Monte Carlo and Empirical Methods for Stochastic Inference

Home Assignment 1 Solution

Alessandro Celoria
Ruoyi Zhao

January 2023

1 Random Number Generation

1.1 Cumulative Distribution Function

We analyze the problem of finding the conditional cumulative probability distribution function $F_{X|X\in I}(x) = \mathbb{P}(X \leq x \,|\, X \in I)$. We start by analyzing the implications of the latter statement:

$$\mathbb{P}(X \leq x \,|\, X \in I) = \frac{\mathbb{P}(X \leq x \bigcap a \leq x \leq b)}{\mathbb{P}(a \leq x \leq b)}$$

In the formula above we can recognize three possible cases:

- $x \leq a \Rightarrow$ The total probability is 0, as this scenario would require X to be simultaneously lower and higher than a[1]
- $x \geq b \Rightarrow$ The total probability is 1 as we know from the prior that $X \leq b$ which automatically satisfies the conditioned probability.
- $x \in (a, b) \Rightarrow$ In this case we have the following:

$$\mathbb{P}(X \leq x \,|\, X \in I) = \frac{\int_a^x f(x)dx}{\int_a^b f(x)dx} = \frac{F(x) - F(a)}{F(b) - F(a)}$$

We therefore conclude that the cumulative probability distribution function of X conditioned on $X \in I$ where $I := (a, b) \mid \mathbb{P}(X \in I) > 0$ is as follows:

$$\mathbb{P}(X \leq x \,|\, X \in I) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{F(x)-F(a)}{F(b)-F(a)} & \text{if } a \leq x \leq b \\ 1 & \text{if } x \geq b \end{cases}$$

1.2 Probability Distribution Function

We know by definition that:

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

Now we recall the first fundamental theorem of calculus, which under sufficient assumptions says that:

$$F(x) = \int_{a}^{x} f(t)dt \;\Rightarrow \frac{d}{dx}F(x) = f(x)$$

[1] X could be exactly equal to a and therefore satisfy both conditions, but that would be a single event among the infinitely many values of the real-valued variable X and therefore the probability of it happening is 0.

Considering that the function derived in the previous section is a valid Cumulative Distribution Function and that the definition of *CDF* is not changed by the conditional probability, we can apply the fundamental theorem of calculus and find that:

$$f_{X|X\in I}(x) = \frac{f_X(x)}{F(b) - F(a)}$$

1.3 Inverse CDF

We remember and apply the definition of general inverse of a Cumulative Distribution Function:

$$F^{-1}(U) = inf\{x \in \mathbb{R} : F(x) \geq U\}$$

$$F_{X|X\in I}(x) \geq U \Rightarrow \frac{F(x) - F(a)}{F(b) - F(a)} \geq U \Rightarrow F(x) \geq U[F(b) - F(a)] + F(a)$$

Since a *CDF* is by definition right continuous and monotonically increasing we have that $inf\{x \in \mathbb{R} : F(x) \geq U\} \ := \ x_0 \mid F(x_0) = U$. Therefore:

$$F(x) = U[F(b) - F(a)] + F(a)$$

With knowledge of $F(x)$ and by rewriting the above equality in an explicit form with respect to x we have found the inverse, which we can use in conjunction with the *inverse method* to sample from an arbitrary distribution starting from an uniformly distributed random variable:

draw U $\sim \mathcal{U}(0,1)$
set X $\leftarrow F^{-1}_{X|X\in I}(U)$
return X

Thus X will be distributed according to $F_{X|X\in I}$.

2 Power Production of a Wind Turbine

2.1 The 95% Confidence Interval

2.1.1 Using Standard Monte Carlo

We begin by using the Standard Monte Carlo method to find the expectation and the 95% confidence interval, the basic Monte Carlo sampler is as follow:

for i $= 1 \to N$
— **draw** $X_i \sim f$
end for
set $\tau_N \to \sum_{i=1}^{N} \phi(X_i)/N$
return X

Where the Weibull probability density function f is known, and due to the law of large numbers as N tends to infinity we have that:

$$\tau_N \overset{def.}{=} \frac{1}{N}\sum_{i=1}^{N} \phi(X_i) \to \tau = E(\phi(X))$$

We Estimate the 95% Confidence Interval through the following equations:

$$I_\alpha = (\tau_N - \lambda_{\alpha/2}\frac{\sigma(\phi)}{\sqrt{N}}, \tau_N + \lambda_{\alpha/2}\frac{\sigma(\phi)}{\sqrt{N}})$$

$$\sigma^2(\phi) \approx \mathbb{E}\{\phi^2(x)\} - \mathbb{E}\{\phi(x)\}^2$$

Where λ_p is the *p-quantile* of the standard normal distribution. In our case with $\alpha = 0.05$ we have that $\lambda_{\alpha/2} = 1.96$. Then, by using the Standard Monte Carlo method to estimate τ_N and $\mathbb{E}\{\phi^2(x)\}$ we get the following results:

Months	Lower Bound (10^6)	Mean (10^6)	Upper Bound (10^6)	Width Bound (10^5)
Jan	4.5791	4.6509	4.7227	1.4360
Feb	4.0447	4.1145	4.1843	1.3954
Mar	3.8046	3.8730	3.9413	1.3669
Apr	2.9223	2.9847	3.0472	1.2494
May	2.7750	2.8364	2.8977	1.2273
Jun	3.0349	3.0983	3.1616	1.2673
Jul	2.7635	2.8244	2.8852	1.2174
Aug	3.0324	3.0959	3.1594	1.2697
Sep	3.6678	3.7350	3.8023	1.3448
Oct	4.1182	4.1897	4.2611	1.4290
Nov	4.5931	4.6649	4.7368	1.4375
Dec	4.6122	4.6839	4.7555	1.4325

