Fractons, Symmetric Gauge Fields and Geometry

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Gapless fracton phases are chracterized by the conservation of certain charges and their higher moments. These charges generically couple to higher rank gauge fields. In this work we study systems conserving charge and dipole moment, and construct the corresponding gauge fields propagating in arbitrary curved backgrounds. The relation between the symmetries of these class of systems and space-time transformations is discussed. In fact, we argue that higher rank symmetric gauge theories are closer to gravitational fields than to a standard gauge theory.

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Symmetries are a fundamental paradigm to organise the degrees of freedom of a system in modern physics. Typically, the symmetries of a physical system can be divided in terms of internal, and space-time symmetries. For instance, electric charge and isospin conservation are consequence of the former, whereas conservation of momentum and angular momentum of the later. Understanding the distinction between the two classes of transformations is vital in order two characterize the force fields associated to interacting charged matter. At the fundamental level, matter charged under internal symmetries interact via gauge fields, whereas gravitational fields carry the 'force' between fields charged respect to space-time symmetries, e.g. energy (mass), and/or momentum. Although physicist have made several attempts to describe gravity as a gauge theory, and a successful algorithm to gauge (relativistic or not) space-time symmetries has been developed [1–4], strictly speaking gravitational fields are not gauge fields.

On the other hand, in recent years, a new class of matter excitations has been proposed named as fractons [5, 6]. The main feature a quasiparticle needs to show to be called fractonic is the property of reduced mobility. In fact, fracton matter can be classified in terms of gapless [7–12] and gapped [13–17] phases. In particular, gapless fracton phases are described by the conservation of certain charges and their higher moments. This peculiar behaviour has as consequence a non-standard continuity equation (see Eq. (3) for the case of charge and dipole conservation, and [18, 19] for more general examples). This class of symmetries naturally arise in the context of spin liquids [10–12], quantum Hall [20], elasticity [21], topological defects [22], and beyond the condensed matter realm in systems with Galileons [23].

For simplicity, we shall focus on the case of systems preserving a scalar charge and its corresponding dipole moment. This symmetry has the form of some generalised $U(1)$ symmetry. Actually, is not hard to conclude that the interaction among such type of charges should be carried by generalized 'electromagnetic gauge fields' with the vector potential substituted by a symmetric tensor field [9, 18]. However, if we try to follow the standard minimal coupling rule to couple charged matter to these gauge fields we run into problems, and the only way out proposed so far is with a non-gaussian theory [19, 24], making analytic computations almost impossible. Nonetheless, certain progress has been made in the hydrodynamic description for such systems in absence of gauge fields [24–28].

Furthermore notice, that similarly to angular momentum, the value of a dipole moment depends on the location of the origin of the coordinates system, making hard to link its conservation to an internal symmetry group, indicating that charge and dipole conservation, should be related to a space-time symmetry group, rather than to a purely internal one. This observation was one of the main motivations for the study presented here. Also notice that the fractonic 'force' field is a symmetric tensor in similarity with the metric which is the responsible for gravitational forces. Actually, in [29] a connection between fractons dynamics and linearized gravity has been discussed. In this paper, we propose a geometric theory where the symmetric gauge fields play the role of vielbeins in a vertical space to the physical space-time. This construction pave the road to a systematic understanding of more generic multipole preserving gauge theories, and opens up a path for the construction of low energy, and possibly microscopic, fractonic matter actions.

Our first main result Eq. (12), is a non-linear action for the spontaneously broken fractonic symmetry. This action describes the embedding of the physical space-time in a Heisenberg space [30, 31]. The isometries of such space agree with global space-time translations, fractonic transformations, and rotations. Within this geometric picture the embedding coordinate is interpreted as the Nambu-Goldstone boson. The second main result Eq. (36) is the action for a fully diffeomorphism and gauge invariant theory. Actually, the construction suggests that symmetric gauge fields will generically become massive on a curved background, and to preserve the gauge invariance we need to introduce a Stueckelberg field. In particular, our theory reduces to the models proposed in [18] once the space-time is assumed to be flat.

