Семинар №2 по квантовой механике

Драчов Ярослав
Факультет общей и прикладной физики МФТИ

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Задача 4. Покажите, что для оператора эволюции

$$\widehat{\mathrm{U}}(t_2,t_1) \equiv Te^{i\int\limits_{t_1}^{t_2} dt\widehat{\mathrm{V}}_0(t)},$$

верно равенство

$$\widehat{\mathrm{U}}(t_2,t_1)\widehat{\mathrm{U}}(t_1,t_0) = \widehat{\mathrm{U}}(t_2,t_0).$$

Решение.

$$\widehat{\mathrm{U}}(t_1,t_0) = Te^{-\frac{i}{\hbar}\int\limits_{t_0}^{t_1} d\tau\widehat{\mathrm{H}}(\tau)} = \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n \int\limits_{t_0}^{t_1} d\tau_1 \ldots \int\limits_{t_0}^{\tau_{n-1}} d\tau_n \widehat{\mathrm{H}}(\tau_1)\ldots\widehat{\mathrm{H}}(\tau_n) =$$

$$= \widehat{1} - \frac{i}{\hbar}\int\limits_{t_0}^{t_1} d\tau\widehat{\mathrm{H}}(\tau) - \frac{1}{\hbar^2}\int\limits_{t_0}^{t_1} d\tau_1 \int\limits_{t_0}^{\tau_1} d\tau_2 \widehat{\mathrm{H}}(\tau_1)\widehat{\mathrm{H}}(\tau_2) +$$

$$+ \frac{i}{\hbar^3}\int\limits_{t_0}^{t_1} d\tau_1 \int\limits_{t_0}^{\tau_1} d\tau_2 \int\limits_{t_0}^{\tau_2} d\tau_3 \widehat{\mathrm{H}}(\tau_1)\widehat{\mathrm{H}}(\tau_2)\widehat{\mathrm{H}}(\tau_3) + \ldots.$$

Проделать самостоятельно формулу 6 из лекций

$$|\psi(t)\rangle = |\psi_0\rangle - \frac{i}{\hbar}\int\limits_{t_0}^{t_1} d\tau\widehat{\mathrm{H}}(\tau)\,|\psi(\tau\rangle\,.$$

$$\widehat{\mathrm{U}}(t_2,t_1) = \widehat{1} - \frac{i}{\hbar}\int\limits_{t_1}^{t_2} d\tau\widehat{\mathrm{H}}(\tau) - \frac{1}{\hbar^2}\int\limits_{t_1}^{t_2} d\tau_1 \int\limits_{t_1}^{\tau_1} d\tau_2 \widehat{\mathrm{H}}(\tau_1)\widehat{\mathrm{H}}(\tau_2) + \ldots.$$

$$\widehat{\mathrm{U}}(t_2,t_1)\widehat{\mathrm{U}}(t_1,t_0)=$$
$$=\left(\widehat{1}-\frac{i}{\hbar}\int\limits_{t_1}^{t_2}d\tau\widehat{\mathrm{H}}(\tau)-\frac{1}{\hbar^2}\int\limits_{t_1}^{t_2}d\tau_2 d\tau_1\int\limits_{t_1}^{\tau_1}d\tau_2\widehat{\mathrm{H}}(\tau_1)\widehat{\mathrm{H}}(\tau_2)+\frac{i}{\hbar^3}\int\limits_{t_1}^{t_2}d\tau_1\int\limits_{t_1}^{\tau_1}d\tau_2\int\limits_{t_1}^{\tau_2}d\tau_3\widehat{\mathrm{H}}(\tau_1)\widehat{\mathrm{H}}(\tau_2)\widehat{\mathrm{H}}(\tau_3)+\ldots\right)\widehat{1}+$$
$$+\frac{i}{\hbar^3}\int\limits_{t_1}^{t_2}d\tau_1\int\limits_{t_1}^{\tau_1}d\tau_2 d\tau_3\widehat{\mathrm{H}}(\tau_1)\widehat{\mathrm{H}}(\tau_2)\widehat{\mathrm{H}}(\tau_3)\left(-\frac{i}{\hbar}\int\limits_{t_0}^{t_1}d\tau'\widehat{\mathrm{H}}(\tau')\right)+\left(\widehat{1}\ldots\right)\left(-\frac{1}{\hbar^2}\int\limits_{t_0}^{t_1}d\tau_1'\int\limits_{t_0}^{\tau_1'}d\tau_2'\widehat{\mathrm{H}}(\tau_1')\widehat{\mathrm{H}}(\tau_2')\right)+$$
$$+\left(\widehat{1}\ldots\right)\left(\left(-\frac{i}{\hbar^3}\right)\int\int\int\right).$$

