Семинар №5

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Нестационарное уравнение Дирака в нековариантной форме

$$i\hbar\frac{\partial}{\partial t}\Psi\left(\mathbf{r},t\right)=\underbrace{\left(c(\widehat{\boldsymbol{\alpha}},\widehat{\mathbf{p}})+\beta mc^2\right)}_{\widehat{\mathrm{H}}_D}\Psi(\mathbf{r},t).$$

$$\widehat{\alpha}_i=\begin{pmatrix} o & \widehat{\sigma}_i \\ \widehat{\sigma}_i & 0\end{pmatrix},\quad \widehat{\beta}=\begin{pmatrix}\widehat{1} & 0\\ 0 & -\widehat{1}\end{pmatrix}.$$

$$\frac{\widehat{\beta}}{c}\times\left(i\hbar\frac{\partial}{\partial t}-c\left(\widehat{\boldsymbol{\alpha}},\widehat{\mathbf{p}}\right)-\widehat{\beta}mc^2\right)\Psi\left(\mathbf{r},t\right)=0.$$

$$\gamma_0=\widehat{\beta},\quad \gamma^i=-\gamma_i=\widehat{\beta}\alpha_i.$$

$$\left(i\hbar\frac{1}{c}\gamma_0\frac{\partial}{\partial t}-\widehat{\gamma}^i\widehat{\beta}_i-mc\right)\Psi(\mathbf{r},t)=0.$$

$$\left(i\hbar\frac{1}{c}\gamma_0\frac{\partial}{\partial t}+\gamma^i i\hbar\frac{\partial}{\partial x^i}-mc\right)\Psi=0.$$

$$\left(i\hbar\underbrace{\left(\frac{\gamma_0}{c}\frac{\partial}{\partial t}+(\boldsymbol{\gamma},\boldsymbol{\nabla})\right)}_{\gamma_\mu\partial^\mu\equiv\widehat{\partial}}i-mc\right)\Psi=0.$$

$$\Psi'=e^{\frac{ief(x,\,t)}{\hbar c}}\Psi.$$

$$\left(i\hbar\widehat{\partial}-mc\right)\Psi=0.$$

$$\Psi(\mathbf{r},t)=e^{-\frac{i}{\hbar}Et}\cdot\Psi(\mathbf{r}).$$

$$\begin{cases}\widehat{\mathrm{H}}_D\psi(\mathbf{r})=E\psi(\mathbf{r})\\ \widehat{\mathbf{p}}\psi(\mathbf{r})=\mathbf{p}\psi(\mathbf{r})\end{cases}.$$

$$\psi(\mathbf{r})=u(\mathbf{p},E)\frac{1}{\sqrt{V}}e^{\frac{i}{\hbar}(\mathbf{p},\mathbf{r})},\qquad\int\psi^\dagger\psi dV=1.$$

$$\left(c\begin{pmatrix}0 & (\widehat{\boldsymbol{\sigma}},\widehat{\mathbf{p}})\\ (\widehat{\boldsymbol{\sigma}},\widehat{\mathbf{p}}) & 0\end{pmatrix}+mc^2\begin{pmatrix}\widehat{1} & 0\\ 0 & -\widehat{1}\end{pmatrix}\right)u\left(\mathbf{p},E\right)e^{\frac{i}{\hbar}(\mathbf{p},\mathbf{r})}=Eu\left(\mathbf{p},E\right)e^{\frac{i}{\hbar}(\mathbf{p},\mathbf{r})}.$$

$$\left(c\begin{pmatrix}0 & (\widehat{\boldsymbol{\sigma}},\widehat{\mathbf{p}})\\ (\widehat{\boldsymbol{\sigma}},\widehat{\mathbf{p}}) & 0\end{pmatrix}+mc^2\begin{pmatrix}\widehat{1} & 0\\ 0 & -\widehat{1}\end{pmatrix}-E\begin{pmatrix}\widehat{1} & 0\\ 0 & \widehat{1}\end{pmatrix}\right)\underbrace{u\left(\mathbf{p},E\right)e^{\frac{i}{\hbar}\left(\mathbf{p},\mathbf{r}\right)}}_{\sim\psi(\mathbf{r})}=0.$$

$$u\left(\mathbf{p},E\right)=\begin{pmatrix}\varphi\left(\mathbf{p},E\right))\\ \chi\left(\mathbf{p},E\right)\end{pmatrix}.$$

