PG4200: Algorithms And Data Structures

Lesson 03: Runtime Analysis and Sorting

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How Long?

- You want **fast** algorithms
- You could just run some "experiments", and check how long your algorithm takes
- But what if algorithm will need to be run on a larger problem than I used in the experiments?
- If the problem is **twice as big**, will my algorithm take just **twice as long**???

```java
public static int sum(int[] array){

    int sum = 0;
    for(int i=0; i<array.length; i++){
        sum += array[i];
    }
    return sum;
}
```

- “Cost” can be measured in number of executed statements
- Given size of array N, the loop will be taken N times
- There is some constant cost independent of N, eg creation of “*int sum*” variable
- If N doubles, would expect function will be *roughly* twice as slow

instructions(N)= 3N + 4

instructions(N=0)=4

```
1. int sum = 0;
2. int i=0;
3. i<array.length;
4. return sum;
```

- Number of instructions depends on size N of the array, plus some constant cost
- Can be represented with a function, eg *f(N)=3N+4*
- For large N, constants are not important

instructions(N=3)=13

```
1.  int sum = 0;
2.  int i=0;
3.  i<array.length;
4.  sum += array[i];
5.  i++
6.  i<array.length;
7.  sum += array[i];
8.  i++
9.  i<array.length;
10. sum += array[i];
11. i++
12. i<array.length;
13. return sum;
```

```java
public static int pairs(int[] array){
    int pairs = 0;

    for(int i=0; i<array.length; i++){
        for(int j=0; j<array.length; j++){
            if(i!=j && array[i] == array[j]){
                pairs++;
            }
        }
    }
    return pairs;
}
```

```
/*
    On my machine, repeated 100 times:

    N=100   seconds=0.005
    N=200   seconds=0.005
    N=400   seconds=0.012
    N=800   seconds=0.072
    N=1600  seconds=0.211
    N=3200  seconds=0.754
    N=6400  seconds=2.829
    N=12800 seconds=11.48
*/
```

- Two nested loops
- Inner loop executed once per each element in array
- So, $N * N = N^2$
- Twice as big is now $2 * 2 = 4$ times as slow!!! (roughly)

Scalability

- When analyzing algorithms, we will not look at the low level optimization details
- N as representation of the problem size (eg, length of array or number of elements in a container)
- How does the algorithm *scale* for larger sizes???
- Example: if my website works fine with a load of 100 users, what will happen with 2,000??? Will I just need 20 times the resources?

Wheat/Rice and Chessboard Problem

- 1 rice grain on first square
- Double at each square
- How many grains on the board?
- 18,446,744,073,709,551,615
- ie, 18 Quintillions

Analysis of Algorithms

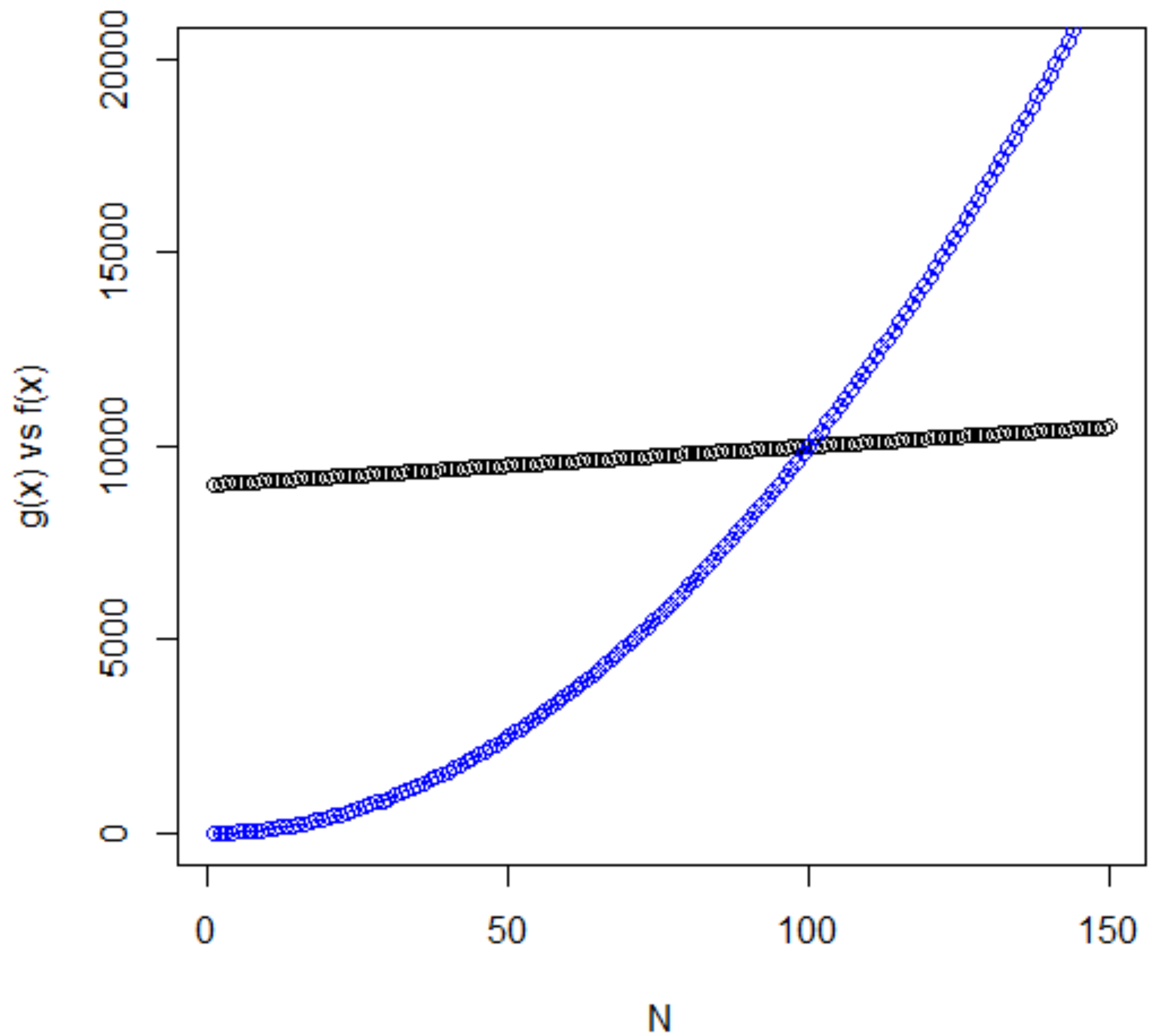
- Mathematically define the cost as a function of the input size
- *Precise* functions can be impractical, so we need approximations
- Usually, we are interested in *upper* and *lower* bounds

Example

- $f(n) = a\,N^2 + b\,N + c$
- Given an algorithm whose performance is described by the polynomial $f(n)$, finding the actual values for a, b, c might be too difficult
- However, can we say something about the **scalability**?
- YES!!! Regardless of $a = 5$ or $a = 400$, still doubling N would result in increase of at least 4 times (roughly…)

Which Is Better?

- $f(n) = n^2$
- $g(n) = 10\,n + 9000$
- For small values $f(n)$ is better, but it become worse from $n > 100$
- We will look at *large* n, so for us $g(n)$ is better



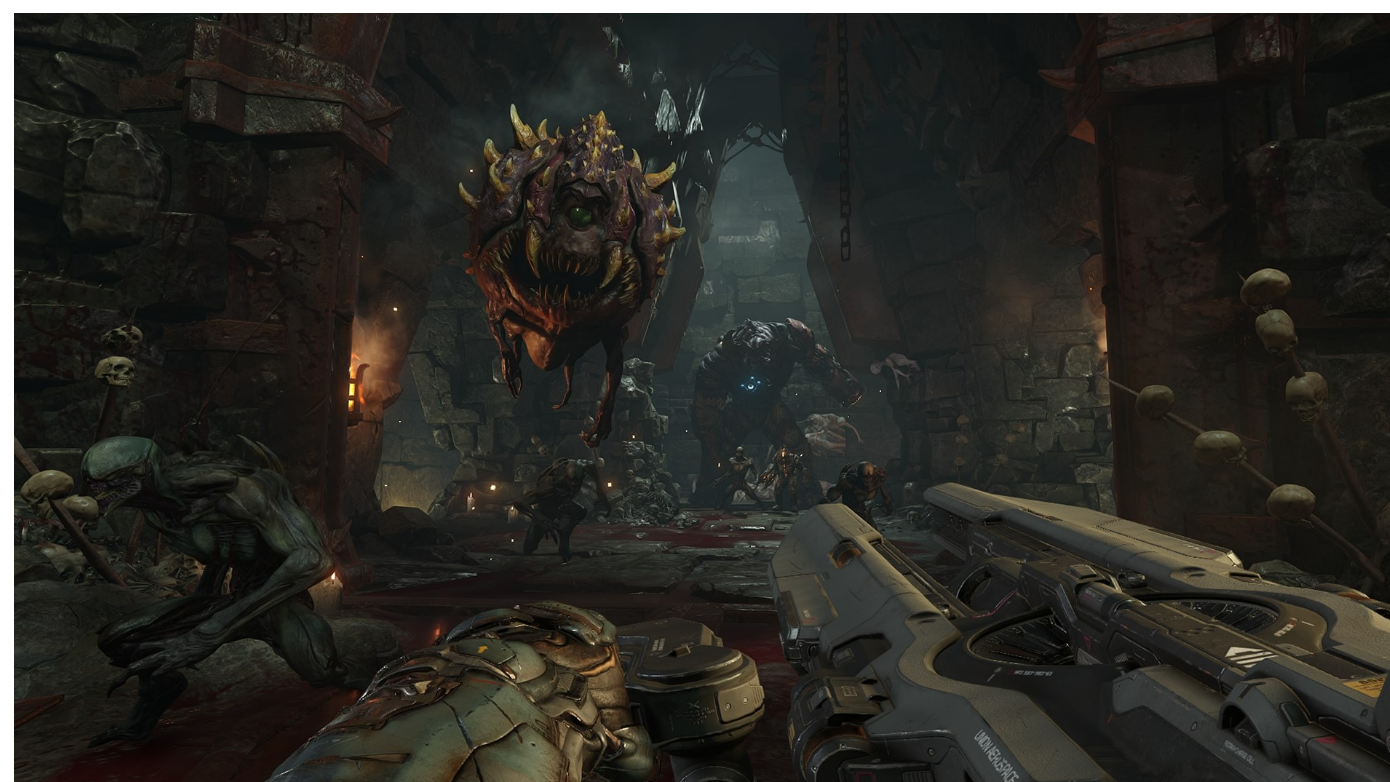
Large N???

