STA3105-01 Bayesian Statistics
Homework 4
DUE Friday, November 4

Copying homework solutions from others lead to a 0 score. No late submission is allowed. Your solution should contain both code and a corresponding explanation for the answer. Submit your HW through LearnUs. You should submit (1) a report file (pdf) and (2) a relevant code file. In this assignment, we will implement MCMC algorithms for a Bayesian spatial hierarchical model as follows (notations follow our lecture slide):

- **Data layer**:
$$\boldsymbol{Y}|\boldsymbol{\eta}, \tau^2 \sim N(\boldsymbol{\eta}, \tau^2 \boldsymbol{I}_n)$$
- **Process layer**:
$$\boldsymbol{\eta}|\boldsymbol{\beta}, \sigma^2, \rho \sim N(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2 \Gamma(\rho))$$
- **Prior layer**:
$$\boldsymbol{\beta} \sim N(0, 1000\boldsymbol{I})$$
$$\sigma^2 \sim IG(0.001, 0.001), \tau^2 \sim IG(0.001, 0.001), \rho \sim G(2, 50)$$

Based on the above model, we will analyze `CAtemps.RData`.

1. (30 points) Based on the above structure, derive the conditional distributions of parameters. More specifically derive
   - $f(\boldsymbol{\eta}|\boldsymbol{\beta}, \sigma^2, \tau^2, \rho, \boldsymbol{Y})$
   - $f(\boldsymbol{\beta}|\boldsymbol{\eta}, \sigma^2, \tau^2, \rho, \boldsymbol{Y})$
   - $f(\sigma^2|\boldsymbol{\eta}, \boldsymbol{\beta}, \tau^2, \rho, \boldsymbol{Y})$
   - $f(\tau^2|\boldsymbol{\eta}, \boldsymbol{\beta}, \sigma^2, \rho, \boldsymbol{Y})$
   - $f(\rho|\boldsymbol{\eta}, \boldsymbol{\beta}, \sigma^2, \tau^2, \boldsymbol{Y})$

   Note that you have a closed form the conditional distributions except for $f(\rho|\boldsymbol{\eta}, \boldsymbol{\beta}, \sigma^2, \tau^2, \boldsymbol{Y})$. For $f(\rho|\boldsymbol{\eta}, \boldsymbol{\beta}, \sigma^2, \tau^2, \boldsymbol{Y})$, just write down the kernel of the distribution.

2. (40 points) In `HW4.txt`, I provide a skeleton code for implementing an MCMC algorithm for the above model. Complete the code and run the MCMC algorithm for 1,000 iterations. Report the effective sample size, trace plot, ACF plot, posterior mean, and 95% HPD intervals for all components. Since the dimension of $\boldsymbol{\eta}$ is 200, just report the results for the first component of the vector (i.e., `eta.obs.samps[1,]`).

3. (30 points) In `HW4.txt`, I provide a skeleton code for implementing Bayesian kriging. Complete the code. Draw the posterior mean surface of $\boldsymbol{\eta}$ (at unobserved locations)

over the California map. Furthermore, draw the posterior standard deviation surface of $\boldsymbol{\eta}$ (at unobserved locations) over the California map. You can use `ploteqc` function to draw these maps.