

# Maxent Example - Self-Energy of a Metal

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## Abstract

This document is a tutorial on the use of Maxent, a program for doing analytical continuation using the maximum entropy method. It will explain how to provide the program with the proper parameter file, data format for particle-hole symmetric data of a self-energy Matsubara space, and understand the output. Included as a supplement is the corresponding interacting Green's function input data. The data provided is of an interacting Hubbard model with  $U = 1, \beta = 2$  for 1 site at half filling. This program uses the ALPSCore libraries[1, 2].

## Contents

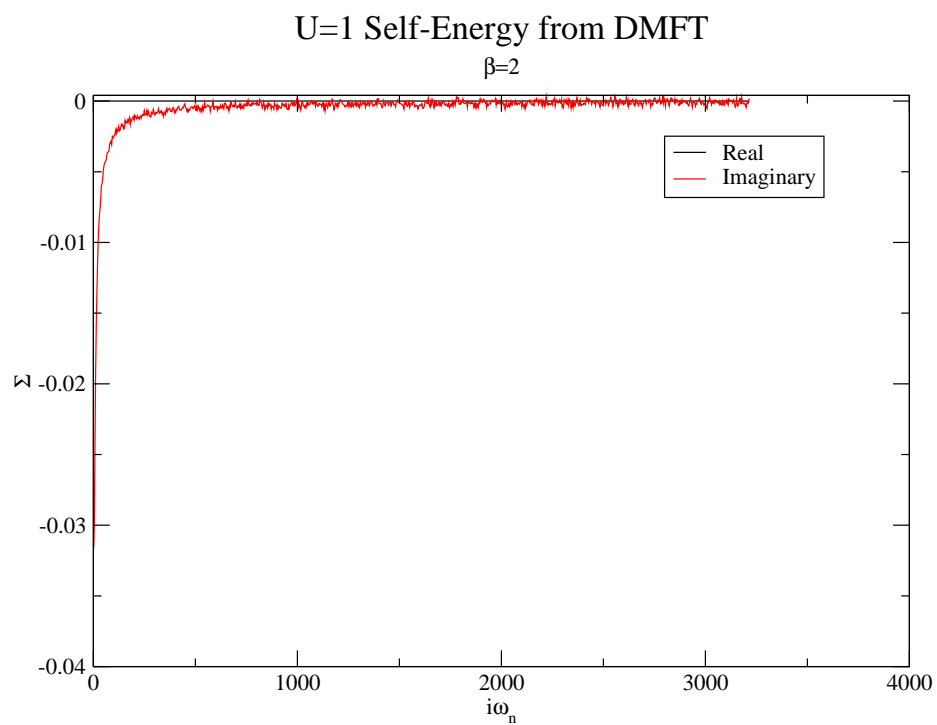
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## 1 Introduction

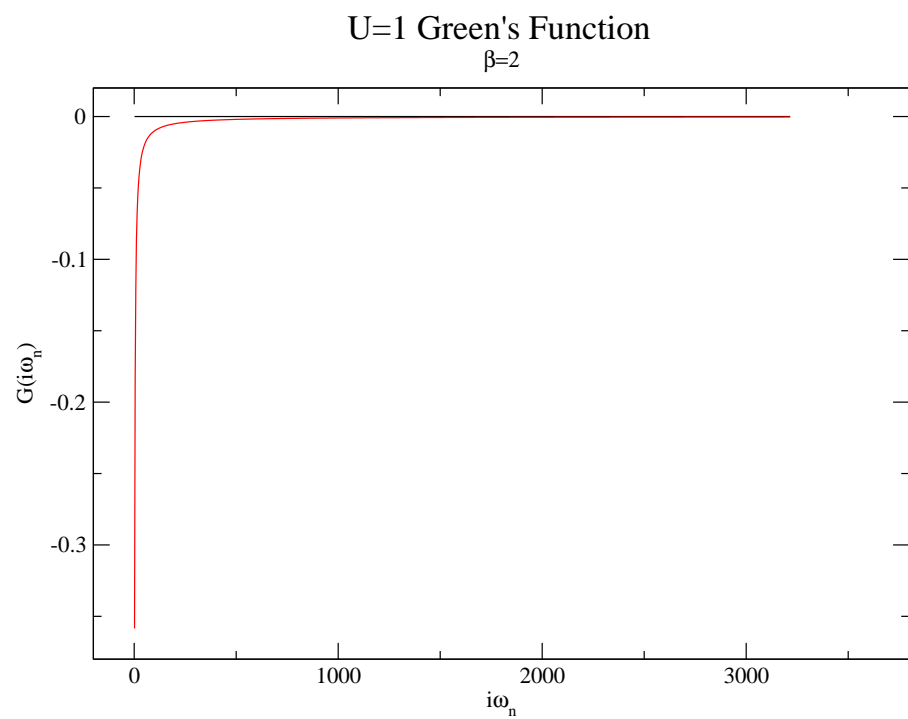
Using DMFT we can set  $U = 1$  for a single site at half-filling to generate the interacting Hubbard model