$$\widehat{\mathrm{p}}\Psi(\mathbf{r})=\mathbf{p}\psi(\mathbf{r}).$$

$$\begin{pmatrix}(mc^2-E)\widehat{1} & c\left(\widehat{\boldsymbol{\sigma}},\mathbf{p}\right)\\ c\left(\widehat{\boldsymbol{\sigma}},\mathbf{p}\right) & (-mc^2-E)\widehat{1}\end{pmatrix}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}=0.$$

$$\begin{vmatrix}(mc^2-E)\widehat{1} & c\left(\widehat{\boldsymbol{\sigma}},\mathbf{p}\right)\\ c\left(\widehat{\boldsymbol{\sigma}},\mathbf{p}\right) & (-mc^2-E)\widehat{1}\end{vmatrix}=0.$$

$$-(mc^2-E)(mc^2+E)\widehat{1}-c^2\left(\widehat{\boldsymbol{\sigma}},\widehat{\mathbf{p}}\right)\left(\widehat{\boldsymbol{\sigma}},\widehat{\mathbf{p}}\right)=0.$$

$$\widehat{1}(E^2-(mc^2)^2)=c^2\left(\widehat{\boldsymbol{\sigma}},\widehat{\mathbf{p}}\right)\left(\widehat{\boldsymbol{\sigma}},\widehat{\mathbf{p}}\right)=c^2\mathbf{p}^2.$$

$$\left(\widehat{\boldsymbol{\sigma}},\widehat{\mathbf{p}}\right)\left(\widehat{\boldsymbol{\sigma}},\widehat{\mathbf{p}}\right)=\widehat{\sigma}_i p_i\widehat{\sigma}_j p_j=p_i p_j\left(\widehat{1}\delta_{ij}+i\varepsilon_{ijk}\widehat{\sigma}_k\right)=\mathbf{p}^2\cdot\widehat{1}+0=\widehat{\mathrm{p}}^2\cdot\widehat{1}.$$

$$E^2=m^2c^4+\mathbf{p}^2c^2.$$

$$\widehat{\Lambda}=\frac{c\left(\widehat{\boldsymbol{\alpha}},\mathbf{p}\right)+\beta mc^2}{E_p}=\frac{\widehat{\mathrm{H}}_D}{\left(\widehat{\mathrm{H}}_D^2\right)^{1/2}}.$$

$$E=\pm\sqrt{m^2c^4+\mathbf{p}^2c^2}=\xi_p E_p.$$

$$\xi_p=+1,\quad E>0.$$

$$\xi_p=-1,\quad E<0.$$

$$\begin{cases}c\left(\widehat{\boldsymbol{\sigma}},\mathbf{p}\right)\chi=(E-mc^2)\varphi\\ c\left(\widehat{\boldsymbol{\sigma}},\widehat{\mathbf{p}}\right)\varphi=(E+mc^2)\chi\end{cases}\implies\chi=\frac{c\left(\widehat{\boldsymbol{\sigma}},\widehat{\mathbf{p}}\right)\varphi}{\xi E_p+mc^2}.$$

$$u\left(\mathbf{p},E\right)=N\begin{pmatrix}\varphi\\ \chi\end{pmatrix}=N\begin{pmatrix}\varphi\\ \frac{c(\widehat{\boldsymbol{\sigma}},\widehat{\mathbf{p}})\varphi}{E+mc^2}\end{pmatrix}.$$

$$\int\psi^\dagger\psi dV\implies u^\dagger\left(\mathbf{p},E\right)u\left(\mathbf{p},E\right)=1.$$

$$N^2\begin{pmatrix}\varphi^\dagger & \frac{\varphi^\dagger c(\widehat{\boldsymbol{\sigma}}^\dagger,\mathbf{p})}{E+mc^2}\end{pmatrix}\begin{pmatrix}\varphi\\ \frac{c(\widehat{\boldsymbol{\sigma}},\widehat{\mathbf{p}})}{E+mc^2}\end{pmatrix}=1.$$

$$N^2\varphi^\dagger\varphi\left(1+\frac{c^2\mathbf{p}^2}{(E+mc^2)^2}\right)=N^2\varphi^\dagger\varphi\left(1+\frac{E^2-m^2c^4}{(E+mc^2)^2}\right)=1.$$

