Space Flight Modelling and Atmospheric Reentry

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1. **Newton’s Gravity Equation and the Two Body Problem**

2. Rescaling

3. Implementation

4. Preview on Plans and Future Talks

2 Body Problem

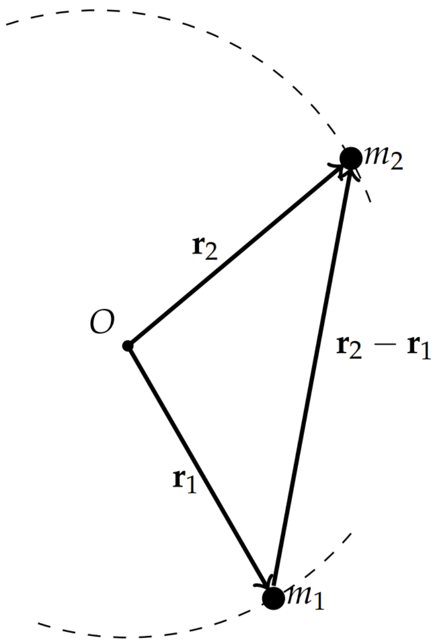
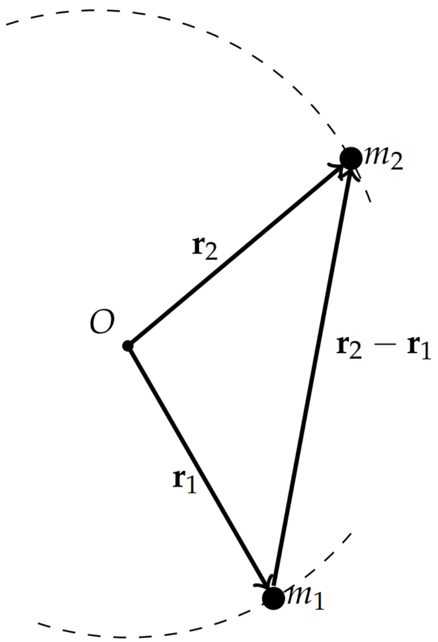


Figure: Scheme of the two body problem [1]

2 Body Problem

- Describes the problem where the future motion of two masses needs to be determined



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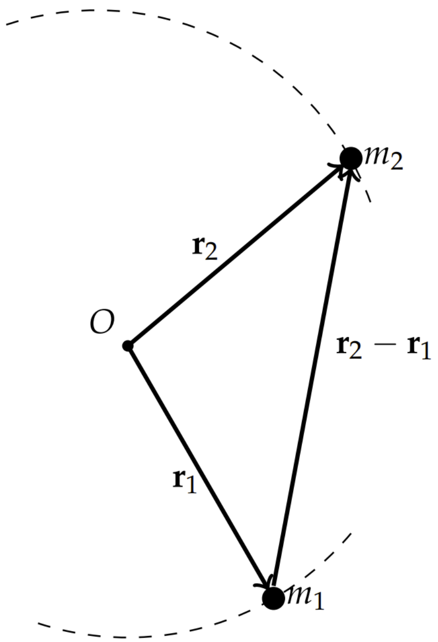
2 Body Problem

- Describes the problem where the future motion of two masses needs to be determined
- $\exists$ plane E s.t. both masses $m_i \in E \Rightarrow$ we can treat the space as 2-dimensional for now



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Newtonian Approach

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$$m_1 \cdot a_1(t) = F_{21} = G \cdot \frac{m_1 \cdot m_2}{||r_2(t) - r_1(t)||^2} \cdot \frac{r_2(t) - r_1(t)}{||r_2(t) - r_1(t)||}$$
$$m_2 \cdot a_2(t) = F_{12} = G \cdot \frac{m_2 \cdot m_1}{||r_1(t) - r_2(t)||^2} \cdot \frac{r_1(t) - r_2(t)}{||r_1(t) - r_2(t)||}$$

$$\ddot{r}_1(t) = \frac{G \cdot m_2}{||r_2(t) - r_1(t)||^3} \cdot (r_2(t) - r_1(t))$$
$$\ddot{r}_2(t) = \frac{G \cdot m_1}{||r_1(t) - r_2(t)||^3} \cdot (r_1(t) - r_2(t))$$