$$H = - \sum_{\langle ij \rangle \sigma} t_{ij} \left( c_{i\sigma}^\dagger c_{j\sigma} + c_{j\sigma}^\dagger c_{i\sigma} \right) + U \sum_i n_{i\uparrow} n_{i\downarrow},$$

This produces a self-energy output in Matsubara space:



Measured separately is the Green's function:



While these functions are on the imaginary axis, we wish to analytically continue them to the real axis. Mathematically, this is equivalent to finding the spectral function  $A(\omega) = \text{Im}[\Sigma(\omega)]$  such that

$$G(X) = -\frac{1}{\pi} \int_{-\infty}^{\infty} K(X, \omega) A(\omega) d\omega,$$

with the kernel  $K$  written as  $K(i\omega_n, \omega) = \frac{1}{i\omega_n - \omega}$  or  $K(\tau, \omega) = \frac{-e^{-\tau\omega}}{1 + e^{-\omega\beta}}$  for this data. The Maxent procedure is one such method of determining  $A(\omega)$ .

## 1.1 Normalization

Note that unlike a Green's function, the normalization (aka high frequency term) is not 1. For a self energy:

$$\Sigma = \Sigma_0 + \frac{\Sigma_1}{i\omega_n} + \dots$$

where  $\Sigma_0 = Un$  (known as the Hartree term) and  $\Sigma_1 = U^2 n(1-n)$  [3]. This normalization is important to Maxent, as it is assumed to be the Green's function normalization of 1. For  $U = 1$  at half-filling ( $n = 0.5$ ), there is a normalization of 0.25.

## 1.2 Errors

In this case, DMFT self-energies are lacking error bars. Because errors go like  $\frac{1}{\sqrt{N}}$  where  $N$  is the number of samples, we can instead use this as an order of magnitude estimate for the error. This data had 4296900 sample iterations, and we therefore choose an estimated error  $\sigma = 0.0005$ .

## 2 File Structure

We've included several files that will be used to generate the remainder of this document:

Filenames and Descriptions		
<ul style="list-style-type: none"> <li>• <math>Selfenergy = \Sigma(i\omega_n)</math> data values               <ul style="list-style-type: none"> <li>– column format: <math>i\omega_n</math> <math>Re[\Sigma(i\omega_n)]</math> <math>Im[G(i\omega_n)]</math></li> </ul> </li> </ul>	Input file - Selfenergy	
	1.5707963267949	5.6703744085571e-16 -0.03158953764285
	4.7123889803847	3.1444695800992e-16 -0.029847318775471
	7.8539816339745	9.4394785839476e-16 -0.023814398782168
<ul style="list-style-type: none"> <li>• <math>Selfin = \Sigma(i\omega_n)</math> input format for Maxent               <ul style="list-style-type: none"> <li>– column format: <math>i\omega_n</math> <math>Im[G(i\omega_n)]</math> <math>\sigma_{I,n}</math></li> </ul> </li> </ul>	Input file - Selfin	
	1.5707963267949	-0.03158953764285 0.0005
	4.7123889803847	-0.029847318775471 0.0005
	7.8539816339745	-0.023814398782168 0.0005
<ul style="list-style-type: none"> <li>• <math>G_{im} = Im[G(i\omega_n)]</math>, also input for Maxent               <ul style="list-style-type: none"> <li>– column format: <math>i\omega_n</math> <math>Im[G(i\omega_n)]</math> <math>\sigma_{I,n}</math></li> </ul> </li> </ul>	Input file - G_im	
	1.5707963267949	-0.3585154015692 0.0005
	4.7123889803847	-0.18353508163336 0.0005
	7.8539816339745	-0.11976613812711 0.0005
<ul style="list-style-type: none"> <li>• <math>G_{re} = Re[G(i\omega_n)]</math> <ul style="list-style-type: none"> <li>– column format: <math>i\omega_n</math> <math>Re[G(i\omega_n)]</math> <math>\sigma_{R,n}</math></li> </ul> </li> </ul>	Input file - G_re	
	1.5707963267949	-2.0386388974517e-17 0.0005
	4.7123889803847	-4.6688456052657e-18 0.0005
	7.8539816339745	-3.4351421175898e-18 0.0005

