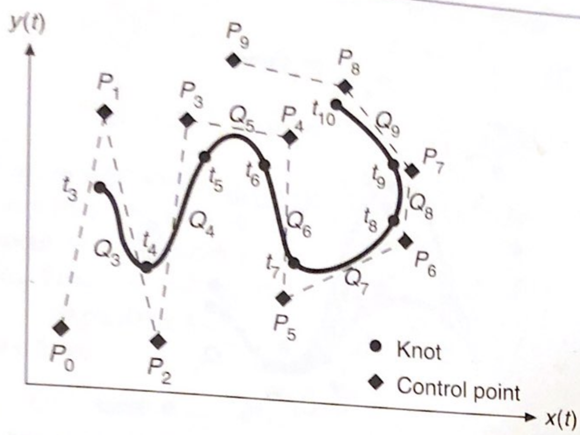

Figure 9.18 A B-spline with curve segments Q_3 through Q_9. This figure and many others in this chapter were created with a program written by Carles Castellsaquè.

and final points at t_3 and t_{m+1} are also called knots, so there are a total of $m - 1$ knots. Figure 9.18 shows a 2D B-spline curve with its knots marked. A closed B-spline curve is easy to create: The control points P_0, P_1, P_2 are repeated at the end of the sequence—$P_0, P_1, \ldots, P_m, P_0, P_1, P_2$.

The term **uniform** means that the knots are spaced at equal intervals of the parameter t. Without loss of generality, we can assume that $t_3 = 0$, and the interval $t_{i+1} - t_i = 1$. Nonuniform nonrational B-splines, which permit unequal spacing between the knots, are discussed in Section 9.2.5. (In fact, the concept of knots is introduced in this section to set the stage for nonuniform splines.) The term **nonrational** is used to distinguish these splines from rational cubic polynomial curves, discussed in Section 9.2.6, where $x(t)$, $y(t)$, and $z(t)$ are each defined as the ratio of two cubic polynomials. The "B" stands for basis, since the splines can be represented as weighted sums of polynomial basis functions, in contrast to the natural splines, for which the weighted-sum property is not true.

Each of the $m - 2$ curve segments of a B-spline curve is defined by four of the $m + 1$ control points. In particular, curve segment Q_i is defined by points P_{i-3}, P_{i-2}, P_{i-1}, and P_i. Thus, the **B-spline geometry matrix** G_{Bs_i} for segment Q_i is

$$G_{Bs_i} = [P_{i-3} \;\; P_{i-2} \;\; P_{i-1} \;\; P_i], \quad 3 \le i \le m. \tag{9.31}$$

The first curve segment, Q_3, is defined by the points P_0 through P_3 over the parameter range $t_3 = 0$ to $t_4 = 1$, Q_4 is defined by the points P_1 through P_4 over the parameter range $t_4 = 1$ to $t_5 = 2$, and the last curve segment, Q_m, is defined by the points P_{m-3}, P_{m-2}, P_{m-1}, and P_m over the parameter range $t_m = m - 3$ to $t_{m+1} = m - 2$. In general, curve segment Q_i begins somewhere near point P_{i-2} and ends somewhere near point P_{i-1}. We shall see that the B-spline blending functions are ... sum to unity, so the curve segment Q_i is constrained

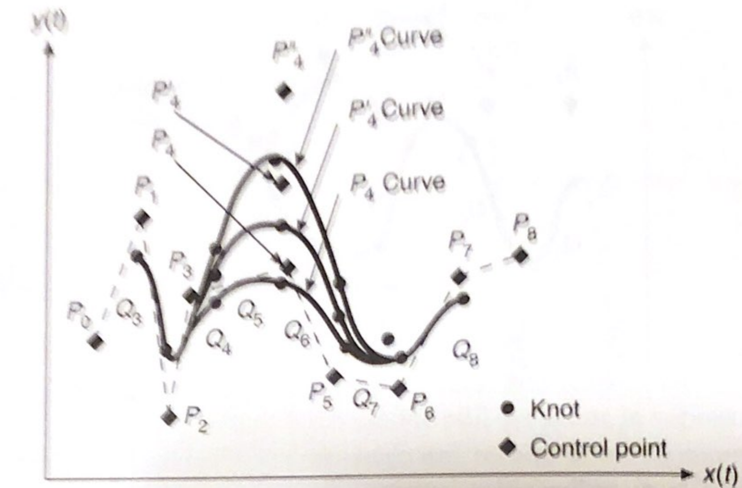

Figure 9.19 A B-spline with control point P_4 in several different locations.

Just as each curve segment is defined by four control points, each control point (except for those at the beginning and end of the sequence $P_0, P_1, \ldots, P_m$) influences four curve segments. Moving a control point in a given direction moves the four curve segments it affects in the same direction; the other curve segments are totally unaffected (see Fig. 9.19). This behavior is the local control property of B-splines and of all the other splines discussed in this chapter.