- How do we define *large*?
    - 10? 50? 1000000000000000000???
- We can't really say… however, things grow so fast… what we think is *large* today, is likely going to be considered *tiny* in few years…
- Today I know how fast my algorithms are, because I run them. But I want to know how they will *scale* to the larger problem instances of tomorrow.
    - Eg, when my apps get more users

FPS… large increase in number of polygons to render…

Doom (1993)

Doom (2016)



Scalability

- $f(n) = 5\,n + 100$
  - If I am interested in scalability, the constants 5 and 100 are *irrelevant*
- $g(n) = 2n^2 + 10n + 7$
  - The constants 2, 10 and 7 are irrelevant. But what about the n compared with n^2??? It is smaller, but maybe still important?

Big O Upper Bound

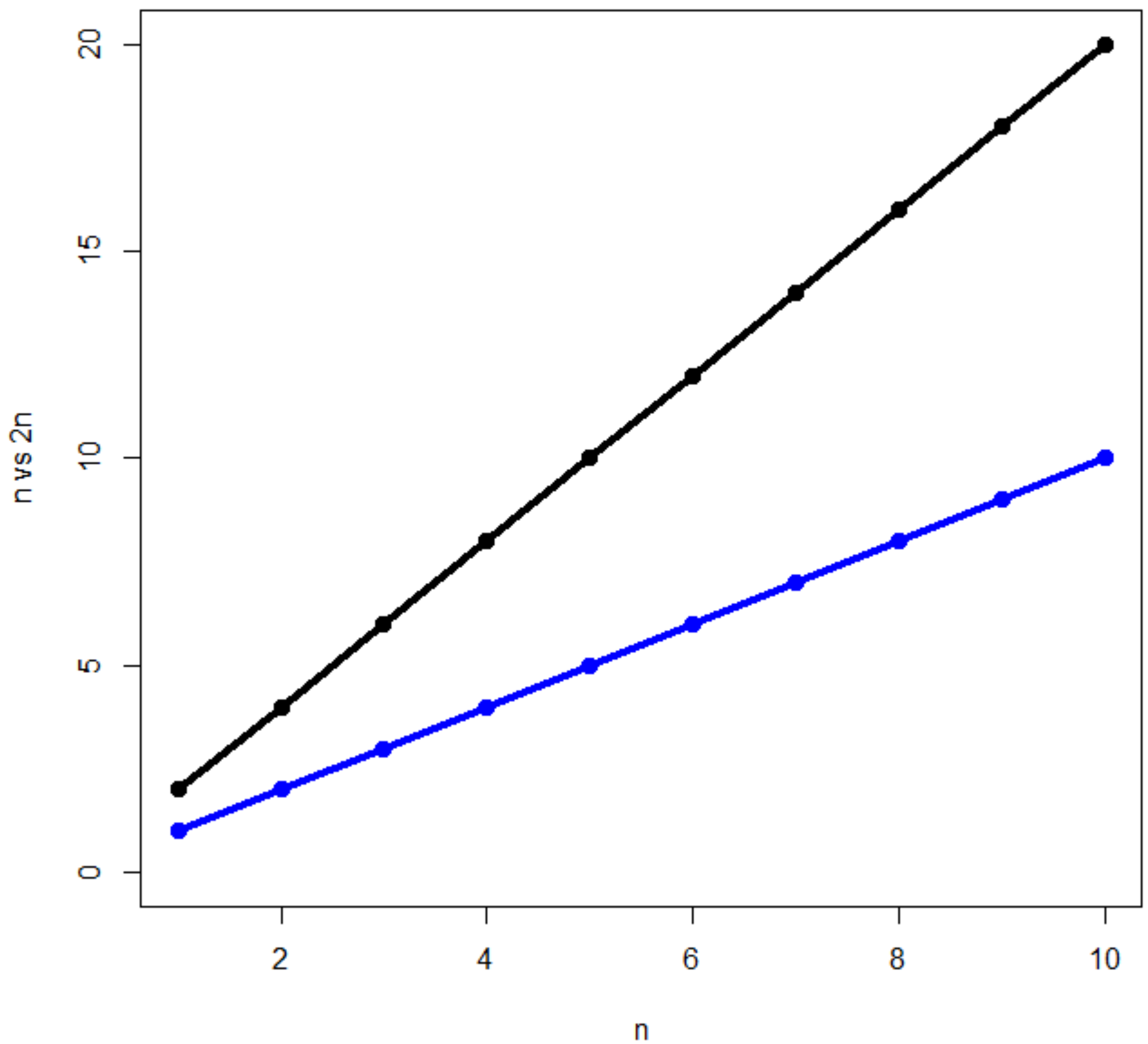
- $f(n) = O(g(n))$

- If there exists positive constants c and n', such that $0 \leq f(n) \leq c * g(n)$ for all $n \geq n'$

- In other words, $c * g(n)$ is an **upper bound** for $f(n)$ for large values of n

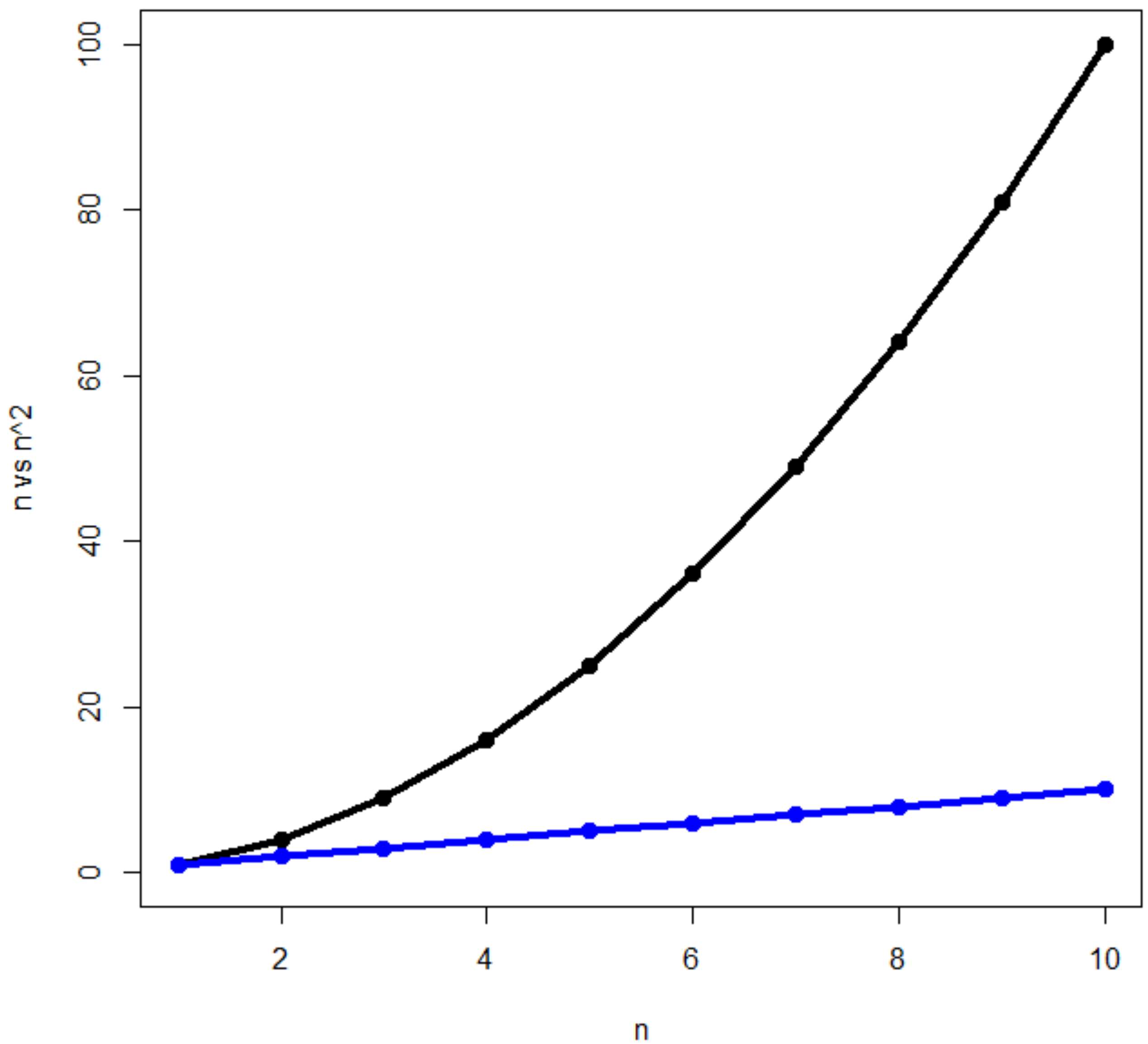
- Useful to consider *worst case scenarios*

$n = O(n)$



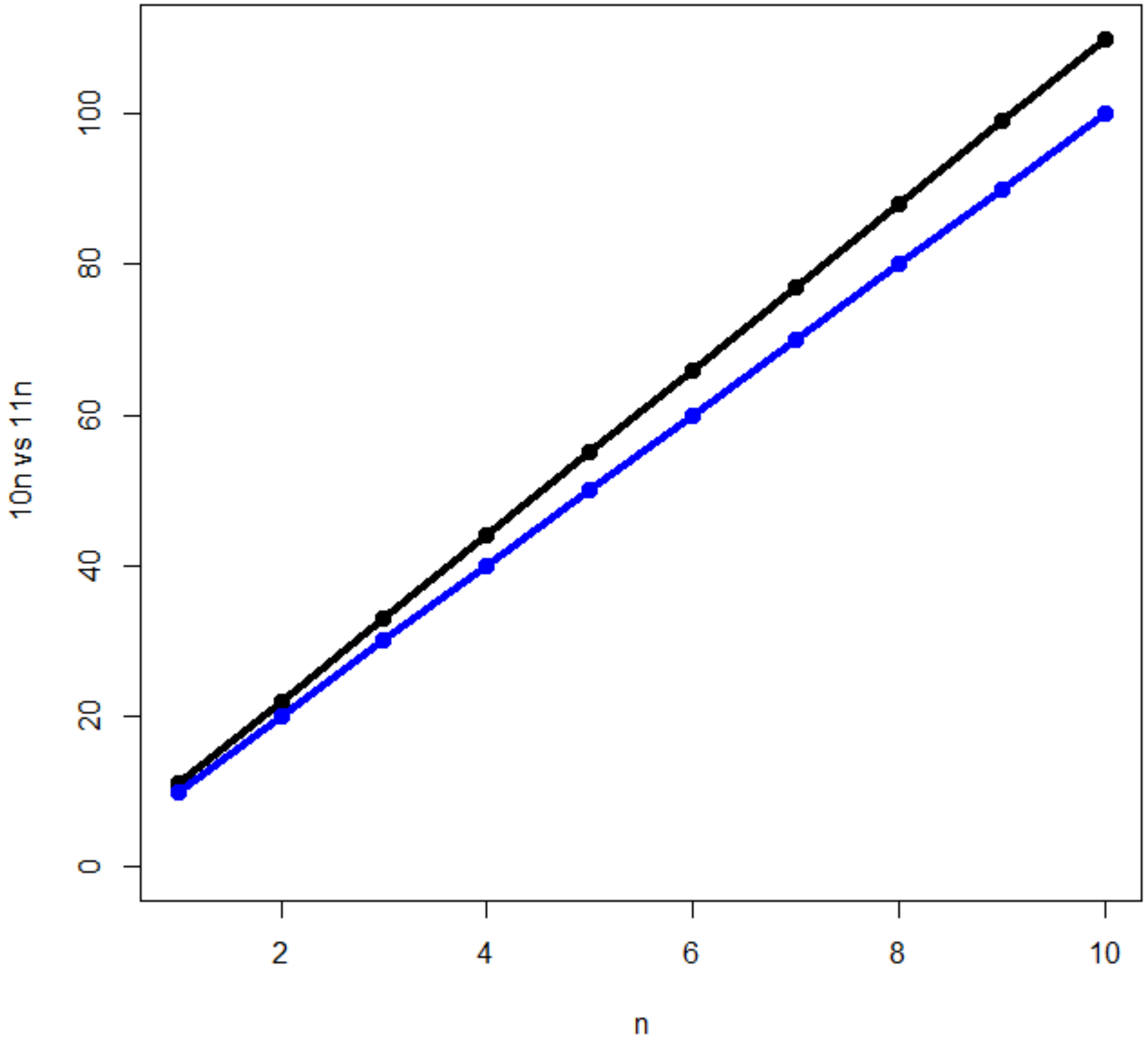
- *g(n) = n*
- Examples: *c = 2, n' = 1*
- *n < 2n* for n >= 1