### 3 Using Maxent

These files are easily used with Maxent. Here is the frequency space input:

Param File in.param	
BETA=2	<i>#inverse temperature</i>
OMEGA_MAX=25	<i>#the spectral function is wider than omega=10</i>
NDAT=1024	<i>#num of data points</i>
NFREQ=1000	<i>#num of output frequencies</i>
DATASPACE=frequency	<i>#G(i<math>\omega</math>)</i>
KERNEL=fermionic	<i>#fermionic/bosonic values</i>
FREQUENCY_GRID=Quadratic	<i>#this grid is better for features away from 0</i>
PARTICLE_HOLE_SYMMETRY=1	<i>#0/1</i>
DATA="Selfin"	<i>#location of data file</i>
SELF=1	<i>#this will output <math>\Sigma(\omega)</math> rather than A(<math>\omega</math>)</i>
NORM=0.25	<i>#self energy norm = <math>U^2 \cdot n(1-n)</math></i>

Maxent then produces the following output:

Maxent output	
Using flat default model	
using kernel fermionic in domain frequency with ph symmetry	
The high frequency limit is not 1!: 2.30495 Check norm?	
Kernel is set up	
# 0	4108.32
# 1	1668.62
# 2	513.561
# 3	126.175
# 4	25.6169
# 5	4.3795
# 6	0.635099
# 7	0.0784661
minimal chi2: 0.098459	
WARNING: Redefinition of parameter NORM: Input (and output) data are assumed to be normalized to NORM.	
alpha it: 0	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 527.241 norm: 1.16962
alpha it: 1	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 481.411 norm: 1.17798
alpha it: 2	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 443.307 norm: 1.18585
alpha it: 3	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 409.139 norm: 1.19303
alpha it: 4	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 378.828 norm: 1.19943
alpha it: 5	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 352.116 norm: 1.20499
alpha it: 6	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 328.667 norm: 1.2097
alpha it: 7	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 308.1 norm: 1.21353
alpha it: 8	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 290.018 norm: 1.21649
alpha it: 9	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 274.026 norm: 1.21858
alpha it: 10	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 259.754 norm: 1.21982
alpha it: 11	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 246.864 norm: 1.22024
alpha it: 12	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 235.057 norm: 1.21985
alpha it: 13	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 224.078 norm: 1.21869
alpha it: 14	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 213.717 norm: 1.21679
alpha it: 15	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 203.807 norm: 1.21418
alpha it: 16	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 194.22 norm: 1.2109
alpha it: 17	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 184.869 norm: 1.20697
alpha it: 18	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 175.7 norm: 1.20245
alpha it: 19	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 166.688 norm: 1.19737
alpha it: 20	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 157.834 norm: 1.19178
alpha it: 21	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 149.256 norm: 1.18533
alpha it: 22	Q = 0.5chi <sup>2</sup> -\ $\alpha$ *entropy: 140.793 norm: 1.17884

```

alpha it: 23      Q = 0.5chi^2-\alpha*entropy: 132.591      norm: 1.17201
alpha it: 24      Q = 0.5chi^2-\alpha*entropy: 124.7        norm: 1.1649
alpha it: 25      Q = 0.5chi^2-\alpha*entropy: 117.171      norm: 1.15758
alpha it: 26      Q = 0.5chi^2-\alpha*entropy: 110.049      norm: 1.15012
alpha it: 27      Q = 0.5chi^2-\alpha*entropy: 103.374      norm: 1.14258
alpha it: 28      Q = 0.5chi^2-\alpha*entropy: 97.1752      norm: 1.13502
