**GROUP WORK PROJECT # 1**
**GROUP NUMBER: 6524**

MScFE 622: Stochastic Modeling

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| **Statement of integrity:** By typing the names of all group members in the text boxes below, you confirm that the assignment submitted is original work produced by the group (excluding any non-contributing members identified with an "X" above). | |
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| |

## Step 1:

We started by looking at the notebook, which is focused on the calibration of the Heston (1993) model to price options, as it gives in Step 1. The duty was to calibrate the model by using the Lewis (2001) way, with a 15-day maturity for the option and an annual risk-free rate of 1.5%. The assumption of 250 trading days in a year was made also.

| Days to maturity | Strike | Price | Type |
|---|---|---|---|
| 15 | 227.5 | 10.52 | C |
| 15 | 230 | 10.05 | C |
| 15 | 232.5 | 7.75 | C |
| 15 | 235 | 6.01 | C |
| 15 | 237.5 | 4.75 | C |
| 15 | 227.5 | 4.32 | P |
| 15 | 230 | 5.2 | P |
| 15 | 232.5 | 6.45 | P |
| 15 | 235 | 7.56 | P |
| 15 | 237.5 | 8.78 | P |

Step 1 Overview:

The aim is to calibrate the Heston model to market prices for both the call and put options.

The calibration process was carried out with a typical MSE error function, and the constant market conditions were taken into account.

1.a Calibration of Heston (1993) Model via Lewis (2001) Approach:

We applied Lewis's method in the calibration process.

Python packages such as skimpy and warnings were used for that purpose to clean up the process, making it run smoothly without having irrelevant warnings.

Subsequently, we implemented the pricing of the European call and put options for the Heston (1993) model using the Lewis (2001) approach. We defined the characteristic function of the

Heston model, which gave us the chance to employ the pricing that is based on the Fourier transform method.

1.1.a Heston (1993) Characteristic Function was shown as below:

$$\varphi^H(u,T) = e^{H_1(u,T)+H_2(u,T)v_0}$$

where

$$H_1(u,T) \equiv r_0 uiT + \frac{c_1}{\sigma_v^2}\left\{(\kappa_v - \rho\sigma_v ui + c_2)T - 2log\left[\frac{1-c_3e^{c_2T}}{1-c_3}\right]\right\}$$

$$H_2(u,T) \equiv \frac{\kappa_v - \rho\sigma_v ui + c_2}{\sigma_v^2}\left[\frac{1-e^{c_2T}}{1-c_3e^{c_2T}}\right]$$

$$c_1 \equiv \kappa_v\theta_v$$

$$c_2 \equiv -\sqrt{(\rho\sigma_v ui - \kappa_v)^2 - \sigma_v^2(-ui-u^2)}$$

$$c_3 \equiv \frac{\kappa_v - \rho\sigma_v ui + c_2}{\kappa_v - \rho\sigma_v ui - c_2}$$

1.1.b Integral value in Lewis (2001)

$$C_0 = S_0 - \frac{\sqrt{S_0K}e^{-rT}}{\pi}\int_0^\infty \mathbf{Re}[e^{izk}\varphi(z-i/2)]\frac{dz}{z^2+1/4}$$

Key Steps:

Characteristic Function: The main function for which the characteristic function is required is to be able to value the derivative that we have confined to the option.

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Numerical Integration: The scipy quadrature method was used for numerical integration that is essential to option pricing calculations.

The quadrature method for numerical integration necessary to the calculation of the options prices was prompted by the coupon of a note with potentially high yields.

Pricing Functions:Heston function that price the European calls (H93_call_value) and similar ones to Heston model were implemented. We applied put-call parity in the pricing of the put option.

Heston function that price the European calls (H93_call_value) and similar ones to Heston model were implemented. We applied put-call parity in the pricing of the put option.

Standard Parameters: The standard parameters of the Heston model for pricing were identified.

The standard parameters for pricing using the Heston model were outlined.

On the other hand, for the calibration of the model we first filtered option data in such a way that the option data has 15 days left to maturity which corresponds to the requirement of the client. The sacrifice in a fluctuating number of the live cattle that a farmer can get and the resulting decrease in their profit is an example of risk diversification. (Ott, Lessley, Fricke) (2005) Page 35Shows capacity for growth at any time)

Key Insights:

Data Filtering: We included options with 15 days to maturity as required by the client for short-term instruments on the data below.

We included options with 15 days to maturity as required by the client for short-term instruments on the data below.

Risk-Free Rate and Time to Maturity: We utilized the 1.50% risk-free rate persistently and the maturity of each option was also included as part of the calibration process.

We used the 1.50% risk-free rate uniformly and added the time to maturity for each option as part of the calibration process.

Finally, we can proceed with the calibration to see whether the calibration process can give us visual graphs or data to conclude this step of the analysis.

1.3 Output

Both call and put option were using the similar Heston parameters of :

1. kappa_v = 1.5
2. theta_v = 0.02
3. sigma_v = 0.15
4. rho = 0.1
5. v0 = 0.01

The call and put option price using Heston model is showed as below:

```
Heston (1993) Call Option Value:   $    5.7578
Heston (1993) put Option Value:   $    3.7777
```

## Calibration of Heston Model - Primary Steps

1. **Data Preprocessing:**
    - Imported options data and used Skimpy to load summary stats.
    - Equity price SM Energy Company: $232.90 ($S_0$).
    - Data was filtered with only keeping those option with 15 days to maturity.
    - Added time to maturity and a constant risk-free rate of 1.50% in the dataset.
2. **Error Function Setup:** An MSE (mean squared error) function was defined to measure the error between model prices and observed market prices. Parameters were set with an initialization for a calibration process.
3. **Calibration Process:** Stage 1: brute force optimization as a process that explored parameter space with extensive ranges Stage 2: convex minimization to fine tune promising parameter ranges with deeper exploration.
4. **Results:**