The paper is structured as follows, in the next section we describe the conservation laws and symmetry algebra of a system conserving energy, mometum, angular momentum, and a scalar charge with its corresponding dipole moment. Next, we discuss the relation of the symmetry group with the so-called Heisenberg group and study the spontaneous symmetry breaking of fracton charges. In the following section, we gauge the full symmetry group obtaining a generalization of the fracton electrodynamic gauge theories on curved background manifolds. Then, we discuss the outputs of our proposal, possible implications, and outlooks.

CHARGE-DIPOLE CONSERVATION

Gapless fracton phases are characterized by the conservation of certain charges and their highest moments. The simplest case, corresponds with the conservation of a charge Q and its dipole Q^a, which in n space dimensions, at the macroscopic level, can be formulated in terms of the charge density ρ as

$$\frac{dQ}{dt} = \frac{d}{dt}\int d^n x\, \rho = 0\,, \tag{1}$$

$$\frac{dQ^a}{dt} = \frac{d}{dt}\int d^n x\, x^a \rho\ = 0\,. \tag{2}$$

In a system with such conservation law, charges are immobile, whereas dipoles can freely move. In fact, similarly to what happens with momentum and angular momentum[1], both charges are conserved once the single (generalized) continuity equation

$$\partial_t \rho + \partial_a \partial_b J^{ab} = 0\,, \qquad a, b = 1, 2, \ldots, n\,, \tag{3}$$

is satisfied. The distinguishing feature in these class of systems is that charge is relaxed via a tensorial current. An immediate consequence of such conservation law, is that a gauged version of the symmetry would require the presence of gauge fields ψ, A_{ab} with the transformation rule $\psi \to \psi - \partial_t \lambda$, and $A_{ab} \to A_{ab} + \partial_a \partial_b \lambda$, and the 'gauge fields' coupling to the fractonic matter as follows

$$S = S_0[\psi, A_{ab}] + \int\, d^{n+1}x \left(\rho\psi + J^{ab} A_{ab}\right)\,. \tag{4}$$

Such type of theories have been proposed as a generalization to electrodynamics [18]. However, due to the unusual transformation law of the fields, is not clear in what sense they are gauge theories. In addition, from this perspective, is not obvious whether is possible to put the theory on a curved manifold without spoiling the gauge symmetry [32].

In order to understand the tension between the space-time transformations and the gauge symmetry introduced above it is useful to notice that the dipole charge Q^a is charged under spatial translations. The main reason, is that its value will change once the origin of the space is shifted, contrary to what happens to the charge Q, which is insensitive to the location of the origin. This is an unusual property for internal symmetries. In fact, a careful analysis of the action of time and space translations, rotations, and the transformations generated by the fracton charges with generators H, P_a, L_{ab}, Q_a, Q respectively, imply that the whole set of transformations form a continuous Lie group $\mathcal{G}$ with its corresponding Lie algebra satisfying the non-vanishing Lie brackets[2]

$$\begin{aligned}
&[P_a, Q^b] = \delta_a^b Q\,, \\
&[P_a, L_{bc}] = \delta_{ac} P_b - \delta_{ab} P_c\,, \\
&[Q^a, L_{bc}] = \delta_c^a Q_b - \delta_b^a Q_c\,, \\
&[L_{ab}, L_{bc}] = \delta_{ac} L_{bd} - \delta_{ad} L_{bc} - \delta_{bc} L_{ad} + \delta_{bd} L_{ac}\,.
\end{aligned} \tag{5}$$

This algebra makes evident that conservation of charge and dipole are consequence of a space-time symmetry group. Contrary to the usual case of charges, which are conserved due to internal groups. A similar example is mass conservation in Galilean invariant theories [4]. Actually, Eqs. (5) show similarities with the Bargmann algebra once the generator of Galilean boosts is identified with the dipole generator Q^a, and mass with charge Q. See also [33, 34] for the similarities with Carroll theories.

GROUP MANIFOLD AND SPONTANEOUS SYMMETRY BREAKING

In this section we shall give a geometric interpretation to the fractonic symmetry in terms of Heisenberg spaces [30, 31]. To do so, we first start by defining a $n-$dimensional vector space $\mathcal{N}_n$ with elements $X = X^a \frac{\partial}{\partial y^a}$ and its dual space $\mathcal{N}_n^*$ with elements $\gamma = z_a dy^a$. Then, we define the cotangent bundle $\mathcal{N}_{2n} = \mathcal{N}_n \oplus \mathcal{N}_n^*$ with coordinates (y^a, z_a), and canonical simplectic form $\mathcal{J} = dy^a \wedge dz_a$. Since $\mathcal{J}$ is closed, we can define the simplectic gauge potential $\theta = -y^a dz_a + d\phi$, such that $\mathcal{J} = -d\theta$.