alpha it: 29      Q = 0.5chi^2-\alpha*entropy: 91.4723      norm: 1.1275
alpha it: 30      Q = 0.5chi^2-\alpha*entropy: 86.2731      norm: 1.12008
alpha it: 31      Q = 0.5chi^2-\alpha*entropy: 81.5754      norm: 1.11281
alpha it: 32      Q = 0.5chi^2-\alpha*entropy: 77.3674      norm: 1.10573
alpha it: 33      Q = 0.5chi^2-\alpha*entropy: 73.6294      norm: 1.09887
alpha it: 34      Q = 0.5chi^2-\alpha*entropy: 70.3353      norm: 1.09227
alpha it: 35      Q = 0.5chi^2-\alpha*entropy: 67.4542      norm: 1.08595
alpha it: 36      Q = 0.5chi^2-\alpha*entropy: 64.9523      norm: 1.07993
alpha it: 37      Q = 0.5chi^2-\alpha*entropy: 62.7942      norm: 1.07421
alpha it: 38      Q = 0.5chi^2-\alpha*entropy: 60.944        norm: 1.0688
alpha it: 39      Q = 0.5chi^2-\alpha*entropy: 59.3668      norm: 1.0637
alpha it: 40      Q = 0.5chi^2-\alpha*entropy: 58.029        norm: 1.05891
alpha it: 41      Q = 0.5chi^2-\alpha*entropy: 56.8992      norm: 1.05441
alpha it: 42      Q = 0.5chi^2-\alpha*entropy: 55.9485      norm: 1.05021
alpha it: 43      Q = 0.5chi^2-\alpha*entropy: 55.1507      norm: 1.04629
alpha it: 44      Q = 0.5chi^2-\alpha*entropy: 54.4826      norm: 1.04263
alpha it: 45      Q = 0.5chi^2-\alpha*entropy: 53.9235      norm: 1.03923
alpha it: 46      Q = 0.5chi^2-\alpha*entropy: 53.4556      norm: 1.03606
alpha it: 47      Q = 0.5chi^2-\alpha*entropy: 53.0636      norm: 1.0331
alpha it: 48      Q = 0.5chi^2-\alpha*entropy: 52.7344      norm: 1.03035
alpha it: 49      Q = 0.5chi^2-\alpha*entropy: 52.457        norm: 1.02779
alpha it: 50      Q = 0.5chi^2-\alpha*entropy: 52.2225      norm: 1.02541
alpha it: 51      Q = 0.5chi^2-\alpha*entropy: 52.0232      norm: 1.02319
alpha it: 52      Q = 0.5chi^2-\alpha*entropy: 51.8531      norm: 1.02112
alpha it: 53      Q = 0.5chi^2-\alpha*entropy: 51.7072      norm: 1.01919
alpha it: 54      Q = 0.5chi^2-\alpha*entropy: 51.5814      norm: 1.01739
alpha it: 55      Q = 0.5chi^2-\alpha*entropy: 51.4727      norm: 1.01571
alpha it: 56      Q = 0.5chi^2-\alpha*entropy: 51.3782      norm: 1.01415
alpha it: 57      Q = 0.5chi^2-\alpha*entropy: 51.2961      norm: 1.01269
alpha it: 58      Q = 0.5chi^2-\alpha*entropy: 51.2245      norm: 1.01133
alpha it: 59      Q = 0.5chi^2-\alpha*entropy: 51.162        norm: 1.01006
Ng: 5.10747
posterior probability of the default model: 1.33065e-33
spectra      max backcont diff      chi^2 value
=====
chispec      0.040974      1053.08
avspec      0.00203636      106.161
maxspec      0.00195274      105.411

```

---

### 3.1 Output Guide

```

Using flat default model
using kernel fermionic in domain frequency with ph symmetry
The high frequency limit is not 1!: 2.30495 Check norm?
Kernel is set up

```

These are the setup messages, confirming your input choices. There is a warning for the high frequency limit, but because our data is very noisy it can be ignored. If this limit was significantly off from 1, then your input NORM should be confirmed. In this case, the last few data points are noisy, leading to an inaccurate high frequency limit warning.

```
# 0 4108.32
# 1 1668.62
# 2 513.561
# 3 126.175
# 4 25.6169
# 5 4.3795
# 6 0.635099
# 7 0.0784661
minimal chi2: 0.098459
```

These represent the eigenvalues that are above precision after the single value decomposition (SVD). The last line represents the smallest  $\chi^2$  value the program thinks it will achieve. If this is  $\gg 1$  there may be something wrong with your input or it is very noisy

```
...
alpha it: 2 Q = 0.5chi^2-\alpha*entropy: 443.307 norm: 1.18585
alpha it: 3 Q = 0.5chi^2-\alpha*entropy: 409.139 norm: 1.19303
...
```

