# какой то семинар

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$$H = L_2\left([0{,}1] \times [0{,}1]\right).$$

$$A = \frac{\partial^2}{\partial x^2} + i\frac{\partial}{\partial y}.$$

$$D(A) = \left\{u \in H : u'_x\mid_{x=0} = u\mid_{x=1} = 0,\ u\mid_{y=0} = -u\mid_{y=1}\right\}.$$

**Задача 1.** 1. Д-ть, что $A$ – сим.

2. Найти в $H$ ортогон. базис из с. ф. $A$
3. Найти обл. опр. и спектр. разл. $\overline{A}$
4. Решение задачи

$$\frac{d}{dt}u(t) + \left(\overline{A}\right)^2 u = e^{it}, \quad t > 0, \quad u(t) \in D\left(\overline{A}^2\right)$$

$$u(+0) = 0.$$

*Решение.* 1. $u, v \in D(A)$

$$(Au, v) \stackrel{?}{=} (u, Av).$$

$$(Av, u) = \int\limits_0^1\int\limits_0^1 (u_{xx} + iu_y)\overline{v}dxdy = \int\limits_0^1 dy\int\limits_0^1 u_{xx}\overline{v}dx + i\int\limits_0^1 dx\int\limits_0^1 u_y\overline{v}dy =$$

$$= \int\limits_0^1 dy\left(\underbrace{u_x\overline{v}\mid_0^1}_{u_x|_{x=1}\cdot\underbrace{\overline{v}\mid_{x=1}}_{=0} - \underbrace{u_x\mid_{x=0}\cdot\overline{v}\mid_{x=0}}}\right)\dots$$

(a) $L_2\left([0{,}1]\times[0{,}1]\right) = L_2[0{,}1] \otimes L_2[0{,}1]$, $\otimes$ — это $\cdot$

(b) $H_1 \otimes H_2$, $\{f_n\}$ — ортог. базис в $H_1$, $\{g_n\}$ — ортог. базис в $H_2$ $\Longrightarrow$ $\{f_n \otimes g_m\}$ — ортог. базис в $H_1 \otimes H_2$

2. (a) $f_(x)$ т.ч. $f'' = \lambda f$, $f'(0) = 0, f(1) = 0$

i. $\lambda > 0$

$$f = A\operatorname{sh}\sqrt{\lambda}x + B\operatorname{ch}\sqrt{\lambda}x$$

$$\left.\begin{aligned} f'(0) = A\sqrt{\lambda} = 0 \to A = 0 \\ f(1) = B\operatorname{ch}\sqrt{\lambda} \to B = 0 \end{aligned}\right. \to \varnothing.$$

ii. $\lambda = 0$

$$f = \underbrace{Ax}_{=0} + B.$$

$$f'(0) = A = 0 f(1) = B = 0 \implies \varnothing.$$

iii. $\lambda < 0$

$$f = A\underbrace{\sin\sqrt{-\lambda}x}_{=0} + B\cos\sqrt{-\lambda}x.$$

$$f'(0) = A\sqrt{-\lambda} = 0, A = 0.$$

$$f(1) = B\cos\sqrt{-\lambda} = 0 \implies \cos\sqrt{-\lambda} = 0 \implies \sqrt{-\lambda} = \frac{\pi}{2} + \pi n.$$

$$\lambda_n = -\left(\frac{\pi}{2} + \pi n\right)^2.$$

$$f_n = \cos\left(\frac{\pi}{2} + \pi n\right)x.$$

(b) $g(y)$ т.ч. $ig' = \mu g$, $g(0) = -g(1)$

$$g' = -i\mu g.$$

$$g = Ae^{-i\mu g}.$$

$$g(0) = A = -g(1) = -Ae^{-i\mu}.$$

$$1 = -e^{i\mu}.$$

$$e^{-i\mu} = -1 = e^{i\pi + 2\pi m i}.$$

$$-i\mu = i\pi + 2\pi m i.$$

$$\mu_m = -\pi - 2\pi m.$$

$$g_m = e^{(\pi + 2\pi m)iy}.$$

$$e_{n,\,m} = \cos\left(\frac{\pi}{2} + \pi n\right)x \cdot e^{(\pi + 2\pi m)iy}.$$

$$\lambda_{n,\,m} = -\left(\frac{\pi}{2} + \pi n\right)^2 - (\pi + 2\pi m).$$

$$\sqrt{-\lambda_n} = \frac{\pi}{2} + \pi n.$$

3.

$$\overline{A} = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{+\infty} \lambda_{n,\,m}\frac{(u, e_{n,\,m})}{(e_{n,\,m}, e_{n,\,m}}e_{n,\,m} \leftarrow \text{спектр. разл.}$$

$$D\left(\overline{A}\right) = \left\{u \in L_2\left([0,1]\times[0,1]\right) : \sum_{n=0}^{\infty}\sum_{m=-\infty}^{+\infty}\lambda_{n,\,m}^2\frac{|(u, e_{n,\,m})|^2}{\|e_{n,\,m}\|^2} < \infty\right\}.$$

4.

$$u = \sum_{n,\,m}^{\infty} T_{n,\,m}(t) e_{n,\,m};\ \overline{A}^2 u = \sum_{n,\,m} \lambda_{n,\,m}^2 T_{n,\,m} e_{n,\,m}.$$

$$1 = \sum_{n,\,m} \alpha_{n,\,m} e_{n,\,m},\ \alpha_{n,\,m} = \frac{(1, e_{n,\,m})}{(e_{n,\,m}, e_{n,\,m})} =$$

$$= \frac{\int\limits_0^1 \int\limits_0^1 \cos\left(\frac{\pi}{2} + \pi n\right) x \cdot e^{-i(\pi+2\pi m)y} dx dy}{\int\limits_0^1 \cos^2\left(\frac{\pi}{2} + \pi n\right) x dx \int\limits_0^1 e^{i(\pi+2m\pi)y} e^{-i(\pi+2\pi m)y} dy} =$$

$$= \sum_{n,\,m} T'_{n,\,m} e_{n,\,m} + \sum_{n,\,m} \lambda_{n,\,m}^2 T_{n,\,m} e_{n,\,m} = \sum_{n,\,m} \alpha_{n,\,m} e_{n,\,m} \cdot e^{it}.$$

$$\begin{cases} T'_{n,\,m} + \lambda_{n,\,m}^2 T_{n,\,m} = \alpha_{n,\,m} e^{it} \\ T_{n,\,m}(0) = 0 \end{cases}$$

$$T_{n,\,m} = C_{n,\,m} e^{-\lambda_{n,\,m}^2 t} + \beta_{n,\,m} e^{it}.$$

$$\beta_{n,\,m} \ldots$$

$$u(t, x, y) = \sum_{n,\,m} \frac{\alpha_{n,\,m}}{i + \lambda_{n,\,m}^2} \left[ -e^{-\lambda_{n,\,m}^2 t} + e^{it} \right] e_{n,\,m}.$$

*Проверки*

$$u \in L_2[0,1] \times [0,1]).$$

$$|u|^2 = \sum_{n,\,m} \frac{|\alpha_{n,\,m}|^2}{|i + \lambda_{n,\,m}^2|^2} \left| -e^{-\lambda_{n,\,m}^2 t} + e^{it} \right|^2 \|e_{n,\,m}\|^2 \leqslant \sum_{n,\,m} \frac{|\alpha_{n,\,m}|^2}{1 + \lambda_{,\,m}^4} \cdot 4 \|e_{n,\,m}\| <$$

$$< \ldots < +\infty.$$