Homework 1

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1 Topological Spaces

1.1 Open Sets and the Definition of a Topology

> **Problem 1.7.** Define a topology on $\mathbb{R}$ (by listing the open sets within it) that contains the open sets $(0,2)$ and $(1,3)$ that contains as few open sets as possible.

$\mathscr{T} = \{\varnothing, (0,2), (1,3), (1,2), (0,3), \mathbb{R}\}$.

1.2 Basis for a Topology

> **Problem 1.10.** Show that $\mathscr{B} = \{[a,b) \subset \mathbb{R} : a < b\}$ is a basis for a topology on $\mathbb{R}$.

1. $\boldsymbol{\varnothing \in \mathscr{T}, \mathbb{R} \in \mathscr{T}.}$ $\varnothing \in \mathscr{T}$ (by the definition of the completion of a basis to a topology).

   Next, we show $\mathbb{R} \in \mathscr{T}$. For all $n \in \mathbb{Z}_{\geq 0}$, $[n-1, n) \in \mathscr{B}$ and $[-n+1, -n) \in \mathscr{B}$. We know that if $b_1, b_2 \in \mathscr{B}$, $b_1 \cup b_2 \in \mathscr{T}$, so these short intervals can be gathered together ("unionized") to produce $\mathbb{R}$:

   $$\bigcup_{n=1}^{\infty} ([n-1, n) \cup [-n+1, -n]) = \mathbb{R},$$

   so $\mathbb{R} \in \mathscr{T}$.

2. **$\mathscr{T}$ contains all finite intersections of elements of $\mathscr{T}$.** Suppose we have two intervals $[a,b)$ and $[c,d)$. Then, we define

   $$\begin{aligned} a' &= \max(a,c) \\ b' &= \min(b,d). \end{aligned}$$

   If $a' > b'$, the intersection $[a,b) \cap [c,d) = \varnothing$, which is in $\mathscr{T}$. Otherwise, the intersection is $[a', b')$, which is an element of $\mathscr{B}$. All elements of the basis are in $\mathscr{T}$, so the intersection of two elements is in the topology.

   Thankfully, the intersection is itself always a basis element, so we can use the same process to show that finite intersections are in $\mathscr{T}$ by induction.

3. **Unions of elements of $\mathscr{T}$ are in $\mathscr{T}$.** By the definition of the completion of a basis to a topology, this is true (all unions of basis elements are included in $\mathscr{T}$).

> **Problem 1.12.** Determine which of the following are open sets in $\mathbb{R}_l$. In each case, prove your assertion.
>
> $$A = [4,5) \quad B = \{3\} \quad C = [1,2] \quad D = (7,8)$$

1. A is open in $\mathbb{R}_l$; $[4,5) \in \mathscr{B}$.

2. B is not an open set in $\mathbb{R}_l$; there is no $[a,b) \subset \mathbb{R}$ where both $b > a$ and $|[a,b)| = 1$ (because $[0,1) \cong \mathbb{R}$, i.e. all intervals contain infinitely many points).

   (Where $\cong$ means "is isomorphic to.")

3. C is not open in $\mathbb{R}_l$ because the upper bound of an open set in $\mathbb{R}_l$ is never inclusive. There is no set of intervals $[a_1, b_1), \ldots$ where the union or intersection of the intervals has an inclusive upper bound.

4. D is open because we can take

   $$D = \lim_{n\to\infty} \left[7 + \frac{1}{n}, 8\right),$$

   where $[7 + 1/n, 8) \in \mathscr{B}$ for any $n \in \mathbb{R}$ with $n \neq 0$.

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> **Problem 1.15.** An arithmetic progression in $\mathbb{Z}$ is a set
>
> $$A_{a,b} = \{\dots, a - 2b, a - b, a, a + b, a + 2b, \dots\}$$
>
> with $a, b \in \mathbb{Z}$ and $b \neq 0$. Prove that the collection of arithmetic progressions
>
> $$\mathscr{A} = \{A_{a,b} : a, b \in \mathbb{Z} \text{ and } b \neq 0\}$$
>
> is a basis for a topology on $\mathbb{Z}$. The resulting topology is called the arithmetic progression topology on $\mathbb{Z}$.

Proof. Let us describe the *minimal form* of an arithmetic progression $A_{a,b}$ to be the progression $A_{a',b'} = A_{a,b}$ with $a', b' > 0$ and the smallest possible a'; in particular, that $a' < b'$.

