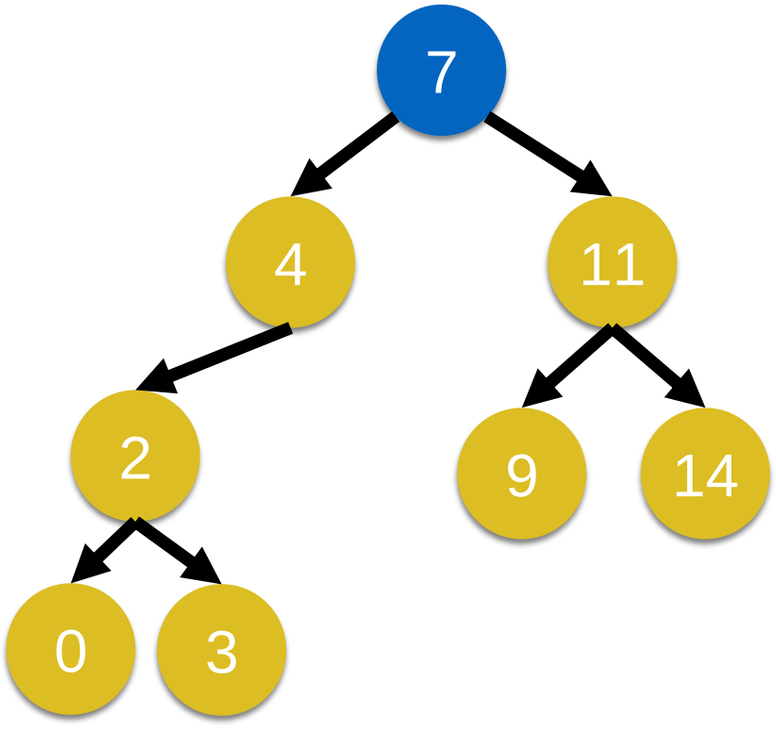




Deletion With 1 Child

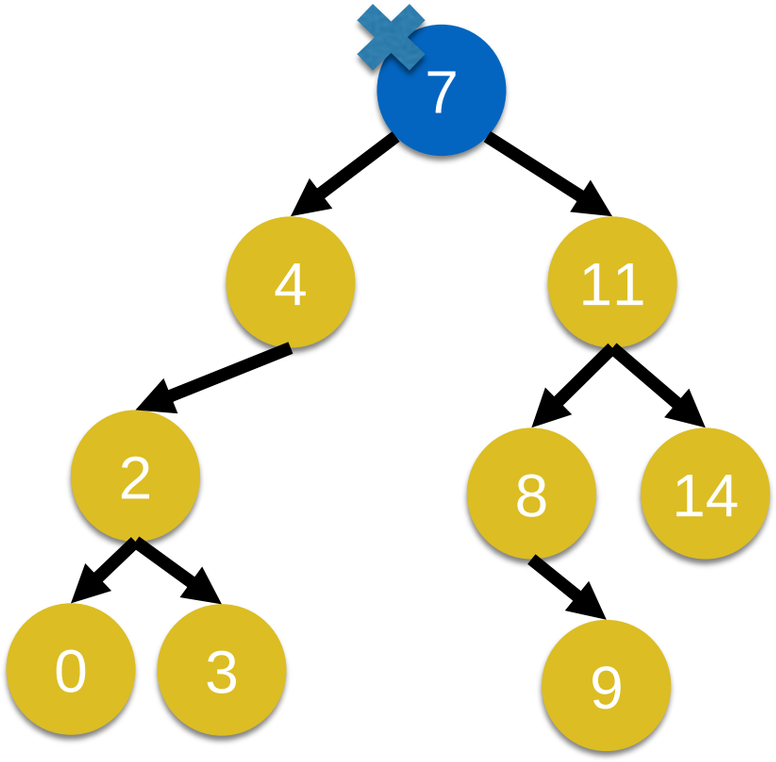
- Eg, delete 8 (or 4)
- Look at the parent of deleted node (ie 11)
- Change the child link of parent to the child of deleted node (ie 9)

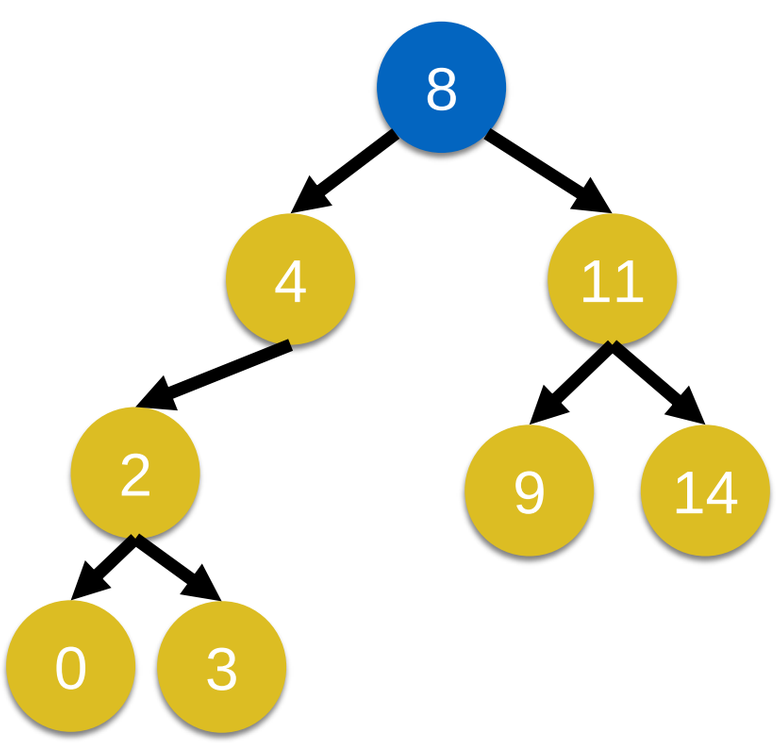




Deletion With 2 Children

- Eg, delete 7 (or 2 or 11)
- Find *minimum* in right subtree (ie 8) and *delete* it
- Replace the content of right-minimum into deleted node





Runtime Complexity

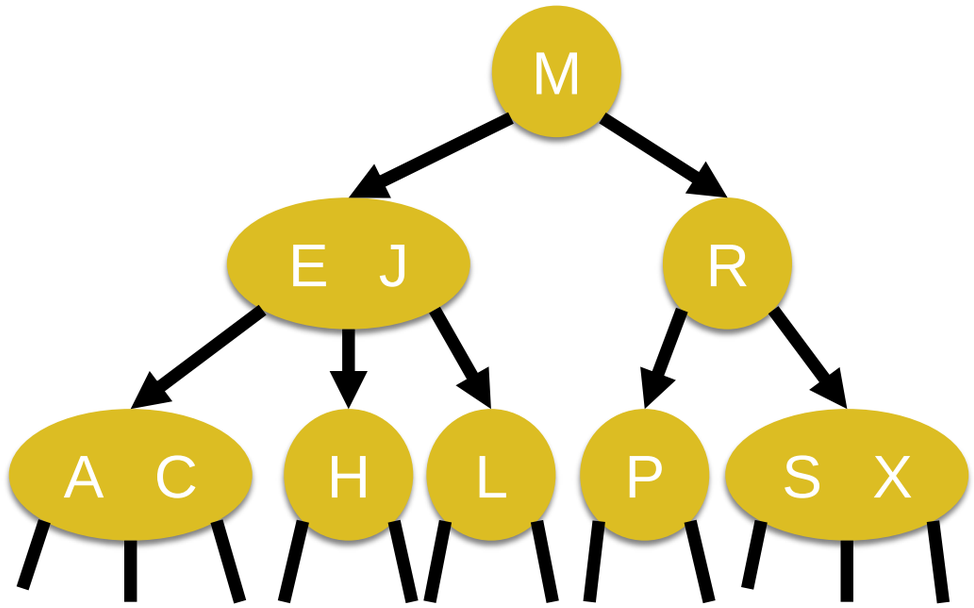
- In worst case, Binary Map degenerates into a list
- All operations have **O(N)** complexity in worst case
- But random trees would be nearly balanced, and operations would hence be in the O(log N) order
  - So good performance on average

Red-Black Trees (RBT)

- Complex type of tree
  - Likely the most complex algorithm we see in this course (and likely in your degree as well…)
  - If you have difficulties in this course, keep this one as last to study/revise, unless you are aiming for an **A** grade…
- Insertions and deletions *always* keep the tree balanced
- Guarantee **O(log N)** on all operations
  - Ie, insertion, deletion and search
- Extremely efficient
  - *Can **guarantee** search into **B**illions of data with just 30-50ish node lookups*

