(2.5) $X = \begin{cases} 1 & \text{Head} \\ 0 & \text{Tail} \end{cases}$ $Y = \begin{cases} 1 & \text{green} \\ 0 & \text{red} \end{cases}$

a)

Y \ X	0	1
0	$\frac{1}{2} \times \frac{4}{10} = 0.2$	$\frac{1}{2} \times \frac{6}{10} = 0.3$
1	$\frac{1}{2} \times \frac{6}{10} = 0.3$	$\frac{1}{2} \times \frac{4}{10} = 0.2$

b) $E[Y] = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$, $P(Y=1) = \frac{1}{2}$

c) $E[Y|X=0] = 0 \cdot \frac{4}{10} + 1 \cdot \frac{6}{10} = \frac{6}{10}$, $E[Y^2|X=0] = \frac{6}{10}$

$E[Y|X=1] = 0 \cdot \frac{6}{10} + 1 \cdot \frac{4}{10} = \frac{4}{10}$, $E[Y^2|X=1] = \frac{4}{10}$

$Var[Y|X=0] = \frac{6}{10} - \left(\frac{6}{10}\right)^2 = \frac{24}{100}$, $Var[Y|X=1] = \frac{4}{10} - \left(\frac{4}{10}\right)^2 = \frac{24}{100}$

$$Var[Y] = E\left[Var[Y|X]\right] + Var\left[E[Y|X]\right]$$
$$= \left(\frac{1}{2} \cdot \frac{24}{100} + \frac{1}{2} \cdot \frac{24}{100}\right) + \left[\left(\frac{6}{10}\right)^2 \cdot \frac{1}{2} + \left(\frac{4}{10}\right)^2 \cdot \frac{1}{2}\right] - \left[\frac{6}{10} \cdot \frac{1}{2} + \frac{4}{10} \cdot \frac{1}{2}\right]^2$$
$$= 0.24 + 0.26 - 0.25 = 0.25$$

d) $P(X=0 \mid Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{0.3}{0.5} = \frac{3}{5}$

(2.6) Given $A \perp B \mid C \Rightarrow P(A \cap B \mid C) = P(A|C) \cdot P(B|C)$

We have $(A \cap B) \cup (A \cap B^c) = A$, $(A \cap B) \cap (A \cap B^c) = \emptyset$.

Thus $P(A \cap B^c \mid C) = P(A|C) - P(A \cap B \mid C) = P(A|C) - P(A|C) \cdot P(B|C) = P(A|C)\{1 - P(B|C)\} = P(A|C)P(B^c|C)$