We can obtain the minimal form of the progression like so:

$$\begin{aligned} a' &= a \bmod b \\ b' &= |b|, \\ A_{a',b'} &= A_{a,b}. \end{aligned}$$

> **Remark:** Two arithmetic progressions have the same elements if their minimal forms are the same; this give an equivalence relation on $\mathscr{A}$.

Now, suppose we have two arithmetic progressions $A_{a,b}$ and $A_{c,d}$. We assume that the progressions are in minimal form without loss of generality. We also assume that $b \leq d$ (by swapping (a, b) with (c, d) if necessary), again without loss of generality.

If $b \mid d$ and $a = c$, we have $A_{a,b} \subset A_{c,d}$. In particular, $A_{a,b} \cap A_{c,d} = A_{c,d}$.

If $b \mid d$ and $a \neq c$, we have $A_{a,b} \cap A_{c,d} = \varnothing$.

If $b \nmid d$, we have a different progression. An intersection is generated by an index (n_1, n_2), where

$$a + bn_1 = c + dn_2.$$

We can then solve for n_1:

$$\begin{aligned} t(n) &= c - a + dn \\ n_1 &= \frac{t(n_2)}{b}. \end{aligned}$$

Next, we have an infinite *set* of possibilities for n_2:

$$n_2 \in \{n \in \mathbb{Z} : t(n) \mid b\}.$$

Sorting the possible values of n_2 by absolute value, let us call the smallest two values i_1 and i_2. Then, the difference between adjacent elements in the intersection progression $A_{a,b} \cap A_{c,d}$ is $i_2 - i_1$.

Let

$$\begin{aligned} a' &= a + bi_1 \\ b' &= i_2 - i_1 \\ A_{a,b} \cap A_{c,d} &= A_{a',b'}. \end{aligned}$$

This isn't super rigorous, admittedly (we're missing some inductive reasoning about the integers to prove that there are an infinite set of valid values of n_2, in particular), but I have some fairly convincing Haskell code. And the missing steps are mostly boilerplate, and it's late at night already...

In all cases, the intersection of two arithmetic progressions is either empty or another arithmetic progression (i.e. either the empty set or another basis element), so the same argument given above for $\mathbb{R}_l$ holds (namely that we can extend this to all finite intersections of elements of $\mathscr{A}$ inductively).

Therefore, finite intersections are in the basis. Unions are in the completion of the basis (again by definition). The special element $\varnothing$ is in the completion (by definition), and $\mathbb{Z} = A_{0,1}$, so $\mathbb{Z} \in \mathscr{B}$. Therefore, $\mathscr{A}$ forms the basis of a topology on $\mathbb{Z}$. $\square$

1.3 Closed Sets

> **Problem 1.27(a).** The infinite comb C is the subset of the plane illustrated in Figure 1.17 and defined by
>
> $$C = \{(x, 0) : 0 \le x \le 1\} \cup \left\{\left(\frac{1}{2^n}, y\right) : n = 0, 1, 2, \ldots \text{ and } 0 \le y \le 1\right\}.$$
>
> Prove that C is not closed in the standard topology on $\mathbb{R}^2$.

Proof. Suppose C is closed in the standard topology on $\mathbb{R}^2$. Then, its complement $C^c = \mathbb{R}^2 \setminus C$ must be an open set.

The point $(0, 1)$ is not in C, so $(0, 1) \in C^c$. Every open ball in $\mathbb{R}^2$ containing $(0, 1)$ also contains a smaller open ball centered about $(0, 1)$. (For example, the open ball about $(-1, 1)$ of radius 1.1 contains the open ball centered about $(0, 1)$ of radius 0.1.)

However, every open ball centered about $(0, 1)$ contains infinitely many points of C; if the ball has radius r, all the comb's "tines" at $x = 1/2^n$ for $n > -\log_2 r$ intersect with the ball.

Therefore, every open ball containing $(0, 1)$ also contains points in C. As a result, C^c is not open, which contradicts our assumption. Therefore, C is not closed. □

> **Problem 1.32.** Prove that intervals of the form $[a, b)$ are closed in the lower limit topology on $\mathbb{R}$.

Proof. Take some interval $[a, b)$. Its complement is given by $(-\infty, a) \cup [b, \infty)$. Given that

$$(-\infty, a) = \bigcup_{n=1}^{\infty} [a - n, a)$$
$$[b, \infty) = \bigcup_{n=1}^{\infty} [b, b + n),$$

the complement of $[a, b)$ is the union of a number of lower-limit intervals in $\mathbb{R}$, i.e. the basis elements. The basis elements and its unions are open sets, so the complement of $[a, b)$ is an open set. Then, by the definition of a closed set, $[a, b)$ is closed in $\mathbb{R}_l$. □