Compact, totally separated and well-ordered types in univalent mathematics

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Our univalent type theory. We work with an intensional Martin-Löf type theory with an empty type $\mathbb{0}$, a one element type $\mathbb{1}$, a type $\mathbb{2}$ with points 0 and 1, a type $\mathbb{N}$ of natural numbers, and type formers $+$ (disjoint sum), Π (product), Σ (sum), W types, and Id (identity type), and a hierarchy of type universes closed under them, ranged over by $\mathcal{U}, \mathcal{V}, \mathcal{W}$. On top of that we add Voevodsky's univalence axiom and a propositional truncation axiom.

A formal version of the development discussed here is available in our github repository TypeTopology, in Agda with the option –without-K. We are considering porting this to cubical Agda, so that no axioms are used and our results get a computational interpretation.

By a *proposition* we mean a type with at most one element (any two of its elements are equal in the sense of the identity type). The *existential quantification* symbol $\exists$ denotes the propositional truncation of Σ. We denote the identity type $\mathrm{Id}\ X\, x\, y$ by $x = y$ with X elided. We assume the notation and terminology of the HoTT Book unless otherwise stated.

Compact types. We consider three notions of exhaustively searchable type. We say that a type X is *compact*, or sometimes Σ-*compact* for emphasis, if the type $\Sigma(x : X), p\, x = 0$ is decidable for every $p : X \to \mathbb{2}$, so that we can decide whether p has a root. We also consider two successively weaker notions, namely $\exists$-*compactness* (it is decidable whether there is an unspecified root) and Π-*compactness* (it is decidable whether all points of X are roots), obtained by replacing Σ by $\exists$ and Π in the definition of compactness.

For the model of simple types consisting of Kleene–Kreisel spaces, these notions of compactness agree and coincide with topological compactness under classical logic, but we reason constructively here, meaning that we don't invoke (univalent) choice or excluded middle.

Finite types of the form $\mathbb{1} + \mathbb{1} + \cdots + \mathbb{1}$ are clearly compact. The compactness of $\mathbb{N}$ is LPO (limited principle of omniscience), which happens to be equivalent to its $\exists$-compactness, and its Π-compactness is equivalent to WLPO (weak LPO), and hence all forms of compactness for $\mathbb{N}$ are not provable or disprovable in our classically/constructively-neutral foundation.

An example of an *infinite* compact type is that of conatural numbers, $\mathbb{N}_\infty$, also known as the *generic convergent sequence* (this was presented in Types'2011 in Bergen). This type, the final coalgebra of $- + \mathbb{1}$, is not directly available in our type theory, but can be constructed as the type of decreasing infinite binary sequences.

We are able to construct plenty of *infinite* compact types, and it turns out they all can be equipped with well-orders making them into ordinals.

Ordinals. An ordinal is a type X equipped with a proposition-valued binary relation $- < - : X \to X \to \mathcal{U}$ which is transitive, well-founded (satisfies transfinite induction), and extensional (any two elements with the same predecessors are equal). The HoTT Book additionally requires the type X to be a set, but we show that this follows automatically from extensionality. For example, the types of natural and conatural numbers are ordinals. By univalence, the type of ordinals in a universe is itself an ordinal in the next universe, and in particular is a set.

Addition is implemented by the type former $- + -$, and multiplication by the type former Σ with the lexicographic order. The compact ordinals we construct are, moreover, *order-compact* in the sense that a minimal element of $\Sigma(x : X), p\,x = 0$ is found, or else we are told that this type is empty. Additionally, we have a selection function of type $(X \to \mathbb{2}) \to X$ which gives the infimum of the set of roots of any $p : X \to \mathbb{2}$, and in particular our compact ordinals have a top element by considering $p = \lambda x.1$.

Discrete types. We say that a type is *discrete* if it has decidable equality.

Totally separated types. It may happen that a non-trivial type has no nonconstant function into the type $\mathbb{2}$ of booleans so that it is trivially compact. For example, this would be the case for a type of real numbers under Brouwerian continuity axioms. Under such axioms, such types are compact, but in a uninteresting way. We say that a type is *totally separated*, again borrowing a terminology from topology, if the functions into the booleans separate the points, in the sense that any two points that satisfy the same boolean-valued predicates are equal. This can be seen as a boolean-valued Leibniz principle. Such a type is automatically a set, or a 0-groupoid, in the sense of univalent mathematics. We construct a totally separated reflection for any type, and show that a type is compact, in any of the three senses, if and only if its totally separated reflection is compact in the same sense.

Interplay between the notions. We show that compact types, totally separated types, discrete types and function types interact in very much the same way as their topological counterparts, where arbitrary functions in type theory play the role of continuous maps in topology, *without* assuming Brouwerian continuity axioms. For instance, if the types $X \to Y$ and Y are discrete then X is Π-compact, and if $X \to Y$ is Π-compact, and X is totally separated and Y is discrete, then X is discrete, too. The simple types are all totally separated, which agrees with the situation with Kleene–Kreisel spaces, but it is easy to construct types which fail to be totally separated (e.g. the homotopical circle) or whose total separatedness gives a constructive taboo (e.g. $\Sigma(x : \mathbb{N}_\infty), x = \infty \to \mathbb{2}$, where we get two copies of the point ∞).

Notation for discrete and compact ordinals. We define infinitary ordinal codes, or expression trees, similar to the so-called Brouwer ordinals, including one, addition, multiplication, and countable sum with an added top point.

We interpret these trees in two ways, getting discrete and compact ordinals respectively. In both cases, addition and multiplication nodes are interpreted as ordinal addition and multiplication. But in the countable sum with a top point, the top point is added with $- + \mathbb{1}$ in one case, and so is isolated, and by a limit-point construction in the other case (given our sequence $\mathbb{N} \to \mathcal{U}$ of types, we extend it to a family $\mathbb{N}_\infty \to \mathcal{U}$ so that it maps ∞ to a singleton type, by a certain universe injectivity construction, and then take its sum).

We denote the above interpretations of ordinal notations ν by Δ_ν and K_ν. The types in the image of Δ are discrete and retracts of $\mathbb{N}$, and those in the image of K are compact, totally separated and retracts of the Cantor type $\mathbb{N} \to \mathbb{2}$. Moreover, there is an order preserving and reflecting embedding $\Delta_\nu \to K_\nu$, which is an isomorphism if and only if LPO holds, but whose image always has empty complement for all ordinal notations ν. An example of such a situation is the evident embedding $\mathbb{N} + \mathbb{1} \to \mathbb{N}_\infty$ (this inclusion is merely a monomorphism, rather than a topological embedding, in topological models – the word *embedding* in univalent mathematics refers to the appropriate notion for ∞-groupoids, which in this example are 0-groupoids). By transfinite iteration of the countable sum, one can get rather large compact ordinals.