Table 1: Estimation results with the Standard Monte Carlo method

2.1.2 The Truncated Version

We repeat the experiment using a truncated distribution and the knowledge acquired through Problem 1: we compute $F^{-1}_{X\in[3.5,25]}$ where in this instance $F(x)$ is the Weibull CDF and then use the *inverse method* and the `wblinv` Matlab function to sample $X \sim f_{X\in[3.5,25]}(x)$ (where f is the Weibull PDF). From there Standard MC is used to find an estimation and 95% Confidence Interval of the wind power:

Months	Lower Bound (10^6)	Mean (10^6)	Upper Bound (10^6)	Width Bound (10^5)
Jan	4.6075	4.6687	4.7299	1.2241
Feb	4.0505	4.1091	4.1677	1.1718
Mar	3.7699	3.8269	3.8839	1.1407
Apr	2.9678	3.0184	3.0689	1.0118
May	2.8151	2.8642	2.9132	0.9803
Jun	3.0333	3.0844	3.1355	1.0223
Jul	2.8144	2.8632	2.9120	0.9759
Aug	3.0433	3.0942	3.1451	1.0183
Sep	3.6890	3.7454	3.8018	1.1283
Oct	4.1594	4.2183	4.2772	1.1784
Nov	4.6120	4.6730	4.7340	1.2198
Dec	4.5877	4.6482	4.7087	1.2108

Table 2: Estimation using the Standard Monte Carlo method through the truncated Weibull distribution

2.1.3 Comparison of the two methods

Comparing the interval we got from the Standard Monte Carlo method and the truncated version respectively (Table 1 and Table 2 as well as Figure 1), it is clear to observe that the width is reduced by the truncated Monte Carlo method, though it's not a great improvement.

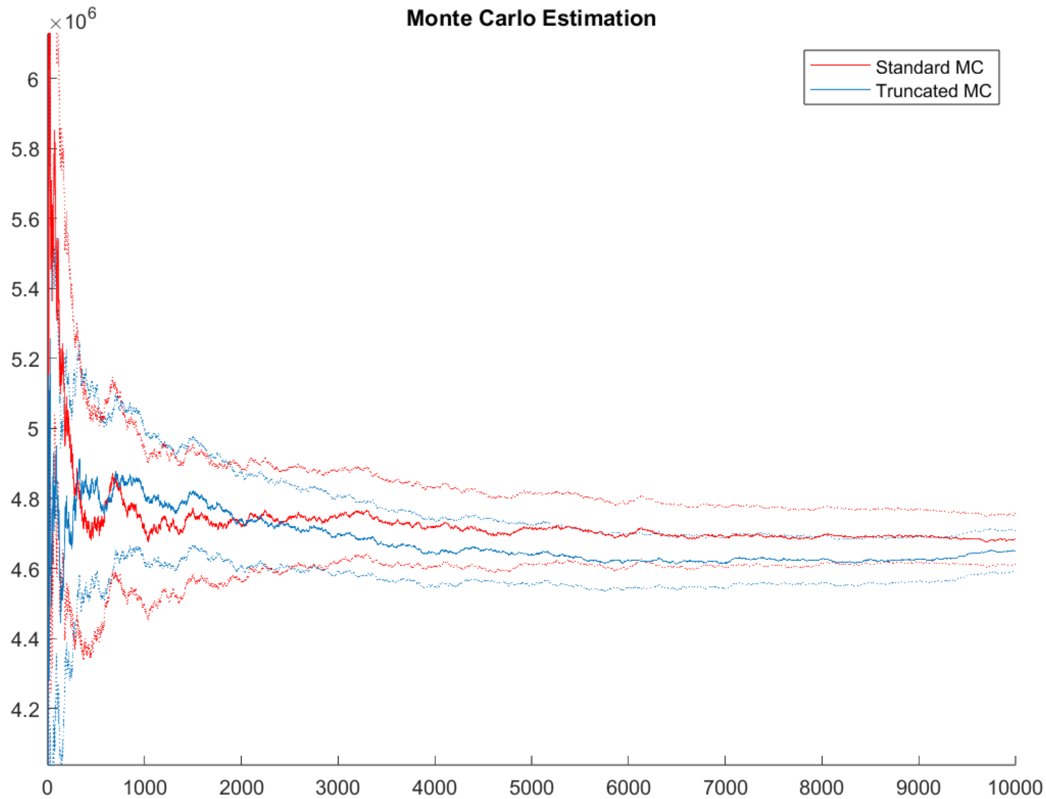


Figure 1: Comparison of the evolution of τ_N for Standard MC and the Truncated Distribution variant for the month of December.

2.2 Control Variate

We use the Monte Carlo Estimator in conjunction with the Control Variate method to reduce the variance and confidence interval width of our estimation. We remember the constraints on the variable Y (our Control Variate) imposed by the method:

- $\mathbb{E}(Y) = m$ is known

- $\phi(X)$ and Y can be simulated at the same complexity as $\phi(X)$

Then for some $\beta \in \mathbb{R}$ we can define Z:

$$Z = \phi(X) + \beta(Y - m)$$

We then have:

$$\mathbb{E}(Z) = \mathbb{E}(\phi(X) + \beta(Y - m)) = \mathbb{E}(\phi(X)) + \beta(\mathbb{E}(Y) - m) = \tau$$

Where $\mathbb{E}(Y) = m$ and the optimal coefficient β^* is $\beta^* = \beta = -\mathbb{C}(\phi(X), Y)/\mathbb{V}(Y)$.