2-3 Search Trees

- Before discussing RBT, let's consider 2-3 Trees



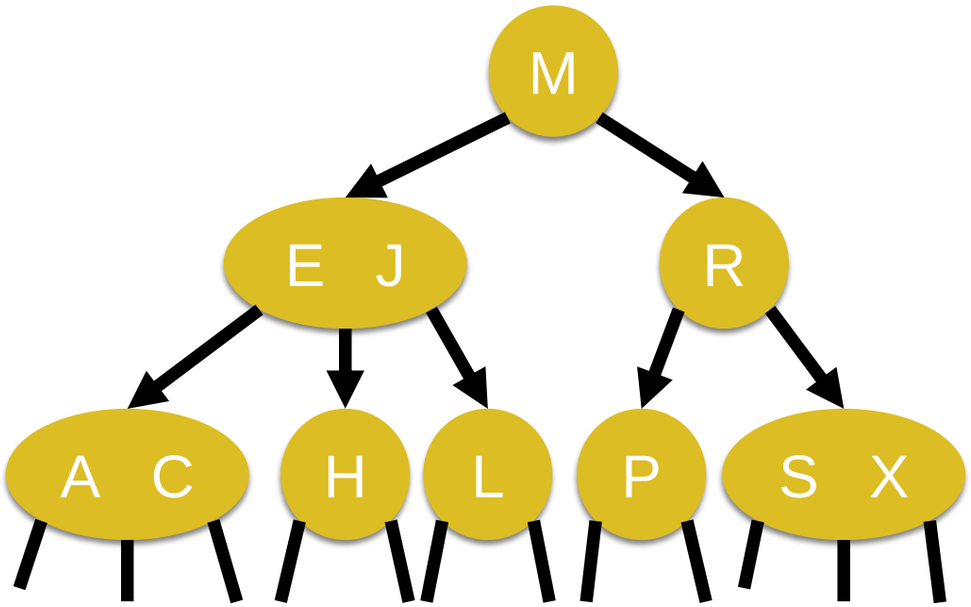
- 2-3 Tree: composed of 2-nodes and 3-nodes

- 2-node: 1 value, 2 children (left and right)

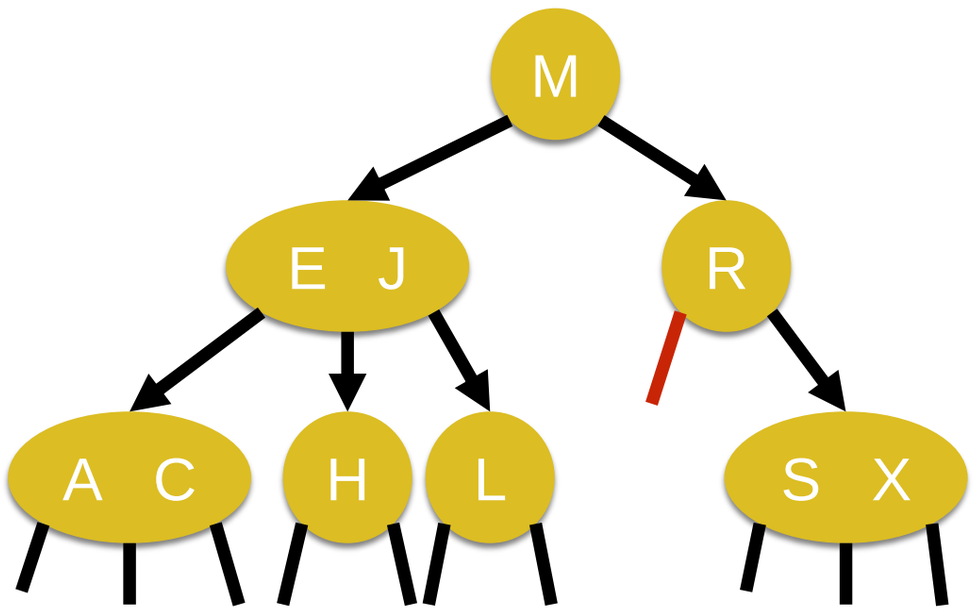
- 3-node: 2 values, 3 children (left, middle and right)

Perfectly Balanced Trees

- When distance from root to any null-child-link is the same
  - Important property to guarantee O(log N)



Balanced



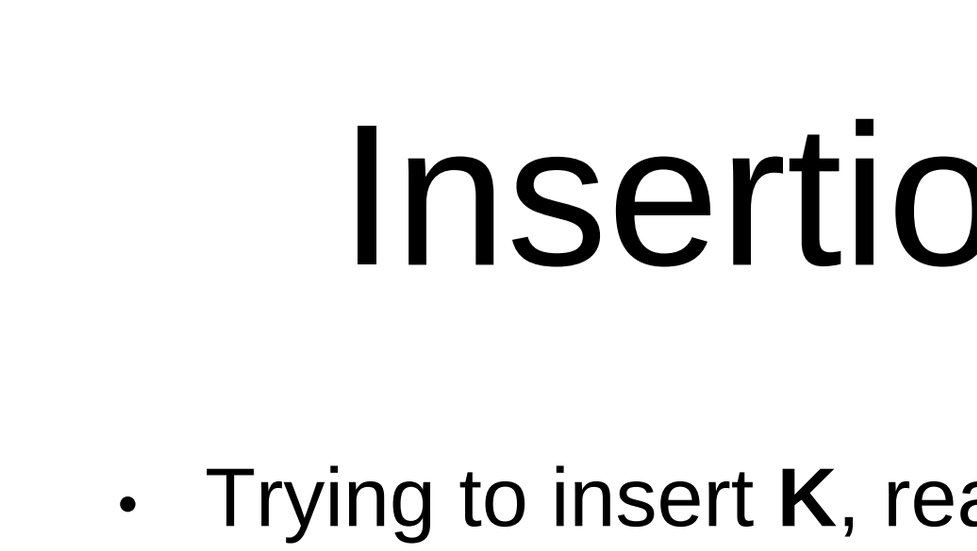
Not Balanced

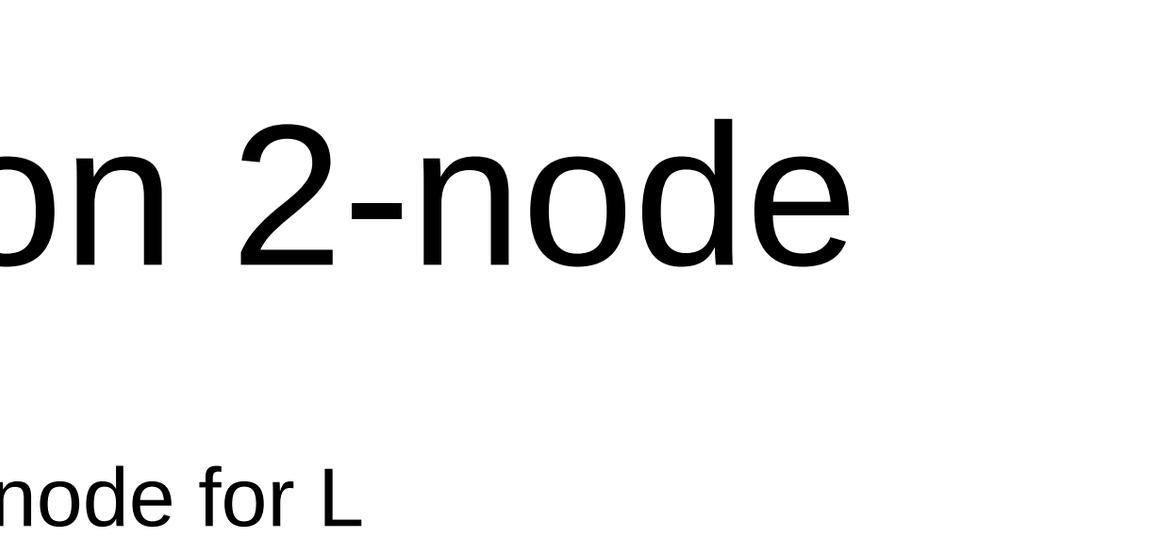
Insertion

- After an insertion we want to keep the tree balanced
- Based on key, search starting from root to the position where to insert
  - This takes O(log N) steps
  - Same concept as Binary Search Trees
- Insertion will depend on position, but many cases to consider
- Insertion on empty tree: just create a 2-node root