System of ODE in 2D

For a two-dimensional model with coordinates $r_1(t) = (x_1(t), y_1(t))^\top$ and velocities $\dot{r}_1(t) = (u_1(t), v_1(t))^\top$ and $\alpha(t) = \frac{G}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}^3}$, we get an ODE-system of 2nd order:

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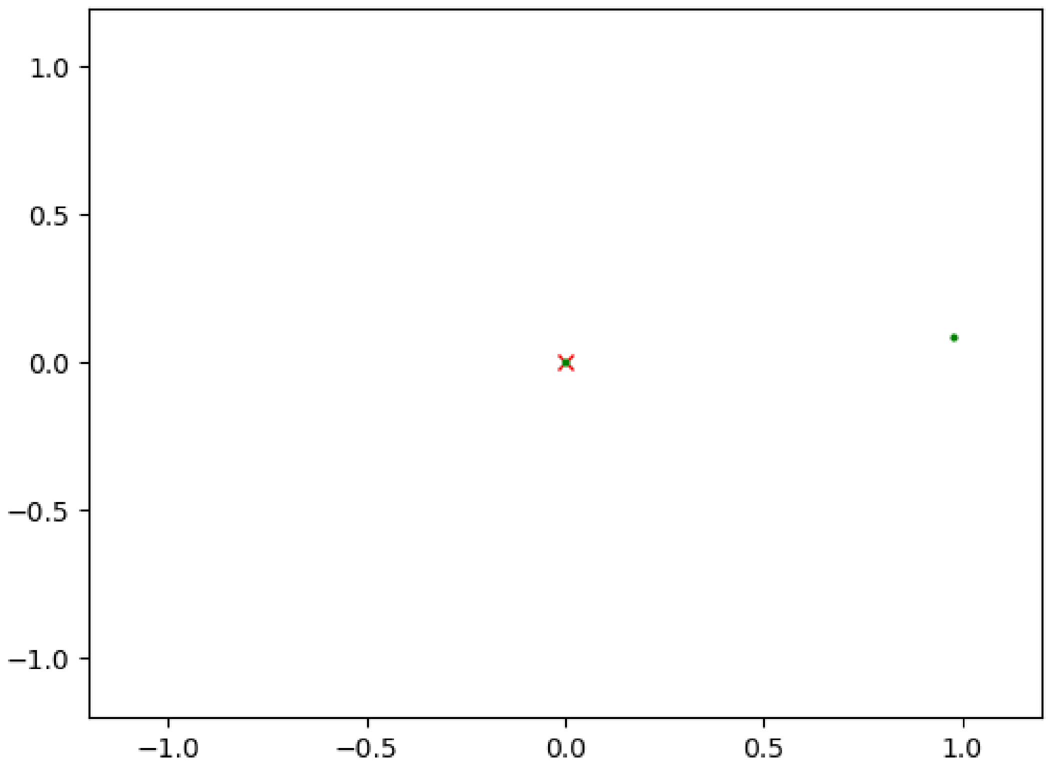
$$\frac{d}{dt}\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \alpha m_2 (x_2 - x_1) \\ \alpha m_2 (y_2 - y_1) \\ \alpha m_2 (x_1 - x_2) \\ \alpha m_2 (y_1 - y_2) \end{pmatrix} \tag{1}$$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0.5
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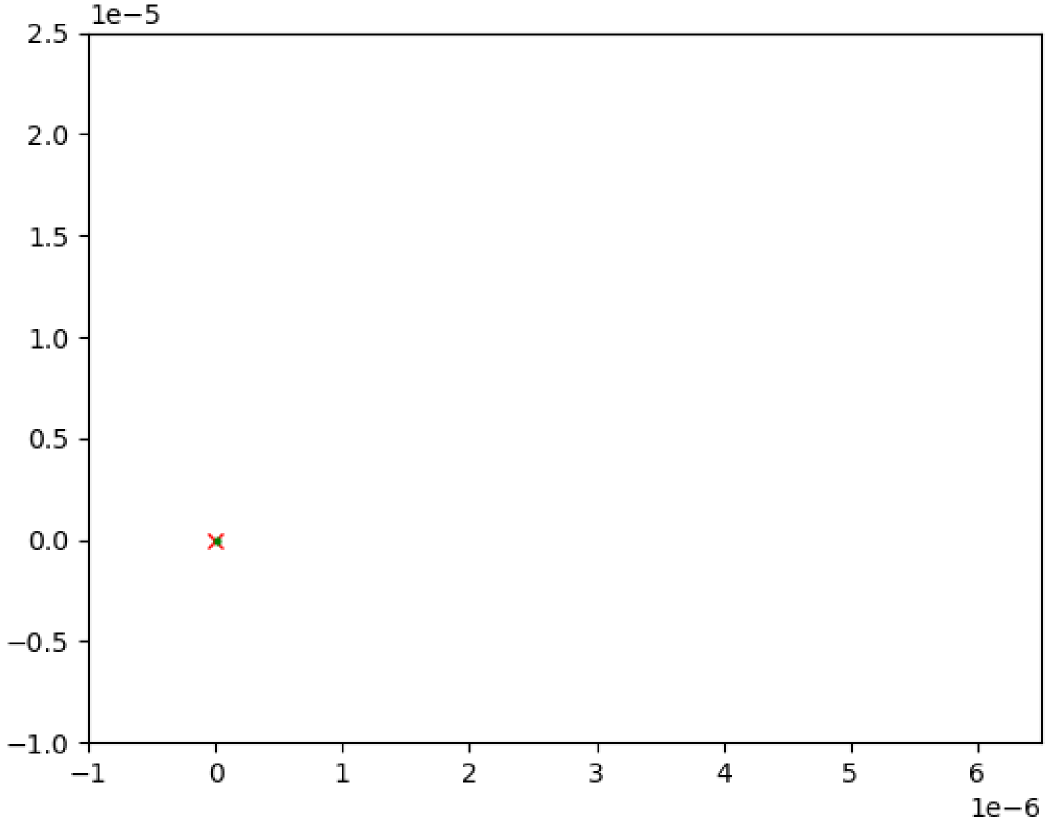
1.0


