Week1 HW

2.5 Urns: Suppose urn H is filled with 40% green balls and 60% red balls, and urn T is filled with 60% green balls and 40% red balls. Someone will flip a coin and then select a ball from urn H or urn T depending on whether the coin lands heads or tails, respectively. Let X be 1 or 0 if the coin lands heads or tails, and let Y be 1 or 0 if the ball is green or red.

a) Write out the joint distribution of X and Y in a table.

b) Find E[Y]. What is the probability that the ball is green?

c) Find Var[$Y|X = 0$], Var[$Y|X = 1$] and Var[Y]. Thinking of variance as measuring uncertainty, explain intuitively why one of these variances is larger than the others.

d) Suppose you see that the ball is green. What is the probability that the coin turned up tails?

$H \begin{cases} G: 40\% \\ R: 60\% \end{cases}$ $\quad T \begin{cases} G: 40\% \\ R: 60\% \end{cases}$ $\quad X \begin{cases} 1: P(X=1) = 0.5 \\ 0: P(X=0) = 0.5 \end{cases}$ $\quad Y \begin{cases} 1: G \\ 0: R \end{cases}$

a) $P(Y=1 | X=1) = 0.4$, $P(Y=0 | X=1) = 0.6$

$P(Y=1 | X=0) = 0.6$, $P(Y=0 | X=0) = 0.4$

$\rightarrow P(Y=1, X=1) = 0.4 \cdot P(X=1) = 0.2$ $\quad P(Y=0, X=1) = 0.6 \cdot P(X=1) = 0.3$

$P(Y=1, X=0) = 0.6 \cdot P(X=0) = 0.3$ $\quad P(Y=0, X=0) = 0.4 \cdot P(X=0) = 0.2$

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b) $E(Y) = E(E(Y|X))$

$= E(Y|X=0) \cdot P(X=0) + E(Y|X=1) \cdot P(X=1)$

$= \{1 \cdot P(Y=1|X=0) + 0 \cdot P(Y=0|X=0)\} \cdot P(X=0)$

$+ \{1 \cdot P(Y=1|X=1) + 0 \cdot P(Y=0|X=1)\} \cdot P(X=1)$

$= (1 \cdot 0.6 + 0 \cdot 0.4) \cdot 0.5$

$+ (1 \cdot 0.4 + 0 \cdot 0.6) \cdot 0.5$

$= 0.3 + 0.2 = 0.5$

c) Find $Var[Y|X=0]$, $Var[Y|X=1]$, $Var[Y]$

$p(Y=1|X=0) = 0.6$, $p(Y=0|X=0) = 0.4$

$\rightarrow E[Y|X=0] = 0.6$

$\rightarrow Var[Y|X=0] = (0.4)^2 \cdot 0.6 + (-0.6)^2 \cdot 0.4 = 0.096 + 0.144 = 0.24$

$p(Y=1|X=1) = 0.4$, $p(Y=0|X=1) = 0.6$

$\rightarrow E(Y|X=1) = 0.4$

$\rightarrow Var[Y|X=1] = (0.6)^2 \cdot 0.4 + (-0.4)^2 \cdot 0.6 = 0.144 + 0.096 = 0.24$

$p(Y=1) = p(Y=1, X=1) + p(Y=1, X=0) = 0.5$

$p(Y=0) = p(Y=0, X=1) + p(Y=0, X=0) = 0.5$

$\rightarrow E(Y) = 0.5$

$\rightarrow Var(Y) = (0.5)^2 \cdot 0.5 + (-0.5)^2 \, 0.5 = 0.125 + 0.125 = 0.25$

$\rightarrow$ $Var(Y|X=0) = Var(Y|X=1) \leq Var(Y)$ why?

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d) $p(X=0|Y=1) = ?$

$$p(X=0|Y=1) = \frac{p(X=0, Y=1)}{p(Y=1)} = \frac{p(Y=1|X=0) \cdot p(X=0)}{p(Y=1, X=0) + p(Y=1, X=1)} = \frac{0.6 \; 0.5}{0.5} = 0.6$$

조건부확률 정의

Bayes's Rule

2.6 Conditional independence: Suppose events A and B are conditionally independent given C, which is written $A \perp B | C$. Show that this implies that $A^c \perp B | C$, $A \perp B^c | C$, and $A^c \perp B^c | C$, where A^c means "not A." Find an example where $A \perp B | C$ holds but $A \perp B | C^c$ does not hold.