Insertion on 2-node

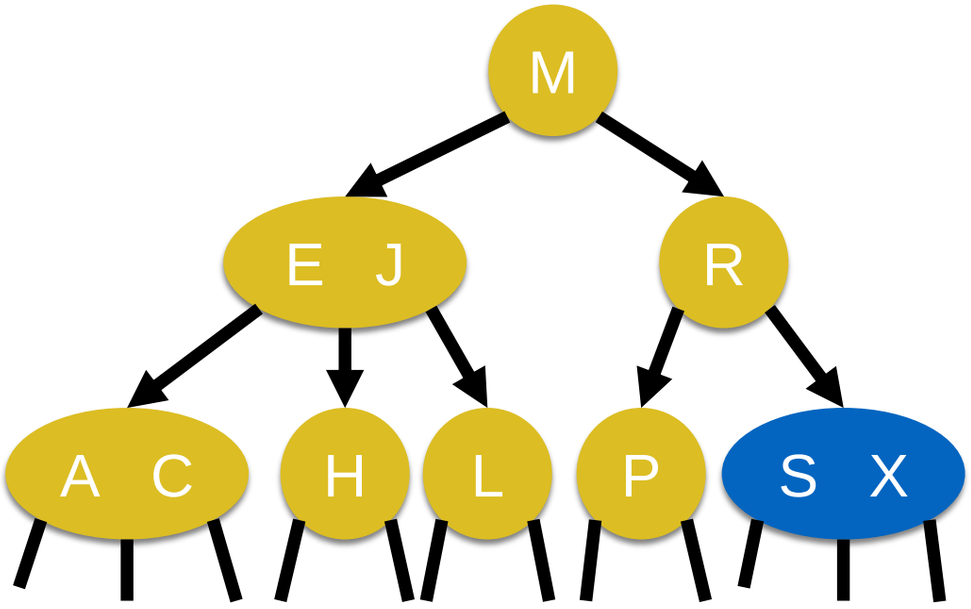
- Trying to insert **K**, reached node for L
- Node for L is a 2-node: easy, just transform into 3-node

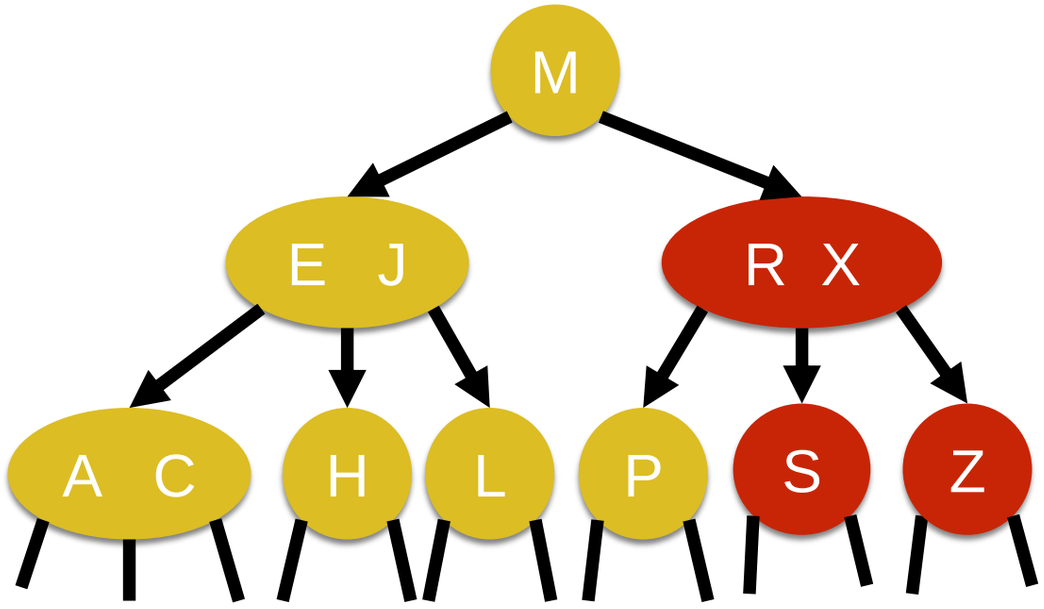




Insertion on 3-node, where Parent is a 2-node

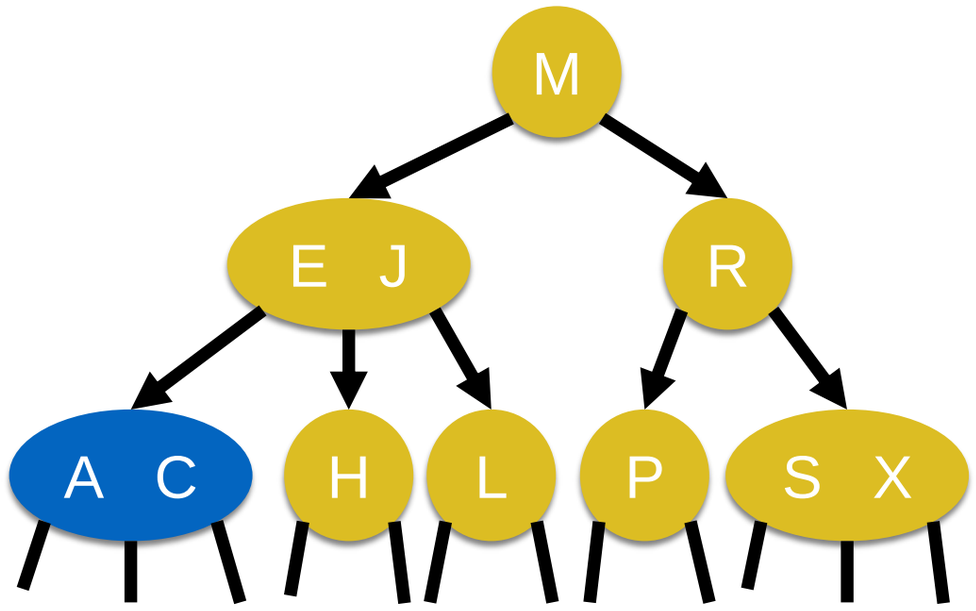
- Inserting **Z**
- 3-node (S,X) with 2-node parent (R)
- Split node, and transform parent into a 3-node