$n = O(n^2)$



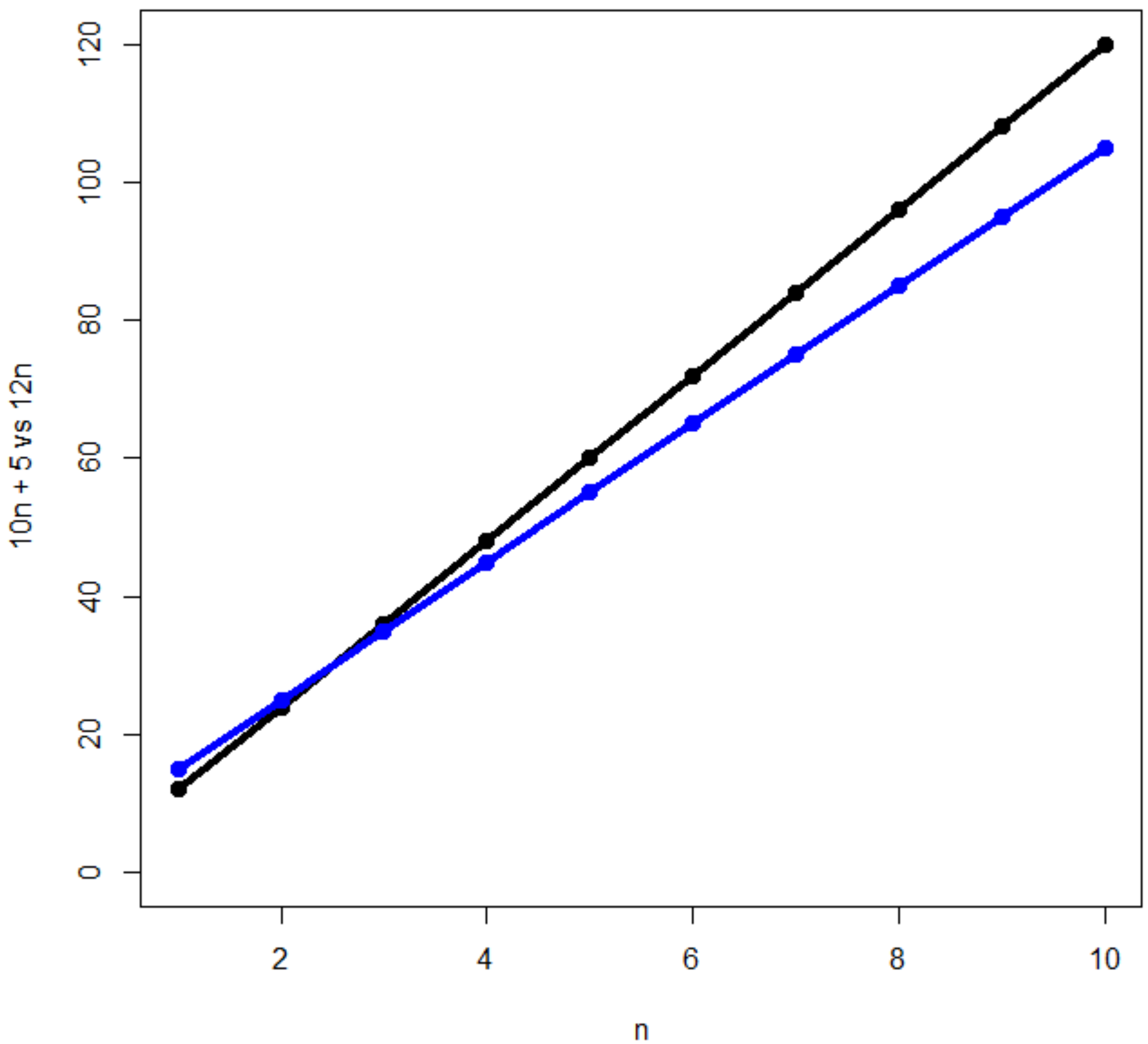
- $g(n) = n^2$
- Examples: $c = 1, n' = 2$
- $n < n^2$ for n >= 2

$10n = O(n)$



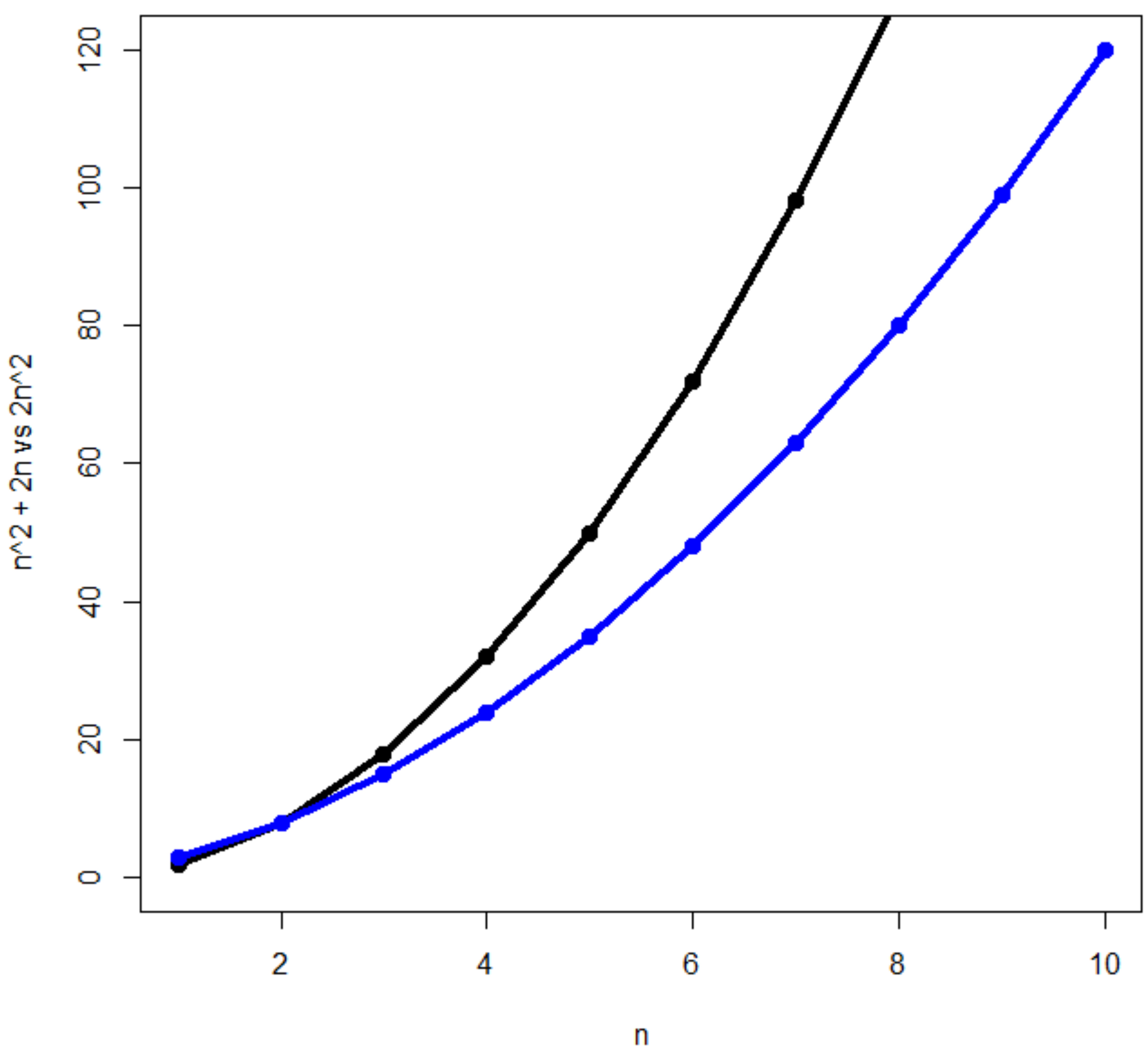
- $g(n) = n$
- Examples: $c = 11$, $n' = 1$
- $10n < 11n$ for $n >= 1$