The root finding procedure will print the iterations through  $\alpha$  values in the range given by the parameters (default: 60 values  $\in [0.01, 20]$ ) If the first two or three do not minimize properly that is ok, as long as the rest continue normally. Notice that the norm stays  $\approx 1$  for all iterations

```
Ng: 5.10747
chi2 max: 105.411
posterior probability of the default model: 1.33065e-33
```

This is posted after completing all  $\alpha$  values and root finding. Ng represents the number of “good input points,” chi2 max is the maximum value of  $\chi^2$  in the  $\alpha$  iterations, and the last line is the probability that the default model is the correct representation of the spectral function. Note that that posterior probability has no known normalization.

spectra	max backcont diff	chi^2 value
=====	=====	=====
chispec	0.040974	1053.08
avspec	0.00203636	106.161
maxspec	0.00195274	105.411

By default, maxent will back-continue, or continue back to the imaginary axis, the spectral function maxent output. Here two useful values are shown, the maximum difference between any of the back-continued points and input data, as well as the  $\chi^2$  value.

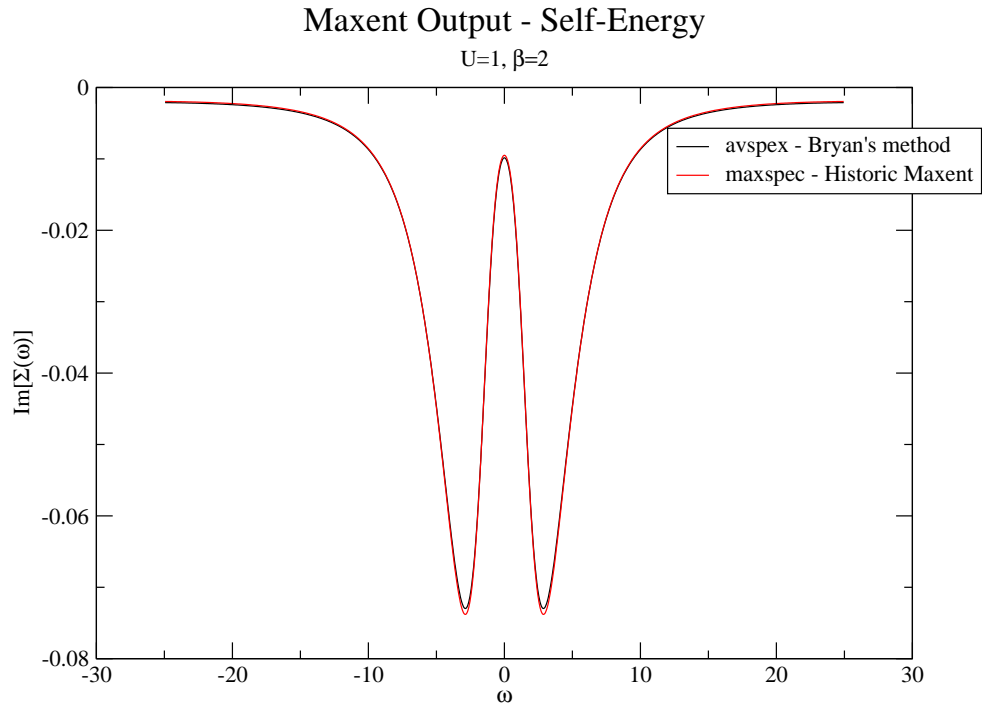
---

If text output is on, Maxent produces 13 files:

name.out.avspec.dat	“Spectral function” using Bayesian Averaging - <b>Bryan’s method</b>
name.out.avspec_back.dat	The avspec spectrum continued back to the imaginary axis
name.out.avspec_self.dat	$\text{Im}[\Sigma(\omega)]$ with the proper sign and normalization; using <b>Bryan’s method</b>
name.out.chi2.dat	Estimated $\chi^2$ for each $\alpha$ value solution
name.out.chispec.dat	“Spectral function” satisfying the best $\chi^2$ - <b>historic Maxent</b>
name.out.chispec_back.dat	The chispec spectrum continued back to the imaginary axis
name.out.fits.dat	Fits of each $\alpha$ value, see comments in file
name.out.maxspec.dat	“Spectral function” with the highest probability - <b>classic Maxent</b>
name.out.maxspec_back.dat	The maxspec spectrum continued back to the imaginary axis
name.out.maxspec_self.dat	$\text{Im}[\Sigma(\omega)]$ with the proper sign and normalization; using <b>classic Maxent</b>
name.out.out.h5	All output data in the hdf5 format
name.out.prob.dat	The posterior probability of each $\alpha$ value
name.out.spex.dat	All spectral functions produced; one for each $\alpha$

