Boolean Operations

Truth Tables

Truth Tables are a tool we use to compute the values of logical expressions over each combination of input value. For example, a truth table for the logical AND of two logic variables, *p* and *q*, would look something like this:

p	**q**	**p ^ q**
true	true	true
true	false	false
false	true	false
false	false	false

not

the *NOT* operation (~) is the logical negation of *p*. ~p = false if p = true and ~p = true if f = false.

Truth Table for ~p:

p	**~p**
true	false
false	true

or

the *OR* operation (∨) holds true when either *p* or *q* are true.

Truth table for p ∨ q:

p	**q**	**p ∨ q**
true	true	true
true	false	true
false	true	true
false	false	false

and

the *AND* operation (∧) holds true only when both *p* and *q* are true.

Truth table for p ∧ q:

p	**q**	**p ∧ q**
true	true	true
true	false	false

false	true	false
false	false	false

xor

The *EXCLUSIVE-OR* operation (⊕) holds true when either *p* or *q* are true, but not both.

Truth table for p ⊕ q:

p	**q**	**p ⊕ q**
true	true	false
true	false	true
false	true	true
false	false	false

Boolean Algebra

<u>Example</u>: Evaluate the boolean function: $F(p,q) = p \wedge (p \oplus q)$

Let us construct a truth table:

p	**q**	**p ⊕ q**	**p ∧ (p ⊕ q)**
false	false	false	false
false	true	true	false
true	false	true	true
true	true	false	false

$F(0,0) = 0$
$F(0,1) = 0$
$F(1,0) = 1$
$F(1,1) = 0$

Boolean Algebra Identities

Identity	AND form	OR form
Identity Laws	$p \wedge 1 = p$	$p \vee 0 = p$
Dominance Laws	$p \wedge 0 = 0$	$p \vee 1 = 1$

Idempotent Laws	$p \wedge p = p$	$p \vee p = p$
Complement Laws	$p \wedge \sim p = 0$	$p \vee \sim p = 1$
Double Complement Laws	$\sim(\sim p) = p$	
Commutative Laws	$p \wedge q = q \wedge p$	$p \vee q = q \vee p$
Associative Laws	$p \wedge (q \wedge r) = (p \wedge q) \wedge r$	$p \vee (q \vee r) = (p \vee q) \vee r$
Distributive Laws	$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$
Absorption Laws	$p \wedge (p \vee q) = p$	$p \vee (p \wedge q) = p$
DeMorgan's Laws	$\sim(p \wedge q) = \sim p \vee \sim q$	$\sim(p \vee q) = \sim p \wedge \sim q$

We can use the identiies to help us evaluate boolean functions. Each law can be proved by either a truth table or by using other laws.

Example:Use a truth table to prove the distributive law over the AND operation:

Proposition: $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
Proof by construction using a truth table:

p	**q**	**r**	**q ∨ r**	**p ∧ q**	**p ∧ r**	**p ∧ (q ∨ r)**	**(p ∧ q) ∨ (p ∧ r)**
false	false	false	false	false	false	false	false
false	false	true	true	false	false	false	false
false	true	false	true	false	false	false	false
false	true	true	true	false	false	false	false
true	false	false	false	false	false	false	false
true	false	true	true	false	true	true	true
true	true	false	true	true	false	true	true
true	true	true	true	true	true	true	true

Example: Use the laws of Boolean Algebra to prove the Absorption Law over the AND operation:

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Proposition: p ∧ ( p ∨ q ) = p

        p ∧ ( p ∨ q ) = (p ∧ p ) ∨ (p ∧ q )      by Distributive Law
                  = p ∨ ( p ∧ q )                by Idempotent Law
                  = p ∧ ( 1 ∨ q )                by Distributive Law
                  = p ∧ 1                        by Dominance Law
                  = p                            by Identity Law
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