$10n + 5 = O(n)$



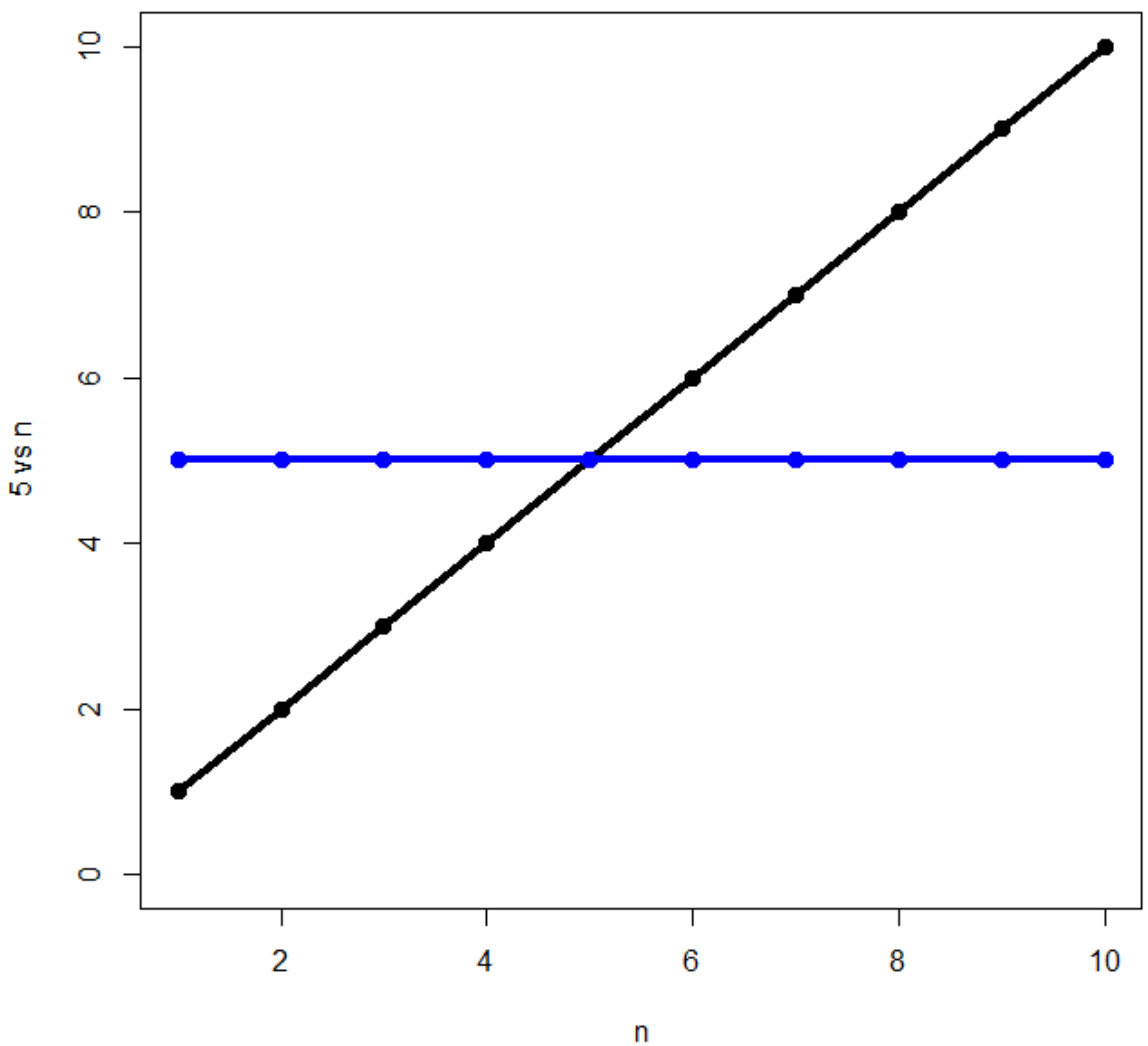
- *g(n) = n*
- Examples: *c = 12, n' = 3*
- *10n + 5 < 12n* for n >= 3
- Eg: *n=3 -> f(n)=35, g(n)=36*
- Note: for *n<=2*, *f(n)* is actually larger

$$n^2 + 2n = O(n^2)$$



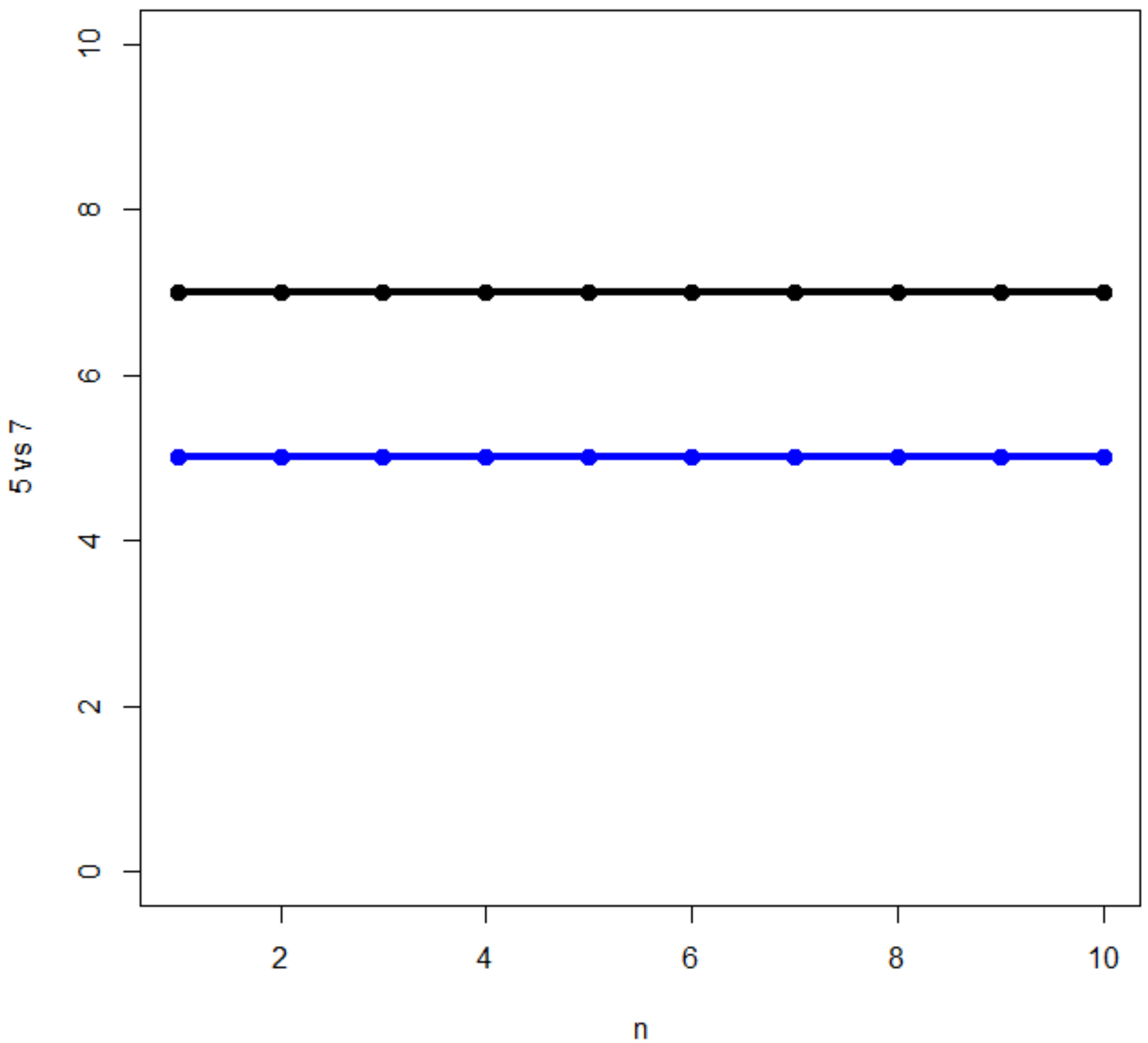
- $g(n) = n^2$
- Examples: $c = 2$, $n' = 3$
- $n^2 + 2n < 2n^2$ for n >= 3
- Eg: $n=3 \to f(n)=15$, $g(n)=18$

$5 = O(n)$



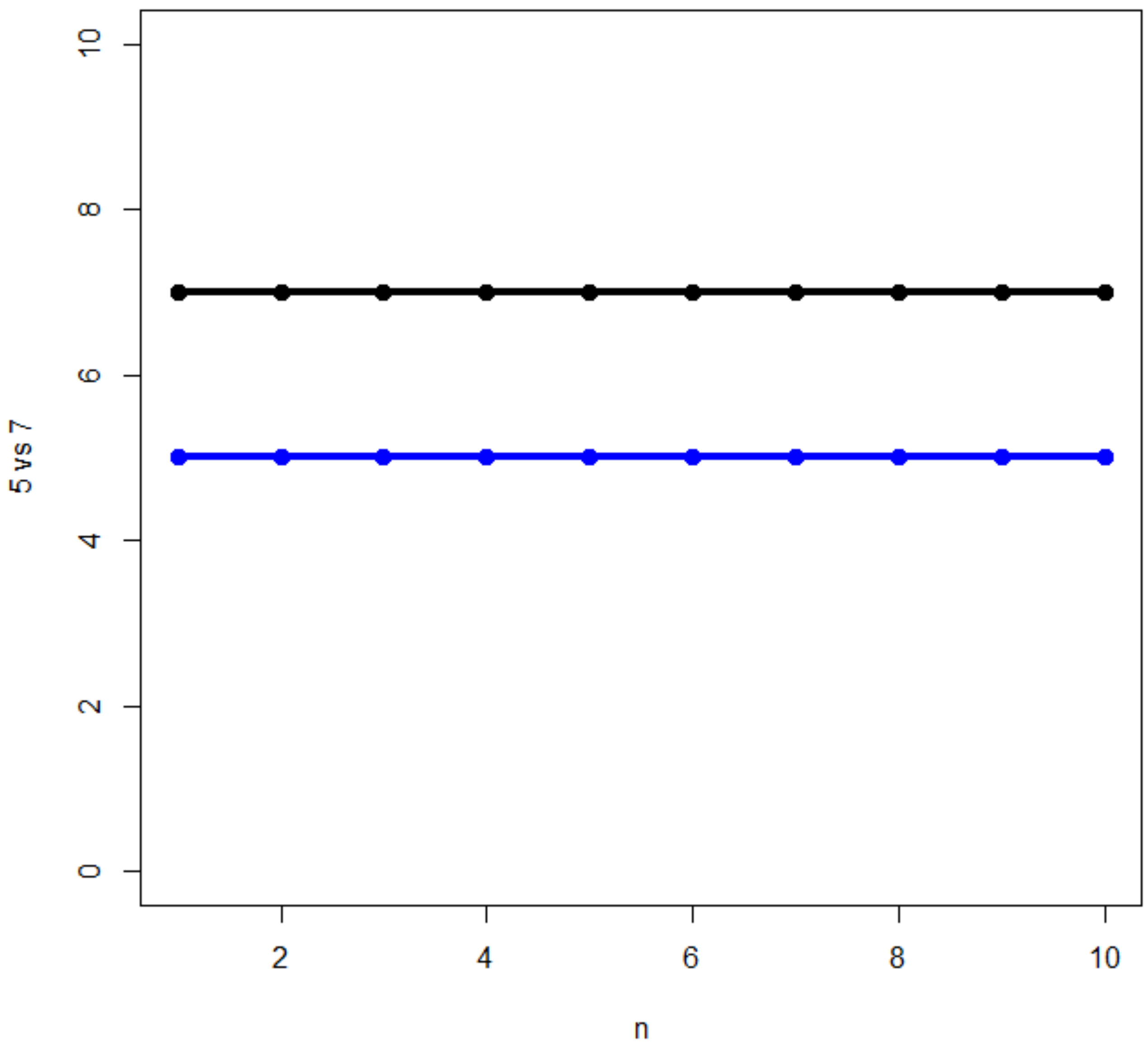
- $g(n) = n$
- Examples: $c = 1$, $n' = 6$
- $5 < n$ for n >= 6

