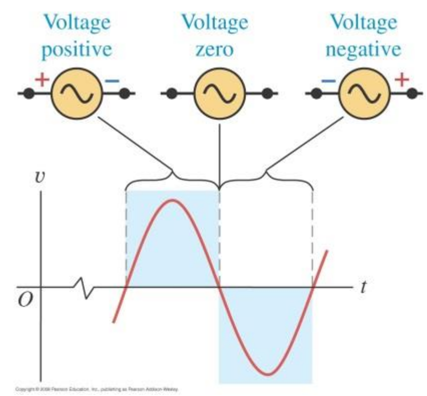
Pre-lecture Notes Section 13:

Alternating Current

I. Introduction to Alternating Current and Voltage

In this section, we will consider alternating current (AC). Our own electric grid runs on AC current and thus it is important to understand the similarities and differences between direct current (DC), which we have studied so far in the course, and AC.



To create an AC current, we need an alternating emf source. As we saw in section 11, this can be created by an alternator (a loop of wire turned at a fixed angular frequency in a uniform magnetic field).

In any AC circuit, the external voltage source will oscillate sinusoidally with some angular frequency ω and amplitude V (see the picture to the right). In response to this voltage source, the current will also oscillate with angular frequency ω, and will have some amplitude I.

By convention, we choose time $t = 0$ to be a time when the current is at its maximum value. Therefore the expression for the current as a function of time must be

$$i(t) = I\cos(\omega t)$$

The voltage must have the same angular frequency but, as we will see, may not be in phase with the current. Therefore the voltage as a function of time is

$$v(t) = V\cos(\omega t + \varphi)$$

φ is called the "phase angle" and its value determines whether and by how much the voltage function "leads" or "lags" the current function.

II. Circuits with Only One Element: Reactance

Our main goal in analyzing AC circuits is to answer the following questions:

1. Given a value of the voltage amplitude, what will the current amplitude in the circuit be?
2. What will be the phase angle of the voltage relative to the current (*i.e. will the voltage lead the current or lag the current, and by how much*)?

Both questions are somewhat formidable for a circuit with all the components we have studied (resistor, capacitor and inductor), so we begin by analyzing the circuit with only one element present at a time.

After we have learned about each component separately, we can combine them to form a series LRC circuit, which is the most complex circuit we will encounter in this section.

We start with the resistor because it is the easiest element to analyze, and then we move on to study the inductor and capacitor.

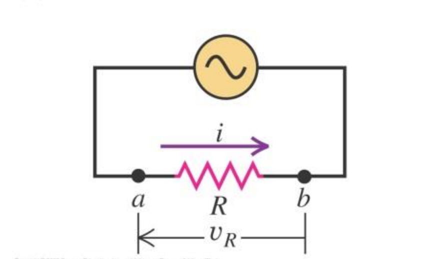
<u>Circuit with Just a Resistor</u>

The circuit with an alternating voltage source and a resistor is shown to the right. With just a resistor, the answers to our two questions are simple. Using Kirchhoff's loop rule, we find

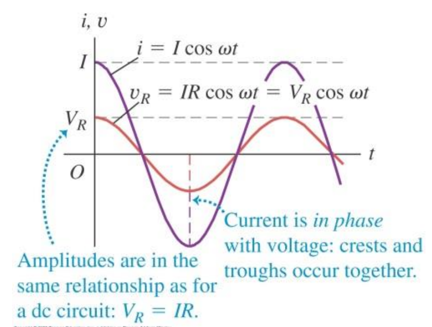
$$I = \frac{V}{R}$$

$$\varphi = 0$$

(a) Circuit with ac source and resistor



(b) Graphs of current and voltage versus time



In words, the current amplitude is very simply related to the voltage amplitude (by the reciprocal of the resistance, just like in a regular DC circuit) and the current and the voltage are in phase (i.e. the voltage does not lead nor lag the current). A graph of the voltage and current functions is shown to the right.