$$N^2\varphi^\dagger\varphi\frac{2E}{E+mc^2}=1\implies N=\sqrt{\frac{E+mc^2}{2E}}.$$

$$\varphi^\dagger\varphi=1.$$

$$\widehat{\boldsymbol{\Sigma}}=\begin{pmatrix}\widehat{\boldsymbol{\sigma}} & 0\\ 0 & \widehat{\boldsymbol{\sigma}}\end{pmatrix}.$$

$$\widehat{\mathbf{s}} = \frac{\hbar}{2}\widehat{\mathbf{\Sigma}} \text{ — спиральность.}$$

$$\frac{\left(\widehat{\mathbf{\Sigma}}, \mathbf{p}\right)}{p}\psi(\mathbf{r}) = s \cdot \psi(\mathbf{r}) \implies \begin{pmatrix} (\widehat{\boldsymbol{\sigma}}, \mathbf{p}) & 0 \\ 0 & (\widehat{\boldsymbol{\sigma}}, \mathbf{p}) \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = s \cdot p \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \implies$$

$$\implies \begin{pmatrix} \widehat{\mathrm{s}}_x & \widehat{\mathrm{s}}_y & \widehat{\mathrm{s}}_z \end{pmatrix} = \begin{pmatrix} \frac{\hbar}{2}\widehat{\Sigma}_x & \frac{\hbar}{2}\widehat{\Sigma}_y & \frac{\hbar}{2}\widehat{\Sigma}_z \end{pmatrix}.$$

$$s = \pm 1.$$

$$\widehat{\Sigma}_x = \begin{pmatrix} \widehat{\sigma}_x & 0 \\ 0 & \widehat{\sigma}_x \end{pmatrix}.$$

$$(\widehat{\boldsymbol{\sigma}}, \widehat{\mathbf{p}})\,\varphi = sp\varphi \implies \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = sp\,(c_1 c_2)\,.$$

$$s = 1 \quad \langle \mathbf{p} \rangle \uparrow\uparrow \langle \mathbf{s} \rangle\,.$$

$$s = -1 \quad \langle \mathbf{p} \rangle \uparrow\downarrow \langle \mathbf{s} \rangle\,.$$

$$\Psi_{\mathbf{p}s\xi_p}(\mathbf{r}, t) = \frac{e^{\frac{i}{\hbar}(-\xi E_p t + \mathbf{pr})}}{2\sqrt{V}} \begin{pmatrix} \sqrt{1 + \xi\frac{mc^2}{E_p}} \left(1 + \frac{sp_z}{p}\right)^{1/2} \\ s\left(1 + \xi\frac{mc^2}{E_1}\right)^{1/2} \left(1 - \frac{sp_z}{p}\right)^{1/2} e^{i\delta} \\ s\xi\left(1 - \xi\frac{mc^2}{E_p}\right)^{1/2} \left(1 + \frac{sp_z}{p}\right)^{1/2} \\ \xi\left(1 - \frac{\xi mc^2}{E_p}\right)^{1/2} \left(1 - \frac{sp_z}{p}\right)^{1/2} e^{i\delta} \end{pmatrix}.$$

Задача 6.

Решение.

$$\Psi\,(\mathbf{r}, t) = \frac{1}{\sqrt{V}} u\,(\mathbf{p}, E)\, C^{\frac{i}{\hbar}(\mathbf{p},\mathbf{r}) - Et}.$$

$$u(\mathbf{p}, E) = N \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = N \begin{pmatrix} \varphi \\ \frac{c(\widehat{\boldsymbol{\sigma}}, \widehat{\mathbf{p}})}{E + mc^2}\varphi \end{pmatrix}, \quad N = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{mc^2}{E}}.$$

$$\langle \mathbf{s} \rangle = \frac{u^\dagger \widehat{\mathbf{s}} u}{u^\dagger u} = \frac{u^\dagger \frac{1}{2}\widehat{\mathbf{\Sigma}} u}{u^\dagger u} = \frac{1}{2}N^2 \begin{pmatrix} \varphi^\dagger & \varphi^\dagger \frac{c(\widehat{\boldsymbol{\sigma}}, \mathbf{p})}{E + mc^2} \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{\sigma}} & 0 \\ 0 & \widehat{\boldsymbol{\sigma}} \end{pmatrix} \begin{pmatrix} \varphi \\ \frac{c(\widehat{\boldsymbol{\sigma}}, \mathbf{p})}{E + mc^2} \end{pmatrix}.$$