$5 = O(7)$



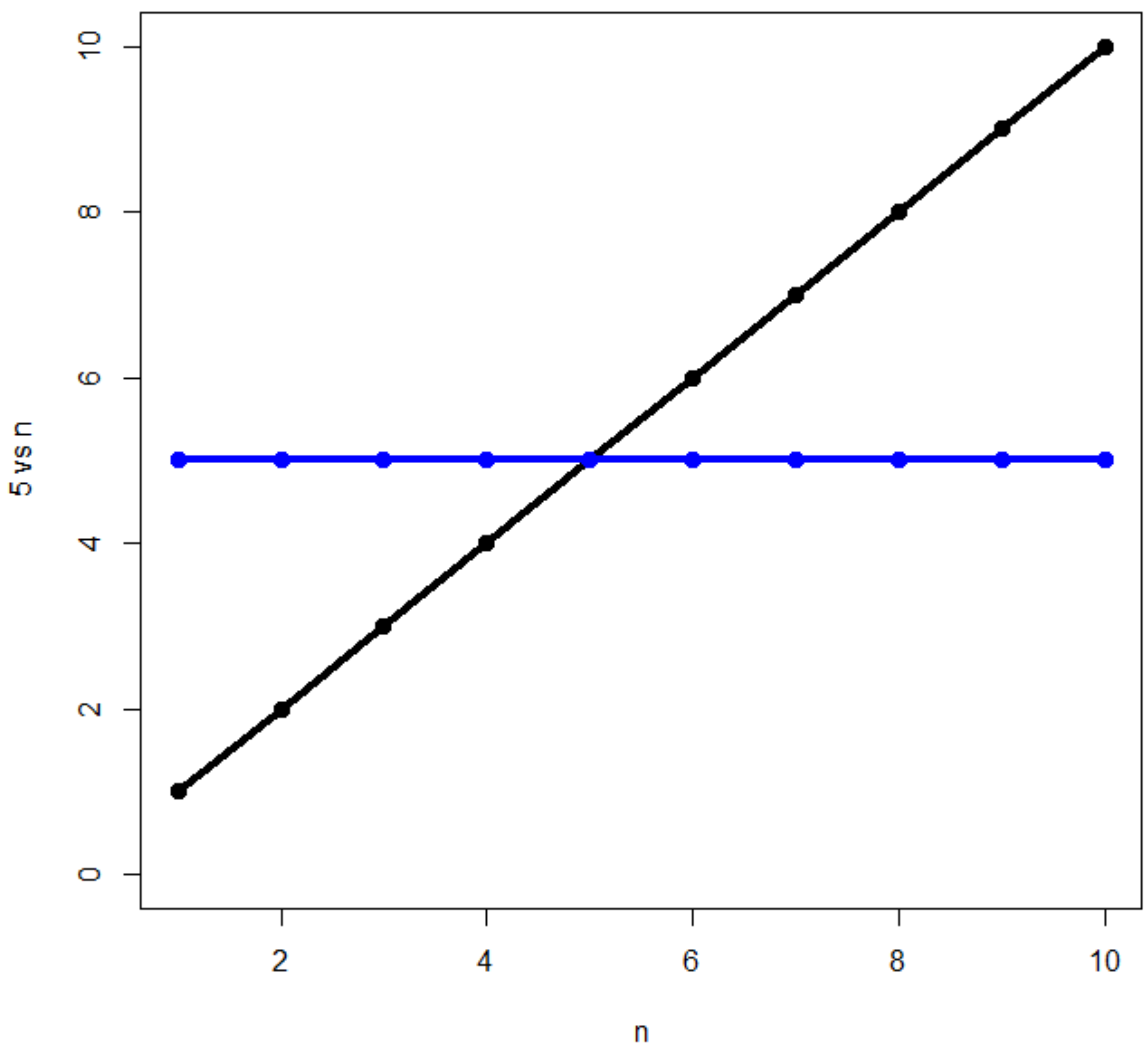
- *g(n) = 7*
- Examples: *c = 1, n' = 1*
- *5 < 7* for regardless of *n*

$5 = O(1)$



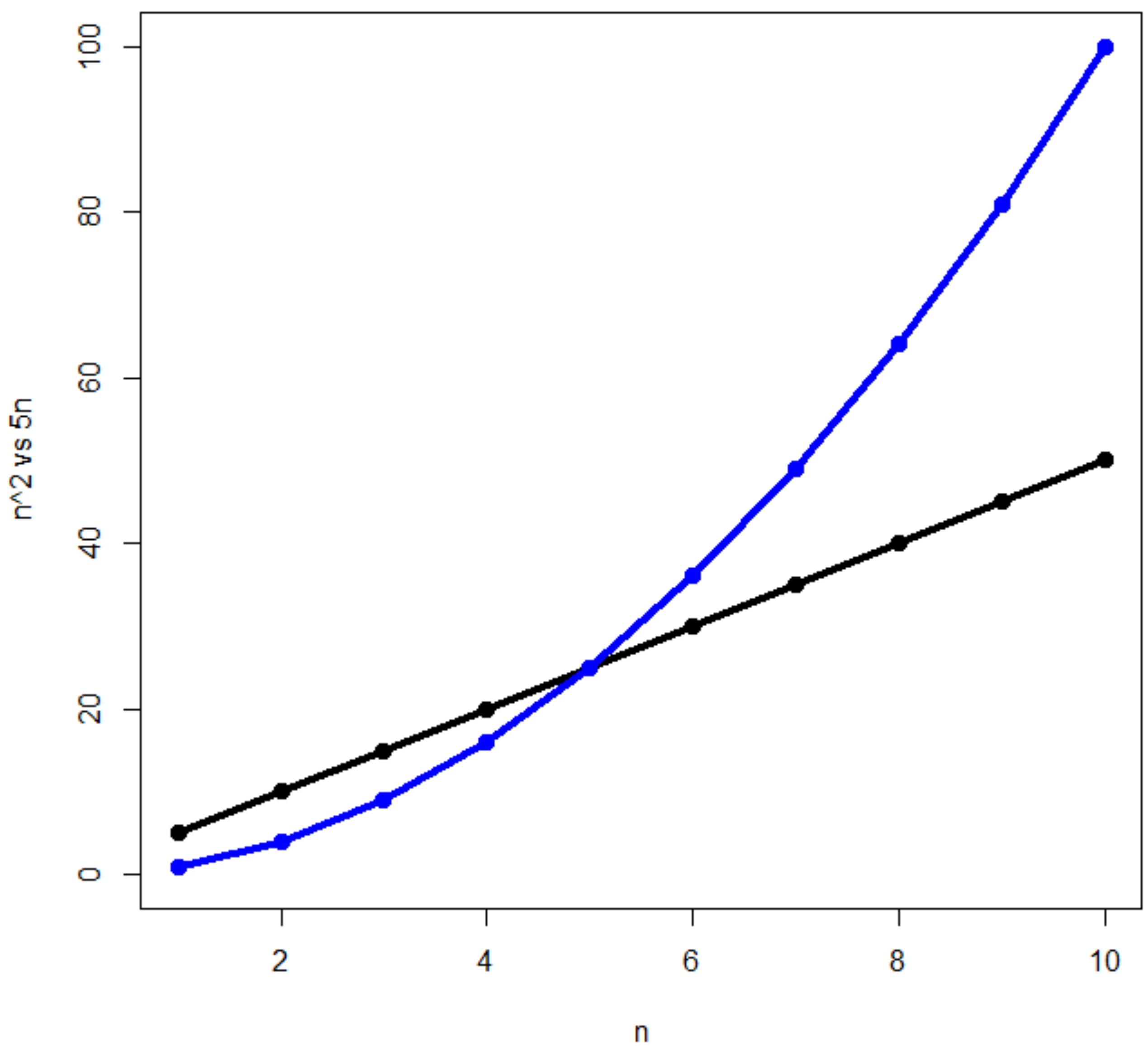
- $g(n) = 1$
- Examples: $c = 7$, $n' = 1$
- $5 < 7$ for regardless of n

$n \neq O(1)$



- *g(n) = 1*
- Whatever *c* I use (eg 5), *f(n)<c* will not hold for *n'>c,* eg *n'=c+1*

$n^2 \neq O(n)$



- $g(n) = n$
- Whatever c I use (eg 5), $f(n)<cn$ will not hold for $n'>c$, eg $n'=c+1$

Big O Examples

- $n = O(n)$
- $n = O(n^2)$
- $10n = O(n)$
- $10n + 5 = O(n)$
- $n^2 + 2n = O(n^2)$
- $5 = O(n)$
- $5 = O(7)$
- $5 = O(1)$
- $n \neq O(1)$
- $n^2 \neq O(n)$

Big O Rules of Thumb

- When you have a polynomial, for upper bound just look at the highest exponent
- $an^3 + bn^2 + cn + d = O(n^3)$
- This means that, when analyzing an algorithm, you can ignore the parts with less impact
- When representing constants independent from the problem size, just use 1 by convention, eg $O(1)$

Notation

- $O(g(n))$ = { *f(n)* : there exists positive constants c and n', such that $0 \leq f(n) \leq c * g(n)$ for all $n \geq n'$}

- *O(g(n))* is actually a *set* of functions

- *f(n) = O(g(n))* is not fully correct as notation, as we use it to represent the fact that *f(n)* is one member of the set *O(g(n))*

- $f(n) \in O\big(g(n)\big)$ would be more precise, but often for simplicity you will see "=" instead of "∈"

Big Ω Lower Bound

- $f(n) = \Omega\big(g(n)\big)$
- If there exists positive constants c and n', such that $\mathbf{0} \ \leq c * g(n) \leq f(n)$ for all $n \geq n'$
- In other words, $g(n)$ is a *lower bound*
- Useful to consider *how expensive* algorithm is even in the *best possible scenario*
- $\Omega(1)$ is a trivial lower bound valid for all functions

Big Θ Tight Bound

- $f(n) = \Theta(g(n))$

- If there exists positive constants $c, \mathbf{d}$ and n', such that $0 \leq c * g(n) \leq f(n) \leq d * g(n)$ for all $n \geq n'$

- In other words, this happens when the lower and upper bounds are asymptotically the same (and just differ by the constant)

```java
public static int sum(int[] array){
    int sum = 0;
    for(int i=0; i<array.length; i++){
        sum += array[i];
    }
    return sum;
}
```

- In this case, the actual cost in number of instructions is $f(n) = 3n + 4$

- Asymptotically, we can say that $3n+4 = \Theta(n)$

- As the number of instructions does not depend on the content of the array, the *best case* and *worst case* for the runtime are the *same*

Order Of Growth Classification

- 1: **constant** (best you can have)
- log(N): **logarithmic** (very, very efficient)
- N: **linear** (OK for most cases)
- N log(N): **linearithmic** (OK for most cases)
- N^2: **quadratic** (bearable, but things start to get expensive)
- N^3: **cubic** (becoming painful)
- 2^N: **exponential** (*completely hopeless*, time to cry in a corner)

Which Scales Best?