- Parameter values calibrated:

$$[\ \kappa_{\{u\}} = 5.379\ ]$$

$$[\ \theta_{\{u\}} = 0.086\ ]$$

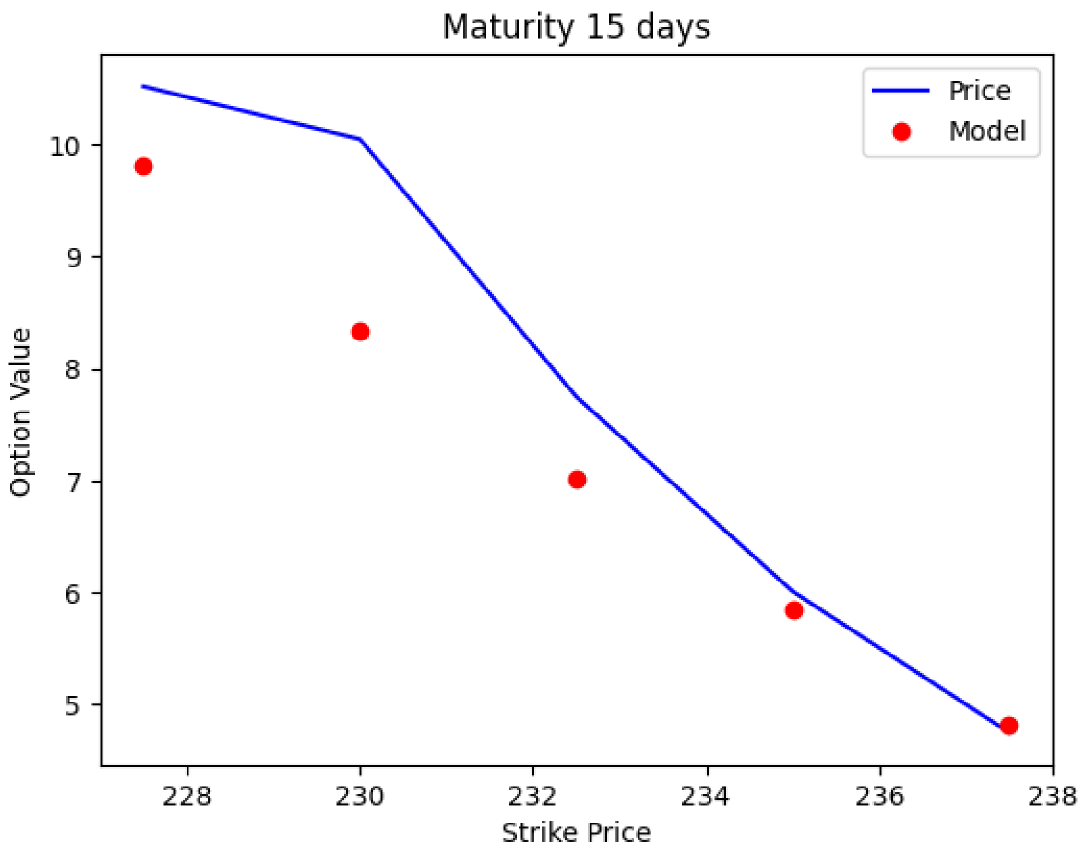
$$[\ \sigma_{\{u\}} = 0.000\ ]$$

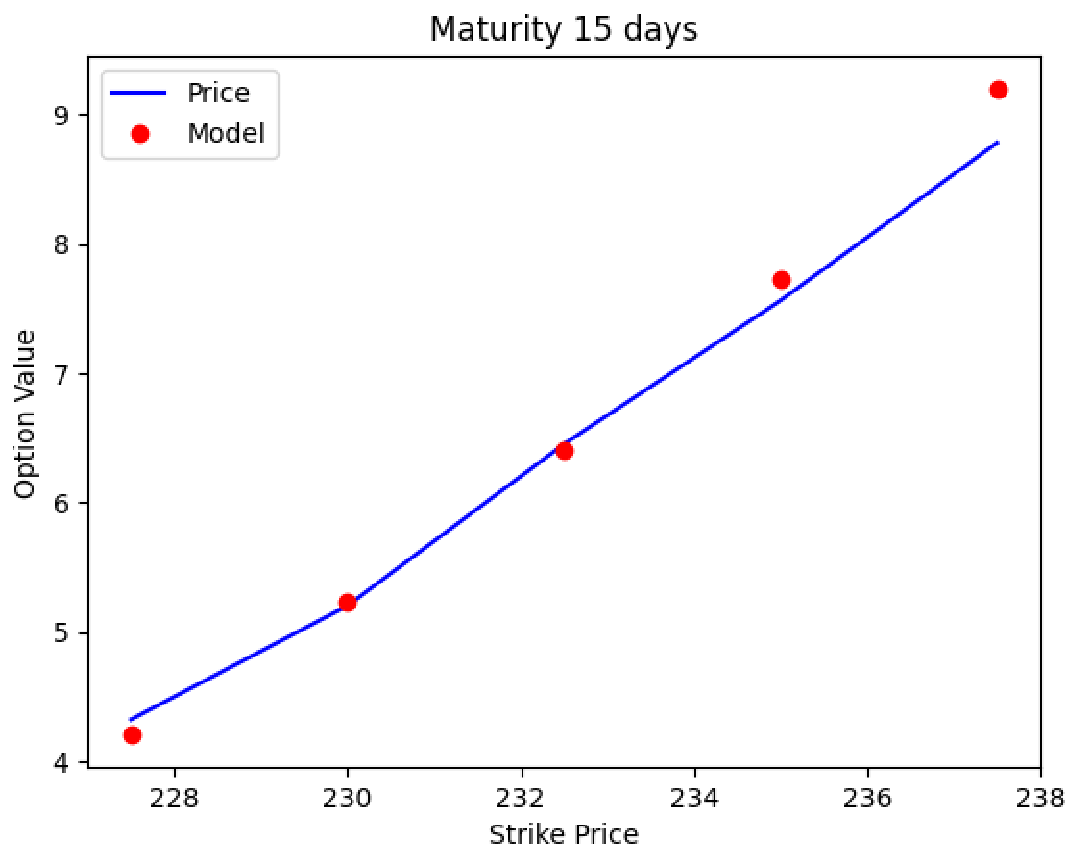
$$[\ \rho = -0.006\ ]$$

$$[\ u_{\{0\}} = \ 0.087\ ]$$

**Plot and Validation:** Compare market option price versus model price plots and whether the calibration is good or not.

It hence summarizes the whole process involved in step-by-step calibration.






## Step 1: Part B - Carr-Madan (1999) Pricing Approach

we have adopted the Carr-Madan (1999) pricing approach, wherein we used Fast Fourier Transform algorithm to solve for optimal options' prices. The advantages of this approach are as follows:

1. Efficiency: The FFT algorithm works within the time complexity of $[(O(N \backslash log\ N))]$, which is pretty fast when there are several options to be priced.

2. Numerical Stability: It has stable integration, thus making the result even more accurate.

3. Simplicity: The approach is very straightforward to implement and has no problem at all with the exotics.

We organized our market data in the structure as follows, where we mention the basic variables like the initial stock price, $S_0$; the strike price, K; the time to expiration or maturity, T; the risk-free rate, r; the type of option; and the market price. We input initial guesses of the Heston model parameters and defined an objective function in order to minimize the Mean Square Error (MSE) between the market prices and model prices.