While $\phi(X)$ and Y have covariance $\mathbb{C}(\phi(X), Y)$ it holds that:

$$\mathbb{V}(Z) = \mathbb{V}(\phi(X) + \beta(Y - m)) = \mathbb{V}(\phi(X)) + 2\beta\mathbb{C}(\phi(X), Y) + \beta^2\mathbb{V}(Y) \tag{1}$$

We choose the wind speed V as our Control Variate, with the following results:

Months	Lower Bound (10^6)	Mean (10^6)	Upper Bound (10^6)	Width Bound (10^4)
Jan	4.6093	4.6425	4.6757	6.6440
Feb	4.1221	4.1478	4.1734	5.1227
Mar	3.8283	3.8514	3.8746	4.6262
Apr	2.9866	3.0061	3.0256	3.9006
May	2.8367	2.8554	2.8742	3.7512
Jun	3.0782	3.0973	3.1164	3.8204
Jul	2.8477	2.8665	2.8852	3.7459
Aug	3.0720	3.0913	3.1106	3.8626
Sep	3.7329	3.7542	3.7756	4.2700
Oct	4.1858	4.2159	4.2460	6.0173
Nov	4.6220	4.6543	4.6866	6.4560
Dec	4.6394	4.6714	4.7033	6.3876

Table 3: Estimation results with the Control Variate method

Results Comparing the interval we got from the truncated Monte Carlo method and the Control Variate method respectively (Table 2 and Table 3 as well as Figure 2), it is clear to observe that the Control Variate converges significantly faster and has a significantly reduced confidence interval.

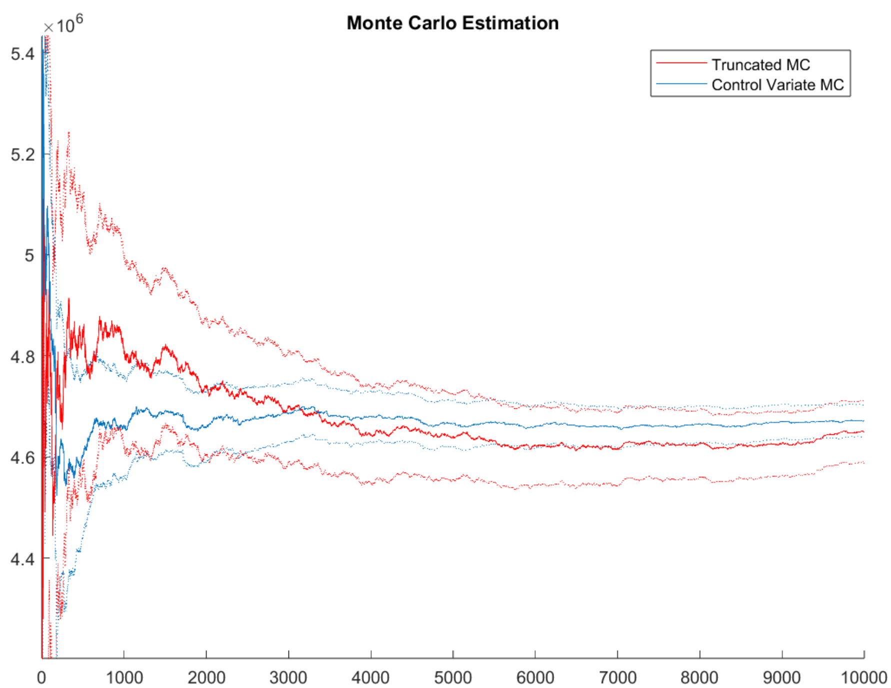


Figure 2: Comparison of the evolution of τ_N for Truncated MC and the Control Variate method for the month of December.

2.3 Importance Sampling

With Importance Sampling method, the objective is, through an instrumental density $g(X)$ such that $g(x) = 0 \to \phi(x)f(x) = 0$, to reduce the variance of the estimation and make it converge faster. To do this, we exploit the following clever rewrites of the Monte Carlo integral:

$$\tau = \mathbb{E}_f(\phi(X)) = \int_{f(x)>0} \phi(x)f(x)dx = \int_{g(x)>0} \phi(x)\frac{f(x)}{g(x)}g(x)dx \ = \ \mathbb{E}_g(\phi(X)\frac{f(X)}{g(X)}) = \mathbb{E}_g(\phi(X)\omega(X))$$

$$\omega : \{x \in X : g(x) > 0\} \ni x \to \frac{f(x)}{g(x)}$$

The variance of the estimation obtained through this method is $\mathbb{V}_g(f(x)\phi(x))$, therefore we try to pick the distribution that minimizes the variation of the function $f(x)\omega(x)$ in the domain of $g(x)$. We accomplish this by calculating the variance $\mathbb{V}\{f(x)\omega(x)\}$ for a handful of differently distribured normal PDFs and choosing the one that minimizes such variance, with the following results:

Months	Lower Bound (10^6)	Mean (10^6)	Upper Bound (10^6)	Width Bound (10^4)
Jan	4.6468	4.6735	4.7003	5.3545
Feb	4.1480	4.1725	4.1971	4.9068
Mar	3.8248	3.8475	3.8702	4.5391
Apr	2.9725	2.9918	3.0112	3.8717
May	2.8486	2.8667	2.8848	3.6246
Jun	3.0681	3.0878	3.1074	3.9295
Jul	2.8455	2.8638	2.8821	3.6645
Aug	3.0717	3.0911	3.1104	3.8693
Sep	3.7568	3.7789	3.8011	4.4324
Oct	4.2091	4.2342	4.2593	5.0238
Nov	4.6417	4.6686	4.6954	5.3707
Dec	4.6349	4.6615	4.6882	5.3288

Table 4: The estimation results with the Importance Sampling method

Results Comparing the interval we got from the Control Variate and the Importance Sampling method respectively (Table 3 and Table 4 as well as Figure 3), the Importance Sampling performance seems level with the Control Variate results, and occasionally marginally better.