$$\widehat{1}-\frac{i}{\hbar}\int\limits_{t_1}^{t_2}d\tau\widehat{\mathrm{H}}(\tau)-\frac{i}{\hbar}\int\limits_{t_0}^{t_1}d\tau'\widehat{\mathrm{H}}(\tau')=-\frac{1}{\hbar^2}\int\limits_{t_1}^{t_2}d\tau_1\int\limits_{t_1}^{\tau_1}d\tau_2\widehat{\mathrm{H}}(\tau_1)\widehat{\mathrm{H}}(\tau_2)+$$
$$+\left(-\frac{i}{\hbar}\right)^2\int\limits_{t_1}^{t_2}d\tau H(\tau)\int\limits_{t_0}^{t_1}d\tau'\widehat{\mathrm{H}}(\tau')+\left(-\frac{i}{\hbar}\right)^2\int\limits_{t_0}^{t_1}d\tau_1\int\limits_{t_0}^{\tau_1}d\tau_2\widehat{\mathrm{H}}(\tau_1)\widehat{\mathrm{H}}(\tau_2)=$$
$$=\left(-\frac{i}{\hbar}\right)^2\int\limits_{t_1}^{t_2}d\tau_1\left[\int\limits_{t_1}^{\tau_1}d\tau_2\widehat{\mathrm{H}}(\tau_1)\widehat{\mathrm{H}}(\tau_2)+\int\limits_{t_0}^{t_1}d\tau_2\widehat{\mathrm{H}}(\tau_1)H(\tau_2)\right]+\ldots=$$
$$=-\left(\frac{i}{\hbar}\right)^2\int\limits_{t_1}^{t_2}d\tau_1\int\limits_{t_0}^{\tau_1}\widehat{\mathrm{H}}(\tau_1)\widehat{\mathrm{H}}(\tau_2)d\tau_2+\left(-\frac{i}{\hbar}\right)^2\int\limits_{t_0}^{t_1}d\tau_1\int\limits_{t_0}^{\tau_1}d\tau_2\widehat{\mathrm{H}}(\tau_1)\widehat{\mathrm{H}}(\tau_2)=$$
$$=\left(-\frac{i}{\hbar}\right)^2\int\limits_{t_0}^{t_2}d\tau_1\int\limits_{t_0}^{\tau_1}d\tau_2\widehat{\mathrm{H}}(\tau_1)\widehat{\mathrm{H}}(\tau_2).$$

Задача 2.

$$\Psi(x,t=0)=Ce^{-\frac{m\omega}{2\hbar}(x-x_0)^2}.$$

Решение.

$$K(t,t_0=0|x,y)=\sqrt{\frac{m\omega}{2\pi i\hbar\sin(\omega t)}}e^{\frac{iS[z_{\mathrm{Cl}}(\cdot)]}{\hbar}}.$$

$$\Psi(t,x)=\int dyK(t|x,y)\Psi_{t=0}(y)=C\int dy\frac{m\omega}{2\pi i\hbar\sin\omega t}e^{\frac{im\omega}{2\hbar\sin\omega t}\left((x^2+y^2)\cos\omega t-2xy\right)}e^{-\frac{m\omega(y-y_0)^2}{2\hbar}}.$$

$$\lambda\to\lambda+i\varepsilon.$$

Задача 7a.