1e−5
2.5
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1
0
1
2
3
4
5
6
1e−6

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$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \\ -\mu\frac{x}{||\mathbf{r}||^3} \\ -\mu\frac{y}{||\mathbf{r}||^3} \end{pmatrix}$$

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Quantity	SI-Unit	Rescaled unit
Mass	$1\,\text{kg}$	$1\,m_{\text{E}} = 5.9772 \cdot 10^{24}\,\text{kg}$
Time	$1\,\text{s}$	$1\,\text{day} = 86400\,\text{s}$
Distance	$1\,\text{m}$	$1\,\text{AU} = 149\,597\,870\,700\,\text{m}$

Rescaling of Dependent Quantities

Dependent Quantities

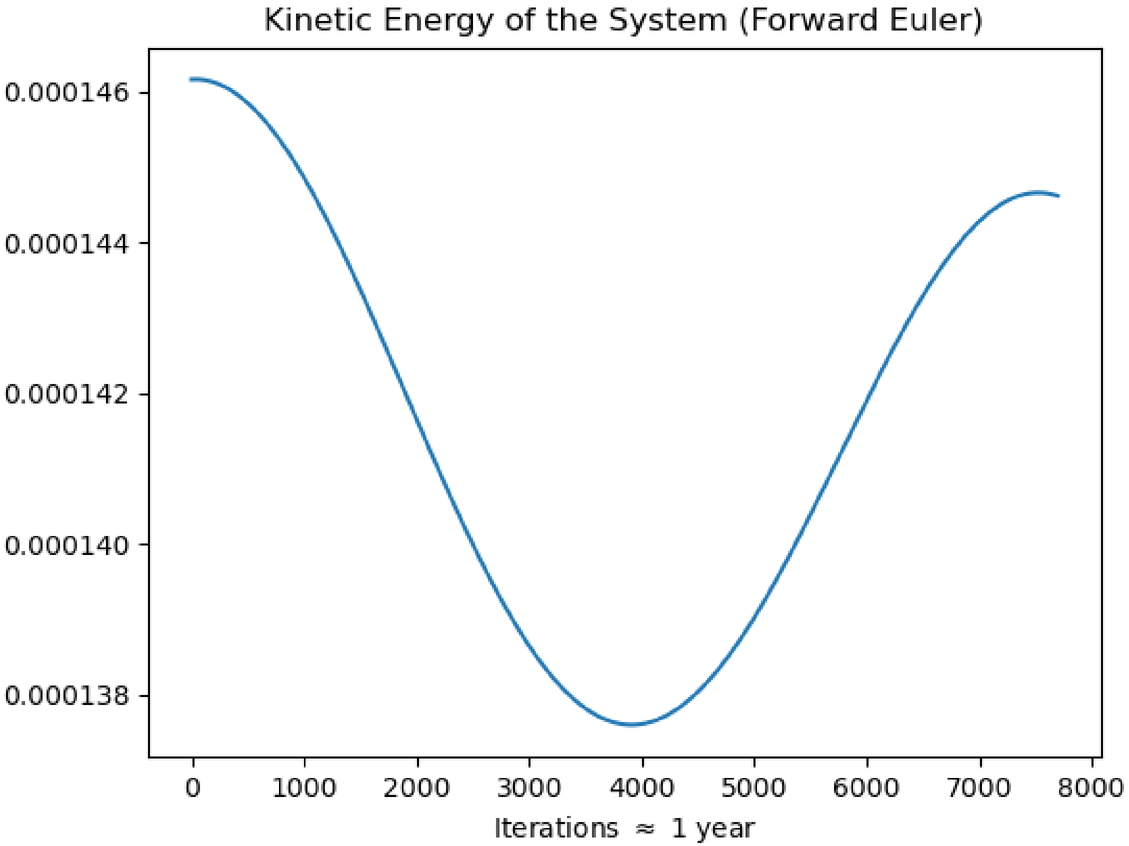
Quantity	SI-Unit	Rescaled unit
velocity	$1\,\frac{m}{\mathrm{s}}$	$1\,\frac{\mathrm{AU}}{\mathrm{day}} \mathrel{\widehat{=}} \frac{1.49\cdot 10^{11}\,\frac{m}{\mathrm{AU}}}{8.64\cdot 10^{4}\,\frac{\mathrm{s}}{\mathrm{day}}} \cdot 1\,\frac{\mathrm{AU}}{\mathrm{day}} \approx 1.72\cdot 10^{6}\,\frac{m}{\mathrm{s}}$