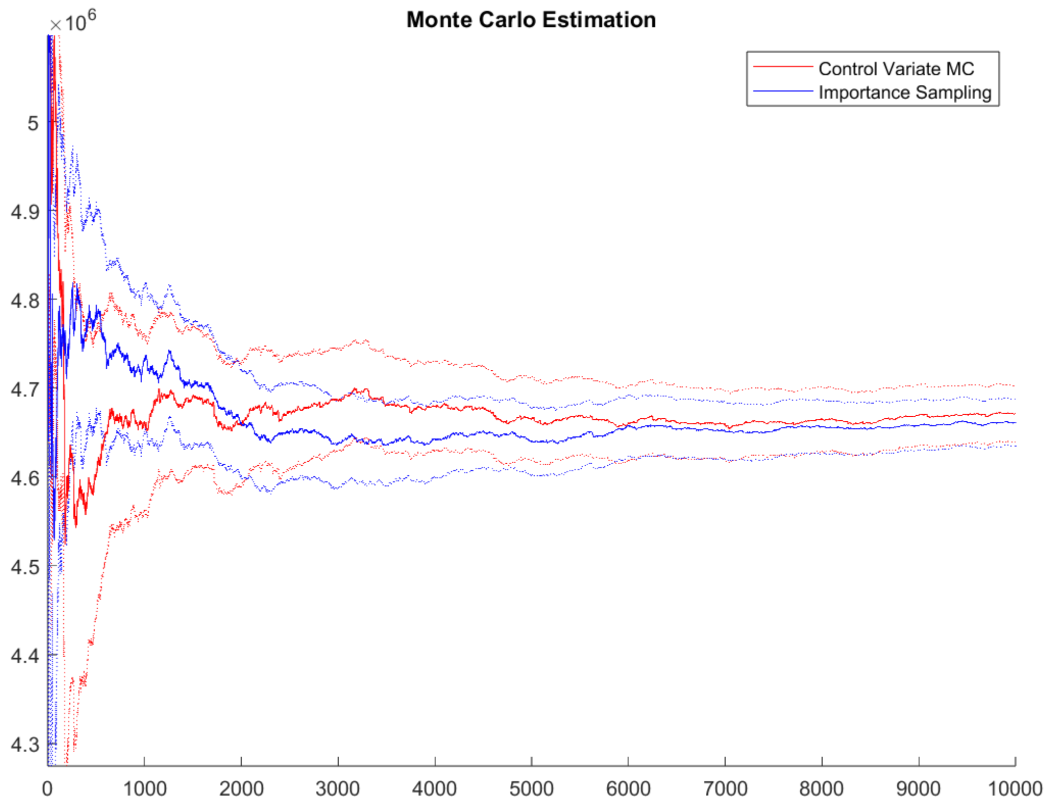


Figure 3: Comparison of the evolution of τ_N for Control Variate and the Importance Sampling method for the month of December.

2.4 Antithetic Sampling

For the Antithetic Sampling, let $V \overset{def.}{=} \phi(X)$, so that $\tau = \mathbb{E}(V)$ and assume that we can generate another variable $\tilde{V}$ such that :

- $\mathbb{E}(\tilde{V}) = \tau$
- $\mathbb{V}(\tilde{V}) = \mathbb{V}(V) = \sigma^2(\phi)$
- $\tilde{V}$ can be simulated at the same complexity as V

Then for

$$W \overset{def.}{=} \frac{V + \tilde{V}}{2}$$

It holds that $\mathbb{E}(W) = \tau$ and

$$\mathbb{V}(W) = \mathbb{V}(\frac{V + \tilde{V}}{2})$$

$$= \frac{1}{4}(\mathbb{V}(V) + 2\mathbb{C}(V, \tilde{V}) + \mathbb{V}(\tilde{V})) = \frac{1}{2}(\mathbb{V}(V) + \mathbb{C}(V, \tilde{V}))$$

To decrease the variance, the antithetic variables V and $\tilde{V}$ need to be negatively correlated. Given a monotonic function $\phi(x)$ and a cumulative distribution function $F(x)$ we have in particular that starting from $U \sim \mathcal{U}(0, 1)$ we can generate two variables $V = \phi(F^{-1}(U))$ and $\hat{V} = \phi(F^{-1}(1 - U))$ such that $\mathbb{C}(V, \hat{V}) \leq 0$. By the inversion theorem we know that V will be distributed according to $f(x) = F^{-1}(x)$ so by choosing F to be the Weibull CDF we can generate V as our wind variable and $\hat{V}$ as our antithetic variable to be used for estimation, with the following results:

Months	Lower Bound (10^6)	Mean (10^6)	Upper Bound (10^6)	Width Bound (10^4)
Jan	4.6501	4.6635	4.6769	2.6799
Feb	4.1230	4.1409	4.1587	3.5668
Mar	3.7974	3.8192	3.8410	4.3622
Apr	3.0008	3.0322	3.0636	6.2830
May	2.8280	2.8599	2.8919	6.3875
Jun	3.0521	3.0831	3.1142	6.2137
Jul	2.8452	2.8774	2.9096	6.4360
Aug	3.0624	3.0929	3.1235	6.1133
Sep	3.7531	3.7759	3.7987	4.5568
Oct	4.2110	4.2295	4.2480	3.7086
Nov	4.6466	4.6597	4.6729	2.6362
Dec	4.6551	4.6679	4.6807	2.5558

Table 5: The estimation results with the Antithetic Sampling method

Results Comparing the interval we got from the Importance Sampling and the Antithetic Sampling method respectively (Table 4 and Table 5 as well as Figure 4), we notice a significant improvement in some months but also a significant degradation in other months, suggesting he method has the potential to reduce significantly the variance of the estimation but has a tight dependence on the shape of the data.