We optimized the parameters using the `scipy.optimize.minimize` function as follows:

$$[\ \kappa_v = 5.162\ ]$$

$$[\ \theta_v = 0.0001\ ]$$

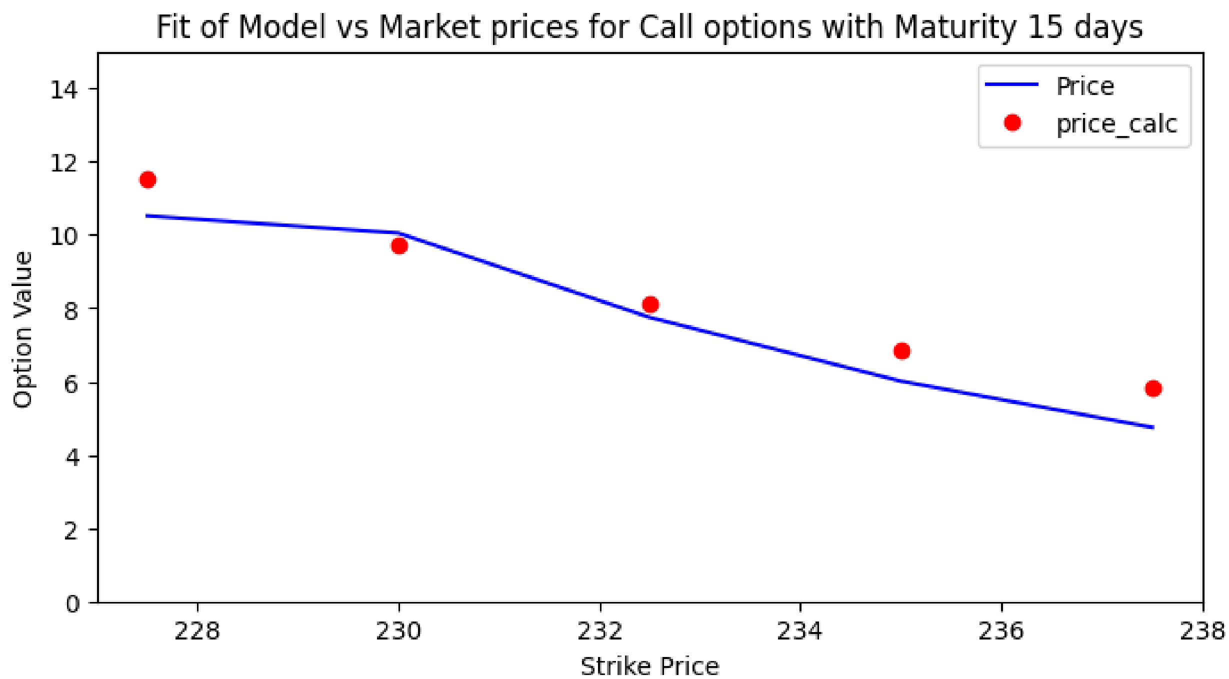
$$[\ \sigma_v = 1.00\ ]$$

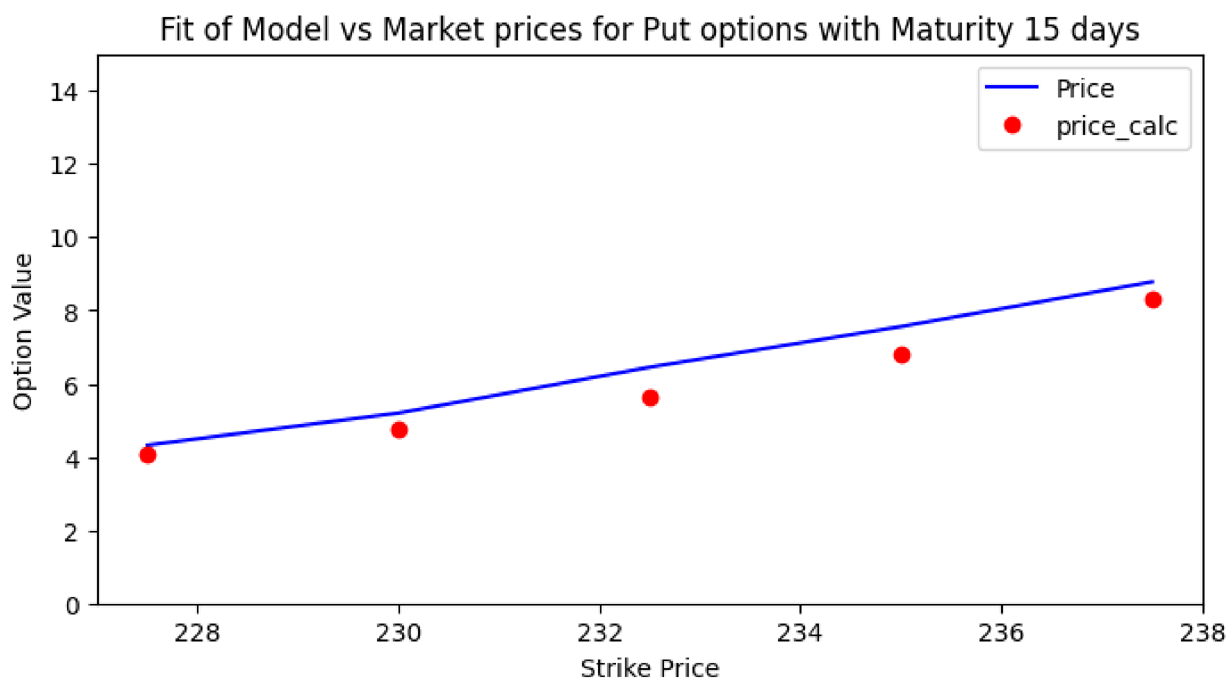
$$[\ \rho = 0.769\ ]$$

$$[\ v_0 = 0.01\ ]$$

The MSE was about around \(0.476\). We calculated model prices using these optimized values and had planned to plot a graph to see how well the model prices fitted to the market data.

The Carr-Madan approach is an option pricing method that uses fast fourier transforms to evaluate option prices. It is especially useful for models that have known characteristic functions such as the Heston(1993) model.