μ	$1\,\frac{m^3}{\mathrm{s}^2}$	$1\,\frac{\mathrm{AU}^3}{\mathrm{day}^2} \mathrel{\widehat{=}} \frac{\left(1.49\cdot 10^{11}\,\frac{m}{\mathrm{AU}}\right)^3}{\left(8.64\cdot 10^{4}\,\frac{\mathrm{s}}{\mathrm{day}}\right)^2} \cdot 1\,\frac{\mathrm{AU}^3}{\mathrm{day}^2} \approx 4.43\cdot 10^{23}\,\frac{m^3}{\mathrm{s}^2}$

Constants

Quantity	SI-Value	Rescaled value
G	$6.67\cdot 10^{-11}\,\frac{m^3}{\mathrm{kg}\cdot\mathrm{s}^2}$	$\tilde{G} = 6.67\cdot 10^{-11}\,\frac{m^3}{\mathrm{kg}\cdot\mathrm{s}^2} \cdot \frac{5.98\cdot 10^{24}\,\frac{\mathrm{kg}}{m_\mathrm{E}}\cdot\left(8.64\cdot 10^{4}\,\frac{\mathrm{s}}{\mathrm{day}}\right)^2}{\left(1.49\cdot 10^{11}\,\frac{m}{\mathrm{AU}}\right)^3} \approx 9.38\cdot 10^{-10}\,\frac{\mathrm{AU}^3}{m_\mathrm{E}\cdot\mathrm{s}^2}$
m_sun	$2\cdot 10^{30}\,\mathrm{kg}$	$\tilde{m}_\mathrm{sun} \mathrel{\widehat{=}} \frac{2\cdot 10^{30}\,\mathrm{kg}}{5.98\cdot 10^{24}\,\frac{\mathrm{kg}}{m_\mathrm{E}}} \approx 334\,448\,m_\mathrm{E}$