$$\mathbf{p} = \begin{pmatrix} 0 & 0 & p \end{pmatrix}, \quad (\widehat{\boldsymbol{\sigma}}, \mathbf{p}) = \sigma_z p.$$

$$\implies \langle \mathbf{s} \rangle = \frac{1}{2}N^2 \varphi^\dagger \left(\widehat{\boldsymbol{\sigma}} + \frac{c^2}{(E + mc^2)^2}\,(\widehat{\boldsymbol{\sigma}}, \mathbf{p})\,\boldsymbol{\sigma}\,(\widehat{\boldsymbol{\sigma}}, \mathbf{p})\right)\varphi = \frac{N^2}{2}\varphi^\dagger \left(\widehat{\boldsymbol{\sigma}} + \frac{p^2 c^2}{(E + mc^2)^2}\widehat{\sigma}_z \widehat{\boldsymbol{\sigma}} \widehat{\sigma}_z\right)\varphi.$$

$$\langle s_x \rangle = \frac{N^2}{2}\varphi^\dagger \left(\widehat{\sigma}_x + \frac{p^2 c^2}{(E + mc^2)^2}\widehat{\sigma}_z \widehat{\sigma}_x \widehat{\sigma}_z\right)\varphi \overset{*}{=}.$$

$$\widehat{\sigma}_z \widehat{\sigma}_x = -\widehat{\sigma}_x \widehat{\sigma}_z = i\widehat{\sigma}_y.$$

$$i\widehat{\sigma}_y \widehat{\sigma}_z = -i\widehat{\sigma}_z \widehat{\sigma}_y = +i\,(-\widehat{\sigma}_z \widehat{\sigma}_y) = i\,(i\widehat{\sigma}_x) = -\widehat{\sigma}_x.$$

$$\stackrel{*}{=} \frac{N^2}{2}\varphi^\dagger\left(\widehat{\sigma}_x - \frac{p^2c^2\widehat{\sigma}_x}{(E+mc^2)^2}\right)\varphi = \frac{1}{2}N^2\varphi^\dagger\widehat{\sigma}_x\varphi\frac{(E+mc^2)^2-p^2c^2}{(E+mc^2)^2} =$$

$$= \frac{1}{2}\frac{1}{2}\frac{E+mc^2}{E}\frac{E^2+2mc^2E+(mc^2)^2-p^2c^2}{(E+mc^2)^2}\varphi^\dagger\widehat{\sigma}_x\varphi = \frac{1}{4}\frac{1}{E}\frac{2(mc^2)^2+2mc^2E}{E+mc^2}\varphi^\dagger\widehat{\sigma}_x\varphi =$$

$$= \frac{mc^2}{2E}\varphi^\dagger\widehat{\sigma}_x\varphi.$$

$$\langle s_y\rangle = \frac{N^2}{2}\varphi^\dagger\left(\widehat{\sigma}_y + \frac{p^2c^2}{(E+mc^2)^2}\underbrace{\widehat{\sigma}_z\widehat{\sigma}_y}_{-\widehat{\sigma}_y\widehat{\sigma}_z}\widehat{\sigma}_z\right)\varphi =$$

$$= \frac{N^2}{2}\varphi^\dagger\left(\widehat{\sigma}_y + \frac{p^2c^2}{(E+mc^2)^2}\left(-\widehat{\sigma}_y\underbrace{\widehat{\sigma}_z\widehat{\sigma}_z}_{\widehat{1}}\right)\right)\varphi = \frac{mc^2}{2E}\varphi^\dagger\widehat{\sigma}_y\varphi.$$

$$\langle s_z\rangle = \frac{N^2}{2}\varphi^\dagger\left(\widehat{\sigma}_z + \frac{p^2c^2}{(E+mc^2)^2}\widehat{\sigma}_z\widehat{\sigma}_z\widehat{\sigma}_z\right)\varphi = \frac{N^2}{2}\varphi{+}\widehat{\sigma}_z\varphi\frac{(E+mc^2)^2+p^2c^2}{(E+mc^2)^2} = \frac{1}{2}\varphi^\dagger\widehat{\sigma}_z\varphi.$$

$$E^2+(mc^2)^2+2mc^2E.$$

$$\varphi^\dagger\widehat{\sigma}_z\varphi \neq 0.$$

$$\frac{mc^2}{E} \ll 1.$$