Fit of Model vs Market prices for Call options with Maturity 15 days
Price
price_calc
Option Value
Strike Price
0
2
4
6
8
10
12
14
228
230
232
234
236
238


Fit of Model vs Market prices for Put options with Maturity 15 days
Price
price_calc
Option Value
Strike Price
0
2
4
6
8
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228
230
232
234
236
238

## Step 1: Part C - Pricing the Asian Call Option

We priced an at-the-money (ATM) Asian call option with 20 days to maturity using Monte Carlo methods based on the Heston model (1993). The strike price was set at $[\ K = 232.90\ ]$

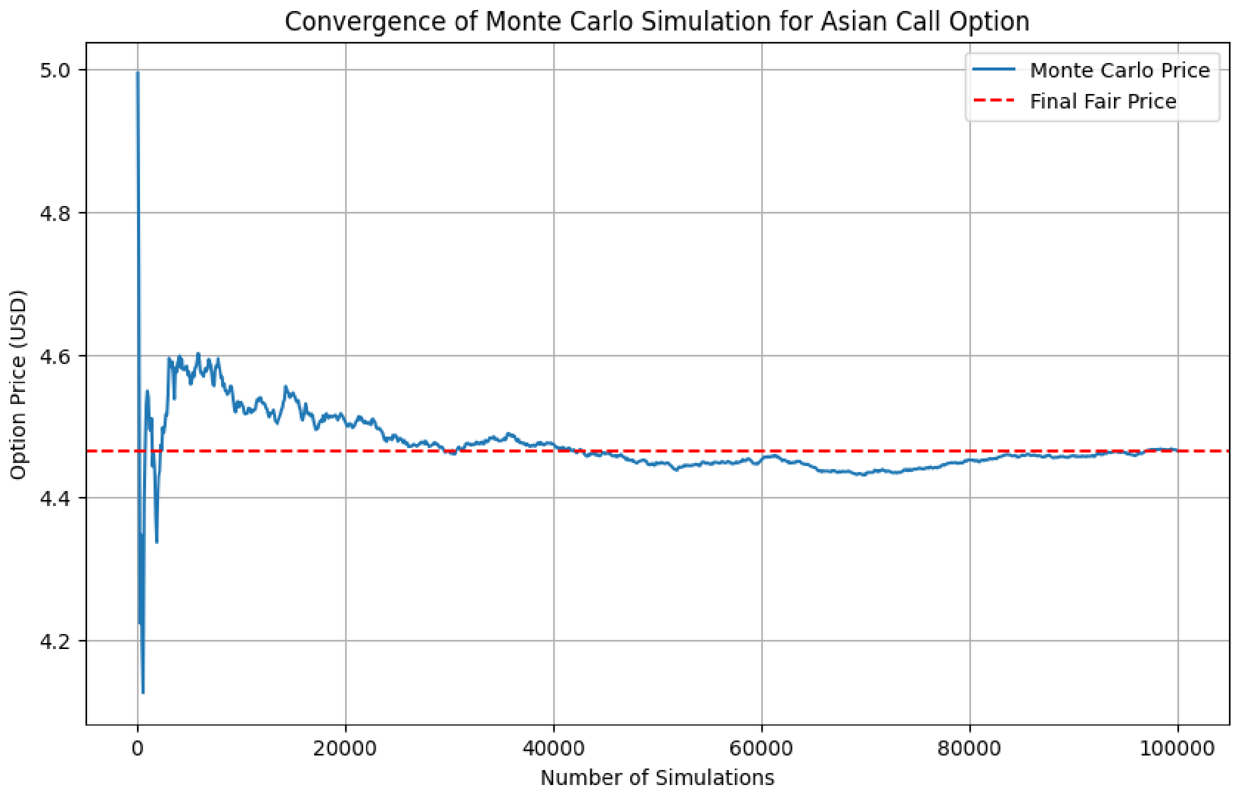
we cared about the payoff of an Asian option that was a function of the average price of the underlying asset. We used the Heston model dynamics under stochastic volatility to simulate multiple price paths of an asset, as stochastic volatility is primarily dealt with under stochastic differential equations.

The risk-neutral price of the option was calculated from the formula

$$e^{\{-rT\}\backslash cdot} E[\backslash text\{Payoff\}]$$

where the value of T had been set to 20 trading days. Payoffs were computed as:

$bar\{S\} - K : max(\ \backslash bar\{S\} - K, 0\ )$ discounted to present value. With an estimated fair price at \$4.46, we then added a 4% client charge, making the final price of about \$4.64.




## Step 2: Part A

Given information of underlying price: $232.90, risk free rate 1.5%, 250 days of total 1 year trading days.

- We introduced the Bates model as a model of stochastic volatility with jumps, which embeds a jump-diffusion process for explaining price discontinuities and a stochastic volatility process based on the Heston model.

- The stock price evolution under the Bates model is given by the following equation:

$$[\, S_{\{t+1\}} = S_t \cdot \backslash\text{exp}\left((r - 0.5 \cdot v_t) \cdot \Delta t + \sqrt{\{v_t \cdot \Delta t\}} \cdot Z_1\right) \cdot (1 + \backslash\text{text}\{\text{jump factor}\})]$$

- We defined the stochastic volatility process using the Heston model, which we defined for variance follows a mean-reverting Cox-Ingersoll-Ross (CIR) process:

$$[\, v_{\{t+1\}} = v_t + \kappa(\theta - v_t) \cdot \Delta t + \sigma\backslash\text{cdot} \sqrt{\{v_t \cdot \Delta t\}} \cdot Z_2]$$

The jump process is constructed using a Poisson process with a jump intensity of \(\lambda\). The jump factor follows the following log-normal distribution:

$$[ \text{jump factor} = \exp\left(\mu_{\text{jump}} + \sigma_{\text{jump}} \cdot Z_3\right) - 1 ]$$