- $f(n) = \text{O}(n)$
- $g(n) = \Omega(n)$
- $t(n) = \Omega\big(n\, log(n)\big)$
- $k(n) = \text{O}(n^2)$
- $\text{z(n)} = \Theta(n)$

The **only** thing that I can say for sure is that $f(n)$ is better than $t(n)$. Why???

	1	$log\ n$	n	$n\ log\ n$	n^2	n^3	2^n
$O(n)$							
$\Omega(n)$							
$\Omega\big(n\ log(n)\big)$							
$O(n^2)$							
$\Theta(n)$							

In Practice

- Proving tight bounds is often infeasible
- Usually, from practical standpoint, the *worst case scenario* is what matters, so most discussions are about O(g(n))
- Often, lower bound is not so interesting, as in happy-day scenario you get $\Omega(1)$

A Big Mistake

- BIG MISTAKE: assuming that an upper bound is tight, eg claiming $f(n) = \mathrm{O}(n)$ is necessarily better than $\mathrm{g}(n) = \mathrm{O}(n^2)$
  - Although you might still want to prefer $f(n)$ if you do not have any other information

- Even if a lower upper bound O exists, it might be too difficult to formally prove it, so $\mathrm{O}(n^2)$ might just be the best approximation we currently have

- However, it is also important to consider that worst case scenario is different from the average one, but proving averages is much more difficult

```java
public static void doSomething(int[] array){

    for(int i=0; i<array.length; i++){
        for(int j=i; j<array.length; j++){
            //... something
        }
    }
}
```

- Note the “int j=i”. What can we say about that function?

- Without digging into the math, we can say that, even in best case, first loop is taken at least once, so $\Omega(n)$

- In worst case, not worse than assuming “int j=0”, so $O(n^2)$
    - In other words, we can mentally consider a more expensive algorithm which does more iterations, but that is easier then to analyze

- Is the true complexity closer to the lower or to the upper bound? Maybe $\Theta(\mathrm{n} \log \mathrm{n})$???

Let's Dig Into the Math…

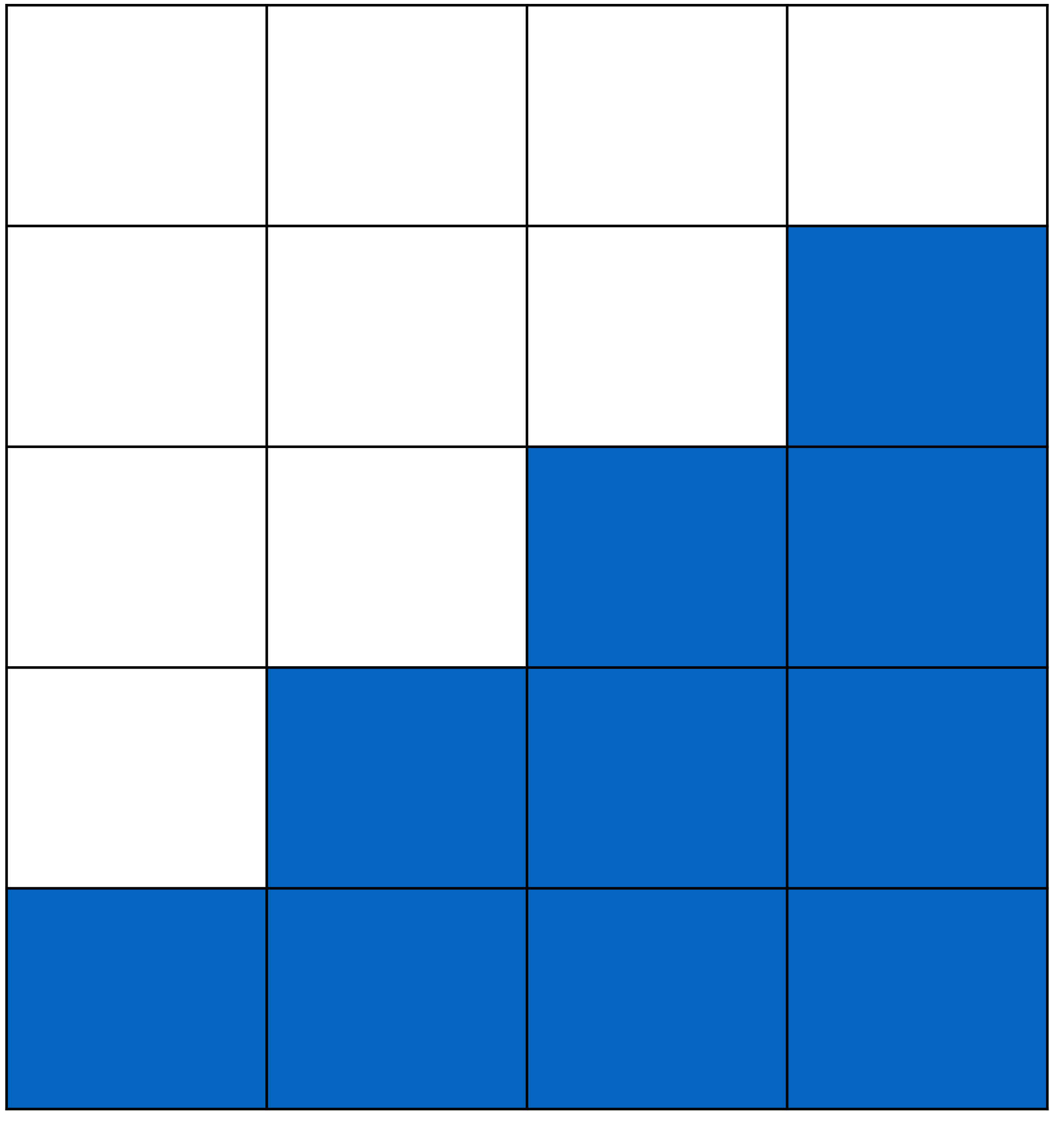
```
public static void doSomething(int[] array){

    for(int i=0; i<array.length; i++){
        for(int j=i; j<array.length; j++){
            //... something
```

- Outer loop is taken N times
- Inner loop is shorter by 1 at each iteration
- $\mathrm{N} + (\mathrm{N}-1) + (\mathrm{N}-2) + \ \ldots + 1 = \sum_{i=0}^{N} i = \frac{1}{2} N(N+1) = \frac{1}{2}(N^2 + N) = \Theta(N^2)$

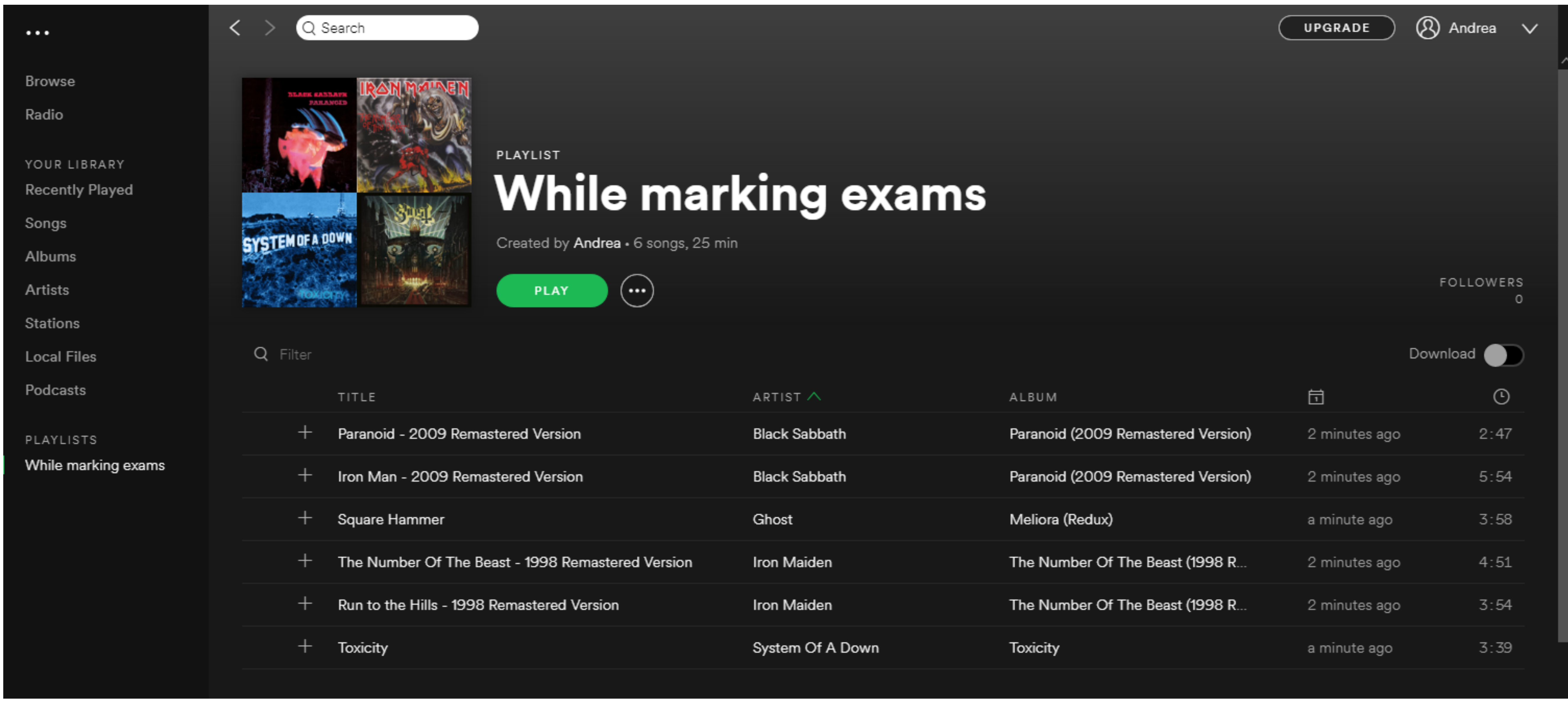
$$\sum_{i=0}^{N} i = \frac{1}{2}N(N+1)$$

- Think about a rectangle with sides N and N+1
- Its area is $N \ * (N+1)$
- But we are interested only in the colored area, so divide by 2
- 1+2+3+4 = 4 * 5 / 2 = 10