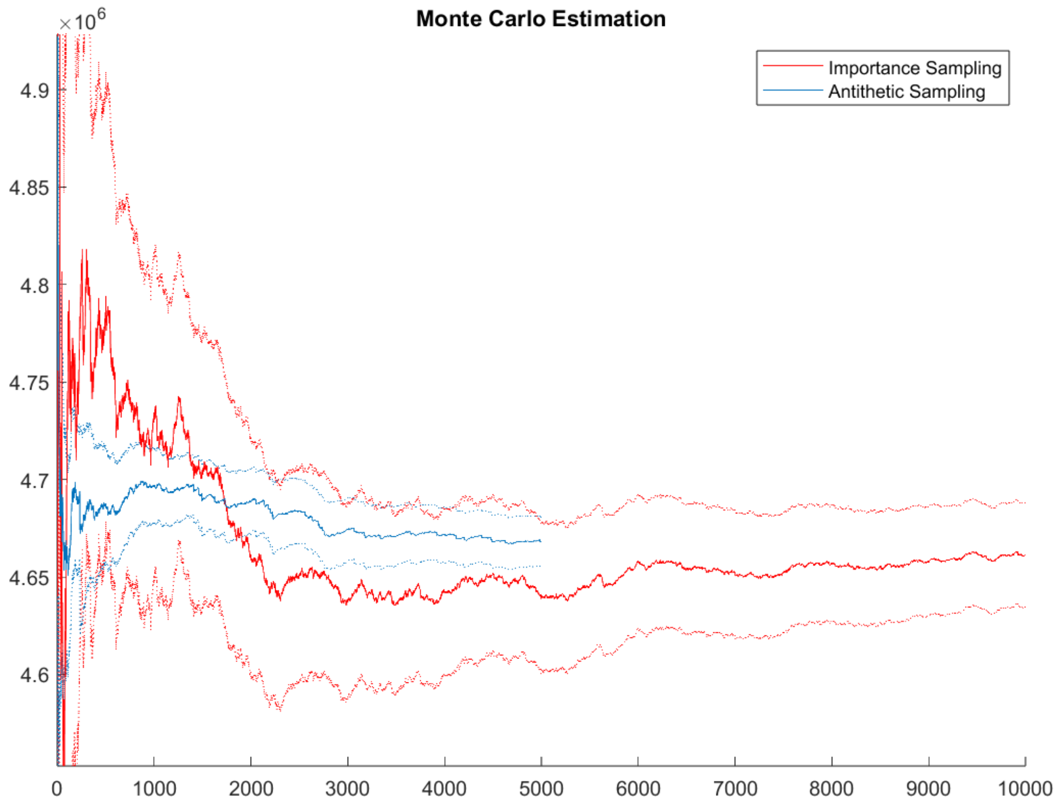


Figure 4: Comparison of the evolution of τ_N for Importance Sampling and the Antithetic Sampling method for the month of December. Notice how the Antithetic Sampling track cuts short at $N = 5000$. This is because Antithetic Sampling effectively uses twice as many variables per iteration, therefore it has been left to run for only half of the iterations.

2.5 Power Production Probability

We notice that $\mathbb{P}(P(V) > 0) = \mathbb{P}(3.5 \leq V \leq 25) = \mathbb{P}(V \geq 3.5 \bigcap V \leq 25) = F_V(25) - F_V(3.5)$ Where F_V is the cumulative probability function of the wind, in this case a Weibull CDF.

Months	Jan	Feb	Mar	Apr	May	Jun
$\mathbb{P}(P(V) > 0)$	0.8929	0.8766	0.8646	0.8121	0.8039	0.8160
Months	Jul	Aug	Sep	Oct	Nov	Dec
$\mathbb{P}(P(V) > 0)$	0.8039	0.8160	0.8620	0.8675	0.8929	0.8929

Table 6: Probability of power production for each month.

2.6 Expected Power Coefficient

Given the following formulations for the total power flowing through the turbine rotor and the expectation of the power of a random variable distributed according to a Weibull distribution:

$$P_{tot}(v) = \frac{1}{2}\rho\pi\frac{d^2}{4}v^3$$
$$\mathbb{E}[V^m] = \Gamma(1 + m/k)\lambda^m \quad , \quad m > -k$$

We can calculate the expectation of P_{tot} as follows:

$$\begin{aligned}
\mathbb{E}P_{tot}(V) &= \int_0^{+\infty} \frac{1}{2}\rho\pi\frac{d^2}{4}v^3 f(v)\, dv \\
&= \frac{1}{2}\rho\pi\frac{d^2}{4}\int_0^{+\infty} v^3 f(v)\, dv \\
&= \frac{1}{2}\rho\pi\frac{d^2}{4}\mathbb{E}[V^3]
\end{aligned}$$

Since $\mathbb{E}\{P(v)\}$ is already known from the estimations above (we are going to use the one obtained through the Antithetic Sampling method as it's the one with the smallest variance) we can just estimate P_{tot} according to these formulas and obtain our result. However some extra math is needed to calculate the 95% confidence interval. Given $\bar{P}_{tot} = \mathbb{E}\{P_{tot}\}$ we have that:

$$\frac{\mathbb{E}\{P(v)\}}{\mathbb{E}\{P_{tot}\}} = \frac{\int_{-\infty}^{+\infty} P(v)f(v)dv}{\bar{P}_{tot}} = \int_{-\infty}^{+\infty} \frac{P(v)}{\bar{P}_{tot}} f(v)dv$$