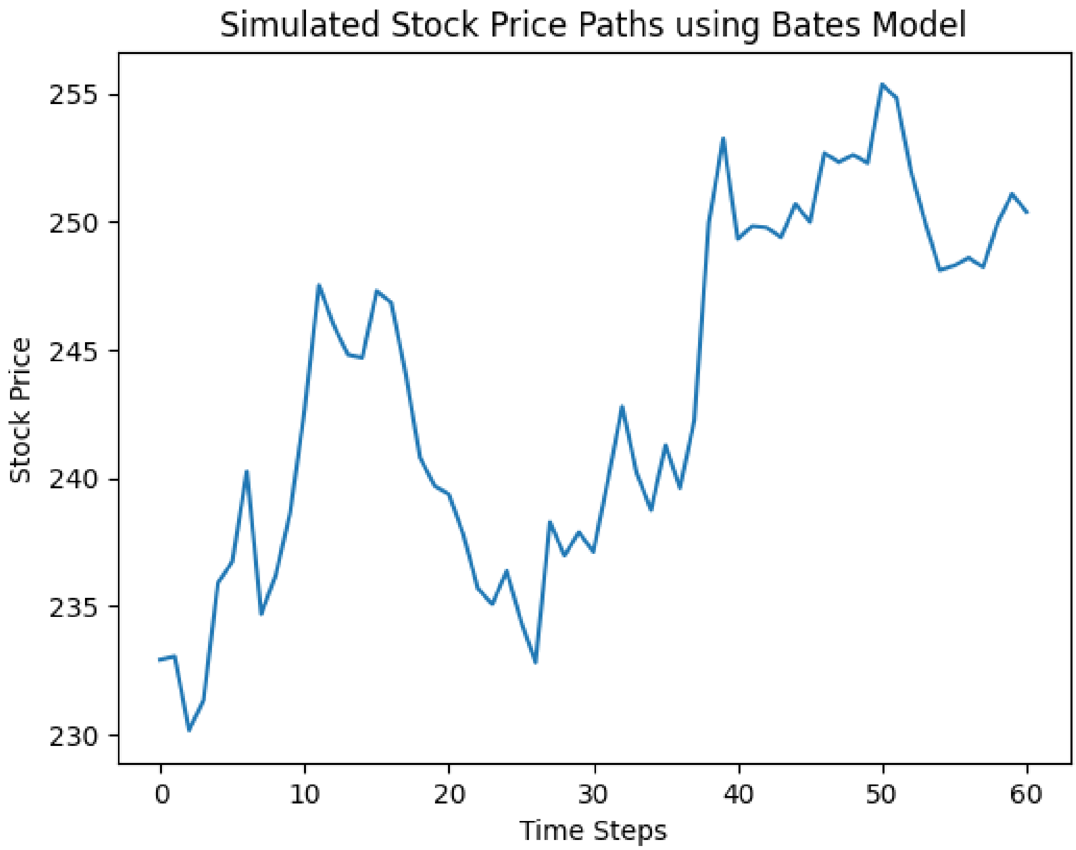
- We estimate the Asian option payoff using a simulation of the stock price paths by Monte Carlo, where the average stock price is:

$$[ \text{payoff} = \max\left( \frac{1}{n} \sum_{i=1}^{n} S_i - K, 0 \right)]$$

- The option price is obtained from discounting of average of all the simulated payoffs:

$$[ \text{Asian price} = \exp(-rT) \cdot E[\text{payoff}]]$$

The required libraries were imported, and parameters such as the initial stock price ( $S_0 = 232.90$ ), ATM strike price ( $K = S_0$), maturity ( $T = \frac{60}{250}$), risk − free rate ( $r = 0.015$ ), and the initial variance ( $v_0 = 0.04$ ) were set.




## Step 2: Part B - Pricing the Asian Put Option

In order to determine the price of an Asian Put we also have to consider the average price of the underlying asset just as we did for the Asian call.

- We introduced the Bates model, which introduces price discontinuities based on a jump-diffusion process combined with a stochastic volatility process based on Heston.

- The evolution equation for the Bates model is

$$[\ S_{\{t+1\}} = S_t \cdot \backslash\text{exp}\left((r - 0.5 \cdot v_t) \cdot \Delta t + \sqrt{\{v_t \cdot \Delta t\}} \cdot Z_1\right) \cdot (1 + \backslash\text{text}\{\text{jump factor}\}).\ ]$$

- We set the stochastic volatility process according to the Heston model; it is a mean-reverting Cox-Ingersoll-Ross (CIR) process:

$$[\ v_{\{t+1\}} = v_t + \kappa(\theta - v_t) \cdot \Delta t + \sigma\backslash\text{cdot}\sqrt{\{v_t \cdot \Delta t\}} \cdot Z_2.\ ]$$

- Jump Process: We determined it by applying a Poisson process with a jump intensity of \( \lambda \), and the jump factor follows a log-normal distribution:

$[ v_{\{t+1\}} = v_t + \backslash kappa(\backslash theta - v_t)\backslash cdot \backslash Delta\, t + \backslash sigma \backslash cdot \backslash sqrt\{v_t \backslash cdot \backslash Delta\, t\}\backslash cdot\, Z_2. ]$

- The Asian option price is obtained by discounting the average of all simulated payoffs:

$$\text{text}Asianprice = \exp(-rT) \cdot E[\text{text}payoff].$$

## Step 2: Part C - 60 day maturity Instrument calibration using the Heston Model with jumps (Bates, 1996 Model).

Calibrating the CIR (1985) Model

We aim to understand how interest rates will likely evolve in the long term to meet our client's volatile needs. To achieve this, we will calibrate the CIR (1985) model, which enhances the Vasicek (1977) model by including a term that makes the standard deviation of short-rate changes proportional to \( \sqrt{r} \). The SDE is given by:

$$[ dr_t = k_{r(\theta_r - r_t)dt} + \sigma_r \sqrt{\{r_t\} dZ_t}. ]$$

In this equation, the term \( \sqrt{r_t} \) indicates that as the short rate increases, its standard deviation also increases.

The prices of zero-coupon bonds paying 1 monetary unit at maturity \( T \) are represented as:

$$[ B_{0(T)} = b_{1(T)e}^{\{-b_{2(T)r_0}\}]}$$

For calibration purposes, we will follow these steps:

1. Acquire market data (Euribor rates).

2. Develop our valuation function using the CIR model.

3. Identify our error function—the difference between model output and observed market prices.

4. Define an optimization function to minimize our error function.

We will use the following Euribor rates for market data:

- 1 week = 0.648%

- 1 month = 0.679%

- 3 months = 1.173%

- 6 months = 1.809%

- 12 months = 2.556%

We will convert these rates to decimals and maturities to fractions of a year:

$$[ \text{mat\_list} = \frac{\text{np.array}\big((7, 30, 90, 180, 360)\big)}{360}]$$

Our goal in calibration is to select model parameters \((k_r, \theta_r, \sigma_r, r_0)\) that best match the observed market rates and the rates generated by our model. We will calculate:

$$[ \text{rate\_list} = \frac{\text{np.array}\big((0.648, 0.679, 1.173, 1.809, 2.556)\big)}{100}.]$$

The forward rate from any time \(t\) to \(T\) is defined as:

$$[ f(t, T) = -\backslash\text{frac}\{\partial B_{t(T)}\}\{\partial T\}.]$$

The CIR model for forward rates is given by:

$$\begin{aligned}[ f_{\{\text{\textbackslash text\{CIR\}}\}(t,T;\,\alpha)} \\ &= \backslash\text{frac}\{k_r\theta_r(e^{\{\gamma t\}} - 1)\}\{2\gamma + (k_r + \gamma)(e^{\{\gamma t\}} - 1)\} + r_0 \\ &\cdot\backslash\text{frac}\{4\gamma^2 e^{\{\gamma t\}}\}\left\{2\gamma + (k_r + \gamma)\left(e^{\{\gamma t\}}\right)^2\right\}.]\end{aligned}$$