Решение.

$$\Psi_{\mathrm{SM}}=\Phi\left(\underbrace{\mathbf{r}_1}_{1e^-},\underbrace{\mathbf{r}_2}_{2e^-}\right)\cdot\chi\left(S_{1z},S_{2z}\right).$$

$$\widehat{\mathrm{S}}=\widehat{\mathrm{S}}_1+\widehat{\mathrm{S}}_2.$$

$$\begin{cases}\widehat{\mathrm{S}}^2(1)\left|\frac{1}{2},m_1\right\rangle=\frac{3}{4}\hbar^2\left|\frac{1}{2},m_1\right\rangle\\ S_z(1)\left|\frac{1}{2},m_1\right\rangle=\hbar m_1\left|\frac{1}{2},m_1\right\rangle\end{cases}$$

$$\begin{cases} \widehat{\mathrm{S}}^2(2)\left|\frac{1}{2}, m_2\right\rangle = \hbar^2 \frac{3}{4}\left|\frac{1}{2}, m_2\right\rangle \\ \widehat{\mathrm{S}}_z(2)\left|\frac{1}{2}, m_2\right\rangle = \hbar m_2\left|\frac{1}{2}, m_2\right\rangle \end{cases}$$

$$\begin{cases} \widehat{\mathrm{S}}^2\Psi_{\mathrm{SM}} = \hbar^2 S(S+1)\Psi_{\mathrm{SM}} \\ \widehat{\mathrm{S}}_z\Psi_{\mathrm{SM}} = \hbar M\Psi_{\mathrm{SM}} \end{cases}$$

Частица №1:

$$\chi_{1/2}(1) \to S_z = \frac{\hbar}{2}.$$

$$\chi_{-1/2}(1) \to S_z = -\frac{\hbar}{2}.$$

Частица №2

$$\chi_{1/2}(2) \to S_z = \frac{\hbar}{2}.$$

$$\chi_{-1/2}(2) \to S_z = -\frac{\hbar}{2}.$$

$$\mathcal{H}(1,2) = \mathcal{H}(1) \otimes \mathcal{H}(2).$$

$$|\Psi_1\rangle = \chi_{1/2}(1) \otimes \chi_{1/2}(2), \quad m_1 = 1/2,\, m_2 = 1/2.$$

$$|\Psi_2\rangle = \chi_{1/2}(1) \otimes \chi_{-1/2}(2) m_1 = 1/2,\, m_2 = -1/2.$$

$$|\Psi_3\rangle = \chi_{-1/2}(1) \otimes \chi_{1/2}(2) m_1 = -1/2,\, m_2 = 1/2.$$

$$|\Psi_4\rangle = \chi_{-1/2}(1) \otimes \chi_{-1/2}(2) m_1 = -1/2,\, m_2 = -1/2.$$

$$\widehat{\mathrm{S}}_z(1)\chi_1 = \hbar m_1 \chi_1, \quad m_1 = \pm\frac{1}{2}.$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$\frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Общие собственные функции: $\Psi_{1,\,1}$, $\Psi_{1,\,-1}$

a)

$$\begin{cases} \widehat{\mathrm{S}}_z\Psi_{1,\,1} = \hbar \cdot 1\Psi_{1,\,-1} \\ \widehat{\mathrm{S}}^2\Psi_{1,\,1} = \hbar^2 \cdot_2 \Psi_{1,\,1} \end{cases}.$$

$$\Psi_{1,\,1} = \chi_{1/2}(1)\chi_{1/2}(2) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad S = 1,\, M = 1.$$

