# Pre-lecture Notes Section 12:

# Inductance

## I. Introduction to Inductance

In this section, we will be applying our newfound knowledge of the principle of electromagnetic induction. Specifically, due to the fact that a change in electric current causes a change in magnetic flux, we will find that in circuits where the current is changing, an emf will be induced in the circuit to oppose the change. This will lead to a wide variety of circuit phenomena and will be the main topic of this section.

## II. Self-Inductance and Inductors

Imagine that we have a wire (which may be looped many times, as with a solenoid) connected to a source of emf. If the emf of the source is changing with time, then the current in the loop changes with time. Thus, the magnetic flux created by the loop will change with time. From Faraday's law, this changing flux means that the loop will induce an emf in itself!!! The phenomenon of a loop of wire inducing an emf in itself due to a change in magnetic flux is called **self-inductance**.

Let's determine what the induced emf should depend on. We know that it is equal to the change in magnetic flux through the loop. The magnetic flux depends on

- the magnetic field
- area of the loop
- orientation of the loop
- the number of turns of the loop

In the scenario described above, we are not changing any of the final three bullet points. We are only changing the magnetic field through the loop. Specifically, the magnetic field is changing because the current through the loop is changing. Thus, from Faraday's law, the induced emf must be proportional to the rate at which the current is changing.

$$\varepsilon_{ind} \propto -\frac{di}{dt}$$

The exact value of the induced emf can only be calculated once the number of loops and shape of the loops are specified. The constant of proportionality which depends on these factors is known as the ***Self-Inductance L***. $L$ is defined such that

$$\varepsilon_{ind} = -L\frac{di}{dt}$$

The SI unit of self-inductance is the Volt·second/Amp. This is known as the Henry.

$$1\ \mathrm{H} = 1\frac{\mathrm{V{\cdot}s}}{\mathrm{A}}$$

Since the induced emf in the loop opposes the change of current through itself, we say that the induced emf is a *back emf*.

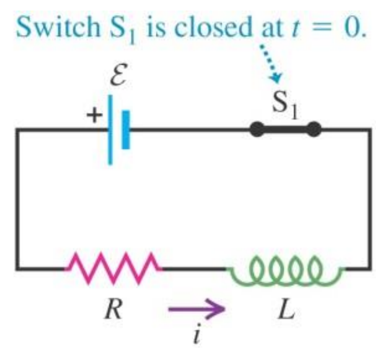
A wire arrangement which is meant to produce a relatively large back emf (often a solenoid, because of the large number of loops) is called an *inductor*. Inductors can be placed in circuits, and along with resistors and capacitors can create an interesting variety of circuit behavior. We will study these types of circuits in the remainder of this section.

Before that, however, let's recap the types of elements we can place in our circuits, along with the voltages across them

| Circuit Element | Voltage |
| --- | --- |
| Battery | $v_{batt} = \varepsilon$ |
| Resistor | $v_{res} = -iR$ |
| Capacitor | $v_{cap} = \frac{q}{C}$ |
| Inductor | $v_{ind} = -L\frac{di}{dt}$ |

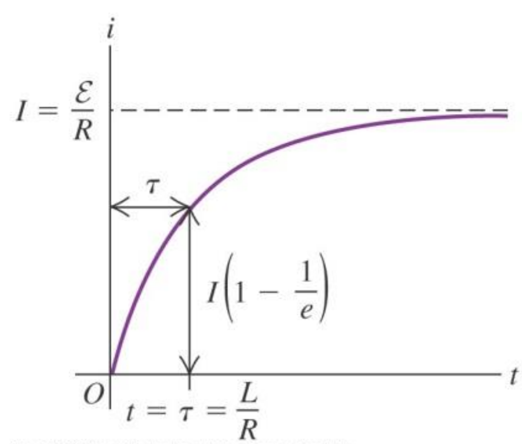
# III. The RL Circuit

*Current Growth in the RL Circuit*




The simplest thing we can do with an inductor is place it in a circuit with a constant source of emf (i.e. a battery) and a resistor. See the picture of the circuit to the right.

Imagine that at time $t = 0$ the battery is connected to the resistor and inductor. Without the inductor, we would expect that as the battery is connected, the current would jump to $i_{max} = \frac{\varepsilon}{R}$ instantaneously. However, with the inductor in the circuit, this cannot happen.



Copyright © 2008 Pearson Education, Inc., publishing as Pearson Addison-Wesley.

