Formula Sheet (Rebecca Turner, 2019-11-12)

$$\mathbf{a}\cdot\mathbf{b} = \sum_i a_i b_i = |\mathbf{a}||\mathbf{b}|\cos\theta.$$

$$\mathbf{a}\times\mathbf{b} = \langle a_2b_3 - a_3b_2,\quad a_3b_1 - a_1b_3,\quad a_1b_2 - a_2b_1\rangle.\quad |\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta.$$

Param. eqns. of line through $\langle x_0, y_0, z_0\rangle$ par. to $\langle a, b, c\rangle$:

$$x = x_0 + at,\quad y = y_0 + bt,\quad z = z_0 + ct.$$

Symm. eqns.: $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$.

Vec. eqn. of plane through $\mathbf{r}$ with $\mathbf{n}$ normal:

$$\mathbf{n}\cdot(\mathbf{r} - \mathbf{r}_0) = 0,\quad \mathbf{n}\cdot\mathbf{r} = \mathbf{n}\cdot\mathbf{r}_0.$$

Length along a vec. fn. $\mathbf{r}(t)$:

$$\textstyle\int_a^b |\mathbf{r}'(t)|\, dt = \int_a^b \sqrt{\sum_i r_i'(t)^2}\, dt,$$

Unit tang. $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$, so curvature of $\mathbf{r}(t)$ w/r/t the arc len. fn. s:

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

Unit normal: $\mathbf{N}(t) = \mathbf{T}'(t)\,/\,|\mathbf{T}'(t)|$

Clairaut's thm.: $f_{xy}(a,b) = f_{yx}(a,b)$

Tan. plane to $z = f(x,y)$ at $\langle x_0, y_0, z_0\rangle$:

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0).$$

Grad.: $\nabla f(x,y) = \dfrac{\partial f}{\partial x}\hat{\imath} + \dfrac{\partial f}{\partial y}\hat{\jmath}.$

Dir. deriv. towards $\mathbf{u}$ at $\langle x_0, y_0\rangle$:

$$D_{\langle a,b\rangle} f(x_0,y_0) = f_x(x,y)a + f_y(x,y)b = \nabla f(x,y)\cdot\mathbf{u}.$$

Max of $D_\mathbf{u} f(\mathbf{x}) = |\nabla f(\mathbf{x})|$. Tan. plane of f at $\mathbf{p}$:

$$0 = f_x(\mathbf{p})(x - \mathbf{p}_x) + f_y(\mathbf{p})(y - \mathbf{p}_y) + f_z(\mathbf{p})(z - \mathbf{p}_z).$$

If f has loc. extrem. at $\mathbf{p}$, then $f_x(\mathbf{p}) = 0$ (& f_y, etc). If so, let

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2.$$

$D = 0$: no information. $D < 0$: saddle pt. $D > 0$: $f_{xx}(\mathbf{p}) > 0 \implies$ loc. min; $f_{xx}(\mathbf{p}) < 0 \implies$ loc. max. (D is the **Hessian mat.**)

Set of possible abs. min and max vals of f in reg. D: f at critical pts. and extreme vals. on the boundary of D.

Lagrange mults.: extreme vals of $f(\mathbf{p})$ when $g(\mathbf{p}) = k$. Find all $\mathbf{x}, \lambda$ s.t.

$$\nabla f(\mathbf{x}) = \lambda\nabla g(\mathbf{x}),\quad g(\mathbf{x}) = k.$$

i.e. $f_x = \lambda g_x$, etc.

$$\iint f(r\cos\theta, r\sin\theta) r\, dr\, d\theta.$$

$$A = \iint_D \left(\sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}\right) dA.$$

Line int.s

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2}\, dt$$

If C is a smooth curve given by $\mathbf{r}(t)$ from $a \le t \le b$,

$$\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Spherical coords: $x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$

$$\operatorname{curl}\mathbf{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle.$$

$$\mathbf{F} = \langle P, Q, R\rangle,\quad \operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F}$$

$\mathbf{F}$ "conservative" $\implies \exists f, \mathbf{F} = \nabla f$.

$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

$$\operatorname{curl}(\nabla f) = \mathbf{0},\quad \operatorname{div}\operatorname{curl}\mathbf{F} = 0$$

If C is a positively-oriented (ccw) closed curve, D is bounded by C, and $\mathbf{n}$ represents the normal,

$$\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \iint_D \operatorname{div}\mathbf{F}(x,y)\, dA.$$

Common derivs: $f(g(x)) \to g'(x)f'(g(x))$,
$b^x \to b^x \ln b$, $f^{-1}(x) \to 1/f'(f^{-1}(x))$,
$\ln x \to 1/x$, $\sin x \to \cos x$, $\cos x \to -\sin x$,
$\tan x \to \sec^2 x$, $\sin^{-1} x \to 1/\sqrt{1-x^2}$,
$\cos^{-1} x \to -(\sin^{-1} x)'$ (etc.),
$\tan^{-1} x \to 1/(1+x^2)$,
$\sec^{-1} x \to 1/(|x|\sqrt{x^2-1})$.

Common ints (don't forget $+C$):

$$x^n \to \frac{x^{n+1}}{n+1} + C \quad \text{when } n \neq -1$$

$$1/x \to \ln|x|$$

$$\tan x \to -\ln(\cos x)$$

$$\int uv'\,dx = uv - \int u'v\,dx \quad \text{(Int. by parts)}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int_{g(a)}^{g(b)} f(u)\,du = \int_a^b f(g(x))g'(x)\,dx \quad u\text{-substitution.}$$

E.x. in $\int 2x\cos x^2\,dx$, let $u = x^2$, find
$du/dx = 2x \implies du = 2x\,dx$, subs.
$\int \cos u\,du = \sin u + C = \sin x^2 + C$.

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

- Integrand contains $a^2 - x^2$, let $x = a\sin\theta$ and use $1 - \sin^2\theta = \cos^2\theta$.
- $a^2 + x^2$, let $x = a\tan\theta$, use $1 + \tan^2\theta = \sec^2\theta$.
- $x^2 - a^2$, let $x = a\sec\theta$, use $\sec^2\theta - 1 = \tan^2\theta$.

$$\lim_{x\to 0} \sin x/x = 1$$

$$\lim_{x\to 0} (1-\cos x)/x = 0$$

$$\lim_{x\to\infty} x\sin(1/x) = 1$$

$$\lim_{x\to 0} (1+x)^{1/x} = e$$

$$\lim_{x\to 0} (e^{ax}-1)/(bx) = a/b$$

$$\lim_{x\to 0^+} x^x = 1$$

$$\lim_{x\to 0^+} x^{-n} = \infty$$

For $0/0$ or $\pm\infty/\infty$, $\displaystyle \lim_{x\to c} f(x)/g(x) = \lim_{x\to c} f'(x)/g'(x)$

For $g(x)$ cont. at L, $\displaystyle \lim_{x\to c} f(x) = L \implies \lim_{x\to c} g(L)$