Section 0: Vector Review

I. Introduction to Vectors

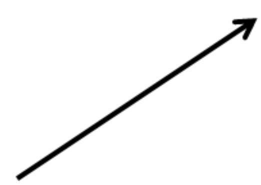
The study of vectors is a purely mathematical topic which we will apply to physical situations often during this course. A vector is a mathematical “object” which can be contrasted with a scalar.

A **scalar** can be thought of as a quantity which is specified by one number. Some particular examples of scalars are mass, temperature, and amount of money in the bank account.

A **vector** can be thought of as a quantity which must be specified by multiple numbers (in this course, usually two). It is easiest conceptually to think about these two numbers as the amount of the quantity and the direction of the quantity in space. For instance, wind velocity is a vector. To fully describe the wind on a given day, you must specify the amount of velocity (the wind speed, say 12 mi/hr) and the direction of the velocity (say, 32 ° North of East). We will encounter many physical quantities in this course which must be described by vectors. Perhaps the most common of these is the quantity of force, which can be thought of as a push or a pull. To fully specify a force, you must not only describe the amount of force (how hard the push or pull is) but also the direction of the force.

For any general vector quantity, the “amount” of the vector is called the **magnitude** of the vector. So, for instance, the magnitude of the wind velocity in the example above is 12 mi/hr. The direction of the vector is simply called the direction.

Representing a Vector



Since a vector is specified by a magnitude and a direction, we can represent it with a directed arrow, as with the picture to the right. The length of the arrow represents the magnitude of the vector and the direction in which the arrow points represents the direction of the vector. Please note that the arrow is only a representation, not the actual quantity itself. The magnitude of the vector is not literally the length on the page. For instance, if the vector in the figure represents wind velocity as discussed previously, it is nonsense to say that the length of the arrow is literally 12 mi/hr, which is not even a length! The picture simply gives us ways to compare different vectors. For instance, if, at another location, the magnitude of the wind velocity is 24 mi/hr, then we would represent that vector with an arrow twice as long as the one above. That is the use of the “arrow” representation.

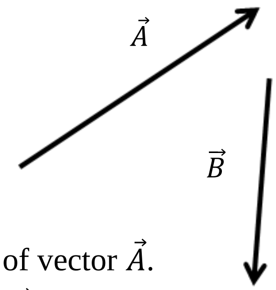
Symbolically, we represent a vector as a letter with an arrow over it. For instance, the vector represented by the arrow in the picture may be symbolically denoted as $\vec{A}$ (*Note: in textbooks you also often see the symbol typed in boldface*). The magnitude of the vector is denoted in one of two ways; either with vertical bars on both sides, or as the letter itself with no arrow. For instance

$$\text{Magnitude of } \vec{A} = |\vec{A}| = A$$

Either of these two ways of representing the magnitude is conventional, although you will probably see the second (letter with no arrow) more often simply because it is simpler to type and write.

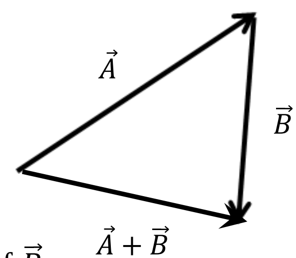
II. Adding and Subtracting Vectors Graphically

Just as we once learned (back in elementary school) to add and subtract scalars, we now need to learn how to add and subtract vectors. There are two general approaches to this problem: the first is graphical (i.e. with pictures) and the second is algebraic. First we will discuss the graphical approach.



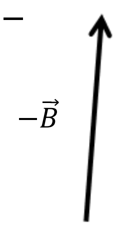
To find the sum of two vectors, $\vec{A}$ and $\vec{B}$, using the graphical method, we must first draw the arrow representation of each vector. Two vectors are represented in the picture to the right. To find the sum of these vectors, we proceed in two steps:

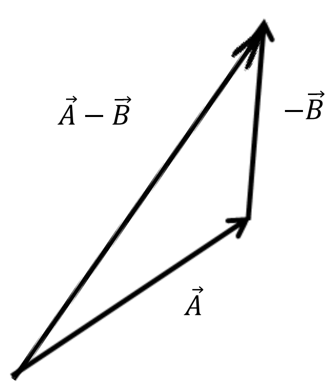
1) Slide the arrow representing vector $\vec{B}$ so that its tail is touching the head of vector $\vec{A}$. You must be careful not to change the length or direction of vectors $\vec{A}$ or $\vec{B}$ during this process. Remember that the length and direction are what specifies the vector itself, so that if you change either, you are no longer representing the same vector!
2) The representation of the sum of $\vec{A} + \vec{B}$ is the arrow which begins at the tail of $\vec{A}$ and ends at the head of $\vec{B}$. See figure to the right.



The graphical method is sometimes called the "tip-to-tail" method of vector addition. As with "regular" addition, you can switch the order in which you add, meaning that $\vec{A} + \vec{B} = \vec{B} + \vec{A}$.

