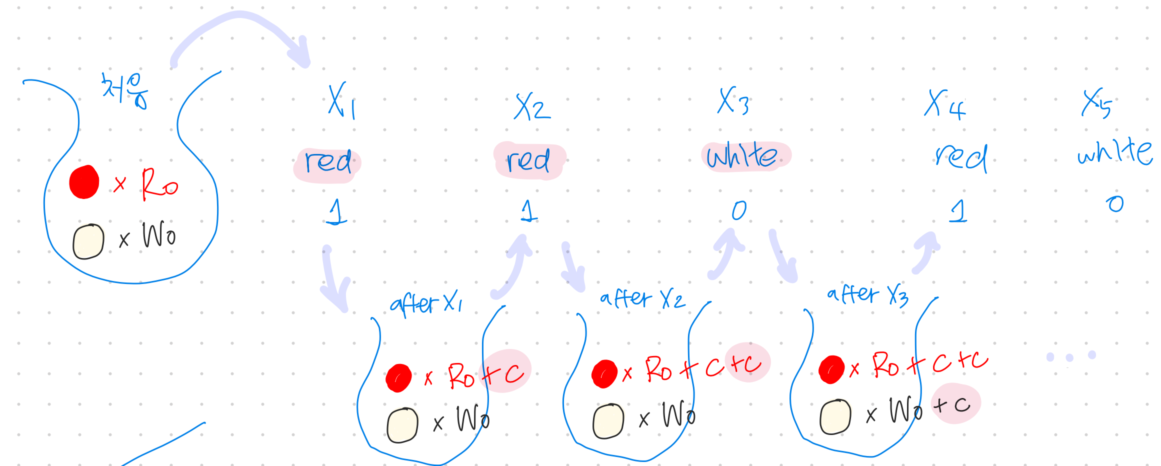
2. Suppose you have an urn containing R_0 red balls and W_0 white balls. Let $c \geq 0$ be a fixed integer. Draw a ball, note the color, replace the ball and put an additional c balls of that color in the urn as well. Rinse and repeat.

Define $X_i = \begin{cases} 1 & \textit{if the ith ball is red} \\ 0 & \textit{otherwise} \end{cases}$

Show that the random variables in the infinite sequence $X_1, X_2, \ldots$ are exchangeable.

Pólya's urn

예를들어...)



$$P(X_1=1, X_2=1, X_3=0, X_4=1, X_5=0)$$

$$= \frac{R_0}{R_0+W_0} \times \frac{R_0+c}{R_0+W_0+c} \times \frac{W_0}{R_0+W_0+2c} \times \frac{R_0+2c}{R_0+W_0+3c} \times \frac{W_0+c}{R_0+W_0+4c}$$

$(X_1=1)$ $(X_2=1)$ $(X_3=0)$ $(X_4=1)$ $(X_5=1)$

분모는 한 단계마다 +c만큼 늘어남

3가지 경우로 나누어 각각의 joint probability 알아보기)

i) $\sum X_i = n$ (다 빨간공만 뽑은 경우)

$P(X_1 = x_1, X_2 = x_2, \cdots X_n = x_n)$

$$= \frac{R_0}{R_0 + W_0} \times \frac{R_0 + c}{R_0 + W_0 + c} \times \frac{R_0 + 2c}{R_0 + W_0 + 2c} \times \cdots \times \frac{R_0 + c(\overbrace{\sum x_i}^{n} - 1)}{R_0 + W_0 + (n-1)c}$$

ii) $\sum X_i = 0$ (다 흰색공만 뽑은 경우)

$P(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n)$

$$= \frac{W_0}{R_0 + W_0} \times \frac{W_0 + c}{R_0 + W_0 + c} \times \frac{W_0 + 2c}{R_0 + W_0 + 2c} \times \cdots \times \frac{W_0 + c(n - \overbrace{\sum x_i}^{0} - 1)}{R_0 + W_0 + (n-1)c}$$

iii) $0 < \sum X_i < n$

$P(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n)$

$$= \frac{\overbrace{[R_0(R_0+c)(R_0+2c)\cdots(R_0+c(\sum x_i - 1))]}^{red} \cdot \overbrace{[W_0(W_0+c)(W_0+2c)\cdots(W_0+c(n-\sum x_i - 1))]}^{white}}{(R_0+W_0)(R_0+W_0+c)(R_0+W_0+2c)\cdots(R_0+W_0+(n-1)c)}$$

$\Rightarrow$ i) ~ iii) 모두 joint probability $P(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n)$이

X_i의 순서와 상관없게 나옴!

$\because$ $\sum x_i$에 대한 식인데, $\sum x_i$는 순서와 관계없이 항상 일정하기 때문에.

$\therefore$

$X_1, X_2, \cdots, X_n$ are exchangeable!