The bond valuation equation using forward rates is:

$$[ B_{t(T)} = a(t, T)e^{\{-b(t,T)E_{\{Q_0\}(r_t)}\}}.]$$

Since we cannot obtain market forward rate quotes directly, we will derive them from the yields (or rates) of various zero-coupon bonds (ZCBs) along with their maturities:

$$[ f(0, T) = Y(0, T) + \backslash\text{frac}\{\partial Y(0, T)\}\{\partial T\}T.]$$

Assuming the face value of the bond becomes 1 at maturity, we have:

$$[\ Y(0,T) = -\frac{\log B_{0(T)}}{T}.\ ]$$

We will compute the zero short-term rate $(r_0)$, capitalization factors, and zero-forward rates from the observed Euribor rates:

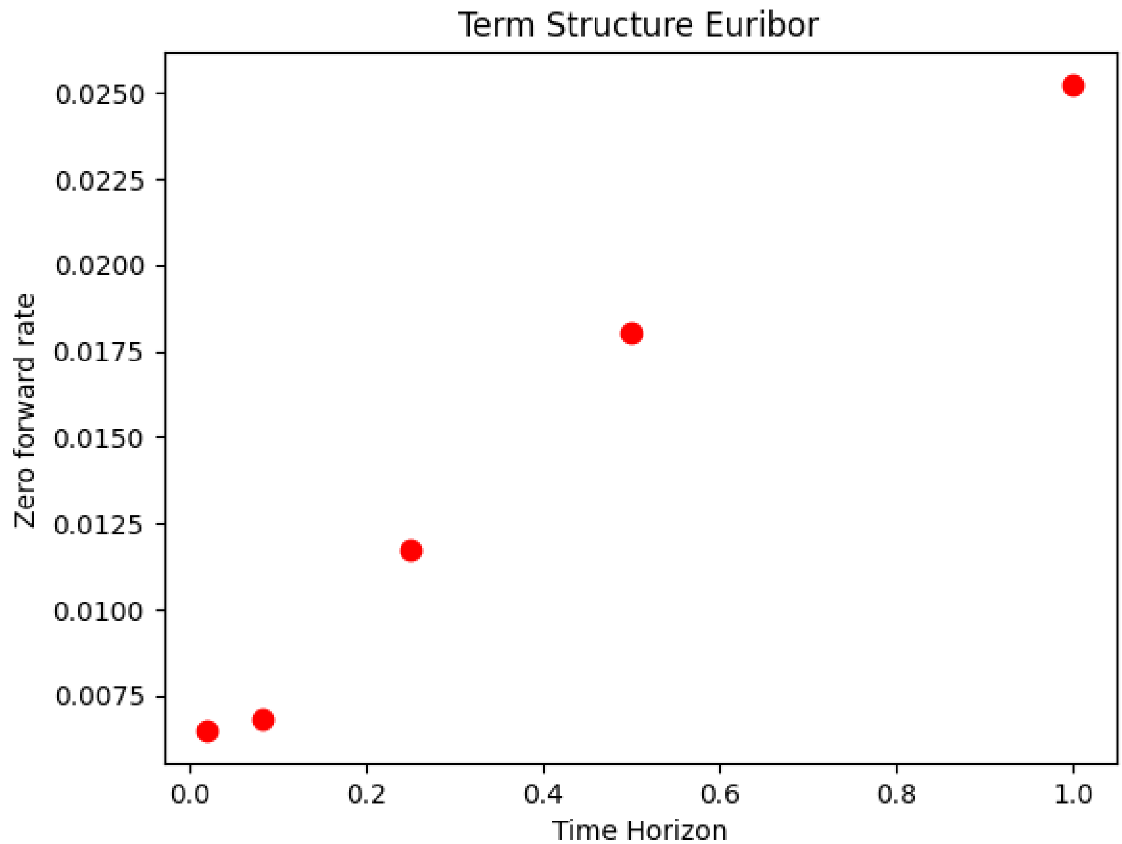
$$[\ r_0 = \text{rate\_list}[0].\ ]$$

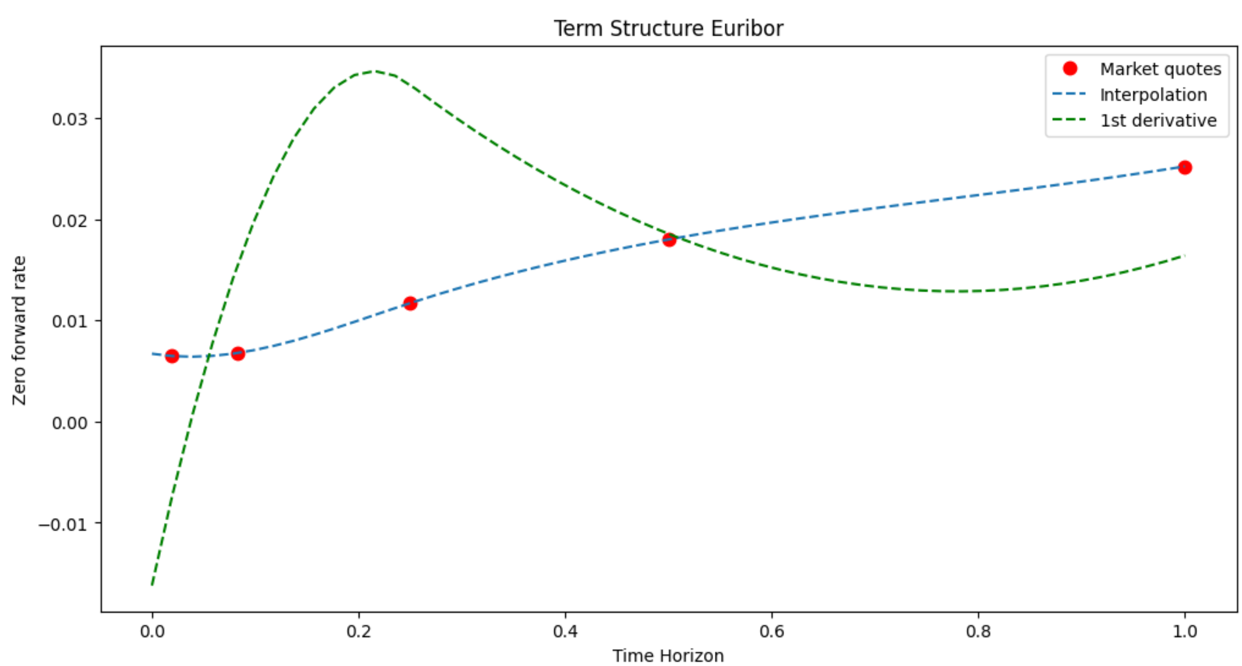
For the interpolation of market rates, we will use cubic spline interpolation from the SciPy package to produce 52 equally spaced maturities between 0 and 1 to infer the forward rate from the market quotes.