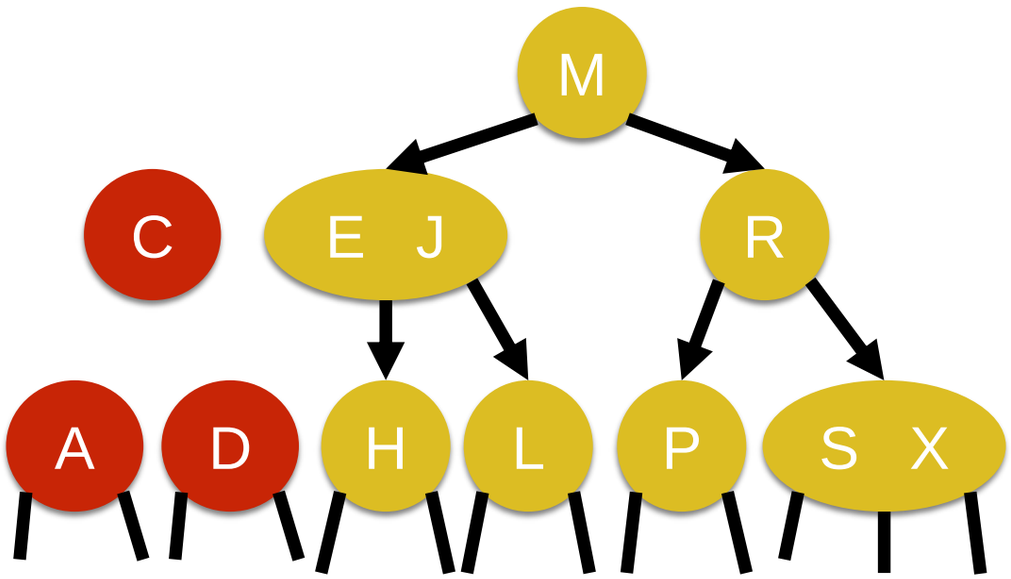


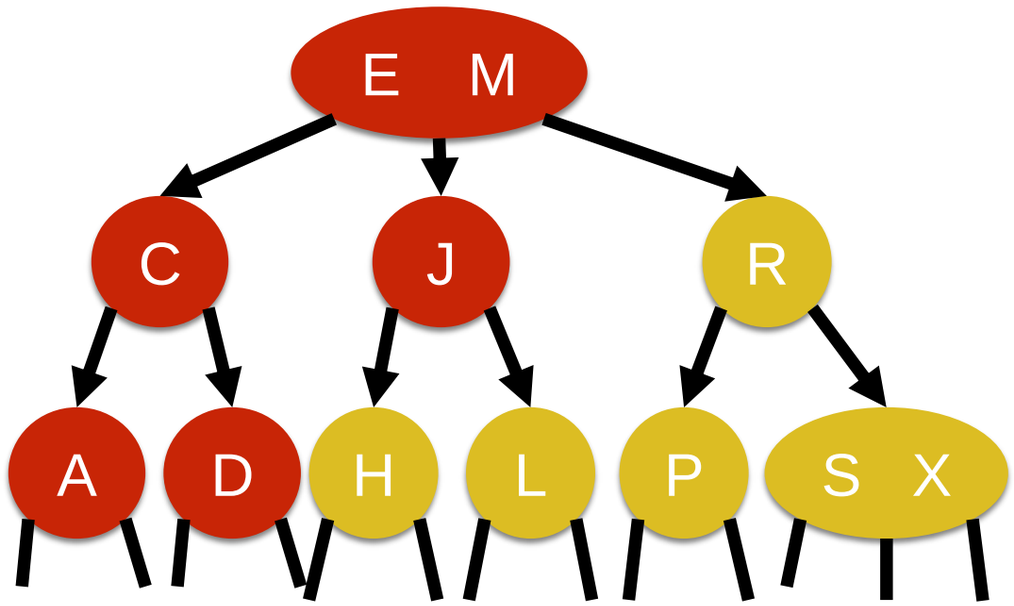


Insertion on 3-node, where Parent(s) are 3-nodes

- Inserting **D**
- 3-node (A,C) with 3-node parent (E,J)
- Split node, add middle to parent, repeat recursively bottom up until find 2-node

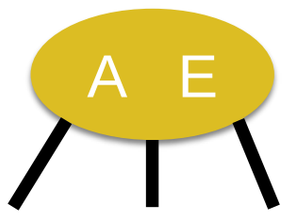


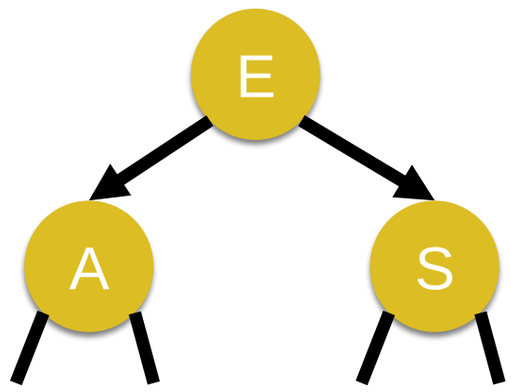




Insertion on 3-node Root, with No Children

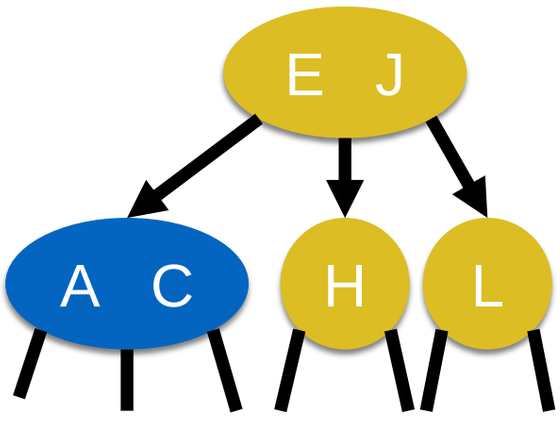
- Inserting a new value, eg **S**
- Split the 3-node, and create new 2-node

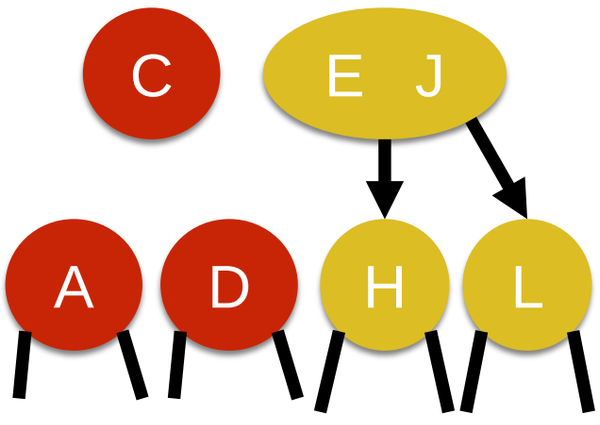


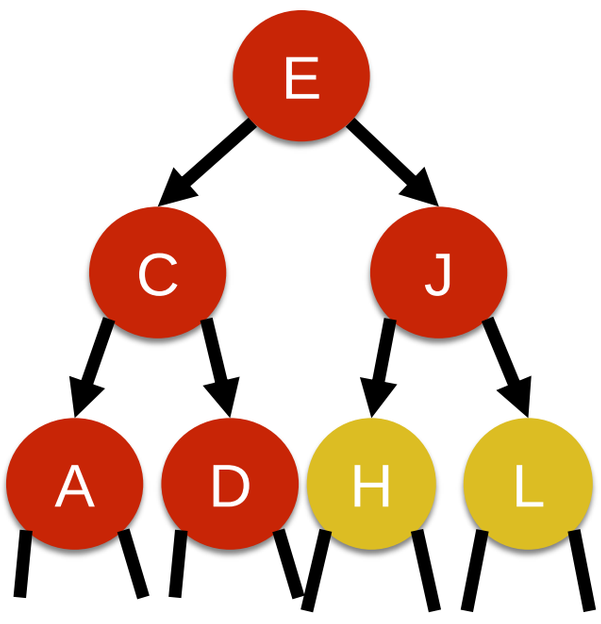


Splitting The Root

- Adding **D**
- What happens when going bottom-up adding recursively to parent we reach a 3-node root?
- Same as root with no children
- Note: this is the only case in which depth increases







Cost

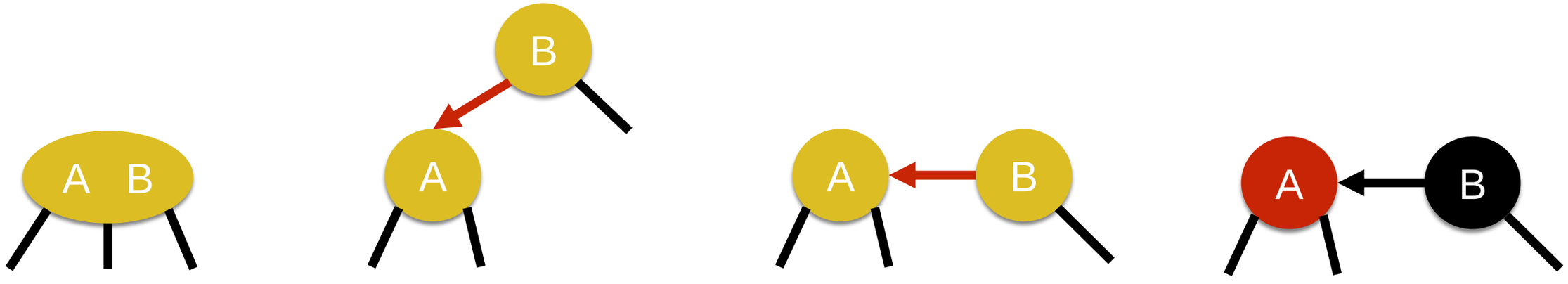
- Searching for position to insert is O(log N)
- Worst case, when inserting, need to go upward till the root, which add a further O(log N)
- So final cost of insertion is O(log N)
  - Recall, constants are not important in asymptotic analyses, so O(log N) + O(log N) = 2O(log N) = O(log N)
- After each insertion, tree is perfectly balanced

From 2-3 Trees to RBT

- Using 2-3 trees is cumbersome, as we need to handle both 2-nodes and 3-nodes
  - And not particularly efficient when transforming 2-nodes into 3-nodes, and vice-versa
- RBT: represents 2-3 Trees by only using 2-nodes
- Each link between nodes has a color: RED or BLACK
  - This is where the RBT name comes from

3-nodes

- Represented with two 2-nodes
- Link with smaller is RED, and RED links can only lean left
- Any other kind of link is BLACK
- For simplicity, using color directly on the node to represent link to parent
- Cannot have two consecutives RED links