Let's first understand why this is so, from a conceptual standpoint. When the battery initially tries to push current through the circuit, there is a change of flux in the inductor, and a back emf is induced which opposes the change. This back emf therefore does not allow the current to reach its maximum value instantaneously. However, as the current grows towards its maximum value, the rate of change in current decreases. Therefore the back emf becomes less and less and finally, when the current reaches its maximum value, it is no longer changing at all and the back emf goes to zero. The circuit will then operate with a constant current of $i_{max}$, as if the inductor is no longer part of the circuit. We thus expect the current to grow from zero to $i_{max}$ in an exponential fashion, as in the graph.

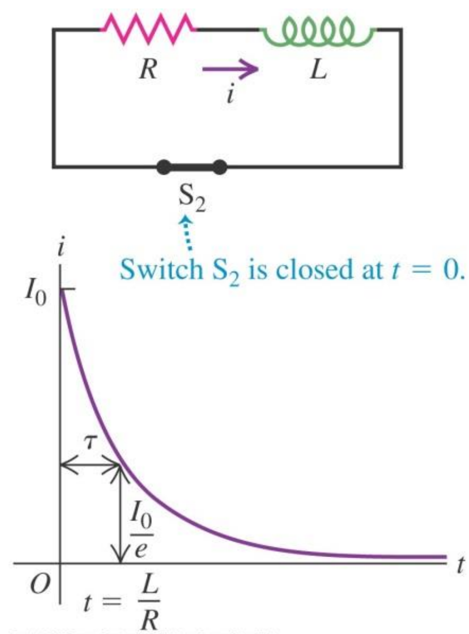
The mathematical solution to the current growth in the RL circuit comes from analyzing Kirchhoff's loop rule, and is given by

$$i(t) = \frac{\varepsilon}{R}\left(1 - e^{-(R/L)t}\right)$$

Indeed, using this equation, it is not hard to see that the initial current is zero, and as the time tends to infinity, the current tends to its final value of $i_{max} \equiv I = \frac{\varepsilon}{R}$. Note that this function matches the graph in the previous figure.

*Current Decay in the RL Circuit*

We now imagine that the battery is disconnected, so that the circuit consists of just an inductor and a resistor, with some initial current $I_0$ flowing in the circuit. See the picture of the circuit to the right.



Copyright © 2008 Pearson Education, Inc., publishing as Pearson Addison-Wesley.

Without an inductor, we would expect that at the instant the battery is removed, the current will drop to zero. However, with the inductor in the circuit, this cannot happen. Qualitatively, this is because the attempted change of current from $I_0$ to zero induces an emf in the inductor which will oppose the change. As the current decreases towards zero, the rate of change of the current decreases, and after a long time, we expect that the current will be approximately zero and will not be changing any longer. At this point, there is no induced emf in the inductor, and the circuit should "come to rest" with zero current.

The solutions for the current as a function of time is again given by analyzing Kirchhoff's loop rule.

$$i(t) = I_0 e^{-(R/L)t}$$

As expected, this is exponential decay, and we can see the graph of the current vs. time above.

*Time Constant for the RL Circuit*

In both cases (current growth and current decay) the same term $\frac{R}{L}$ appears in the exponential. We thus can identify the constant $\frac{L}{R}$ as the time constant of the RL circuit.

$$\tau_{RL} = \frac{L}{R}$$

The larger the inductance and the smaller the resistor, the longer the current takes to grow and to decay.

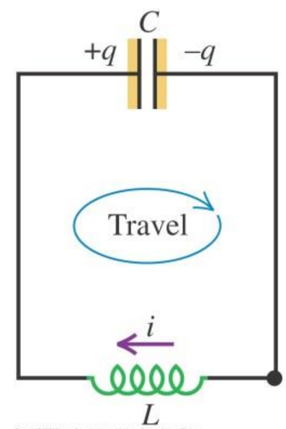
## IV. Magnetic Field Energy

We would now like to analyze the energy in the circuit as it pertains to the inductor. We will show in class that the energy stored in the magnetic field inside the inductor is

$$U_{ind} = \frac{1}{2}Li^2$$

As the current builds up, the inductor stores more and more energy. The maximum energy is stored when the current is a maximum. When the battery is turned off, and the current begins to decay, the energy in the inductor decreases. This energy is supplied back to the circuit, ultimately being dissipated in the resistor.

## V. The LC Circuit




Next, we imagine that we place a fully charged capacitor with capacitance $C$ in series with an inductor with inductance $L$. We assume no resistance in the circuit. See the figure to the right.