б)

$$\begin{cases} \widehat{\mathrm{S}}_z \Psi_{1,\,-1} = \hbar(-1)\Psi_{1,\,-1} \\ \widehat{\mathrm{S}}^2_{1,\,-1} = \hbar^2 \cdot 2\Psi_{1,\,-1} \end{cases}.$$

$$\Psi_{1,\,-1} = \chi_{-1/2}(1)\chi_{-1/2}(2) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad M=-1,\, S=1.$$

$$\widehat{\mathrm{S}}_+ \Psi_{SM} = \hbar\sqrt{S(S+1) - M(M+1)}.$$

$$\Big(\widehat{\mathrm{S}}_+(1) + \widehat{\mathrm{S}}_+(2)\chi_{1/2}(1)\chi_{-1/2}(2)\Big) = \widehat{\mathrm{S}}_+(1)\chi_{-1/2}(1)\chi_{-1/2}(2) + \chi_{-1/2}(1)\widehat{\mathrm{S}}_+(2)\chi_{-1/2}(2) = \\ = \hbar\left[1\cdot\chi_{1/2}(1)\chi_{-1/2}(2) + \chi_{-1/2}(1)\chi_{1/2}(2)\right].$$

$$\Psi_{0,\,0} = \frac{1}{\sqrt{2}}\left(\chi_{1/2}(1)\chi_{-1/2}(2) - \chi_{-1/2}(1)\chi_{1/2}(2)\right).$$

$$\begin{cases} \sum\limits_{m_1,\,m_2} C^{SM}_{s_1m_1s_2m_2} C^{S'M'}_{s_1m_1s_2m_2} = \delta_{SS'}\delta_{MM'} \\ \sum\limits_{SM} C^{SM}_{s_1m_1s_2m_2} C^{SM}_{s_1m_1's_2m_2'} = \delta_{m_1m_1'}\delta_{m_2m_2'} \end{cases}$$

$$C^{1\,1}_{\frac12\,\frac12\,\frac12\,\frac12} = 1.$$

$$C^{1\,-1}_{\frac12\,-\frac12\,\frac12\,-\frac12} = C^{1\,1}_{\frac12\,\frac12\,\frac12\,\frac12} = 1.$$

$$C^{1\,0}_{\frac12\,-\frac12\,\frac12\,\frac12} = \frac{1}{\sqrt{2}}.$$

$$C^{1\,0}_{\frac12\,\frac12\,\frac12\,-\frac12} = \frac{1}{\sqrt{2}}.$$

$$C^{0\,0}_{\frac12\,-\frac12\,\frac12\,\frac12} = \frac{1}{\sqrt{2}}.$$

$$C^{0\,0}_{\frac12\,\frac12\,\frac12\,-\frac12} = \frac{1}{\sqrt{2}}.$$

$$\widehat{\mathrm{S}}_+\Psi_{0,\,0} = \widehat{\mathrm{S}}_-\Psi_{0,\,0} = 0.$$

Задача 7б.

Решение.

$$\widehat{\mathbf{j}} = \widehat{\mathbf{l}} + \widehat{\mathbf{s}}.$$

$$\widehat{\mathrm{j}^2},\, \widehat{\mathrm{j}_\mathrm{z}},\, \widehat{\mathrm{l}^2},\, \widehat{\mathrm{s}^2}.$$

$$\widehat{\mathrm{j}_\mathrm{z}}\Psi_{\text{об}} = m_j\Psi_{\text{об}}.$$

$$\widehat{\mathrm{j}_\mathrm{z}} = \widehat{\mathrm{l}_\mathrm{z}} + \widehat{\mathrm{s}_\mathrm{z}} = -i\widehat{1}\frac{\partial}{\partial\varphi} + \frac{1}{2}\widehat{\sigma}_\mathrm{z} = \begin{pmatrix} -i\frac{\partial}{\partial\varphi} + \frac12 & 0 \\ 0 & -i\frac{\partial}{\partial\varphi} - \frac12 \end{pmatrix}.$$