Forward Euler

$$
\begin{aligned}
\alpha_n &= \frac{\mu}{\sqrt{x_{n-1}^2+y_{n-1}^2}^3} \\
x_n &= x_{n-1} + \Delta t \cdot u_{n-1} \\
y_n &= y_{n-1} + \Delta t \cdot v_{n-1} \\
u_n &= u_{n-1} - \Delta t \cdot \alpha_n x_{n-1} \\
v_n &= v_{n-1} - \Delta t \cdot \alpha_n y_{n-1} \\
t_n &= t_{n-1} + \Delta t
\end{aligned}
$$


Kinetic Energy of the System (Forward Euler)
0.000146
0.000144
0.000142
0.000140
0.000138
0
1000
2000
3000
4000
5000
6000
7000
8000
Iterations ≈ 1 year

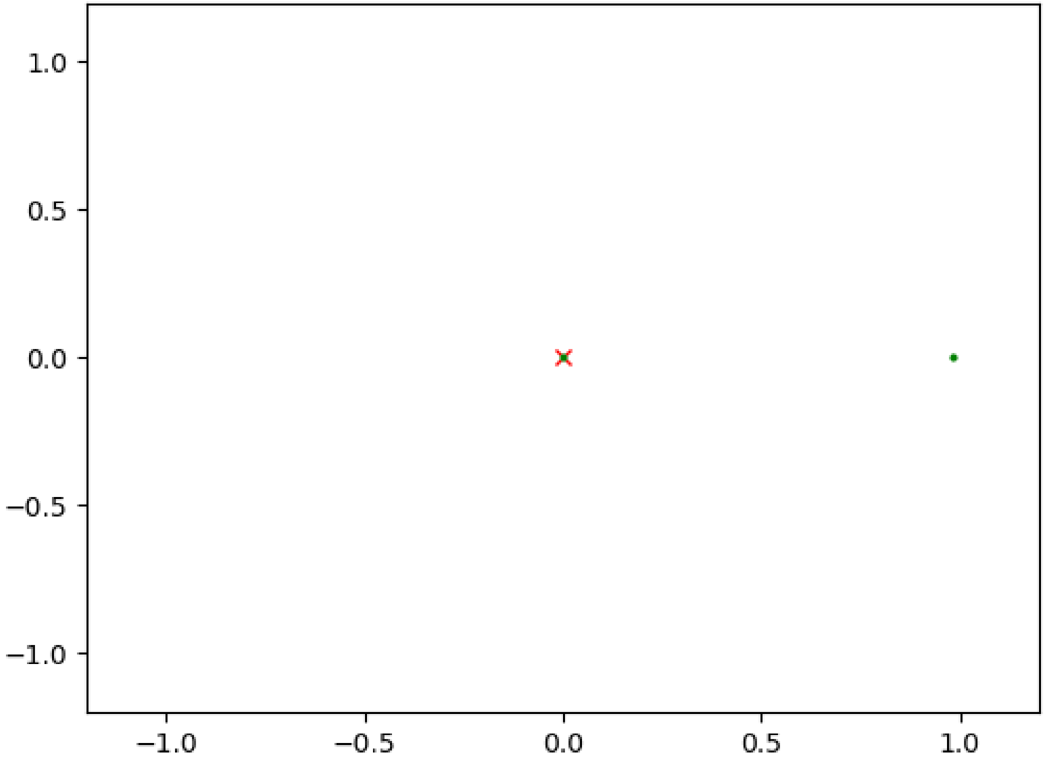
Conservation of Energy

Energy

$$
\begin{aligned}
E_{tot}(t) &= E_{kin_1}(r_1(t)) + E_{kin_2}(r_2(t)) + E_{pot}(\mathbf{r}(t)) \\
&= \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2 - \frac{Gm_1m_2}{||\mathbf{r}||} \\
&= \frac{1}{2}\left(m_1(u_1^2+v_1^2) + m_2(u_2^2+v_2^2)\right) - \frac{Gm_1m_2}{||\mathbf{r}||} \overset{!}{=} const.
\end{aligned}
$$