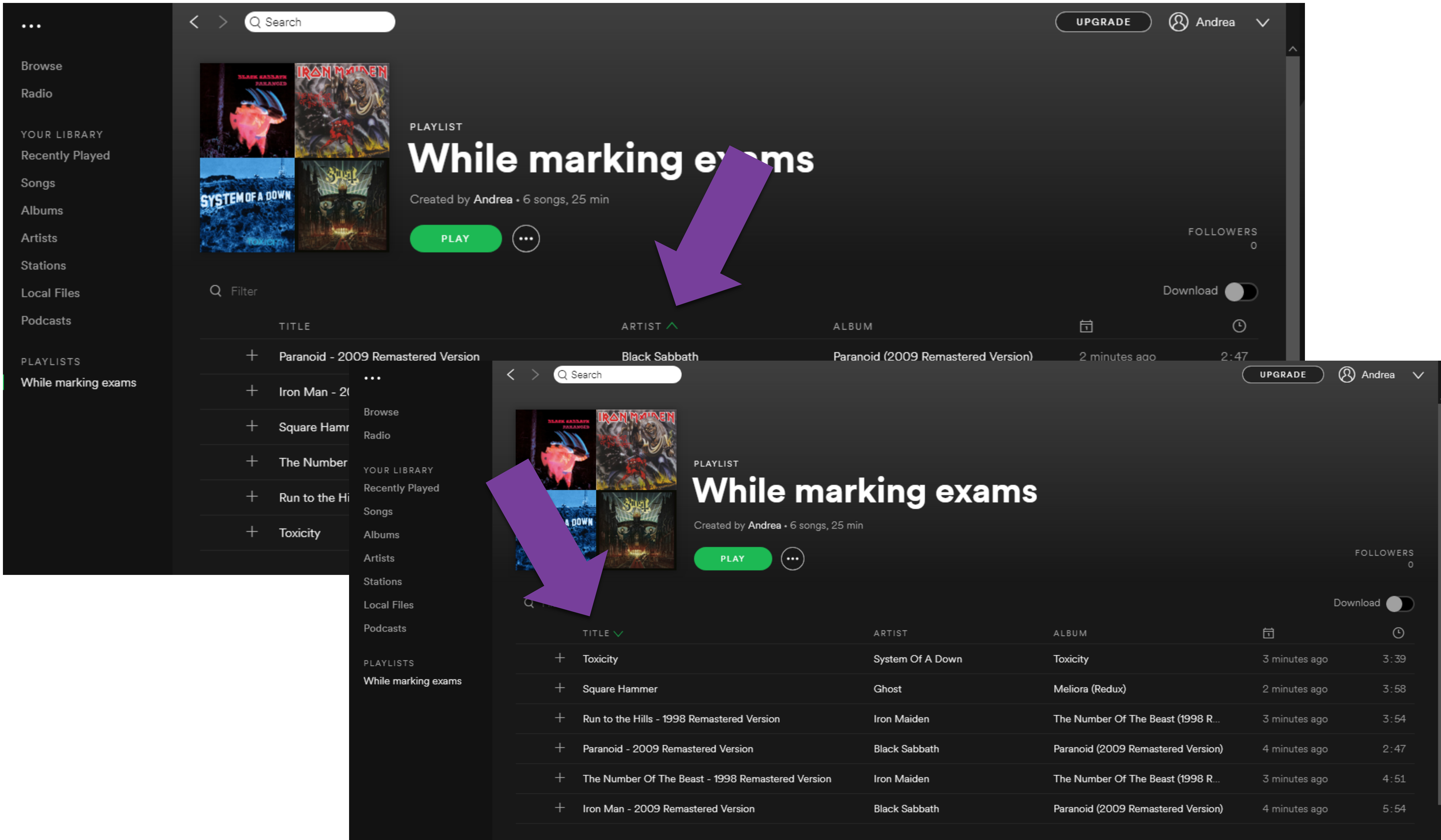


Sorting

Consider a Playlist



Sort by Title or Artist, Ascending or Descending



How do you sort when playing cards?

Sorting Algorithms

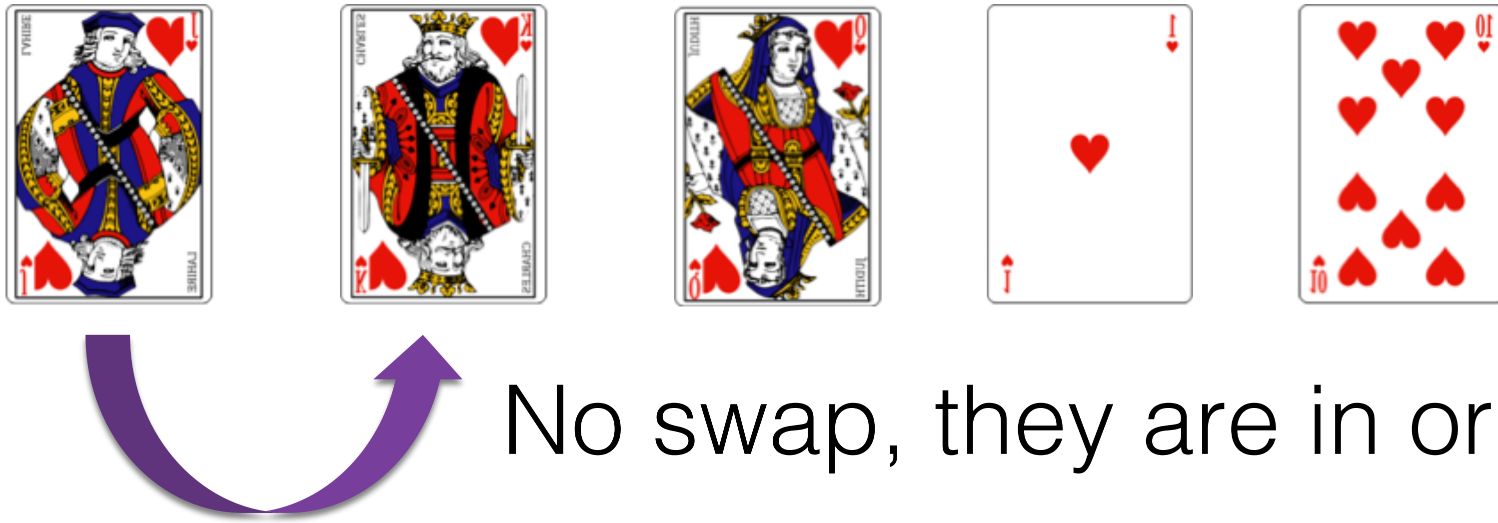
- Many different sorting algorithms, with different properties
- Given two items *A* and *B*, just need a *comparator* that can state which one is greater or equal
  - easy to say that 5 greater than 2, but what does it mean that song *A* is greater than song *B*? e.g., could look at alphabetic ordering of titles or artist names
- Most language APIs provides good defaults
  - Unless very large data, default will be fine 99% of the cases
- Sorting is very popular in programming
  - Important to understand how it works under the hood
- Tractable mathematically
  - So good example to show how to analyze algorithms

Sorting Algorithms

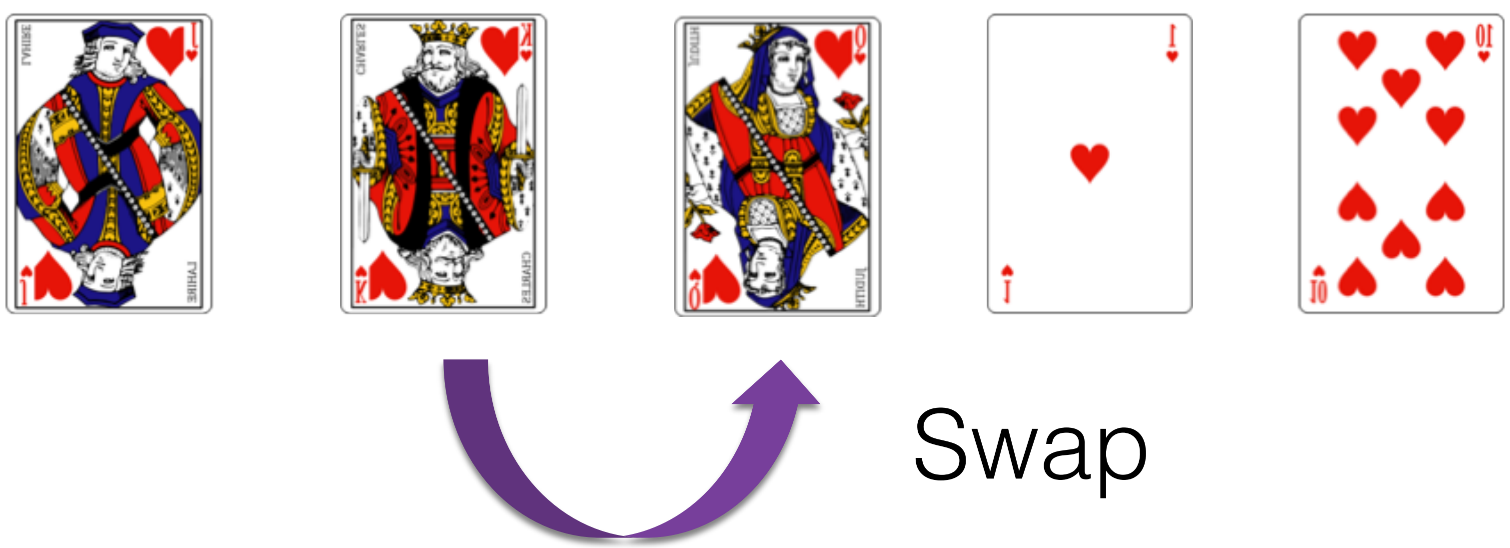
- Bubble Sort
- Insertion Sort
- Merge Sort (next class)
- Quick Sort (next class)
- There are more, but those are the most famous that you need to know
- Good way to see a problem been solved in many different ways