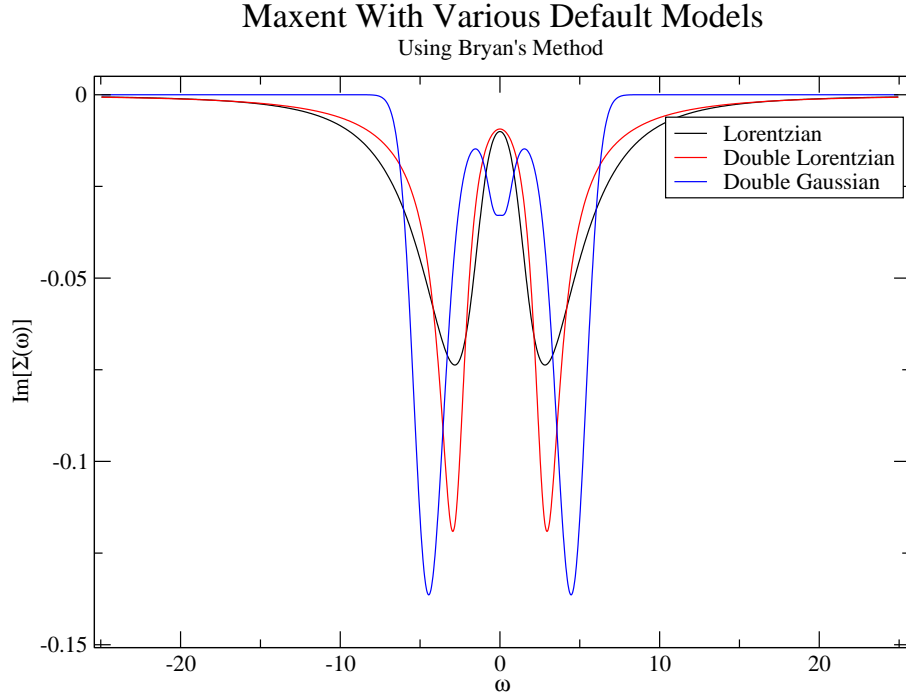


Because this is a self-energy, Maxent treats the input as a Green's function and finds a spectral function associated with it, but the spectral function output itself is meaningless. In our example here are the self-energy outputs with a flat default model:



## 4 Fine-Tuning Output

Different default models shouldn't change the results much, but sometimes end up doing so. Here are a variety of models from the above example:



With  $\sigma = 1, \Gamma = 1, \mu(\text{shift}) = 2.8$ . The default model gives a spectral function most similar to a double Lorentzian. When provided with a double Gaussian, Maxent attempts to fit the center peak, but is overcome with the entropy from the Gaussian model underneath.

## References

- [1] B Bauer et al. The ALPS project release 2.0: open source software for strongly correlated systems. *Journal of Statistical Mechanics: Theory and Experiment*, 2011(05):P05001, 2011. doi:[10.1088/1742-5468/2011/05/P05001](https://doi.org/10.1088/1742-5468/2011/05/P05001).
- [2] Alexander Gaenko, Emanuel Gull, Andrey E. Antipov, Lukas Gamper, and Gabriele Carcassi. ALPSCore: Version 0.4.5. May 2015. doi:[10.5281/zenodo.17398](https://doi.org/10.5281/zenodo.17398).
- [3] Xin Wang, Emanuel Gull, Luca de' Medici, Massimo Capone, and Andrew J. Millis. Antiferromagnetism and the gap of a mott insulator: Results from analytic continuation of the self-energy. *Phys. Rev. B*, 80:045101, Jul 2009. URL: <http://link.aps.org/doi/10.1103/PhysRevB.80.045101>, doi:[10.1103/PhysRevB.80.045101](https://doi.org/10.1103/PhysRevB.80.045101).