$A \perp B | C \Leftrightarrow P(A|C) = P(A|C \cap B)$ ②, $P(A \cap B | C) = P(A|C) \cdot P(B|C)$ ①

Theorem of Cond. Ind

$P(A^c|C) = 1 - P(A|C)$ ($\because P(A|C) + P(A^c|C) = 1$)

① ? $P(A^c|C) \cdot P(B|C) = \{1 - P(A|C)\} \cdot P(B|C) = P(B|C) - P(A|C) \cdot P(B|C)$

$= P(B|C) - P(A \cap B | C)$

$= P(A^c \cap B | C)$ ($\because P(B|C) = P(A \cap B|C) + P(A^c \cap B|C)$)

② $P(A|C) = P(A|C \cap B)$, $1 - P(A|C) = 1 - P(A|C \cap B)$

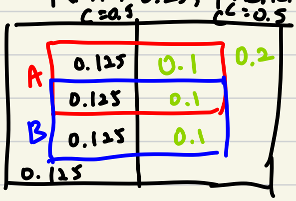
$\to P(A^c|C) = P(A^c|C \cap B) \Leftrightarrow A^c \perp B | C$

$A \perp B | C \not\Leftrightarrow A \perp B | C^c$

↳ $P(A|C) \cdot P(B|C) = P(A \cap B | C)$

let $P(A|C) = 0.5$, $P(B|C) = 0.5$, $P(A \cap B|C) = 0.25$, $P(C) = 0.5$ } satisfies $A \perp B | C$!

$\to P(A \cap C) = 0.25$, $P(B \cap C) = 0.25$, $P(A \cap B \cap C) = 0.125$



let $P(A \cap C^c) = 0.2$, $P(B \cap C^c) = 0.2$, $P(A \cap B \cap C^c) = 0.1$

$\to P(A|C^c) = 0.4$, $P(B|C^c) = 0.4$, $P(A \cap B|C^c) = 0.2$

$\to P(A|C^c) \cdot P(B|C^c) = 0.16 \neq P(A \cap B | C^c) = 0.2$

$\not\Leftrightarrow A \perp B | C^c$

2\. i) let $C=0$ $\rightarrow$ easy to say it's exchangeable ($\because$ Sampling with Replacement $\rightarrow X_i \sim$ i.i.d

ii) let $C=1$ $\rightarrow$ $p(X_1=1) = \frac{R_0}{R_0+W_0}$, $p(X_1=0) = \frac{W_0}{R_0+W_0}$

$$p(X_2=1) = \frac{R_0+1}{R_0+W_0+1}\cdot p(X_1=1) + \frac{R_0}{R_0+W_0+1}\cdot p(X_1=0) = \frac{2R_0W_0+R_0^2}{(R_0+W_0+1)(R_0+W_0)}$$

$$p(X_2=0) = \frac{W_0}{R_0+W_0+1}\cdot p(X_1=1) + \frac{W_0+1}{R_0+W_0+1}\cdot p(X_1=0) = \frac{2R_0W_0+W_0^2}{(R_0+W_0+1)(R_0+W_0)}$$

$$p(X_3=0) = \frac{R_0+2}{R_0+W_0+2}\cdot p(X_1+X_2=2) + \frac{R_0+1}{R_0+W_0+2}\cdot p(X_1+X_2=1) + \frac{R_0}{R_0+W_0+2}\cdot p(X_1+X_2=0)$$

$$p(X_1=1,\ X_2=1,\ X_3=1,\ X_4=0) = \frac{R_0}{R_0+W_0}\cdot\frac{R_0+1}{R_0+W_0+1}\cdot\frac{R_0+2}{R_0+W_0+2}\cdot\frac{W_0}{R_0+W_0+3} = \frac{R_0(R_0+1)(R_0+2)W_0}{(\qquad)}$$

$$p(X_1=1,\ X_2=1,\ X_3=0,\ X_4=1) = \frac{R_0}{R_0+W_0}\cdot\frac{R_0+1}{R_0+W_0+1}\cdot\frac{W_0}{R_0+W_0+2}\cdot\frac{R_0+2}{R_0+W_0+3} = \text{''}$$

$p(X_1=1,\ X_2=0,\ X_3=1,\ X_4=1) =$ '' $=$ ''

$p(X_1=0,\ X_2=1,\ X_3=1,\ X_4=1) =$ '' $=$ ''

$\rightarrow$ iii) let $C \geq 1$, fixed int.

then for any $X_1, X_2, X_3, \cdots, X_n$ that satisfies $\sum_{i=1}^{n} X_i = k$,

$$p(X_1, X_2, X_3, \cdots, X_n) = \frac{\left[R_0\cdot(R_0+c)\cdot\cdots\cdot(R_0+(k-1)\cdot c)\right]\cdot\left[W_0\cdot(W_0+c)\cdot\cdots\cdot(W_0+(n-k-1)\cdot c)\right]}{(R_0+W_0)\cdot(R_0+W_0+c)\cdot\cdots\cdots(R_0+W_0+(n-1)\cdot c)}$$

$\rightarrow$ Regardless