Using the gauge freedom in the definition of θ, we extend the space $\mathcal{N}_{2n} \to \mathcal{N}_{2n+1}$ using ϕ as the vertical coordinate, and define the non-commutative sum

$$\xi \oplus X = X + \mathcal{S}[\xi, \cdot]\,, \tag{6}$$

with $\mathcal{S} = dy^a \otimes \frac{\partial}{\partial y^a} + dz_a \otimes \frac{\partial}{\partial z_a} + \theta \otimes \frac{\partial}{\partial \phi}$. In particular, given a vector field $\xi = \zeta^a \frac{\partial}{\partial y^a} + \beta_a \frac{\partial}{\partial z_a} + \lambda \frac{\partial}{\partial \phi}$ the sum Eq.

[1] In a system with both momentum and angular momentum conservation once momentum is conserved, conservation of angular momentum follows.

[2] A possible representation of the group on a scalar field is the following $\lambda Q[\phi] = c$, $\beta_a Q^a[\phi] = \beta_a x^a$, $\zeta^a P_a[\phi] = \zeta^a \partial_a \phi$, and $\beta^{ab} L_{ab}[\phi] = \beta^{ab} x^a \partial_b \phi$.

(6) induces the non-commutative translational symmetry $g(\zeta^a, \beta_a, \lambda) : (y^a, z_a, \phi) \to (y^a+\zeta^a, z_a+\beta_a, \phi+\lambda-\beta_a y^a)$. Given the action of such symmetry group, we can define the invariant basis ($e^a = dy^a, \omega_a = dz_a, v = d\phi + z_a dy^a$) with dual (invariant) vector fields

$$\mathcal{E}_a = \frac{\partial}{\partial y^a} - z_a \frac{\partial}{\partial \phi}\,, \quad \bar{\mathcal{E}}^a = \frac{\partial}{\partial z_a}\,, \quad V = \frac{\partial}{\partial \phi}\,, \tag{7}$$

The non-vanishing Lie brackets between the basis elements satisfy $[\mathcal{E}_a, \bar{\mathcal{E}}^b] = \delta^b_a V = \mathcal{J}[\mathcal{E}_a, \bar{\mathcal{E}}^b]V$. In addition, it is convenient to define the invariant metric $\bar{G} = e^a e^a + \omega_a \omega_a + v^2$. Notice, that the Lie algebra between the basis vectors agrees with the brackets between momentum and dipole charge in Eqs. (5), which allow us to identify this space (at least locally) with the charge-dipole-momentum symmetry group, after the identification $\mathcal{E}_a \to P_a$, $\bar{\mathcal{E}}^a \to Q^a$, and $V \to Q$.

On the other hand, an element Ω of the coset group $\mathcal{G}/SO(n)$ can be expanded as

$$\Omega = e^{y^0 H + y^a P_a} e^{z_a Q^a} e^{\phi Q} \tag{8}$$

and the action of $g = e^{\zeta^0 H + \zeta^a P_a} e^{\beta_a Q^a} e^{\lambda Q}$ on Ω produces the new element

$$\Omega' = e^{(y^0+\zeta^0)H + (y^a+\zeta^a)P_a} e^{(z_a+\beta_a)Q^a} e^{(\phi+\lambda-\beta_a y^a)Q}\,, \tag{9}$$

in complete agreement with the structure of the space constructed above, once time is included as an extra coordinate. Therefore, we interpret such quotient space $T_x\mathcal{M}_{n+1} = \mathcal{G}/SO(n)$ as the tangent space at the point x^μ of the physical manifold[3] $\mathcal{M}_{n+1}$. This can be seen as a non-trivial embedding with coordinates coordinates $y^0(x^\mu)$, $y^a(x^\mu)$, $z_a(x^\mu)$, and $\phi(x^\mu)$. Within this picture the transverse coordinates are the Nambu-Goldstone bosons of the system. On the other hand, identifying the points $(y^0, y^a, z_a, \phi) \sim (y^0, y^a, z_a+\beta_a, \phi+\lambda-\beta_a y^a)$ would correspond with the symmetric phase.