Backward Euler

$$
\begin{aligned}
\alpha_n &= \frac{\mu}{\sqrt{x_n^2+y_n^2}^3} \\
t_n &= t_{n-1} + \Delta t \\
x_n &= x_{n-1} + \Delta t \cdot u_{\mathbf{n}} \\
y_n &= y_{n-1} + \Delta t \cdot v_{\mathbf{n}} \\
u_n &= u_{n-1} - \Delta t \cdot \alpha_n x_{\mathbf{n}} \\
v_n &= v_{n-1} - \Delta t \cdot \alpha_n y_{\mathbf{n}}
\end{aligned}
$$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Expansion to the n-Body Problem

Expansion for n bodies

$$\ddot{r}_i(t) = G \cdot \sum_{j \neq i} \frac{m_j}{||r_j(t) - r_i(t)||^3}(r_j(t) - r_i(t)) \quad i = 1, ..., n$$

System Energy

$$E_{tot}(t) = \sum_{i=1}^{N} E_{kin_i}(t) + E_{pot}(t)$$

Goals

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- Implement higher-ordered solver, i.e., Runge-Kutta
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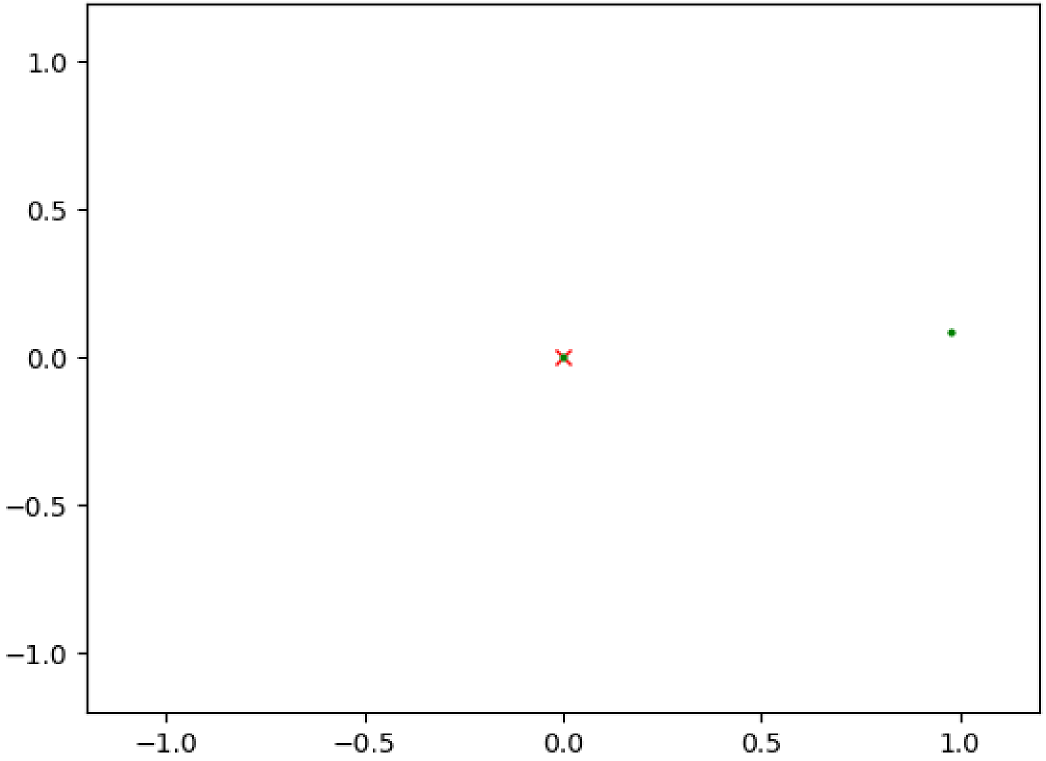
- Discuss leaving entering the atmosphere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