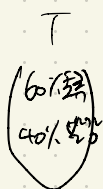
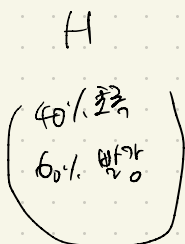


2.5 Urns: Suppose urn H is filled with 40% green balls and 60% red balls, and urn T is filled with 60% green balls and 40% red balls. Someone will flip a coin and then select a ball from urn H or urn T depending on whether the coin lands heads or tails, respectively. Let X be 1 or 0 if the coin lands heads or tails, and let Y be 1 or 0 if the ball is green or red.

- Write out the joint distribution of X and Y in a table.
- Find $E[Y]$. What is the probability that the ball is green?
- Find $\text{Var}[Y|X=0]$, $\text{Var}[Y|X=1]$ and $\text{Var}[Y]$. Thinking of variance as measuring uncertainty, explain intuitively why one of these variances is larger than the others.
- Suppose you see that the ball is green. What is the probability that the coin turned up tails?

※ fair coin game.



H: 동전 앞면

$X=1: H$ $Y=1: \text{green}$

T: 동전 뒷면

$X=0: T$ $Y=0: \text{red}$

a)

	$X=1$	$X=0$
$Y=1$	0.2	0.3
$Y=0$	0.3	0.2

$$b) E(Y) = 1 \times 0.5 + 0 \times 0.5 = 0.5$$

$$\text{공통 확률} = P_r(Y=1) = 0.5$$

$$c) \text{Var}[Y|X=0] = E[Y^2|X=0] - (E[Y|X=0])^2$$

$$= 1^2 \times 0.6 + 0^2 \times 0.4 - (1 \times 0.6 + 0 \times 0.4)^2$$

$$= 0.6 - 0.36$$

$$= 0.24$$

$$\text{Var}(Y) > \text{Var}[Y|X=0]$$

$$\text{Var}[Y|X=1] = 0.4 - 0.16 = 0.24$$

$X=0, 1$ 모두 24%
Y의 uncertainty가 더 크다!

$$\text{Var}(Y) = 0.5 - 0.25 = 0.25$$

$$d) P(\text{Tail} | \text{green}) = P(X=0 | Y=1) = 0.6$$

2.6 Conditional independence: Suppose events A and B are conditionally independent given C , which is written $A \perp B | C$. Show that this implies that $A^c \perp B | C$, $A \perp B^c | C$, and $A^c \perp B^c | C$, where A^c means "not A ." Find an example where $A \perp B | C$ holds but $A \perp B | C^c$ does not hold.

Conditional independence

$$A \perp B | C : P(A | B, C) = P(A | C) \\ P(B | A, C) = P(B | C)$$

$$1 - P(A | B, C) = 1 - P(A | C)$$

$$\Rightarrow P(A^c | B, C) = P(A^c | C)$$

$$\Rightarrow A^c \perp B | C \quad \text{①}$$

$$1 - P(B | A, C) = 1 - P(B | C)$$

$$\Rightarrow P(B^c | A, C) = P(B^c | C)$$

$$\Rightarrow B^c \perp A | C \quad \text{②}$$

$$\Rightarrow P(A | B^c, C) = P(A | C)$$

$$\Rightarrow 1 - P(A | B^c, C) = 1 - P(A | C)$$

$$\Rightarrow P(A^c | B^c, C) = P(A^c | C)$$

$$\Rightarrow A^c \perp B^c | C \quad \text{③}$$

$A \perp B | C$ but $A \not\perp B | C^c$ (x)

예시

A: 감온이냐 감동이냐가 맞았을
감아준다.

B: 누가 라가냐다.

C: 누가 감온이냐 사귀지
않느냐.

2. Suppose you have an urn containing R_0 red balls and W_0 white balls. Let $c \geq 0$ be a fixed integer. Draw a ball, note the color, replace the ball and put an additional c balls of that color in the urn as well. Rinse and repeat.

Define $X_i = \begin{cases} 1 & \text{if the } i\text{th ball is red} \\ 0 & \text{otherwise} \end{cases}$

Show that the random variables in the infinite sequence X_1, X_2, \dots are exchangeable.

Pólya's urn

$$\begin{aligned}
 P(X_1, \dots, X_n) &= P(X_{\pi_1}, \dots, X_{\pi_n}) \quad \pi_1, \dots, \pi_n : \text{permutation result of } 1, \dots, n \\
 &= \frac{R_0 \cdot (R_0 + c) \cdots (R_0 + (n_{\text{red}} - 1)c) \cdot W_0 \cdot (W_0 + c) \cdots (W_0 + (n_{\text{white}} - 1)c)}{(R_0 + W_0) \cdot (R_0 + W_0 + c) \cdots (R_0 + W_0 + (n-1)c)}
 \end{aligned}$$