With such picture in mind, we introduce first the space-time vielbeins $e^I = (\tau, e^a)$, and their corresponding inverse vielbeins $E_I = (t, E_a)$[4]. They satisfy $E^\mu{}_I e_\mu{}^J = \delta^J_I$, and $E^\mu{}_I e_\nu{}^I = \delta^\mu_\nu$. Actually, without lost of generality we can pick coordinates $y^0 = \delta^0_\mu x^\mu$, $y^a = \delta^a_\mu x^\mu$. The spatial metric $h = e^a e^a$ is not invertible, therefore, we dote the full space-time with the non-degenerate metric $G = p\tau^2 + h$, with p a function that will be conveniently fixed below. In fact, p assigns t a norm. Translational invariance, fixes the vielbeins to be $e^I = (dx^0, dx^a)$, and set p as a real constant.

However, notice that $\mathcal{J}[\partial_a, \partial_b] = 0$ implies the embedding of v on the physical space should have the form $v \equiv v_0\tau = \partial_0\phi\tau$, in order for this relation to be satisfied the embedding field z_a necessarily has to be $z_a = -\partial_a\phi$. The previous geometric constraint implies the embedding of the field ω_a takes the form

$$\omega_a \equiv i_t\omega_a\tau + A_{ab}e^b = -\partial_0\partial_a\phi\tau - \partial_a\partial_b\phi e^b\,. \tag{10}$$

From a different view point, we can notice that a deformation of the embedding coordinates of the form $\delta\phi = \gamma_a x^a$ can be undone by a constant shift $-\gamma_a$ in the z_a directions, justifying previous constraint. Actually, this reduction in the number of Nambu-Goldstones is generically common when space-time symmetries are spontaneously broken, and it is known as inverse Higgs constraint [35–37]. Thus, an invariant action must be a generic functional of $\partial_0\phi$, and $(\partial_a\partial_b\phi)^2$.

It is interesting to notice that via the embeding in the Heisenberg space, the tangent space has a natural induced metric $\bar{G} = p(dx^0)^2 + \delta_{ab}dx^a dx^b + (v_\mu dx^\mu)^2 + (\omega_{\mu a}dx^\mu)^2$, which differ from the space-time metric G. That suggest, the dynamic of the fracton system under consideration could be related to bi-gravity theories. Then, for convenience we introduce the relative metric

$$B_{\mu\nu}(x) = v_\mu(x)v_\nu(x) + \omega_{\mu a}(x)\omega_{\nu a}(x)\,. \tag{11}$$

This field carries the information on how the Heiseberg space is projected down to the physical space-time. Therefore, we find natural to request a geodesic embedding, and tempted to propose the following Born-Infeld action

$$S_{SSB} = -\alpha \int d^{n+1}x \sqrt{|\det\,(G_{\mu\nu} + B_{\mu\nu})\,|} \tag{12}$$

as a non-linear candidate for the spontaneously broken fractonic system considered in this paper. If we assume gradients are small, and introduce the derivative expansion $\partial_0 \sim \nabla^2$ the action reads

$$S_{SSB} \approx -\alpha \int \left(1 + \frac{p}{2}(\partial_0\phi)^2 + \frac{1}{2}(\partial_a\partial_b\phi)^2 + \dots\right)\,, \tag{13}$$

notice that setting $p = -1$, will guarantee a possitive definite energy for the linearized theory. In next section, we will see that such condition will also give the right signs in the action for the symmetric gauge fields.

From now on, we will call $G_{\mu\nu}$ the space-time metric and $B_{\mu\nu}$ the fracton metric, and will generalise this construction for the more general case of curved space-times, and local fracton symmetry.

