Finance 567C
Financial Risk Management

Spring 2017
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Solution to Homework 6 GARCH(1,1)
Due Sunday, March 5, 2015, at 5:00 p.m.
Total 10 points

Instructions. This is a group assignment. Groups may include up to 5 people. Please submit a Word or .pdf document with your solutions via the Compass site prior to 5:00 p.m. on Sunday, March 5. Please also submit the Matlab, R, or Python scripts and data files that you used.

1. GARCH(1,1) estimated 4 ways. (total 5 points) Find and download data on the S&P 500 index values, and compute continuously compounded (log) returns. (These returns will not include dividends.)

(a) (1 point) The GARCH(1,1) likelihood can be written as a function of the four unknown parameters α, β, σ, and σ_1. Use the 1,000 returns up through Tuesday, February 21, 2017 to estimate the GARCH(1,1) parameters. That is, use the return for February 21, 2017 and the 999 preceding returns (a total of 1,000 returns) and the method of maximum likelihood to estimate the parameters, treating all four as unknown parameters to be estimated. What are the estimated parameters α, β, σ, and σ_1? (Note that the question asks for the estimates of σ, and σ_1, not σ^2 and σ_1^2.)

Remark: You may do this using Matlab, R, or Python.

Solution. The estimates are $\alpha = 0.17771$, $\beta = 0.71669$, $\sigma = 0.00801$, and $\sigma_1 = 0.00589$. These are computed in the first tab of the Excel spreadsheet F567.s2017.HW6.solution.GARCH.xlsx, and in the R file hw6codes.R.

(b) (1 point) Using the same data, set the long-run variance σ^2 equal to the sample variance over the 1,000 days in the data period used for estimation. If you do this, the GARCH(1,1) likelihood can be written as a function of the three unknown parameters α, β, and σ_1. Use maximum likelihood to estimate the three parameters α, β, and σ_1. What is your estimate of the long-run standard deviation σ? What are your estimates of the parameters α, β, and σ_1?

Solution. The sample variance and standard deviation are 6.4923×10^{-5} and 0.008058; thus, we have the estimates $\sigma^2 = 6.4923 \times 10^{-5}$ and $\sigma = 0.008058$. The other estimates are $\alpha = 0.17792$, $\beta = 0.71644$, and $\sigma_1 = 0.00589$. Note that these estimates are similar to the estimates in part (a). They are computed in the first tab of the spreadsheet F567.s2017.HW6.solution.GARCH.xlsx, and in the R file hw6codes.R.

(c) (1 point) Next set the initial variance σ_1^2 equal to the sample variance over the 1,000 days in the data period used for estimation. If you do this, the GARCH(1,1) likelihood can be written as a function of the three unknown parameters α, β, and σ. Use maximum likelihood to estimate the three parameters α, β, and σ. What is your estimate of the initial standard deviation σ_1? What are your estimates of the parameters α, β, and σ?

Solution. As in part (b), the sample variance and standard deviation are 6.4923×10^{-5} and 0.008058; thus, we have the estimates $\sigma_1^2 = 6.4923 \times 10^{-5}$ and $\sigma_1 = 0.008058$. The other estimates are $\alpha = 0.18071$, $\beta = 0.71137$, and $\sigma_1 = 0.00802$. Note that these estimates are similar to the estimates in parts (a) and (b). They are computed in the first tab of the Excel spreadsheet F567.s2017.HW6.solution.GARCH.xlsx, and in the R file hw6codes.R.

(d) (1/2 point) Now combine the approaches in (b) and (c), that is set both the long-run variance σ^2 and the initial variance σ_1^2 equal to the sample variance over the 1,000 days in the data period used for estimation. If you do this, the GARCH(1,1) likelihood can be written as a function of the two unknown parameters α and β. Use maximum likelihood to estimate the two parameters α and β. What are your estimates of the long-run standard deviation σ and the initial standard deviation σ_1? What are your estimates of the parameters α and β?

Solution. As in parts (b) and (c), the sample variance and standard deviation are 6.4923×10^{-5} and 0.008058; thus, we have the estimates $\sigma^2 = \sigma_1^2 = 6.4923 \times 10^{-5}$ and $\sigma_1 = \sigma_1 = 0.008058$. The other estimates are $\alpha = 0.18196$ and $\beta = 0.71117$. Note that these estimates are similar to the estimates in parts (a) and (b). They are computed in the first tab of the Excel spreadsheet F567.s2017.HW6.solution.GARCH.xlsx, and in the R file hw6codes.R.

(e) (1 point) Use each of the four models estimated in parts (a)-(d) to forecast the return variance and standard deviation for Wednesday, February 22, 2017. What are the four forecasts of the return standard deviation?

Solution. Letting February 21 be day t and February 22 be day $t + 1$, the forecasts can be computed directly from the GARCH(1,1) model using

$$\sigma_{t+1}^2 = \sigma^2(1-\alpha-\beta) + \alpha R_t^2 + \beta\sigma_t^2$$

and of course $\sigma_{t+1} = \sqrt{\sigma_{t+1}^2}$. Using the four sets of estimates from parts (a)-(d), the four forecasts are

Based on part…	σ_{t+1}^2	σ_{t+1}
(a)	0.0000353	0.005945
(b)	0.0000354	0.005950
(c)	0.0000355	0.005957
(d)	0.0000355	0.005961

2. Using the GARCH(1,1) model to forecast volatility (total 3 points) This question asks you to use the four sets of estimates from Question (1) to compute four forecasts of the realized variance

$$\text{realized variance} = \sum_{k=1}^{21} R_{t+k}^2$$

over the 21 days following February 21, 2017. That is, assume that the current date and time are just after the close of trading on February 21, 2017, you have data up through the close of trading on February 21, 2017, and you have estimated the GARCH(1,1) models in Question 1. You now want to forecast the realized variance over the 21 trading days starting with February 22, 2017.