To see what will happen in this circuit, we need to analyze the circuit as the capacitor is starting to discharge. As it does so, it will produce a current in the circuit. This will induce a back emf in the inductor. This will "slow down" the discharge. At some later time, the current will be at some maximum value.

However, as the current decreases after reaching this maximum value (as the capacitor continues to move towards its uncharged, equilibrium state), the inductor will produce an emf which will keep the current flowing! This is because the induced emf always opposes the change, and in this case the change is negative, so the inductor will try to “get back” to its previous, high current! This induced current will cause the capacitor to start charging up in the opposite orientation. This charging and discharging will continue indefinitely!

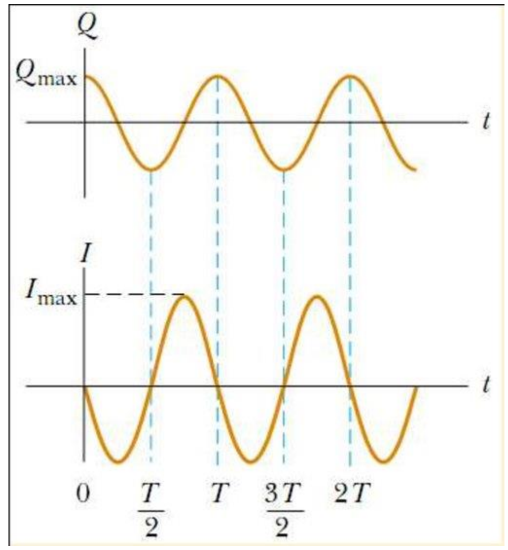
This is our first look at oscillating circuit behavior. The charge on the capacitor and the current in the circuit will oscillate sinusoidally with time.

By applying Kirchhoff’s loop rule, we can find solutions for both the charge on the capacitor and the current in the circuit as functions of time. Given that the charge and current are oscillating, it is not surprising to see that these functions are cosines and sines. The functions are

$$q(t) = Q_{max}\cos(\omega t + \varphi)$$

$$i(t) = \frac{dq}{dt} = -\omega Q_{max}\sin(\omega t + \varphi)$$

The phase constant $\varphi$ tells you how much charge you have at time $t = 0$. For instance, if you take time $t = 0$ to be when the capacitor is fully charged, then $\varphi = 0$.




A graph of the charge vs. time and the current vs. time is shown in the figure to the right. Note that both oscillate with a period $T$ which is related to the angular frequency by the usual definition. $\omega = \frac{2\pi}{T}$.

*Angular Frequency for the LC Circuit*

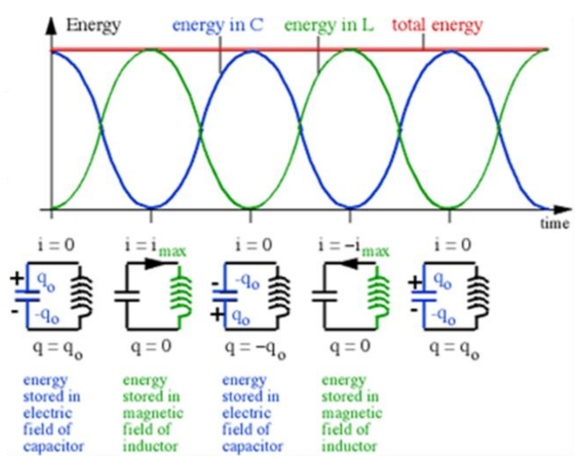
The angular frequency $\omega$ of the oscillation can be proven to be

$$\omega = \sqrt{\frac{1}{LC}}$$

The angular frequency thus depends on the inductance and the capacitance in the circuit. As the product of the two grows, the angular frequency decreases (i.e. the capacitor charges and discharges fewer times per second).

*Energy in the LC Circuit*

An important aspect of the LC circuit is that no power is dissipated by the inductor or the capacitor. Instead, the energy in the circuit simply oscillates between the electric field (when the capacitor is fully charged) and the magnetic field (when the inductor has maximum current flowing through it). The energy is never dissipated because there is no resistor in the circuit to convert the electric and magnetic energy to thermal energy. Thus, the total electromagnetic energy in the circuit is conserved!




See the graph, which shows the energy in the capacitor, inductor and total energy as a function of time.

There is a deep analogy between the LC circuit and a mass-spring with no friction. In both cases, energy oscillates (in the mass-spring it oscillates between potential and kinetic energy, while in the LC circuit it oscillates between electric energy and magnetic energy). You will investigate the deep mathematical and physical similarities between the LC circuit and the mass-spring in the pre-lecture video and in class.