THE GAUGE THEORY

Previous formulation naturally allows us to gauge the fractonic symmetry in full analogy with gravitational the-

[3] Notice that we are considering the time and space coordinates $(x^0, \mathbf{x})$ collectively as x^μ, therefore, $\mu = 0, 1, \dots, n$

[4] Do not confuse the time coordinate x^0 with the vector field t.

ories[5]. This procedure should provide a consistent field theory for fractonic gauge fields in curved backgrounds. In order to do so, we will follow a standard recipe for the gauging of space-time symmetries. In particular, we will follow the method described in [1]. Using this technique the space-time coordinates are interpret as the Stueckelberg fields associated to local translations breaking. Nonetheless, in our system we have embedded the space-time in a larger space, therefore, in full analogy we could expect that local fracton translations might be broken with Stueckelberg fields $(z_a(x), \phi(x))$. In such regime the connection must be defined as $\mathcal{A} = \Omega^{-1}(d + \tilde{\mathcal{A}})\Omega$, with Ω defined in Eq. (8), and $\tilde{\mathcal{A}}$ being the gauge field. Since the components of $\tilde{\mathcal{A}}$ along the algebra directions are independent, we find convenient to parameterize it as

$$\tilde{\mathcal{A}} = \tilde{\tau} H + \tilde{e}^a P_a + \tilde{\omega}_a Q^a + (\tilde{v} + x^a \tilde{\omega}_a) Q + \frac{1}{2}\tilde{\omega}^{ab} L_{ab} \,. \tag{14}$$

By construction $\mathcal{A}$ will transform as a gauge connection respect to the unbroken generators, and reads

$$\mathcal{A} = \tau H + e^a P_a + \omega_a Q^a + vQ + \frac{1}{2}\omega^{ab} L_{ab} \,, \tag{15}$$

with $\omega^{ab} = \tilde{\omega}^{ab}$, $\tau = dx^0 + \tilde{\tau}$, $e^a = Dx^a + \tilde{e}^a$, $\omega_a = Dz_a + \tilde{\omega}_a$, and $v = d\phi + z_a e^a + \tilde{v}$. Notice that the gauge fields associated to local space-time and internal translations[6] redefine the definition of the (embedded) basis $(\tau(x), e^a(x), \omega_a(x), v(x))$, and $\omega^{ab}(x)$ is the spin connection. Also notice, that fields with tangent space indices a, b, c transform as vectors (tensors) respect to $SO(n)$ transformations (see Eqs. (5)). Therefore, the covariant exterior derivative is defined as $Ds^a = ds^a - \omega^{ab} \wedge s^b$. All this implies that (τ, v) and (e^a, ω_a) are invariant under translations, and transform as scalars and vectors respect to local rotations respectively.

Inspire by the results of previous section, we define the space-time metric $G_{\mu\nu} = e_\mu{}^a e_\nu{}^a - \tau_\mu \tau_\nu$ and inverse vielbeins t^μ, $E^\mu{}_a$. Using the internal vielbeins we also define the fracton metric $B_{\mu\nu} = v_\mu v_\nu + \omega_{\mu a}\omega_{\nu a}$. In particular, the field v not only can be interpreted as one of the internal vielbeins, but also as the simplectic gauge potential satisfying $\mathcal{J} = -dv$.

Under an infinitesimal internal translation generated with the group element $g(x) = 1 + \beta_a(x) Q^a + (\lambda(x) + x^a \beta_a(x))Q$, it can be proven the Stueckelberg and gauge fields transform as

$$\delta\phi = \lambda \,, \quad \delta z_a = \beta_a \,, \quad \delta\omega^{ab} = 0 \,, \tag{16}$$

$$\delta\tilde{\omega}_a = -D\beta_a \,, \quad \delta\tilde{v} = -d\lambda - e^a \beta_a \,, \tag{17}$$

whereas a local rotation will generate the transformation

$$\delta e^a = -\beta^{ab} e^b \,, \tag{18}$$

$$\delta\tilde{\omega}_a = \tilde{\omega}_b \beta^b{}_a \,, \tag{19}$$

$$\delta\omega^{ab} = -D\beta^{ab} \,. \tag{20}$$

As it happened in previous section the two form $\mathcal{J}$ acting on space-like vector fields must vanish, therefore

$$\mathcal{J}[E_a, E_b] = 0 \implies v = v_0 \tau \,, \tag{21}$$

this condition is satisfied after the gauge fixing $z_a = -i_{E_a} d\phi$, and $\tilde{v} = \psi\tau$. Such constraint reduces the gauge redundancy to internal translations satisfying $\beta_a = -i_{E_a} d\lambda$. After doing so, $v = (i_t d\phi + \psi)\tau$.