Black-links Balance

- In RBT there is only guaranteed balance for the black-links
- Every path from root to a black-null-link has same number of black links
- So, depth is still bounded by O(log N)

Rotations

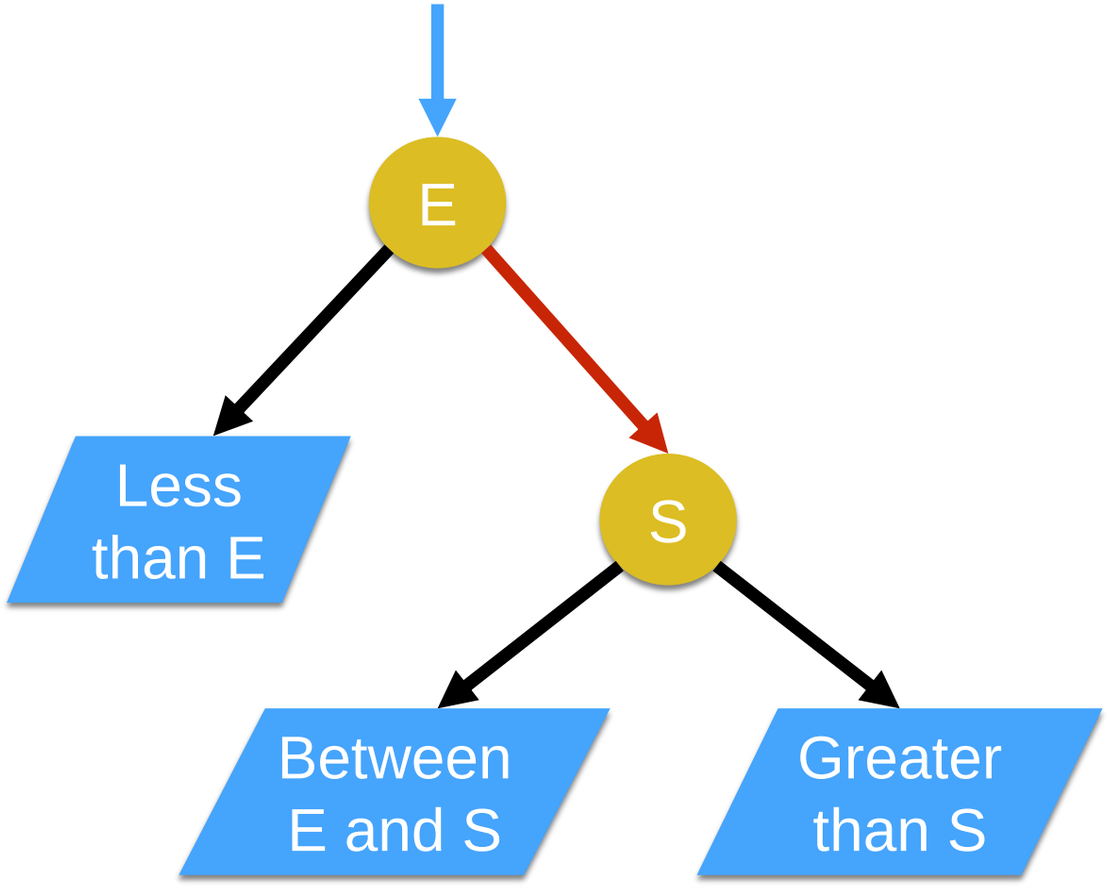
- When inserting/deleting, need to maintain RED-link properties, ie, most 1 leaning left

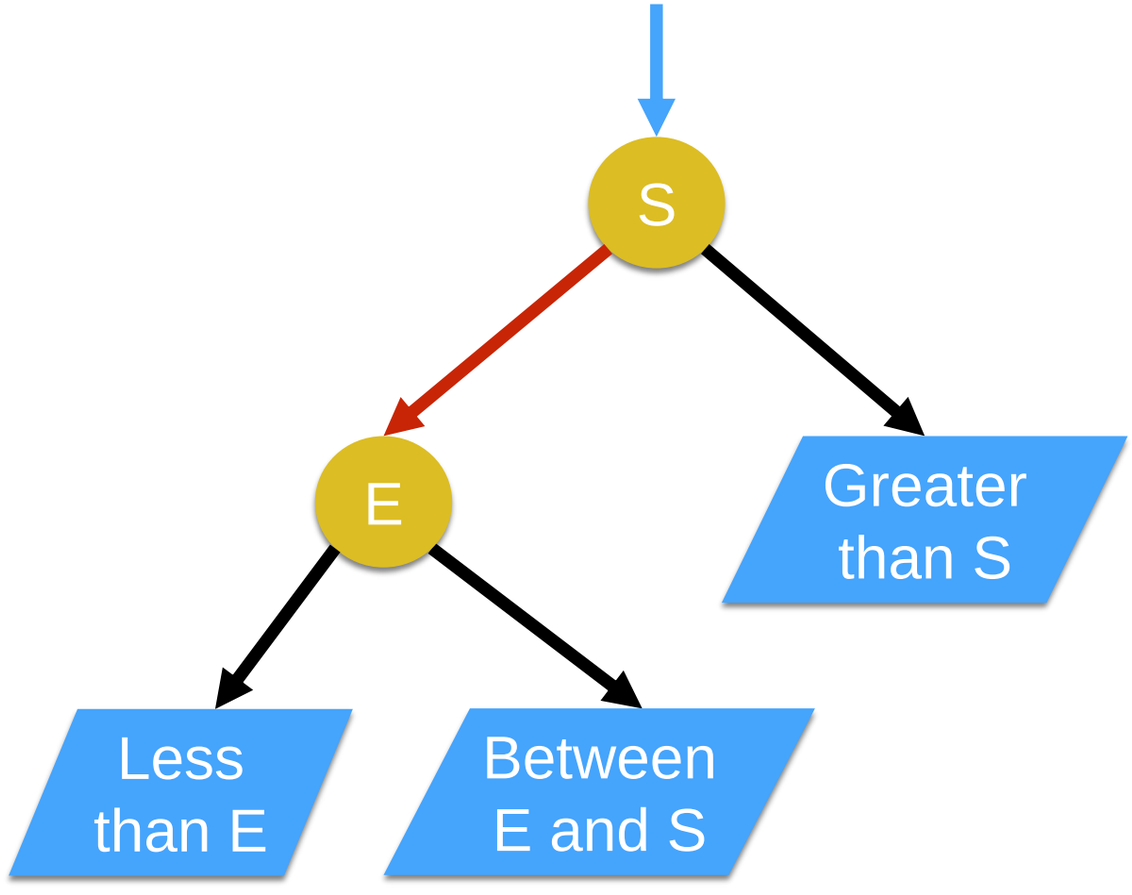
- So need to have transformation operators on the tree, which are used to maintain the tree balance

- These operations have O(1) cost

Left Rotation

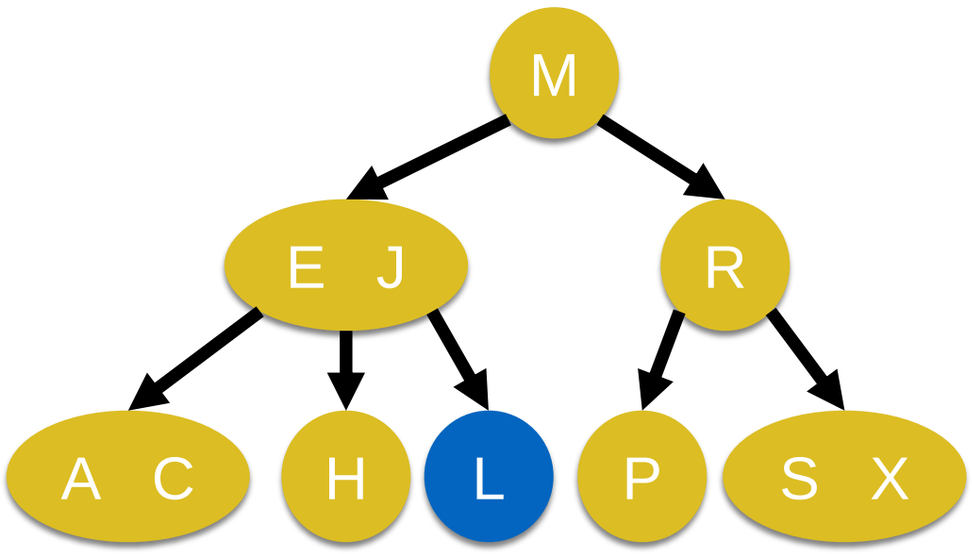
- S is moved up
- Still RED link between E and S, but now E is child of S
- Left child of S now becomes right child of E
- Color (RED/BLACK) of link toward E is maintained in new rotated root S
- Right-rotation does the exact opposite