For the purpose of calculating the 95% confidence interval we can thus assume $\phi'(x) = P(x)/\bar{P}_{tot}$. Thus we have

$$I_\alpha = \tau \pm \lambda_{\alpha/2}\sqrt{\frac{\mathbb{V}\{\phi'(x)\}}{N}} \Rightarrow I_\alpha = \tau \pm \lambda_{\alpha/2}\sqrt{\frac{\mathbb{V}\{\phi(x)/\bar{P}_{tot}\}}{N}} = \tau \pm \lambda_{\alpha/2}\sqrt{\frac{\mathbb{V}\{\phi(x)\}}{\bar{P}_{tot}^2 N}} = \tau \pm \lambda_{\alpha/2}\sqrt{\frac{\mathbb{V}\{\phi(x)\}}{N}} \cdot \frac{1}{\bar{P}_{tot}}$$

Which gives us as a result that the 95% confidence interval of the Power Coefficient is the 95% confidence interval of $\mathbb{E}\{P(v)\}$ divided by $\mathbb{E}\{P_{tot}(v)\}$. With this observation in mind we can now calculate our confidence interval with the following results (in particular we notice an increase in efficiency in the period going from April to August):

Months	Lower Bound	Mean	Upper Bound	Width Bound
Jan	0.2259	0.2266	0.2273	0.0015
Feb	0.2624	0.2635	0.2646	0.0023
Mar	0.2835	0.2851	0.2867	0.0032
Apr	0.3205	0.3238	0.3271	0.0066
May	0.3280	0.3318	0.3355	0.0075
Jun	0.3130	0.3162	0.3193	0.0063
Jul	0.3262	0.3300	0.3337	0.0075
Aug	0.3143	0.3175	0.3207	0.0064
Sep	0.2887	0.2905	0.2923	0.0035
Oct	0.2381	0.2391	0.2402	0.0020
Nov	0.2269	0.2275	0.2282	0.0013
Dec	0.2272	0.2278	0.2285	0.0013

Table 7: Power Coefficient for each month

2.7 Capacity and Availability Factor

The Capacity Factor is computed as the ratio of the actual energy output over a time period and the maximum possible output during the time, which is $t \cdot 9.5MW$ (where t is the length of the time period taken into consideration). This corresponds to the estimated power output divided by 9.5MW.

The Availability Factor is computed as the portion of a time period in which the turbine generates electricity, which corresponds to $\mathbb{P}\{P(v) > 0\}$.

Months	Jan	Feb	Mar	Apr	May	Jun
Capacity Factor	0.4905	0.4356	0.4028	0.3169	0.3032	0.3259
Availability Factor	0.8929	0.8766	0.8646	0.8121	0.8039	0.8160
Months	Jul	Aug	Sep	Oct	Nov	Dec
Capacity Factor	0.3030	0.3228	0.3961	0.4440	0.4912	0.4904
Availability Factor	0.8039	0.8160	0.8620	0.8675	0.8929	0.8929

Table 8: Probability of power production for each month.

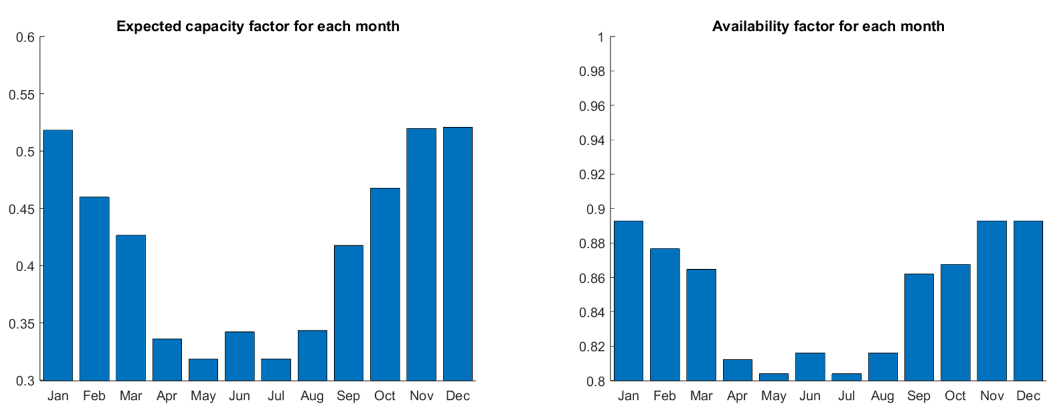


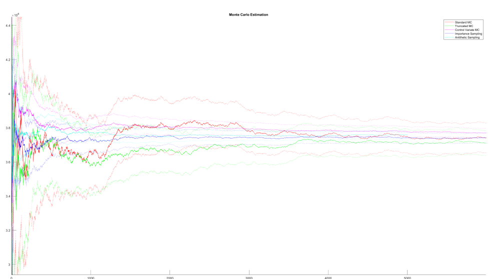
Figure 5: Per-month plot of the expected Availability Factor and Capacity Factor

2.8 Final Remarks

Having experimented with the Monte Carlo estimation of the performance of this turbine we report the following impressions on the different methods used thorought:

- The **Standard Monte Carlo Estimator** is the most basic one and needs approximately *N=6000* iterations to converge to a stable value.
- **All other estimators** have a comparable level of variance which is *much* smaller than the one of the Standard MC estimator.
- The **Truncated MC Estimator** offers a slightly narrower confidence interval, but no other significant change from the Standard MC. However, it is to be noted that this estimator is of *extreme* importance in situations where only the conditional probability is available (for example because the data has been gathered from observation).
- Application of the **Control Variate** variance reduction method yields our first noticeable decrease in convergence time and variance, as well as CI width.
- **Importance Sampling** yields a similar result to the Control Variate method.
- **Antithetic Sampling** is sometimes on par with Importance Sampling, but sometimes *significantly* better in terms of variance and CI width.

In regards to the performance of the turbine, after computing its average capacity and availability factor across all months we obtain a Capacity Factor of 0.3937 and an Availability Factor of 0.8501. Good wind turbines typically have a Capacity Factor of 20%−40% and an availability greater than 90%, so our conclusion is that the turbines perform exceptionally well (at the top of what would be expected) but their availability is below the typical level. As such, it is difficult to give a clear answer on how effective this positioning would be with respect to other alternative placements in the same region of competence, however the deficiency of the Availability Factor of this setup must absolutely be taken into account as its reliability could prove unsatisfactory.