The curvature two-form is then defined as $\mathcal{R} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ which can be expanded as

$$\mathcal{R} = T^0 H + T^a P_a + \bar{T}_a Q^a + \frac{1}{2} R^{ab} L_{ab} + \bar{T} Q \,, \tag{22}$$

where $R^{ab} = D\omega^{ab}$ is the curvature associated to the spin connection, $T^0 = d\tau$, $T^a = De^a$ are the torsion of the space-time, and

$$\bar{T}_a = D\tilde{\omega}_a + z_b R^b{}_a \equiv F_a + z_b R^b{}_a \,, \tag{23}$$

$$\bar{T} = d\tilde{v} + e^a \wedge \tilde{\omega}_a + z_a T^a \equiv f + z_a T^a, \tag{24}$$

the internal torsion fields. By construction, they are invariant under local translations, and transform covariantly under rotations

$$\delta T^a = -\beta^{ab} T^b \tag{25}$$

$$\delta\bar{T}_a = F_b \beta^b{}_a \,, \tag{26}$$

notice that F_a is not invariant under internal translations if the curvature R^{ab} is not vanishing, which justify the presence of the Stueckelberg fields. From this perspective, the breaking of dipole conservation has the same origin as the breaking of translational invariance once we allow the space-time to be curved [1, 2], although in odd space-time dimensions there are exceptions. In that case, gravitational theories can be related to Chern-Simons theories with Poincaré as gauge group [39], and no broken translations. Studying the generalization to our problem, escape from scope of this paper and will be left for future studies.

For simplicity, we fix the space-time to be torsionless ($T^0 = T^a = 0$). Such constraint, fixes τ to be a closed form, and allow us express the spin connection in terms of the vielbeins. On the other hand, if we interpret v as the simplectic gauge potential, we must require $dv = \omega_a \wedge e^a$, which translates into $\bar{T} = 0$. That imposes the constraint

$$e^a \wedge \tilde{\omega}_a = -d\psi \wedge \tau \,, \tag{27}$$

this condition fixes $\tilde{\omega}^a$ modulo a symmetric tensor, and set it to be

$$\tilde{\omega}_a = -i_{E^a} d\psi\tau + \tilde{A}_{ab} e^b \,, \tag{28}$$

[5] See [34, 38] for similar constructions, although different physical systems.

[6] We will refer to fracton gauge transformations with parameters $\beta_a(x)$ and $\lambda(x)$ as internal translations.

with $\tilde{A}_{ab} = \tilde{A}_{ba}$. The gauge transformations Eqs. (17) imply

$$\delta\tilde{A}_{ab} = i_{E_a} D i_{E_b} d\lambda \,, \tag{29}$$

if $i_{E_a} d i_t d\lambda = i_t D i_{E_a} d\lambda$. Also notice that the symmetry property of $\tilde{A}_{ab}$ requires $i_{E_b} D i_{E_a} d\lambda = i_{E_a} D i_{E_b} d\lambda$, which are guaranteed thanks to the torsionless condition imposed above.

Thus, we have a fully consistent set of constraints and gauge fixing. Therefore, the final set of independent fields are the close form τ indicating the direction of time flow, the spatial vielbeins e^a, and the internal vielbeins

$$v = (i_t d\phi + \psi)\tau \tag{30}$$

$$\omega_a = -i_{E_a} dv + A_{ab} e^b \,, \tag{31}$$

with $A_{ab} = \tilde{A}_{ab} - i_{E_a} D i_{E_b} d\phi$. The transformation of the symmetric gauge field Eq. (29) can be written in components as

$$\delta\tilde{A}_{ab} = E^\mu{}_a E^\nu{}_b \nabla_\mu \nabla_\nu \lambda \,, \tag{32}$$

with the diffeomorphism covariant derivative $\nabla_\mu \zeta_\nu = \partial_\mu \zeta_\nu - \Gamma^\alpha_{\mu\nu}\zeta_\alpha$, and the connection $\Gamma^\alpha_{\mu\nu} = t^\alpha \partial_\mu \tau_\nu + E^\alpha{}_a D_\mu e_\nu{}^a$. This covariant derivative satisfy[7]

$$\nabla_\mu \tau_\nu = 0, \quad \nabla_\mu t^\nu = 0 \,, \tag{33}$$

$$\nabla_\mu E^\nu{}_a + E^\nu{}_b \omega_\mu{}^b{}_a = 0 \,, \tag{34}$$

$$\nabla_\mu e_\nu{}^a - \omega_\mu{}^a{}_b e_\nu{}^b = 0 \,, \tag{35}$$

these set of relations imply the covariant derivative is metric compatible satisfying $\nabla_\mu G_{\alpha\beta} = 0$. The last necessary ingredient is an invariant volume form, that we defined as $vol \equiv {}^\star 1 = \sqrt{|G|} d^{n+1}x$.