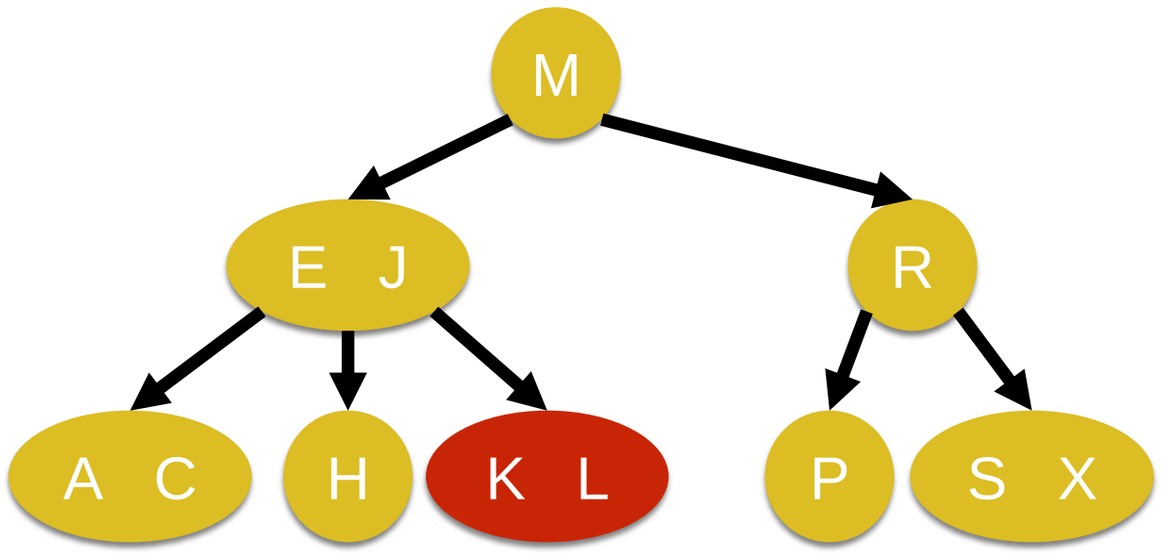




Recall Insertion on 2-node

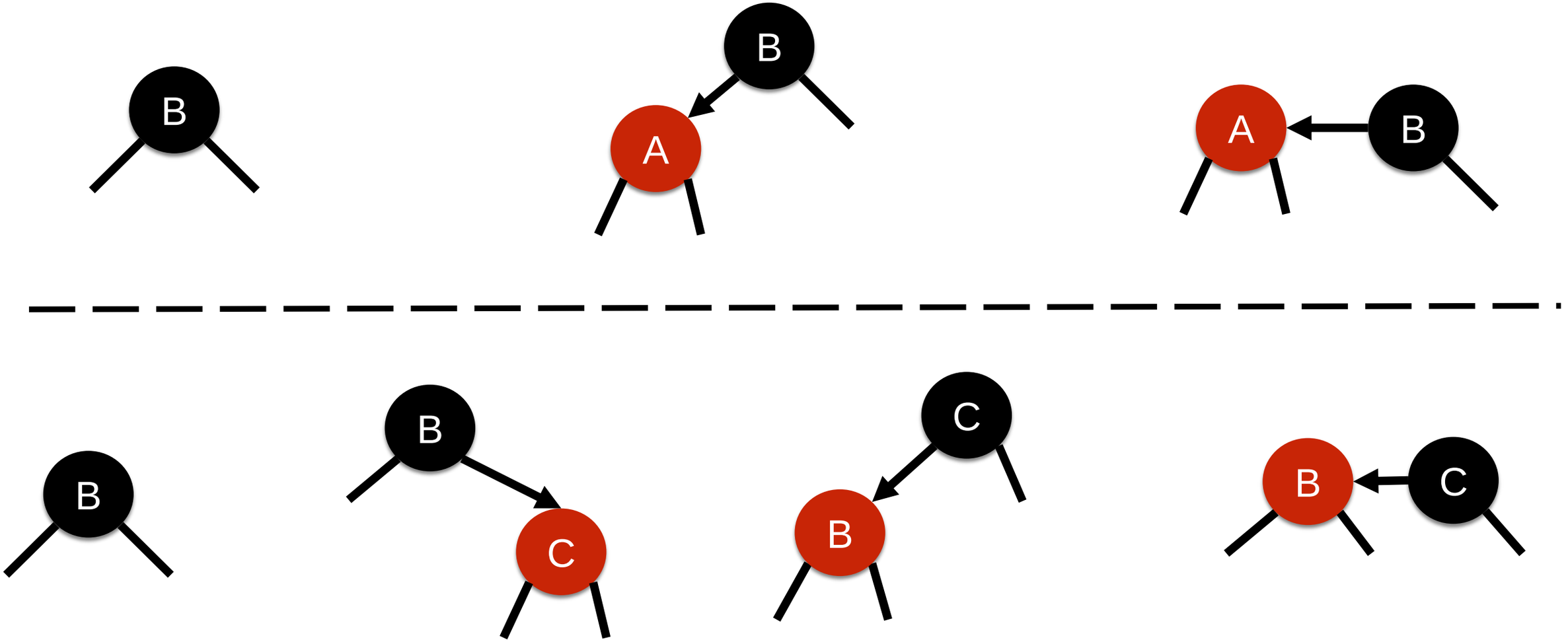
- 2-3 Trees
- 2-node transformed into 3-node





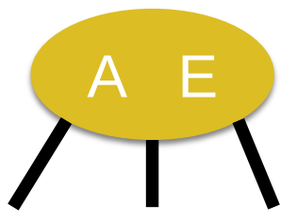
Insertion into 2-node

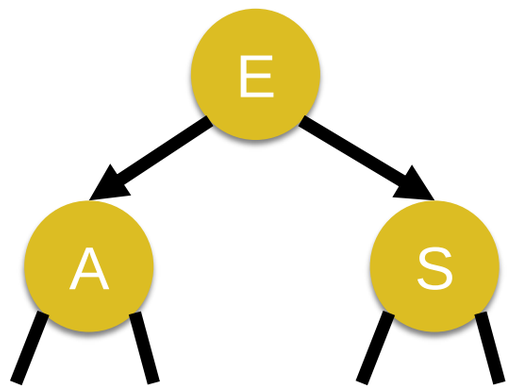
- 2-node will need to transformed into a 3-node
- In RBT, need to have new node with RED link
- Need left rotation if new value is greater



Recall Insertion on 3-node Root, with No Children

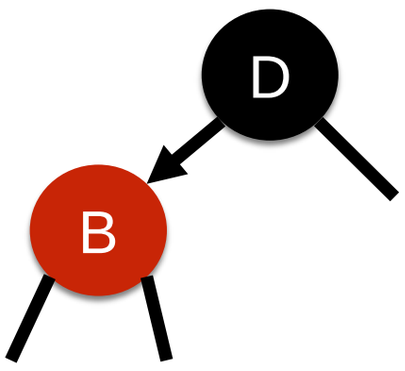
- 2-3 Trees
- Split the 3-node, and create new 2-node

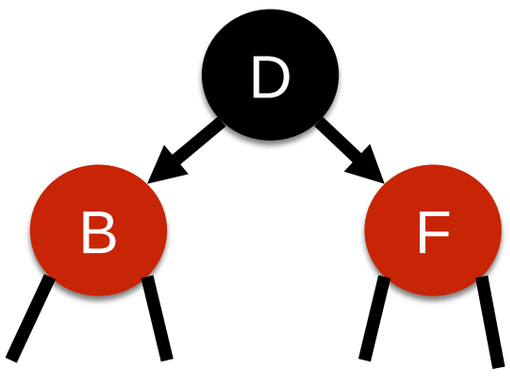


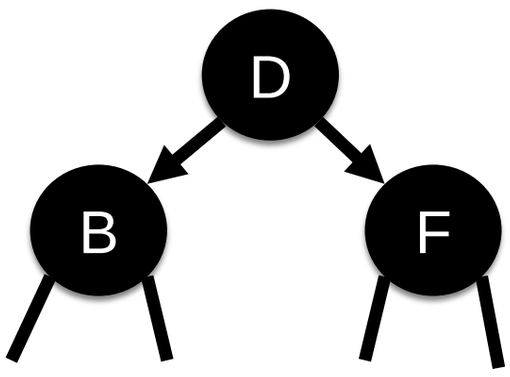


Insertion in Single 3-node

- 3 cases, based on whether new value is larger, smaller or between
- Simple case: new value is greater, eg **F**
- Flip color from RED to BLACK once inserted