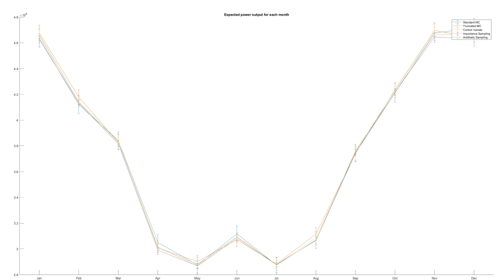


Figure 6: Left: Estimation of power with respect to N for the month of January - comparison of all methods. Right: Estimation of power for all months and methods used.

3 Combined Power Production of Two Wind Turbines

3.1 Assessment of the situation

We have two identical wind turbines generating power according to the intensity of the wind that passes through them with velocity v_1 and v_2 respectively. These winds are not independent and are instead jointly distributed according to the formula:

$$f(v_1, v_2) = f(v_1)f(v_2)[1 + \alpha(1 - F(v_1)^p)^{q-1}(1 - F(v_2)^p)^{q-1}(F(v_1)^p(1 + pq) - 1)(F(v_2)^p(1 + pq) - 1)]$$

With $\alpha = 0.638$, $p = 3$, $q = 1.5$ - and considering that

$$f(v_1) = f(v_2) = f(v) = \frac{k}{\lambda}\left(\frac{v}{\lambda}\right)^{k-1}\ exp\left(-\left(\frac{v}{k}\right)^k\right) \text{ with } k = 1.96,\ \lambda = 9.13$$

We want to find the expected value of a number of different parameters of this wind turbine setup, namely we want to find:

- $\mathbb{E}\{P(v_1) + P(v_2)\}$
- $\mathbb{C}\{P(v_1), P(v_2)\}$
- $\mathbb{V}\{P(v_1) + P(v_2)\}$
- $\mathbb{D}\{P(v_1) + P(v_2)\}$
- $\mathbb{P}\{P(v_1) + P(v_2) > 9.5\ MW\}$
- $\mathbb{P}\{P(v_1) + P(v_2) < 9.5\ MW\}$

3.2 Estimation

The quantities described above can be broken down into formulas involving simpler expectations, some of which can be reused to estimate more than one quantity which is optimal for the sake of efficiency. We identified six basic expectations that we need to estimate in order to answer the questions above:

- $\mathbb{E}(P(v_1))$
- $\mathbb{E}(P(v_2))$
- $\mathbb{E}(P(v_1)P(v_2))$
- $\mathbb{E}((P(v_1) + P(v_2))^2)$
- $\mathbb{P}(P(v_1) + P(v_2) > 9.5)$
- $\mathbb{P}(P(v_1) + P(v_2) < 9.5)$

We decided to use a multivariate normal distribution as our instrumental function for all six estimators. Due to the increased size of the problem we deemed the previously used approach of automatic identification of the optimal distribution not viable, resorting to manual fine-tuning of the parameters. We tried to have $g(x)$ match as closely as possible $\phi(x)f(x)$ over the domain $[3.5, 25] \times [3.5, 25]$, increasing the variance slightly to make sure $g(x)$ stays at the top as the two distributions taper off towards zero to avoid the product shooting up to infinity near the edges of the considered interval.

3.3 The Results

We got the following results out of the importance sampling process:

Quantity	Instrumental Distribution	Estimation Result
$\mathbb{E}(P(v_1))$	$\mathcal{N}\{[12,10],[28,6;6,28]\}$	$3.7376 \cdot 10^6$
$\mathbb{E}(P(v_2))$	$\mathcal{N}\{[10,12],[28,6;6,28]\}$	$3.7538 \cdot 10^6$
$\mathbb{E}(P(v_1)P(v_2))$	$\mathcal{N}\{[12,12],[16,1;1,16]\}$	$2.1038 \cdot 10^{13}$
$\mathbb{E}((P(v_1)+P(v_2))^2)$	$\mathcal{N}\{[12,12],[20,2;2,20]\}$	$9.4676 \cdot 10^{13}$
$\mathbb{P}(P(v_1)+P(v_2)>9.5)$	$\mathcal{N}\{[10,10],[28,2;2,26]\}$	0.3741
$\mathbb{P}(P(v_1)+P(v_2)<9.5)$	$\mathcal{N}\{[5.6,5.6],[20,0;0,20]\}$	0.6175

Table 9: Probability of power production for each month.

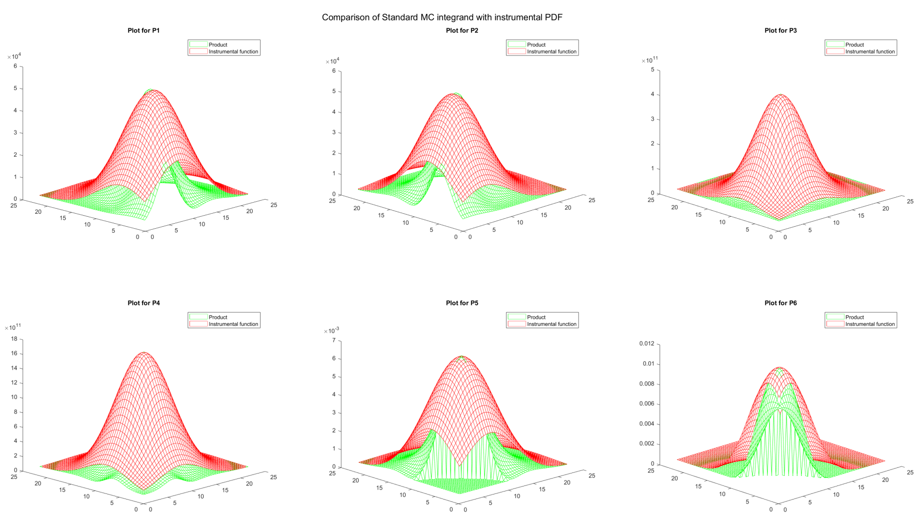


Figure 7: Comparison of MC integrands with their corresponding (and rescaled) instrumental PDF