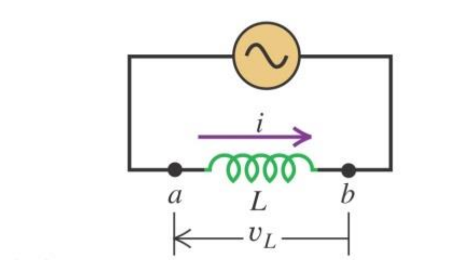
<u>Circuit with Just an Inductor</u>

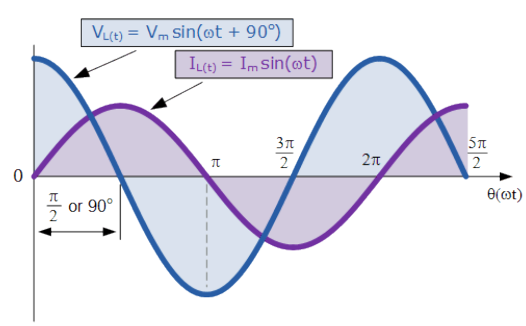
We next tackle a circuit with just an inductor. The values of I and φ can be found with Kirchhoff's loop rule, and are

$$I = \frac{V}{\omega L}$$

$$\varphi = \frac{\pi}{2}$$

(a) Circuit with ac source and inductor





Regarding the phase angle, we see that the voltage is ahead, or "leads", the current by a quarter of a cycle! This is shown in the graph to the right. This is because the inductor "resists" the change of current, so the current rises after the voltage does!

With regards to the current amplitude, we see that it is inversely proportional to the product of the angular frequency and the inductance. In other words, the larger the inductance and frequency, the smaller the current amplitude for a given voltage amplitude.

We therefore define a quantity called the inductive reactance X_L, defined so that

$$I = \frac{V}{X_L}$$

The inductive reactance therefore must be

$$X_L = \omega L$$

The inductive reactance "acts" like a resistance, in that the larger the inductive reactance, the smaller the current. Unlike resistance, however, the inductive reactance is not only dependent on the properties of the inductor itself, but also the angular frequency of the voltage source!

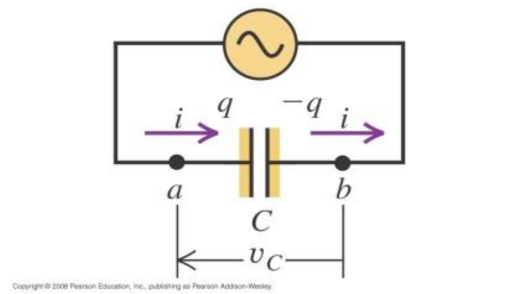
<u>Circuit with Just a Capacitor</u>

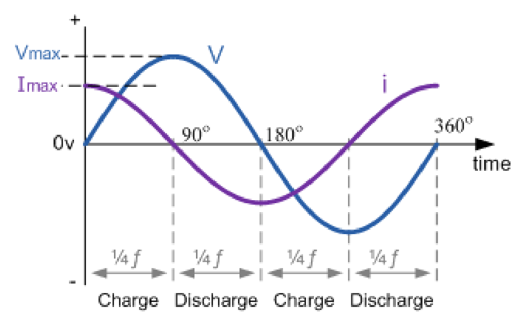
Finally, we investigate a circuit with just a capacitor using Kirchhoff's loop rule. We find

$$I = V\omega C$$

$$\varphi = -\frac{\pi}{2}$$

(a) Circuit with ac source and capacitor





Regarding the phase angle, we see that the voltage is behind, or "lags", the current by a quarter of a cycle! This is shown in the graph to the right.

With regards to the current amplitude, we see that it is directly proportional to the product of the angular frequency and the capacitance. In other words, the smaller the capacitance and angular frequency, the smaller the current amplitude for a given voltage amplitude.

We therefore define a quantity called the capacitive reactance X_C, defined so that

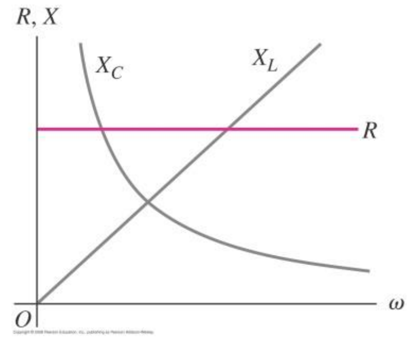
$$I = \frac{V}{X_C}$$

The capacitive reactance therefore must be

$$X_L = \frac{1}{\omega C}$$

Again, the capacitive reactance "acts" like a resistance, in that the larger the reactance, the smaller the current. Again, however, the capacitive reactance is not only dependent on the properties of the capacitor itself, but also the angular frequency of the voltage source!