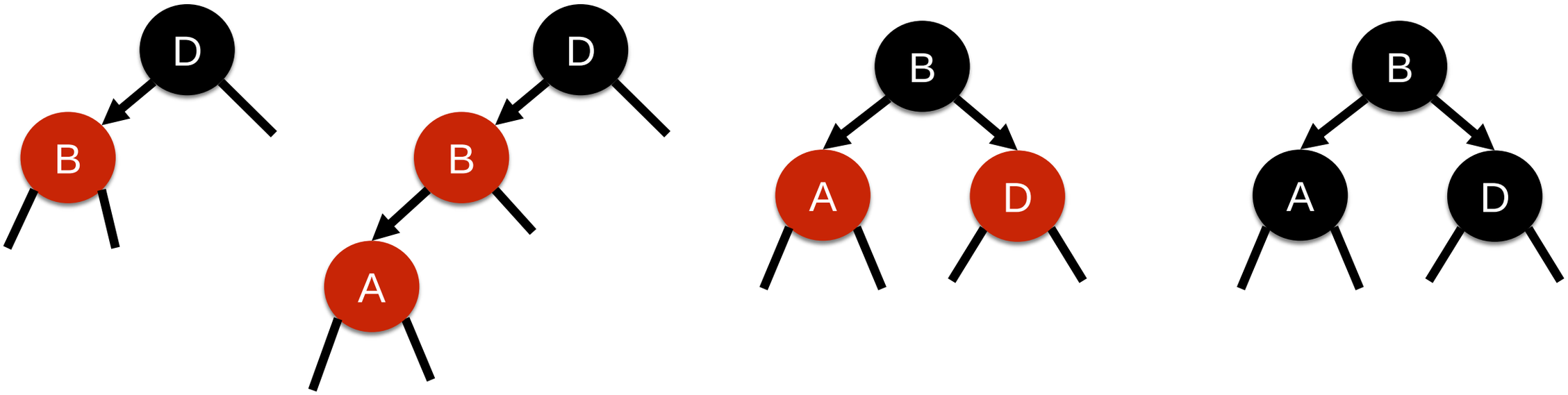






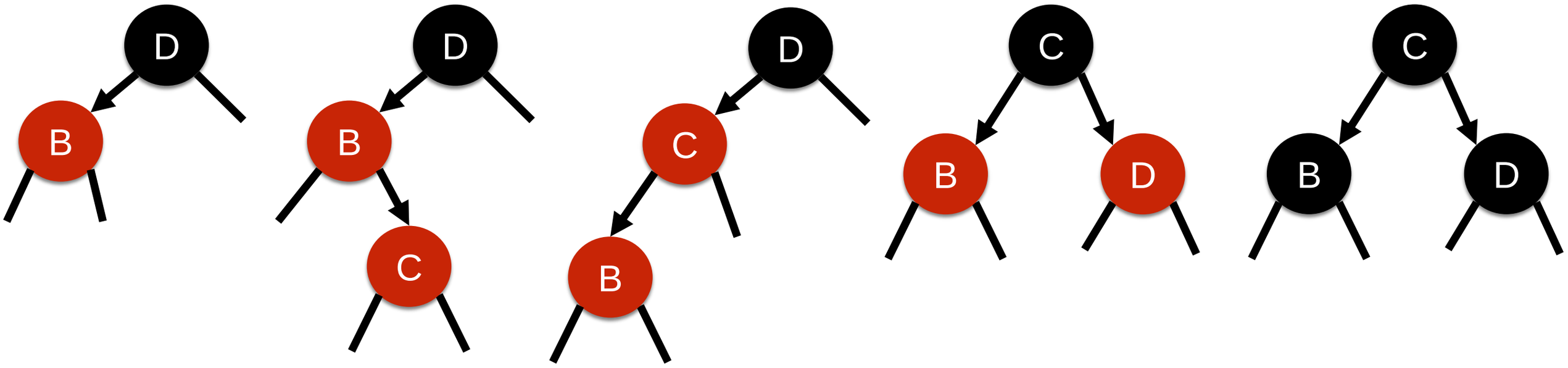
Insertion in 3-node: Smaller Case

- Smaller value, eg adding **A**
- After insertion, need right rotation
- Finally, need flipping of RED colors



Insertion in 3-node: Between Case

- Between value, eg adding **C**
- After insertion, need left rotation, followed by right rotation
- Finally, need flipping of RED colors

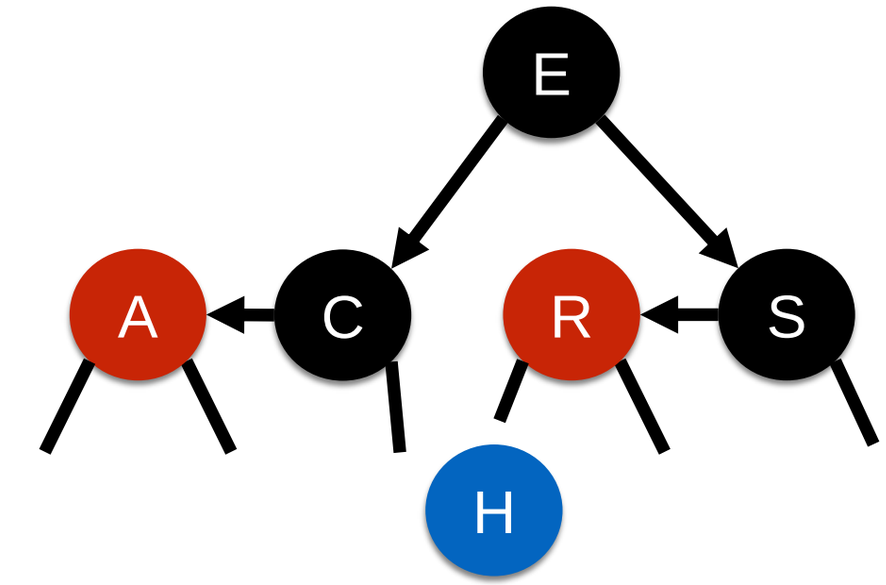


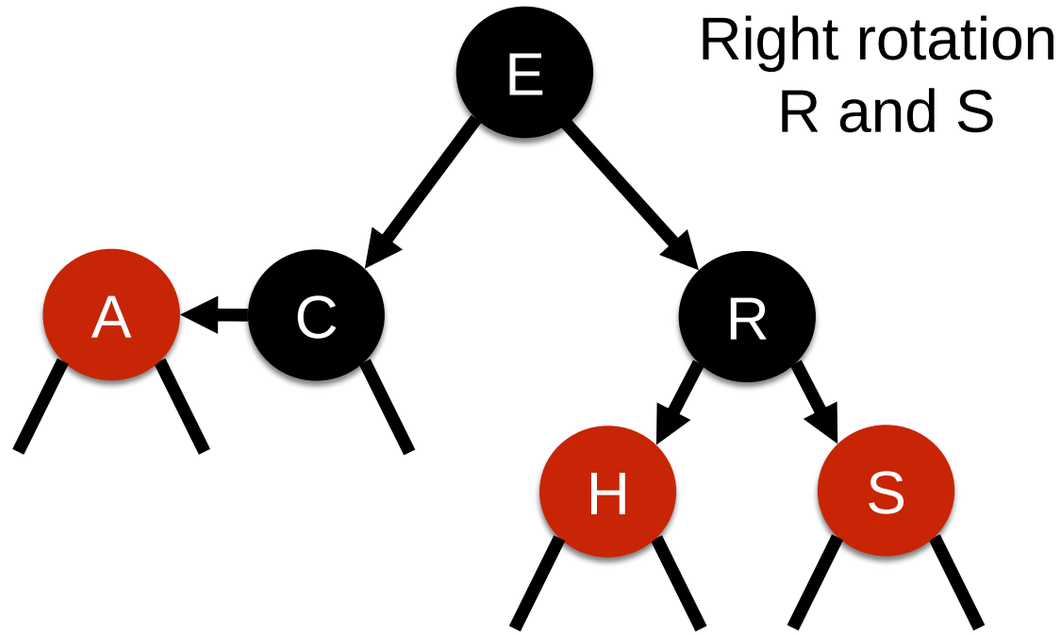
Cost of 3-node Insertion

- 3 cases, based on inserted key
- Larger: color flip
- Between: left rotation, right rotation, color flip
- Smaller: right rotation, color flip
- These are constant costs