Finally, the diffeomorphism and gauge invariant quadratic action for the theory can be expressed as

$$S = -\frac{1}{2}\int {}^\star \bar{T}_a \wedge \bar{T}_a + S_{SSB} \,. \tag{36}$$

In fact notice, that if we take the flat space limit ($e^a = dx^a$, $\tau = dx^0$) the internal torsion field $\bar{T}_a$ becomes z_a independent (see Eq. (24)), which allows for masless fracton gauge fields, and we can safely write an invariant theory under the full fractonic gauge group

$$S = -\frac{1}{2}\int {}^\star F_a \wedge F_a = \int \left[F_{0ab} F_{0ab} - \frac{1}{2} F_{abc} F_{abc} \right] \,, \tag{37}$$

which has the form of a generalized electrodynamics theory, with the electric and magnetic fields being

$$F_{0ab} = \partial_0 \tilde{A}_{ab} + \partial_a \partial_b \psi \,, \tag{38}$$

$$F_{abc} = \partial_a \tilde{A}_{bc} - \partial_b \tilde{A}_{ac} \,, \tag{39}$$

in full agreement with previous results [18].

[7] Notice that we have naturally obtained a set of vielbein postulates by writing the symmetric gauge field transformations in components.

DISCUSSION

We have given a geometric interpretation to the symmetry group associated to the conservation of charge and dipole charge. In this picture the group is associated with a $2n + 2$ dimensional space with the actual physical space-time of dimension $n + 1$ understood as a hypersurface embedded in the larger space. The Nambu-Goldstone mode $\phi(t, \mathbf{x})$ appearing in the system when the symmetry is spontaneously broken is the breathing modes of the physical space inside the larger one. The advantage of that picture, is that allow us to construct consistently a gauge theory associated to the symmetry under discussion, in either flat or curved space-time. Our results, explain the incompatibility of the fractonic symmetry with space curvature, since a fully invariant theory requires a Stueckelberg field. Therefore, we conclude that a fractonic system on a curved manifold will generically suffer spontaneous symmetry breaking due to curvature effects. This analysis pave the road for a more systematic analysis of theories preserving charges and their corresponding higher moments. In addition, it may help in the construction of low energy effective fracton theories, because it provides a receipe to construct diffeomorphism and 'gauge' invariant actions. Invariance will allow to derive covariant Ward identities for fracton charge, energy and momentum conservation. For instance, the knowledge of such Ward identities are fundamental to systematically construct fracton hydrodynamics.

The geometric interpretation given in this paper, to the gauge fields, could clarify the issues emerging when trying to minimally couple matter fields to them. Our proposal strongly suggest that the realization of the symmetry on charged fields could not be the currently used $\psi \to e^{i(\lambda + \beta_a x^a)}\psi$. In general, a field does not need to transform with unitary representations of the symmetry group. If the transformations keep the space-time points fixed, the necessary condition is a finite dimensional representation.

In the context of elasticity, this construction may help in going beyond the current fractons/elasticity duality [12]. In fact, it would be interesting to explore within our geometric context the recently proposed generalization of such duality to the case of quasi-crystals [40].

On the other hand, in the context of quantum Hall systems, volume preserving diffeomorphisms have been related to the fractonic symmetry group discussed here [20]. In fact, since the entire symmetry group preserve the simplectic form $\mathcal{J}$, it seems possible to connect our approach with volume preserving diffeos. However, an important difference between the two approaches is that the symmetric gauge field in [20] are directly interpreted as a metric field, whereas, our fracton metric $B_{\mu\nu}$ depends quadratically on $\tilde{A}_{ab}$. Another interesting direction would be the construction of Chern-Simons actions.

We leave the study of all these aspects for future investigations.

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