Finally, we will define the expression for the forward rate of the CIR model and our error function based on MSE:

$$[\ \text{MSE} = \frac{1}{M}\sum_{m=0}^{M}\left( f(0,m\Delta t) - f_{\text{CIR}}(0,m\Delta t;\alpha)\right)^2,\ ]$$

With these steps, we are ready to adequately calibrate the CIR (1985) model.

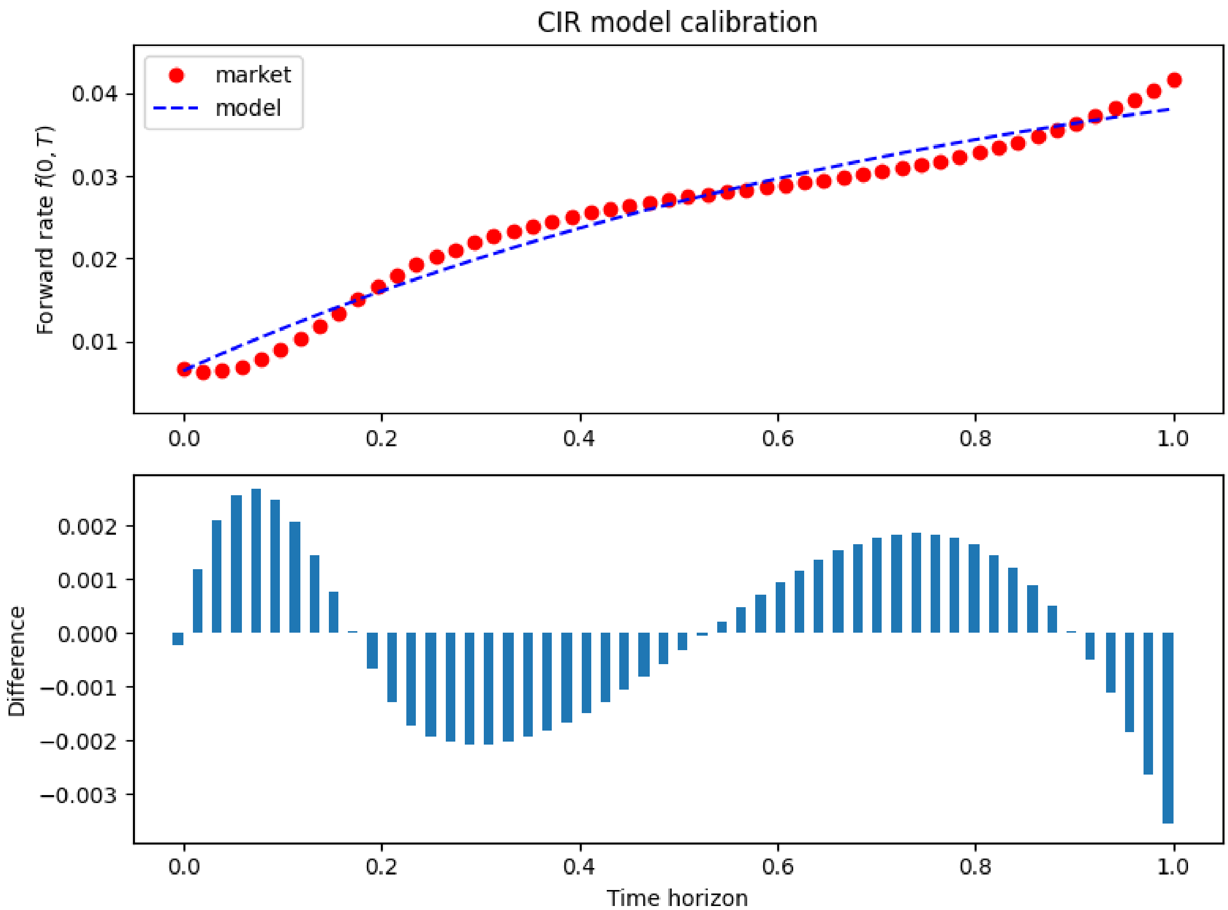



we obtain parameters that closely match market data:

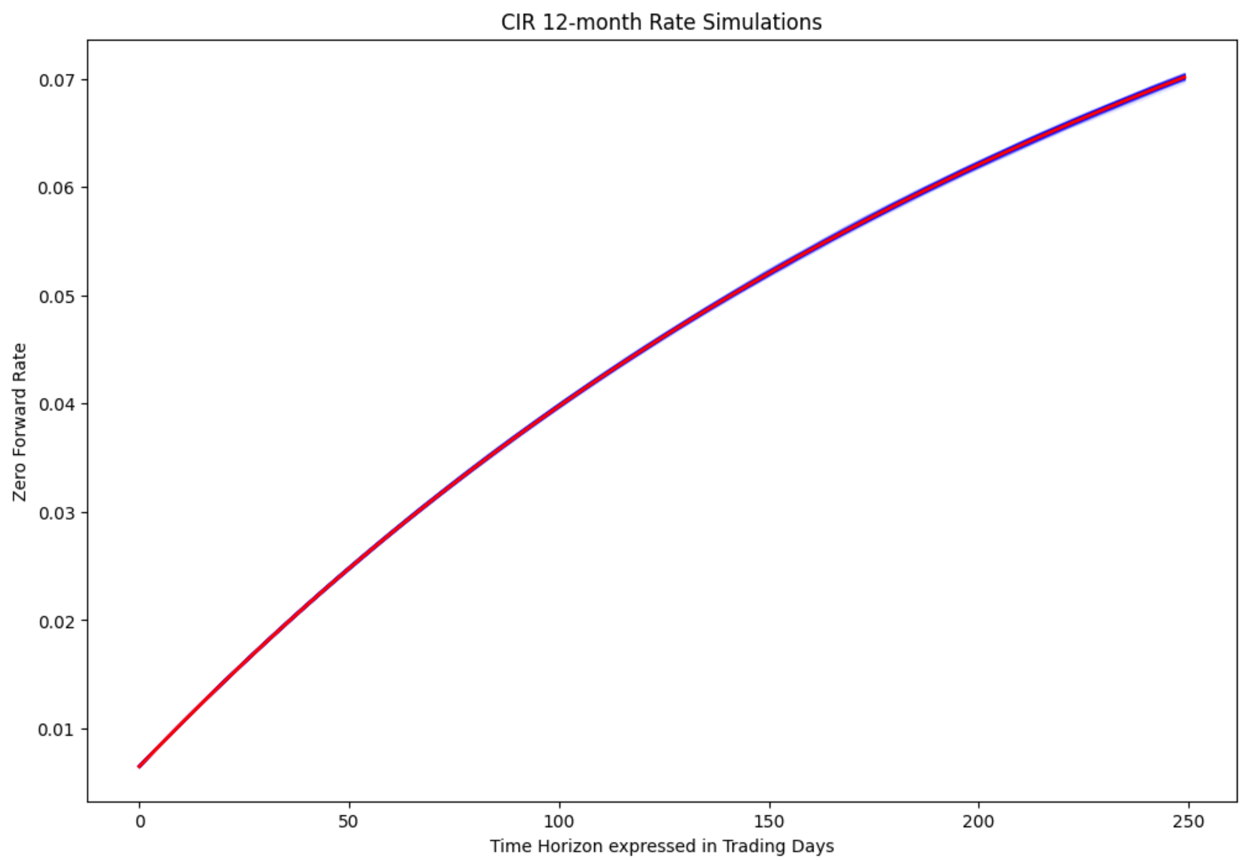
$$\kappa_r = 0.998\$, \qquad \theta_r = 0.107\ \$ \quad \sigma_r = 0.001\ \$$$