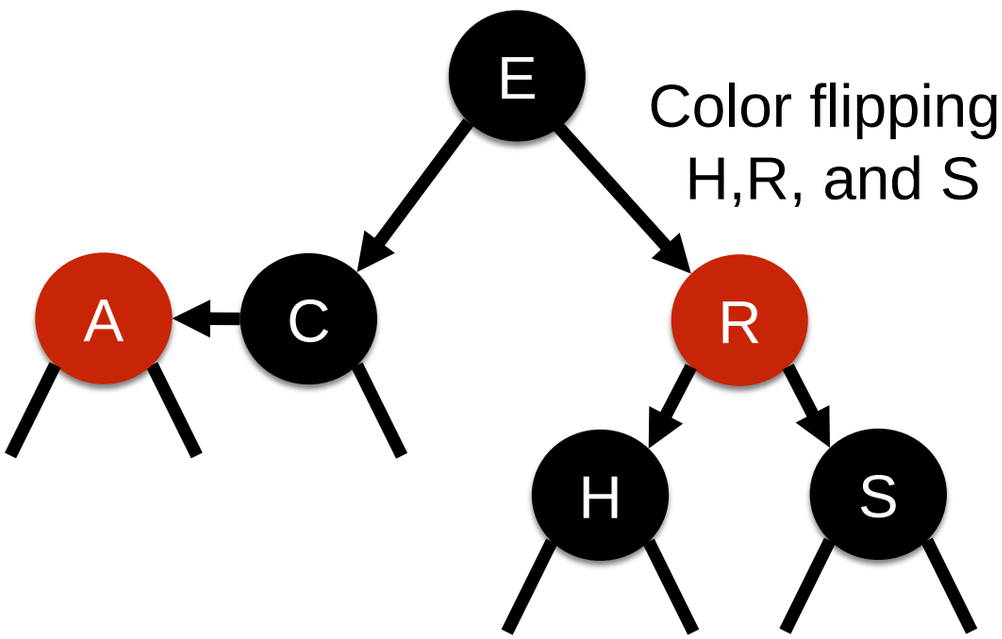
Insertion in Leaf 3-Node

- Flipping color makes parent RED
- Root is always put back to BLACK once insertion completed
- Inserting **H**, which “smaller” value in the 3-node (R,S)

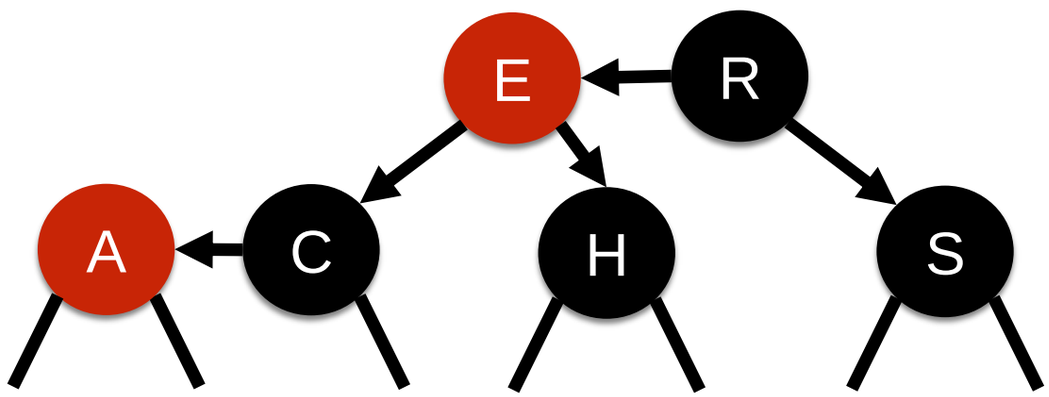




Right rotation:
R and S



Color flipping:
H,R, and S

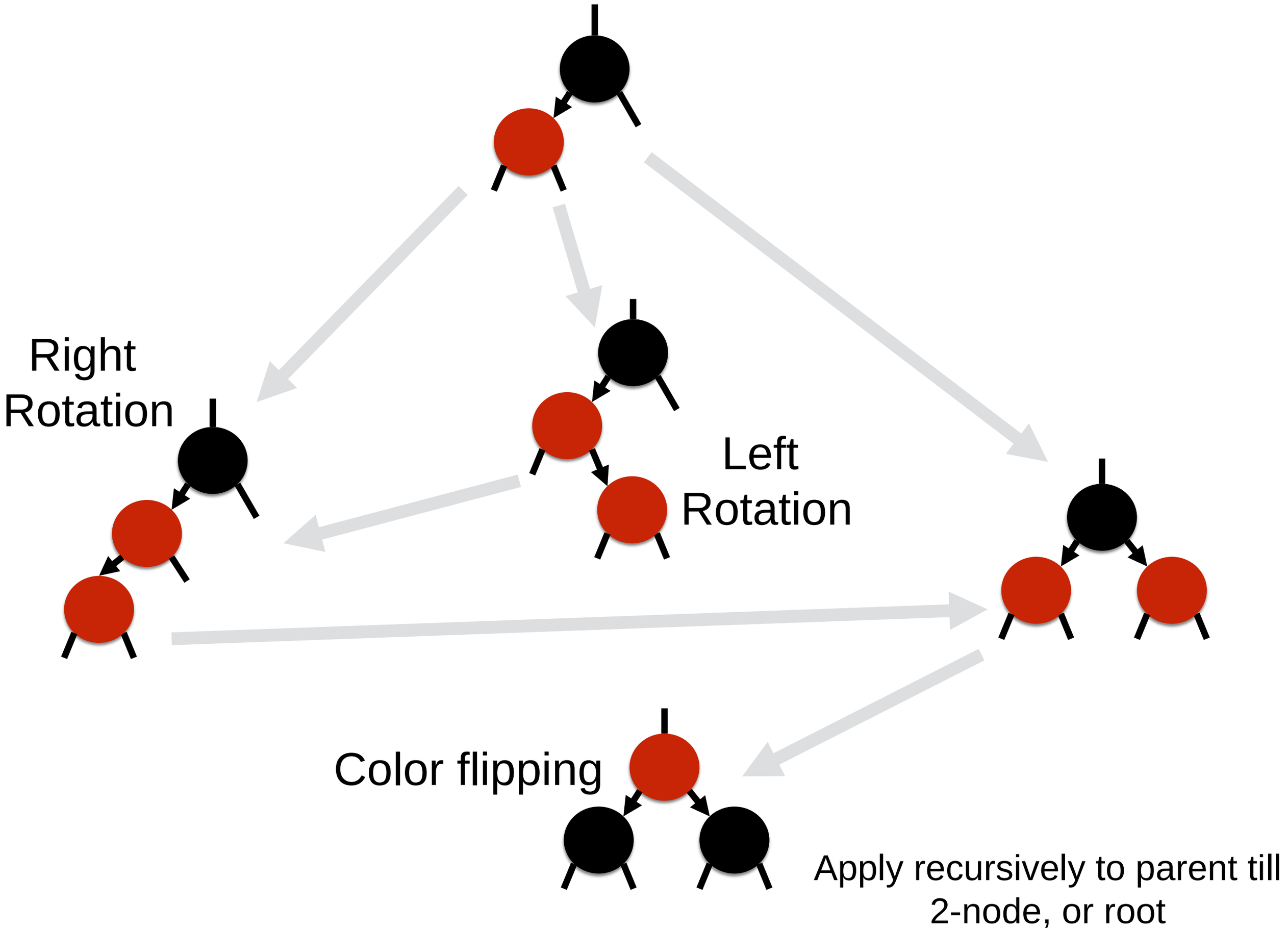


Left rotation: E and R

Recursion

- When flipping colors, need to fix parent
- If parent is a 3-node, we need to recursively handle the propagation of RED color up to a 2-node or the root

- Handling a RED child in a 3-node is equivalent to handling insertion of new RED node


Right
Rotation
Left
Rotation
Color flipping
Apply recursively to parent till
2-node, or root

Deletion

- "Insertion" in a RBT is the *easy* part…
- … the *fun* begins with the "Deletion" operator
- As Insertion is already complicated enough, we will not see Deletion in this course
  - But look at book/code if you are curious

Homework

- Study Book Chapter 3.1, 3.2 and 3.3
- Study code in the *org.pg4200.les05* package
- Do exercises in *exercises/ex05*
- Extra: do exercises in the book