Bubble Sort

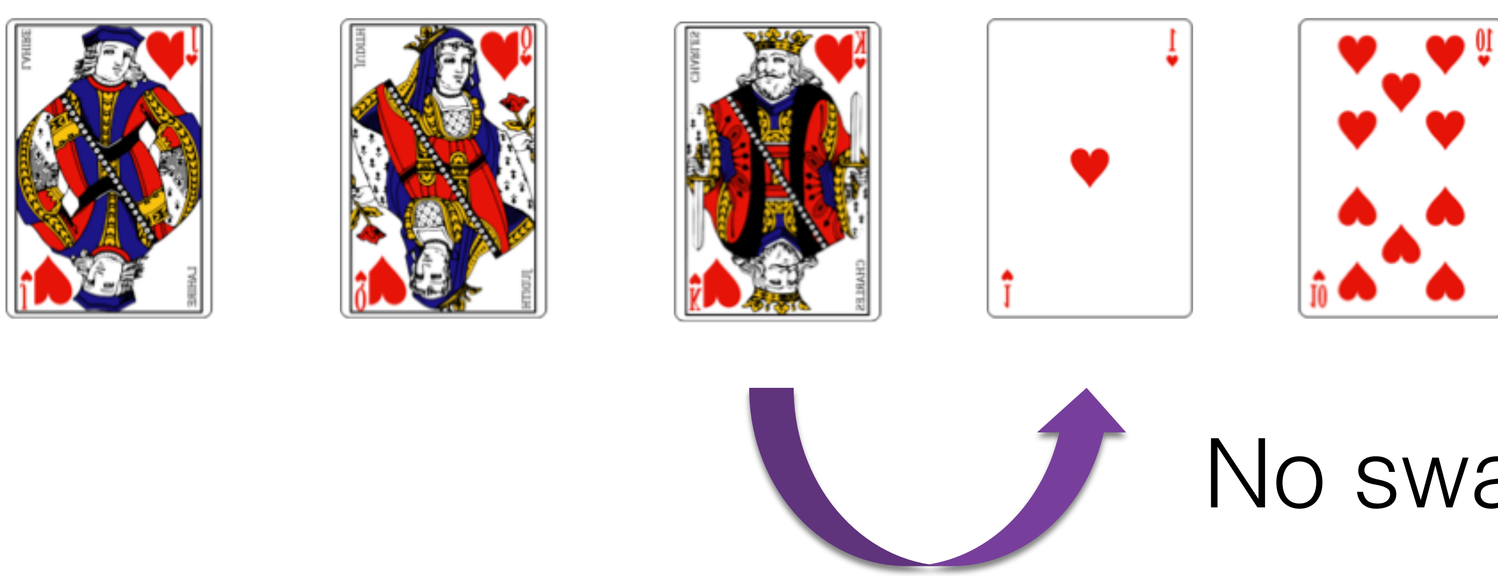
- *Easiest* sorting algorithms
- From left to right
- Look at adjacent cards, and swap them if not in order
- Repeat from left to right till no more swap



No swap, they are in order



Swap



No swap, they are in order

Swap

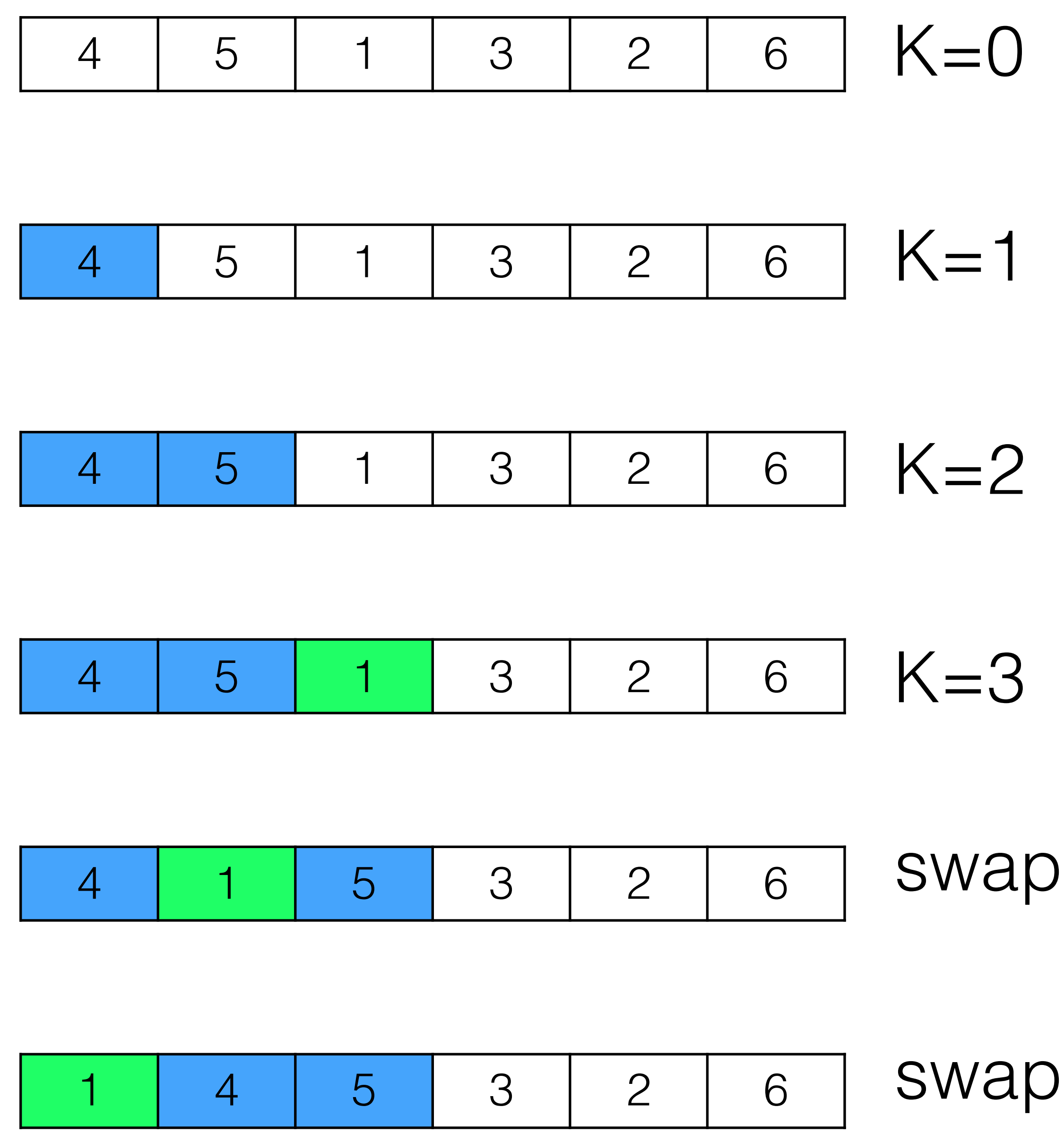
- Restart from beginning.
- At each iteration, at least one card will be in right position, as it *bubbles up* to the to top.

Runtime of Bubble Sort

- To sort N cards, need *at most* N iterations, in which you check *at most* N-1 pairs
- Even if already sorted, need to check each of N-1 pairs at least once, to see if indeed sorted
- $\Omega(N)$ and $O(N^2)$ pair comparisons

Insertion Sort

- An array of size 0 or 1 is always considered sorted
- From left to right, till length N
- K-leftmost values are sorted
- Position K+1 is not sorted, insert it in the first K
  - by swapping adjacent elements, like in Bubble Sort



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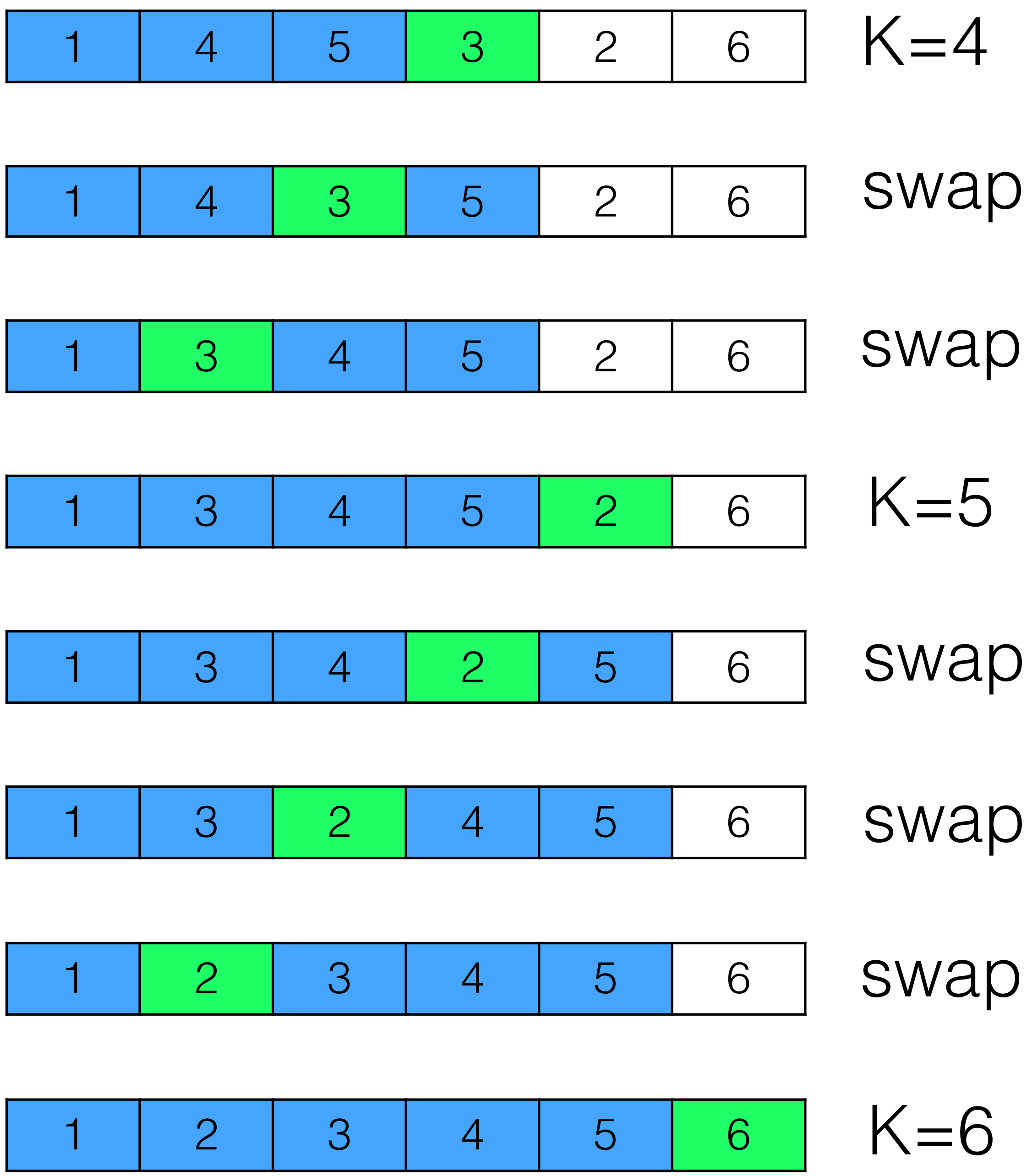

- Best case: already sorted, e.g., 1-2-3-4-5-6, need to do N-1 comparisons, so $\Omega(N)$

- Worst case: opposite order, e.g, 6-5-4-3-2-1, each element needs to be compared and swapped with all previous K ones, so $O(N^2)$

Best vs. Worst Case

- “Best case” is not when **N=0**
- Runtime might not only depend on size **N**, but also on the **content** of the input
  - an array already sorted will likely take less time to sort than a shuffled one
  - on the other hand, summing all integers in an array will likely have no difference between Best and Worst case, ie $\Theta(N)$
- Considering all possible inputs in a mathematical formula would not be viable
  - that is why we talk about BEST and WORST cases in relation to **N**

Homework

- Study Book Chapter 1.4 and 2.1
- Study code in the *org.pg4200.les03* package
- Do exercises in *exercises/ex03*
- Extra: do exercises in the book