# Jacobian Matrix for Permanent Magnet Tracking

## Problem Definition

In the tracking problem, a cylindrical permanent magnet moves freely in space. Its state is described by seven parameters: the three components of the magnet center, the three raw components of the orientation vector (normalized internally), and a scale parameter representing the effective magnetization strength.  
  
Each magnetic sensor, located at a known position, measures the three Cartesian components of the magnetic flux density. For a single sensor this yields a vector of length three. For an array of sensors the measurements are concatenated into a vector of length three times the number of sensors. The inverse task is to estimate the unknown magnet state from this measurement vector. To apply gradient-based optimization methods such as Gauss–Newton or Levenberg–Marquardt, the Jacobian matrix of the forward model with respect to the state parameters is required.

## Forward Model with Surface Discretization

The cylindrical magnet is represented by discretizing its two circular end faces into many small surface elements. Each element carries an equivalent surface charge and contributes to the magnetic field at a sensor position. The total field at a sensor is obtained by summing over all element contributions:

Here, σₖ denotes the effective surface charge density, ΔSₖ the area of the element, rₖ the relative vector between sensor and element, and pₖ the world coordinates of the element.

## Structure of the Jacobian

For one sensor, the Jacobian has three rows (corresponding to the measured field components) and seven columns (corresponding to the state parameters). For an array of sensors, the individual 3×7 Jacobian blocks are stacked vertically, resulting in a matrix of size 3N × 7, where N is the number of sensors.

The columns correspond to the derivatives with respect to:

1. the three center coordinates,

2. the three orientation vector components,

3. the scale parameter.

## Parameter-Specific Contributions

Center coordinates:

Translating the magnet center shifts all surface elements equally. The relative vectors from sensors to elements therefore change linearly. The derivative is the negative spatial derivative of the kernel function:

Orientation vector:

The orientation vector defines the local orthonormal basis of the magnet. A small change in orientation rotates this basis and thus displaces all surface elements. The derivatives of the basis with respect to internal spherical angles are computed, and the chain rule is applied to map them to the raw orientation vector components.

Scale parameter:

The scale enters linearly. Hence the derivative with respect to scale is simply:

## Application in Inverse Tracking

The measured field vector is compared to the predicted field, and the residual is formed:

Here θ denotes the parameter vector. Using the Jacobian J = ∂B/∂θ, the Gauss–Newton update step is:

Iterations continue until the residual norm is below the sensor noise level or a maximum number of iterations is reached.

## Practical Considerations

- Numerical stability: denominators with sine terms may become small; lower bounds avoid division by zero.  
- Computational complexity: scales with sensor count × number of surface elements.  
- Interpretability: each Jacobian column represents the linearized effect of one parameter on all sensor readings.