To find the difference of two vectors, $\vec{A} - \vec{B}$, you simply add $\vec{A}$ to the negative of $\vec{B}$. Mathematically, $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$. The negative of a vector is simply a vector of the same magnitude in the opposite direction. See the vector $-\vec{B}$ to the right and also the vector $\vec{A} - \vec{B}$ below.

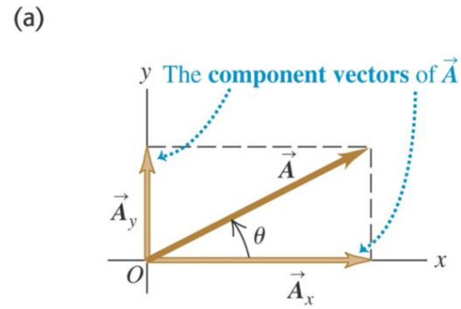




III. Vector Components

In the previous sections, we described a vector as a quantity that has a magnitude and a direction. There is another way to describe a vector quantity which gives us equivalent information about the vector, but can make vector addition and subtraction (as well as multiplication, which we'll get to later) much easier.

This other way of describing a vector is called the method of vector **components**. The basic idea here is to imagine laying the vector down on a Cartesian coordinate system (with coordinates usually labeled x-, y-, and z-). We then represent the vector as a sum of a vector purely in the x-direction and a vector purely in the y-direction (if we are dealing with a vector in three dimensions, we also have to include the z-direction, but for now we'll stick to two dimensions). These vectors are called the x-component vector and y-component vector, respectively. Notice how the vector $\vec{A}$ in the picture is simply the sum of the two component vectors. The component vectors, as in the picture, are labeled $\vec{A}_x$ and $\vec{A}_y$, respectively. Thus the vector $\vec{A}$ can be written as

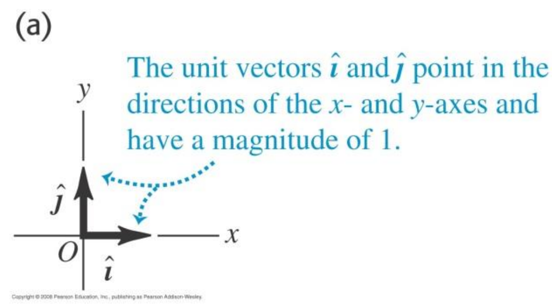


$$\vec{A} = \vec{A}_x + \vec{A}_y$$

Conventionally, the magnitudes of the component vectors, A_x and A_y, are simply called the x- and y-components, respectively, of the vector $\vec{A}$. These components are extremely easy to find using trigonometric identities if the magnitude of $\vec{A}$ as well as its direction θ are known. Conversely, if one knows the x- and y-components of $\vec{A}$, then it is again a simple matter of applying some trigonometric identities to find the magnitude and direction of $\vec{A}$.

Unit Vectors

It is useful to define and label three specific vectors. Each of these three vectors has a magnitude of one, and each of them points directly along one of the coordinate axes. The vector of magnitude one which points in the x-direction is labeled $\hat{\imath}$, the vector of magnitude one which points in the y-direction is labeled $\hat{\jmath}$, and the vector of magnitude one which points in the z-direction is labeled $\hat{k}$. These vectors are called **unit vectors**, because they have a magnitude of one. Notice that each unit vector is given a "hat" symbol instead of an "arrow" symbol. This is simply to designate the fact that a unit vector must have magnitude one. The x- and y- unit vectors are shown in the picture to the right.



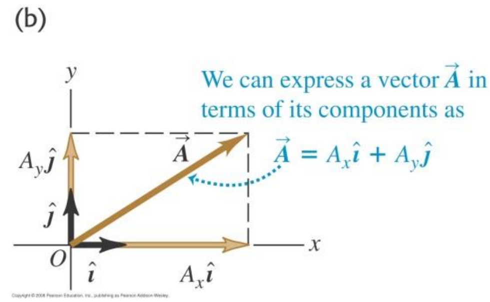
This is useful because we may now express the component vectors neatly as

$\vec{A}_x = A_x\hat{\imath}$ and $\vec{A}_y = A_y\hat{\jmath}$

With this new way of writing component vectors, we may represent $\vec{A}$ as

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$$

See the figure for a pictorial representation of this equation.



IV. Addition and Subtraction Using Vector Components

Previously we learned the "graphical" method of addition. This gave us an arrow, which is a good pictorial representation of what a sum of (or difference between) two vectors looks like. However, if we want a precise numerical value for the magnitude and direction of the sum or difference, the pictorial representation is not very useful. Instead, we will perform addition and subtraction with the aid of components. Since component vectors always point along either the x- or y- axis, adding component vectors is as easy as simply adding numbers!

