d) Put it all together to show Bayes' rule, as described above.

2.5 Urns: Suppose urn H is filled with 40% green balls and 60% red balls, and urn T is filled with 60% green balls and 40% red balls. Someone will flip a coin and then select a ball from urn H or urn T depending on whether the coin lands heads or tails, respectively. Let X be 1 or 0 if the coin lands heads or tails, and let Y be 1 or 0 if the ball is green or red.

a) Write out the joint distribution of X and Y in a table.

b) Find $\mathrm{E}[Y]$. What is the probability that the ball is green?

c) Find $\mathrm{Var}[Y|X=0]$, $\mathrm{Var}[Y|X=1]$ and $\mathrm{Var}[Y]$. Thinking of variance as measuring uncertainty, explain intuitively why one of these variances is larger than the others.

d) Suppose you see that the ball is green. What is the probability that the coin turned up tails?

※ fair coin 가정.

H: 40% 초록, 60% 빨강

T: 60% 초록, 40% 빨강

H: 동전 앞면 $X=1$: H $Y=1$: green

T: 동전 뒷면 $X=0$: T $Y=0$: red.

a)

	$X=1$	$X=0$
$Y=1$	0.2	0.3
$Y=0$	0.3	0.2

b) $E(Y) = 1\times 0.5 + 0\times 0.5$
$= 0.5$

공 초록 확률 $= Pr(Y=1) = 0.5$

c) $$\mathrm{Var}(Y|X=0) = E[Y^2|X=0] - (E[Y|X=0])^2$$
$$= 1^2\times 0.6 + 0^2\times 0.4 - (1\times 0.6 + 0\times 0.4)^2$$
$$= 0.6 - 0.36$$
$$= 0.24$$

$\mathrm{Var}[Y|X=1] = 0.4 - 0.16 = 0.24$

$\mathrm{Var}(Y) = 0.5 - 0.25 = 0.25$

$\mathrm{Var}(Y) > \mathrm{Var}[Y|X=0]$, $\mathrm{Var}[Y|X=1]$

$X=0,1$ 주어졌을 때 Y의 uncertainty 가 더 적다!

d) $P(\text{Tail}|\text{green}) = P(X=0|Y=1) = 0.6.$

2.6 Conditional independence: Suppose events A and B are conditionally independent given C, which is written $A \perp B | C$. Show that this implies that $A^c \perp B | C$, $A \perp B^c | C$, and $A^c \perp B^c | C$, where A^c means "not A." Find an example where $A \perp B | C$ holds but $A \perp B | C^c$ does not hold.

Conditional independence

$$A \perp B | C : \quad P(A | B, C) = P(A | C)$$
$$P(B | A, C) = P(B | C)$$

$$1 - P(A | B, C) = 1 - P(A | C)$$

$$\Rightarrow P(A^c | B, C) = P(A^c | C)$$

$$\Rightarrow A^c \perp B | C \quad \text{①}$$

$$1 - P(B | A, C) = 1 - P(B | C)$$

$$\Rightarrow P(B^c | A, C) = P(B^c | C)$$

$$\Rightarrow B^c \perp A | C \quad \text{②}$$

$$\Rightarrow P(A | B^c, C) = P(A | C)$$

$$\Rightarrow 1 - P(A | B^c, C) = 1 - P(A | C)$$

$$\Rightarrow P(A^c | B^c, C) = P(A^c | C)$$

$$\Rightarrow A^c \perp B^c | C \quad \text{③}$$

$A \perp B | C$ but $A \perp B | C^c$ (X)
예시

A: 갑순이가 갑돌이 깻잎을 잡아준다.
B: 내가 화가난다.
C: 내가 갑순이랑 사귀지 않는다.

2. Suppose you have an urn containing R_0 red balls and W_0 white balls. Let c ≥ 0 be a fixed integer. Draw a ball, note the color, replace the ball and put an additional c balls of that color in the urn as well. Rinse and repeat.

Define $X_i = \begin{cases} 1 & \text{if the ith ball is red} \\ 0 & \text{otherwise} \end{cases}$

Show that the random variables in the infinite sequence $X_1, X_2, \ldots$ are exchangeable.

Pólya's urn

$P(X_1 \ldots X_n) = P(X_{\pi_1}, \ldots X_{\pi_n})$ $\quad \pi_1 \ldots \pi_n$: permutation result of $1, \ldots, n$

$$= \frac{R_0 \cdot (R_0+C) \cdots (R_0 + (n_{red}-1)C) \cdot W_0 \cdot (W_0+C) \cdots (W_0+(n_{white}-1)c)}{(R_0+W_0) \cdot (R_0+W_0+C) \cdots R_0+W_0+(n-1)C}$$