Tutorial #1: Introduction to Algebra

1 Exercise 1:

Proof:

$$\sum_{i=1}^{n} i^2 = \frac{(n)(n+1)(2n+1)}{6} \tag{1}$$

Base case: When $n = 1$, the left hand side (LHS) of (1) is $1^2 = 1$, and the right hand side (RHS) of (1) is $\frac{1\times(1+1)\times(2\times1+1)}{6} = 1$. Thus, (1) is true for $n = 1$

Induction step:

Let $k \in \mathbb{N}$ be given and suppose (1) is true for $n = k$. Then:

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \frac{(k+1)(k+2)(2k+3)}{6} \\
\sum_{i=1}^{k} i^2 + (k+1)^2 &= \frac{(k+1)(k+2)(2k+3)}{6} \\
\frac{(k)(k+1)(2k+1)}{6} + (k+1)^2 &= \frac{(k+1)(k+2)(2k+3)}{6} \\
\frac{(k+1)(2k^2+k+6(k+1))}{6} &= \frac{(k+1)(k+2)(2k+3)}{6} \\
\frac{(k+1)(2k^2+7k+6)}{6} &= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}$$

Thus, (1) holds to $n = k+1$ and the proof of induction step is complete.

Conclusion: By the principle of induction, (1) is true for all $n \in \mathbb{N}$

2 Exercise 2:

Proof:

$$n! > 2^n \; for \; n \geq 4 \tag{2}$$

Base case: When $n = 4$, $LHS = 4! = 24, RHS = 2^4 = 16$. Thus, (2) holds for $n = 4$.
Induction step: Let $k \in \mathbb{N}$ be given and suppose (2) is true for $n = k$. Then:

$$\begin{aligned} (k+1)! &> 2^{k+1} \\ (k+1)! = k! \times (k+1) &> 2k! > 2^{k+1} \end{aligned}$$

Thus, (2) holds to $n = k+1$ and the proof of induction step is complete.
Conclusion: By the principle of induction, (2) is true for all $n \geq 4$

3 Exercise 3:

Proof: $10^{n+1} + 10^n + 1$ is divisible by 3 $\forall n \in \mathbb{N}$.
As $10^{n+1} + 10^n + 1$ is represented as 11000.....001. Thus, the total of digit values is 3, which is divisible for 3. Thus, $10^{n+1} + 10^n + 1$ is divisible by 3 $\forall n \in \mathbb{N}$.

4 Exercise 4:

Proof:

$$\sum_{i=0}^{n} 2^n = 2^{n+1} - 1 \tag{3}$$

Base case: When $n = 1$, LHS $= 2^0 + 2^1 = 3$, RHS $= 2^{1+1} - 1 = 3$. Thus, (3) holds for $n = 1$.
Induction step: Let $k \in N$ be given and suppose (2) is true for $n = k$.

Then:

$$\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$$
$$\sum_{i=0}^{k} 2^i + 2^{k+1} = 2^{k+2} - 1$$
$$2^{k+1} - 1 + 2^{k+1} = 2^{k+2} - 1$$
$$2^{k+1} + 2^{k+1} = 2^{k+2}$$

Thus, (3) holds for $n = k+1$ and the proof of induction step is complete.
Conclusion: By the principle of induction, (3) is true $\forall n \in \mathbb{N}$

5 Exercise 5:

a)

$$3x \equiv 2 \ (mod\ 7)$$
$$3x \times 5 \equiv 2 \times 5 \ (mod\ 7)$$
$$x \equiv 3 \ (mod\ 7)$$

b)

$$5x + 1 \equiv 13 \ (mod\ 23)$$
$$5x \equiv 12 \ (mod\ 23)$$
$$5x \times 14 \equiv 12 \times 14 \ (mod\ 23)$$
$$x \equiv 7 \ (mod\ 23)$$

c

$$2x \equiv 1 \ (mod\ 6)$$

Since there are not exist such k that $2k = 1$, (c) has not solution.