For instance, if we have to vectors, $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$, then the sum of $\vec{A}$ and $\vec{B}$ is simply the sum of their component vectors.

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{\imath} + \left(A_y + B_y\right)\hat{\jmath}$$

This means that if we know the x- and y- components of each of the two vectors to be added, we can very easily find the x- and y-components of their sum! Once we have the x- and y-components of their sum, we can work "backwards" using trigonometric identities, to find the magnitude and direction of that sum. Everything works exactly the same for subtraction, except that you subtract the components of the vectors rather than add them.

V. Vector Multiplication

Now that we have learned how to add and subtract, we can learn to multiply. There are actually two ways to define the multiplication of two vectors. Both types of multiplication will have applications to real physical situations, as we will see later. The two types of vector multiplication are

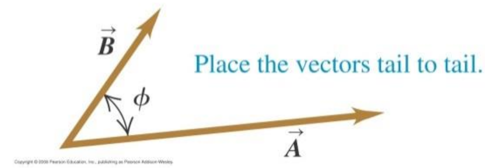
1) Scalar multiplication, also called the "dot product", denoted as $\vec{A} \cdot \vec{B}$
    - The result of scalar multiplication is… a scalar! In other words, the dot product of two vectors produces a scalar, i.e. a number with no direction!
2) Vector multiplication, also called the "cross product", denoted as $\vec{A} \times \vec{B}$
    - The result of vector multiplication is… a vector! In other words, the cross product of two vectors produces another vector, with a magnitude and direction!

We now proceed to define and briefly analyze each of these types of multiplication.

Scalar Product

The scalar product of vectors $\vec{A}$ and $\vec{B}$ is defined as

(a)



$$\vec{A} \cdot \vec{B} \equiv |\vec{A}||\vec{B}| \cos\varphi = AB\cos\varphi$$

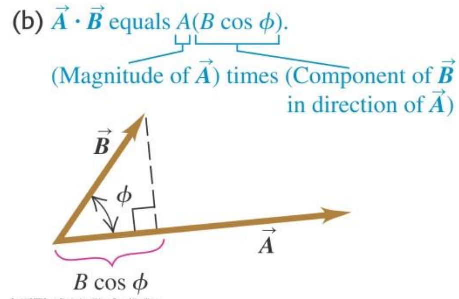
where φ is defined as the opening angle between the two vectors. (See the figure)

In words, the scalar product of two vectors is equal to the product of the magnitudes of the vectors multiplied by the cosine of the angle between them. Notice that the scalar product does indeed produce a number with no direction (i.e. a scalar).

The scalar product can also be written another way. It can be proven that the definition given above is equivalent to the following:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

The original form is preferable if the magnitudes and directions of the two vectors are known, whereas the form involving components is preferable if the components of the two vectors are known.



The scalar product has a specific interpretation, which will be useful in later chapters. **The scalar product can be thought of as the magnitude of $\vec{A}$ multiplied by the component of $\vec{B}$ which is parallel to $\vec{B}$ (or vice-versa)**. See figure. *In particular, if vector $\vec{A}$ has no component parallel to $\vec{B}$ (i.e. the vectors are perpendicular), then their scalar product is zero!*

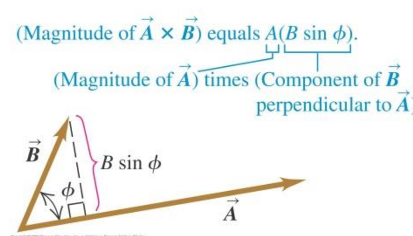
Vector Product

Since the vector product produces a vector, it is slightly more complicated than the scalar product. We first define the magnitude of the vector product, and then we will define the direction.

The magnitude of the vector product is defined as

$$|\vec{A} \times \vec{B}| \equiv |\vec{A}||\vec{B}| \sin\varphi = AB\sin\varphi$$

(a)



The magnitude of the vector product has a specific interpretation, which will be useful in later chapters. **The magnitude of the vector product can be thought of as the magnitude of $\vec{A}$ multiplied by the component of $\vec{B}$ which is perpendicular to $\vec{B}$ (or vice-versa)**. See figure. *In particular, if vector $\vec{A}$ has no component perpendicular to $\vec{B}$ (i.e. the vectors are parallel), then their vector product is zero!*

With the magnitude taken care of, we must define the direction of the vector product. The direction of the vector product is always defined to be in a plane perpendicular to <u>both</u> of the vectors being multiplied. There are always two such directions, and so we must choose one of them. By convention, we choose the direction based on the **Right Hand Rule**.

The right hand rule is applied as follows: Point your fingers straight along the first vector ($\vec{A}$ in our example) and then curl your fingers towards the second vector ($\vec{B}$ in our example). Your thumb should be sticking out, and the direction of your thumb is the direction of the vector product! See the figures to the right.