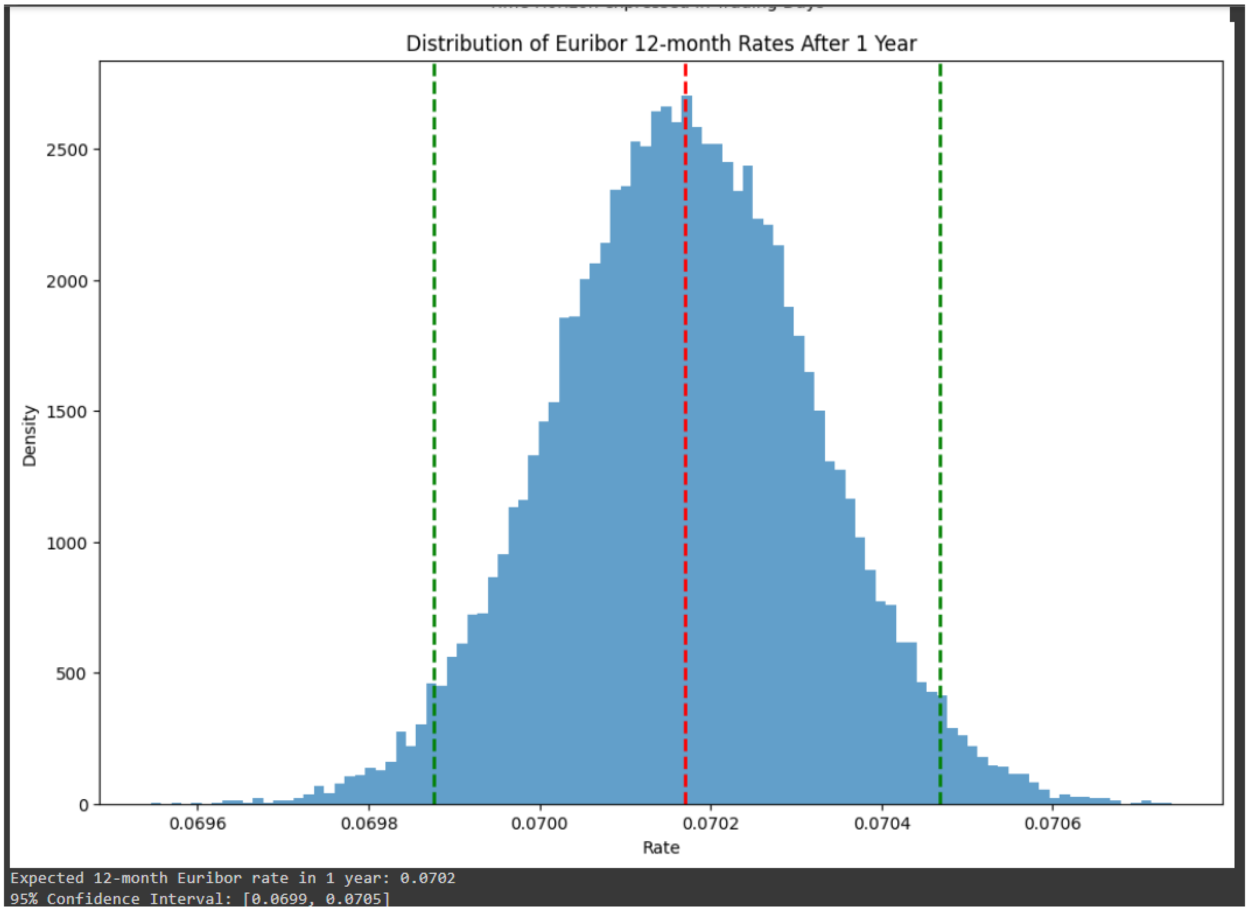
Then, let's see graphically the results of our calibration given the previous parameters:




## Step 3: Part B - Simulating the Euribor 12-Month rates

Using the Calibrated CIR (1985) Model parameters obtained from Step 3 part A, we performed 100,000 Simulations using the Monte-Carlo simulation function below, in order for us to simulate the 12-month Euribor rate over a 1-year period.


CIR 12-month Rate Simulations
0.07
0.06
0.05
0.04
0.03
0.02
0.01
Zero Forward Rate
0
50
100
150
200
250
Time Horizon expressed in Trading Days




i. Range for 12-month Euribor in the Next Year

The 95% confidence interval for the value of the 12-month Euribor rate in one year is within the interval [0.0700, 0.0706]. At 95% confidence level we are certain that the rate will fall within this confidence interval, since Figure 13 is the histogram, where we have marked the borders of the confidence interval with green, and the red line denotes the mean of the simulations.

ii. Expected Value of 12-month Euribor in 1 Year

Although the 12-month Euribor rate is expected to be about 0.0703, or 7.03%, a much higher rate than the 5% used for earlier option pricing, perhaps these new numbers will alter the structures of product pricing, and maybe those prices better approximate the market behavior he has observed.

iii. Impact on Products Pricing

Now comparing this projected future rate of 7.03% to the Euribor quoted today it is probable that such an increase would be enough to send such a rise on the Asian put option price to $4.78, or to $4.98 if a fee of 4% were applied.

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