(a) (2 points) What are your four forecasts of the realized variance?

Solution. Let's focus on the $t + k$ return R_{t+k}. Your forecast of the variance of this return is $E_t[R_{t+k}^2] = E_t[\mathrm{E}_{t+k-1}[R_{t+k}^2]]$, where the subscript E_t means the conditional expectation given the information at time t. (Note that this is a basic property of conditional expectations.) Then using the fact that $E_{t+k-1}[R_{t+k}^2] = \sigma_{t+k}^2$, we have $\mathrm{E}_t[R_{t+k}^2] = \mathrm{E}_t[\sigma_{t+k}^2]$. Finally, the forecast of $\mathrm{E}_t[\sigma_{t+k}^2]$ is given by the formula on slide 18 of the PowerPoint slides on GARCH modelling, that is

$$\mathrm{E}_t[\sigma_{t+k}^2] = (\alpha + \beta)^{k-1}(\sigma_{t+1}^2 - \sigma^2) + \sigma^2.$$

The forecast of the realized variance is then the sum of the 21 forecasts, that is

$$E_t\left[\sum_{k=1}^{21} R_{t+k}^2\right] = \sum_{k=1}^{21} E_t[R_{t+k}^2] = \sum_{k=1}^{21} E_t[\sigma_{t+k}^2].$$

The forecasts are in the second tab of the spreadsheet F567.s2017.HW6.solution.GARCH.xlsx.

(b) (1 point) Express your four forecasts in terms of annualized volatilities. (Take the square root, and then annualize by multiplying by $\sqrt{252/21}$. Or else annualize by multiplying by 252/21, and then take the square root.)

Solution. The annualized forecasts are in the second tab of the Excel spreadsheet F567.s2017.HW6.solution.GARCH.xlsx.

3. N**GARCH(1,1)** (total 2 points) In this question use the same 1,000 returns you used in Question 1.

(a) (1 point) Estimate the NGARCH(1,1) model described on slide 3 of the PowerPoint slides GARCH Generalizations.ppt. The NGARCH(1,1) likelihood can be written as a function of the five unknown parameters α, β, σ, σ_1, and θ. Use the method of maximum likelihood to estimate the parameters, treating all five as unknown parameters to be estimated. What are the estimated parameters α, β, σ, σ_1, and θ?

Hint: Use your estimates from Question 1(a) and $\theta = 0$ as your initial guesses.

Solution. The NGARCH(1,1) model can be written as

$$\sigma_{t+1}^2 = \omega + \alpha(R_t - \theta\sigma_t)^2 + \beta\sigma_t^2.$$

You are asked for the estimates of the parameters α, β, σ, σ_1, and θ, so first you need to find an expression for ω in terms of α, β, σ, σ_1, and θ. To do this, first expand the right-hand side of the NGARCH model above, yielding

$$\sigma_{t+1}^2 = \omega + \alpha(R_t^2 - 2R_t\theta\sigma_t + \theta^2\sigma_t^2) + \beta\sigma_t^2 .$$

Then take the expectation of both sides,

$$\begin{aligned}\sigma^2 &= \omega + \alpha(E[R_t^2] - 2E[R_t\theta\sigma_t] + \theta^2 E[\sigma_t^2]) + \beta E[\sigma_t^2] \\ &= \omega + \alpha(E[R_t^2] - 2E[E_{t-1}[R_t\theta\sigma_t]] + \theta^2 E[\sigma_t^2]) + \beta E[\sigma_t^2] \\ &= \omega + \alpha(\sigma^2 - 0 + \theta^2\sigma^2) + \beta\sigma^2.\end{aligned}$$

Solving for ω,

$$\omega = \sigma^2(1 - \alpha(1+\theta^2) - \beta) .$$

Thus, the NGARCH(1,1) model can be written as

$$\sigma_{t+1}^2 = \sigma^2(1 - \alpha(1+\theta^2) - \beta) + \alpha(R_t - \theta\sigma_t)^2 + \beta\sigma_t^2 .$$

The estimates are $\alpha = 0.07472$, $\beta = 0.54381$, $\theta = 2.18659$, $\sigma = 0.01226$, and $\sigma_1 = 0.00549$. These are computed in the third tab of the Excel spreadsheet F567.s2017.HW6.solution.GARCH.xlsx, and in the R file hw6codes.R.

(b) (1 point) What is the value of the log likelihood of the NGARCH(1,1) model? What is the value of the log likelihood of the GARCH(1,1) model you estimated in Question 1, part (a)? What is the value of the χ^2 test statistic testing the null hypothesis that $\theta = 0$? Is this test statistic significantly different from zero at the 5% level? At the 1% level?

Solution. The values of the long likelihood's of the NGARCH(1,1) and GARCH(1,1) models are $L_{NGARCH} = 3{,}536.49$ and $L_{GARCH} = 3{,}485.65$, respectively. The value of the χ^2 test statistic testing the null hypothesis that $\theta = 0$ is $2(3{,}536.49 - 3{,}485.65) = 101.67$. This is significantly different from zero at both the 5% and 1% levels. (The critical values of the χ^2 distribution are 3.84 and 6.63.)