6 Exercise 6:

$$\begin{cases} 3x + 7y \equiv 4 \pmod{11} \\ 8x + 6y \equiv 1 \pmod{11} \end{cases}$$

Cayley table for $\mathbb{Z}_{11}$

$\mathbb{Z}_{11}$	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	1	3	5	7	9
3	3	6	9	1	4	7	10	2	5	8
4	4	8	1	5	9	2	6	10	3	7
5	5	10	4	9	3	8	2	7	1	6
6	6	1	7	2	8	3	9	4	10	5
7	7	3	10	6	2	9	5	1	8	4
8	8	5	2	10	7	4	1	9	6	3
9	9	7	5	3	1	10	8	6	4	2
10	10	9	8	7	6	5	4	3	2	1

$$\begin{cases} 3x + 7y \equiv 4 \pmod{11} \\ 8x + 6y \equiv 1 \pmod{11} \end{cases} \iff \begin{cases} 8x + 4y \equiv 7 \pmod{11} \\ 8x + 6y \equiv 1 \pmod{11} \end{cases}$$

$$\iff \begin{cases} 8x + 4y \equiv 7 \pmod{11} \\ 2y \equiv 5 \pmod{11} \end{cases} \iff \begin{cases} 8x + 4 \times 8 \equiv 7 \pmod{11} \\ y \equiv 8 \pmod{11}) \end{cases}$$

$$\iff \begin{cases} 8x \equiv 8 \pmod{11} \\ y \equiv 4 \pmod{11} \end{cases} \iff \begin{cases} x \equiv 1 \pmod{11} \\ y \equiv 8 \pmod{11} \end{cases}$$

7 Exercise 7:

$$\begin{cases} 3x + 5y - 7z \equiv 8 \pmod{83} \\ 8x - 9y + 13z \equiv 13 \pmod{83} \\ 7x + 4y + 5z \equiv 15 \pmod{83} \end{cases} \iff \begin{cases} x + 57y + 53z \equiv 58 \pmod{83}\ (\times 28) \\ 8x - 9y + 13z \equiv 13 \pmod{83} \\ 7x + 4y + 5z \equiv 15 \pmod{83} \end{cases}$$

$$\iff \begin{cases} x + 57y + 53z \equiv 58 \pmod{83} \\ -56y + 29z \equiv -29 \pmod{83} \\ -6y + 19z \equiv -1 \pmod{83} \end{cases} \iff \begin{cases} x + 57y + 53z \equiv 58 \pmod{83} \\ 27y + 29z \equiv -29 \pmod{83} \\ -6y + 19z \equiv -1 \pmod{83} \end{cases}$$

$$\iff \begin{cases} x + 57y + 53z \equiv 58 \pmod{83} \\ y + 81z \equiv 2 \pmod{83}\ (\times 40) \\ -6y + 19z \equiv -1 \pmod{83} \end{cases} \iff \begin{cases} x + 37y - 2z \equiv 26 \pmod{83} \\ y + 81z \equiv 2 \pmod{83} \\ 7z \equiv 15 \pmod{83} \end{cases}$$

$$\iff \begin{cases} x + 57y + 53z \equiv 58 \pmod{83} \\ y + 81z \equiv 2 \pmod{83} \\ z \equiv 14 \pmod{83} \end{cases} \iff \begin{cases} x \equiv 23 \pmod{83} \\ y \equiv 30 \pmod{83} \\ z \equiv 14 \pmod{83} \end{cases}$$

Any mistakes might be considered later ;)

8 Exercise 8:

Cayley table for $\mathbb{Z}_7$

$\mathbb{Z}_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

$$A = \begin{bmatrix} 5 & 2 \\ 6 & 3 \end{bmatrix} \Rightarrow [A|I] = \begin{bmatrix} 5 & 2 & 1 & 0 \\ 6 & 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 3 & 0 \\ 6 & 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 3 & 0 \\ 0 & 2 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 & 3 & 0 \\ 0 & 1 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 5 & 4 \end{bmatrix} = [I|A^{-1}]$$

Thus, the inverse of A is $A^{-1} = \begin{bmatrix} 1 & 4 \\ 5 & 4 \end{bmatrix}$