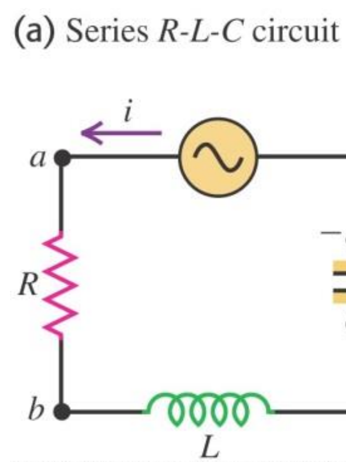
Comparing Inductive Reactance, Capacitive Reactance and Resistance



One very important fact about the inductive and capacitive reactances is that they depend on the angular frequency in completely different ways. The inductive reactance is directly proportional to the angular frequency and the capacitive reactance is inversely proportional to the angular frequency (the resistance, of course, is independent of the angular frequency). In other words, the inductor displays a strong "resistance" to high frequency voltage sources, while the capacitor displays a strong "resistance" to low frequency sources.

Another consequence of this behavior is that there is always one (and only one) angular frequency for which the inductive reactance and the capacitive reactance will be equal. This is shown in the plot of the reactances vs. angular frequency. (*This "special" value of angular frequency will be important later*).

III. The LRC Alternating Circuit



Now we would like to analyze what happens when we put all of the elements in the circuit together! See the circuit diagram to the right. It is much more difficult in this case to find the current amplitude and phase angle. We will learn in class how to use a method called the "phasor method" to solve this system. The results are

$$I = \frac{V}{\sqrt{R^2 + (X_L - X_C)^2}}$$

$$\varphi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

We define a new quantity, called the impedance Z, such that

$$I = \frac{V}{Z}$$

The impedance designates the "resistance" of the circuit to current flow, such that the larger the value of the impedance, the smaller the current amplitude for a given voltage amplitude. From the above definition, we see that

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

Regarding the phase angle, we see that if the inductive reactance is greater than the capacitive reactance, the inductor "wins" and the phase angle is positive, so that the voltage leads the current. Conversely, if the capacitive reactance is greater than the inductive reactance, the capacitor "wins" and the phase angle is negative, so that the voltage leads the current.

IV. Resonance

To understand the phenomenon of resonance in an AC circuit, we want to go back to our mass-spring analogy from the previous section. In a mass-spring with friction, the oscillations dissipate unless we introduce an oscillating, external source of energy to the spring (for instance, I could hit the spring with my finger at a fixed frequency).

It turns out that there is always some frequency of oscillation of the external source that will maximize the amplitude of the mass-spring. What frequency is this?

It is the "natural" frequency of the system, which in the case of a mass-spring is $\omega_0 = \sqrt{\frac{k}{m}}$. In other words, if the external source produces a frequency which exactly matches the natural frequency of oscillation of the system, the amplitude of oscillations will be the greatest. This is known as the phenomenon of resonance.

We can apply this to our LRC circuit. The resonance frequency is the frequency which maximizes the amplitude of oscillations (the current) in the circuit. From the equation

$$I = \frac{V}{Z}$$

we see that the current is maximized when Z is minimized. Looking at the formula for the impedance, we see that it is minimized when $X_L = X_C$. Solving this for ω, we find that

$$\omega_{res} = \omega_0 = \sqrt{\frac{1}{LC}}$$

Thinking back to the LC circuit in section 12, we see that this is simply the "natural frequency" of the LC circuit!

Thus, resonance again occurs when the external source's frequency matches the natural frequency of the circuit. At this frequency, the inductor and capacitor are fully "in-sync" such that they cancel each other out! It is not difficult to prove that not only is the current amplitude largest at this frequency, but the power produced by the source (and dissipated by the resistor) is also maximized at this frequency.

Note in the graph to the right that the impedance is minimized, and thus the current and power maximized, when the voltage frequency ω is equal to the natural frequency ω_0.

Impedance, current, and phase angle as functions of angular frequency