If we define T_i as the column vector $[(t-t_i)^3 \quad (t-t_i)^2 \quad (t-t_i) \quad 1]^T$, then the B-spline formulation for curve segment i is

$$Q_i(t) = G_{\mathrm{Bs}_i} \cdot M_{\mathrm{Bs}} \cdot T_i, \quad t_i \le t < t_{i+1}. \tag{9.32}$$

We generate the entire curve by applying Eq. (9.32) for $3 \le i \le m$.

The **B-spline basis matrix,** M_{Bs}, relates the geometrical constraints G_{Bs} to the blending functions and the polynomial coefficients:

$$M_{\mathrm{Bs}} = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}. \tag{9.33}$$

This matrix is derived in [BART87].

The B-spline blending functions B_{Bs} are given by the product $M_{\mathrm{Bs}} \cdot T_i$, analogously to the previous Bézier and Hermite formulations. Note that the blending functions for each curve segment are exactly the same, because, for each segment i, the values of $t - t_i$ range from 0 at

$$B_{Bs} = M_{Bs} \cdot T = [B_{Bs_{-3}} \quad B_{Bs_{-2}} \quad B_{Bs_{-1}} \quad B_{Bs_0}]^{T}$$
$$= \frac{1}{6}[-t^3 + 3t^2 - 3t + 1 \quad 3t^3 - 6t^2 + 4 \quad -3t^3 + 3t^2 + 3t + 1 \quad t^3]^{T}$$
$$= \frac{1}{6}[(1 - t)^3 \quad 3t^3 - 6t^2 + 4 \quad -3t^3 + 3t^2 + 3t + 1 \quad t^3]^{T}, \quad 0 \le t < 1. \tag{9.34}$$

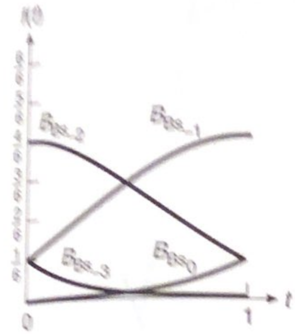

Figure 9.20
The four B-spline blending functions from Eq. (9.34). At $t = 0$ and $t = 1$, just three of the functions are nonzero.

Figure 9.20 shows the B-spline blending functions B_{Bs}. Because the four functions sum to 1 and are nonnegative, the convex-hull property holds for each curve segment of a B-spline. See [BART87] to understand the relation between these blending functions and the Bernstein polynomial basis functions.

Expanding Eq. (9.32), again replacing $t - t_i$ with t at the second equal-to sign, we have

$$Q_i(t - t_i) = G_{Bs_i} \cdot M_{Bs} \cdot T_i = G_{Bs_i} \cdot M_{Bs} \cdot T$$
$$= G_{Bs_i} \cdot B_{Bs} = P_{i-3} \cdot B_{Bs_{-3}} + P_{i-2} \cdot B_{Bs_{-2}} + P_{i-1} \cdot B_{Bs_{-1}} + P_i \cdot B_{Bs_0}$$
$$= \frac{(1-t)^3}{6} P_{i-3} + \frac{3t^3 - 6t^2 + 4}{6} P_{i-2} + \frac{-3t^3 + 3t^2 + 3t + 1}{6} P_{i-1}$$
$$+ \frac{t^3}{6} P_i, \quad 0 \le t < 1. \tag{9.35}$$

It is easy to show that Q_i and Q_{i+1} are C^0, C^1, and C^2 continuous where they join. The additional continuity afforded by B-splines is attractive, but it comes at the cost of less control of where the curve goes. We can force the curve to interpolate specific points by replicating control points; this is useful both at endpoints and at intermediate points on the curve. For instance, if $P_{i-2} = P_{i-1}$, the curve is pulled closer to this point because curve segment Q_i is defined by just three different points, and the point $P_{i-2} = P_{i-1}$ is weighted twice in Eq. (9.35)—once by $B_{Bs_{-2}}$ and once by $B_{Bs_{-1}}$.

If a control point is used three times—for instance, if $P_{i-2} = P_{i-1} = P_i$—then Eq. (9.35) becomes

$$Q_i(t) = P_{i-3} \cdot B_{Bs_{-3}} + P_i \cdot (B_{Bs_{-2}} + B_{Bs_{-1}} + B_{Bs_0}). \tag{9.36}$$

Q_i is clearly a straight line. Furthermore, the point P_{i-2} is interpolated by the line at $t = 1$, where the three weights applied to P_i sum to 1, but P_{i-3} is not in general interpolated at $t = 0$. Another way to think of this behavior is that the convex hull for Q_i is now defined by just two distinct points, so Q_i has to be a line. Figure 9.21 shows the effect of multiple control points at the interior of a B-spline.

Another technique for interpolating endpoints, **phantom vertices**, is discussed in [BARS83; BART87]. We shall see in the next section that, with nonuniform B-splines, endpoints and internal points can be interpolated in a more natural way than they can with the uniform B-splines.

9.2.5 Nonuniform, Nonrational B-Splines

Nonuniform, nonrational B-splines differ from the uniform, nonrational B-splines discussed in Section 9.2.4 in that the parameter interval between successive