3.4 Derivations

We now aim at deriving the quantities that have been requested to us:

3.4.1 Sum of powers

We can rewrite the expected total power as follows:

$$\mathbb{E}\{P(v_1)+P(v_2)\} = \mathbb{E}\{P(v_1)\} + \mathbb{E}\{P(v_2)\} = 7.4914 \cdot 10^6$$

It is worth noting that the expectations $\mathbb{E}(P(v_1))$ and $\mathbb{E}(P(v_2))$ are computed for two identically distributed random variables, using the same objective function, through an importance sampling process involving two instrumental distributions with the same mean and symmetrical covariance matrix (therefore same shape too), which means that the two powers have the same expectation and the formula could then be rewritten as $\mathbb{E}((P(v_1)+P(v_2))^2) = 2\mathbb{E}(P(v_1)) = 2\mathbb{E}(P(v_2))$, and similarly for all other formulas involving the individual powers or their sum. The two powers have been calculated (and used) independently just to demonstrate this very point and for the sake of completeness, but in a less dimostrative and more performance-focused environment the computation of $\mathbb{E}(P(v_2))$ could have been skipped entirely.

3.4.2 Covariance of Powers

We can rewrite the expected covariance of the power of the two turbines as follows:

$$\mathbb{C}\{P(v_1), P(v_2)\} = \mathbb{E}\{P(v_1)P(v_2)\} - \mathbb{E}\{P(v_1)\}\mathbb{E}\{P(v_2)\} = 7.0083 \cdot 10^{12}$$

3.4.3 Variance and Standard Deviation of the total power

We can rewrite the expected variance and standard deviation of the total power as follows:

$$\mathbb{V}\{P(v_1) + P(v_2)\} = \mathbb{E}\{(P(v_1) + P(v_2))^2\} - \mathbb{E}\{P(v_1) + P(v_2)\}^2 = 3.8555 \cdot 10^{13}$$
$$\mathbb{D}\{P(v_1) + P(v_2)\} = \sqrt{\mathbb{V}\{P(v_1) + P(v_2)\}} = 6.2092 \cdot 10^6$$

3.4.4 Probabilities

For probabilities we use a little trick to convert them to something that can be estimated using Monte Carlo methods: in principle the probability of an event happening is the integral of the (joint) PDF of whatever stochastic event we are measuring over the interval in which the event occurs, but this can also be rewritten as the integral over the whole stochastic space of the PDF multiplied by an indicator function which is 1 when the conditions are satisfied and 0 when they are not. This would be the continuous equivalent to dividing the number of positive causes by the number of total cases in discrete probability and can be estimated through Monte Carlo if we consider the indicator function our objective function. The indicator function, multiplied by the joint PDF of v_1 and v_2, can be observed in the last two graphs of Figure 7 (notice the abrupt cut from the PDF to zero when the probabilistic boundary is reached).

To calculate the 95% confidence interval for the two estimated probabilities we exploit the fact that the square of the indication function (which we remember is our objective function for the purpose of estimating probabilities through Monte Carlo methods) is the same as the indicator function itself:

$$\mathbb{I}^2 = \mathbb{I} \Rightarrow \sigma^2(\mathbb{I}) = \mathbb{E}\{\mathbb{I}^2(x)\} - \mathbb{E}\{\mathbb{I}(x)\}^2 = \mathbb{E}\{\mathbb{I}(x)\} - \mathbb{E}\{\mathbb{I}(x)\}^2 \approx \tau_N - \tau_N^2$$

We therefore have this data:

	Lower Bound	Mean	Upper Bound
$\mathbb{P}(P(v_1) + P(v_2) > 9.5)$	0.3607	0.3741	0.3875
$\mathbb{P}(P(v_1) + P(v_2) > 9.5)$	0.6177	0.6312	0.6042

Table 10: Probabilities of the power production of the two wind turbines with the confidence interval.

We notice how the estimated means of the two probabilities do not add up to 1. This could be written off as a result of this being just an estimation, but it's not the case: normally the probability of a continuous variable of assuming an exact value is null, as that would be the probability of *exactly one* value being picked from a pool of infinitely many values, resulting in the probability being $\frac{1}{\infty}$ which tends to zero. But in this case there is not a single combination of (v_1, v_2) that yields $P(v_1) + P(v_2) = 9.5\ MW$. Indeed, there are infinitely many such combinations, arranged in an open curve over the probability space of this problem. Since the integral of a continuous variable in a continuous space along a curve is not necessarily zero (as the curve itself contains infinitely many points) the probability $\mathbb{P}(P(v_1) + P(v_2) = 9.5\ MW)$ can be larger than zero, provided the joint distribution is non-zero for at least some segments of that curve. Indeed for the problem at hand we can estimate:

$$\mathbb{P}(P(v_1) + P(v_2) = 9.5\ MW) \approx 1 - [\mathbb{P}(P(v_1) + P(v_2) > 9.5) + \mathbb{P}(P(v_1) + P(v_2) > 9.5)] = 0.0083$$

This probability is very small and the (comparatively) large confidence interval of the two estimated probabilities does not help, thus a high degree of uncertainty is associated with this probability.
Which is, we can consider these two random variable expectations as the same one, according to they will eventually converge to the same value. So this actually is a one-dimensional problem.