STA3105-01 Bayesian Statistics
Homework 6
DUE Friday, December 2

Copying homework solutions from others lead to a 0 score. No late submission is allowed. Your solution should contain both code and a corresponding explanation for the answer. Submit your HW through LearnUs. You should submit (1) a report file (pdf) and (2) relevant code files. We will implement the MCMC algorithm for a Bayesian regression as follows:

$$\mathbf{Y} \sim \text{Bernoulli}(\mu(\mathbf{X}))$$

$$\log\left(\frac{\mu(\mathbf{X})}{1-\mu(\mathbf{X})}\right) = \mathbf{X}\beta$$

$$\beta_j \overset{\text{ind}}{\sim} \lambda_j N(0,\sigma_1^2) + (1-\lambda_j)N(0,\sigma_2^2) \text{ for } j = 1,\cdots,10$$

$$\lambda_j \overset{\text{ind}}{\sim} \text{Bernoulli}(1/2)$$

$$\sigma_1^2 \sim \text{IG}(1,20), \quad \sigma_2^2 \sim \text{G}(1,20)$$

The notations follow our lecture notes.

1. (10 points) Simulate the dataset as follows.
   (a) Let $\mathbf{X}_i \in \mathbb{R}^{10}$ be the predictors for ith observation. For $i = 1,\cdots,10,000$, simulate $\mathbf{X}_i \sim N(0,\mathbf{I})$ independently. Here $\mathbf{I}$ is an identity matrix.
   (b) For $i = 1,\cdots,10,000$, simulate $Y_i \sim \text{Bernoulli}(\mu(\mathbf{X}_i))$ independently. Set the true regression coefficient value as $\beta = (0.5,-0.5,1,-1,0.7,0,0,0,0,0)$.
2. (45 points) Implement the MCMC algorithm using the simulated dataset in Problem 1. Here, you should write down the code without using any packages (e.g., `nimble`, `adaptMCMC`). Report followings:
   - Trace plots, density plots, 95% HPD intervals, posterior mean, acceptance probability, and effective sample size for $\beta_1,\cdots,\beta_{10},\sigma_1^2,\sigma_2^2$.
   - Posterior inclusion probability for all β_j. Based on the posterior inclusion probabilities, discuss whether we have to include jth predictor or not. Compare these results with the simulated truth $\beta = (0.5,-0.5,1,-1,0.7,0,0,0,0,0)$.
3. (45 points) Implement the MCMC algorithm using the simulated dataset in Problem 1. Here, you should use `nimble` package to implement the algorithm. Report followings:
   - Trace plots, density plots, 95% HPD intervals, posterior mean, acceptance probability, and effective sample size for $\beta_1,\cdots,\beta_{10},\sigma_1^2,\sigma_2^2$.
   - Posterior inclusion probability for all β_j. Based on the posterior inclusion probabilities, discuss whether we have to include jth predictor or not. Compare these results with the simulated truth $\beta = (0.5,-0.5,1,-1,0.7,0,0,0,0,0)$.