Tutorial #3.4: Cosets

Exercise 1:

- $< 8 >$ in $(\mathbb{Z}_{24}, +)$
  Since $< 8 >= \{0, 8, 16\}$, the left cosets of $< 8 >$ in $\mathbb{Z}_{24}$ is

$$\begin{aligned}
0+ < 8 > &= \{0, 8, 16\} = 8+ < 8 >= 16+ < 8 > \\
1+ < 8 > &= \{1, 9, 17\} = 9+ < 8 >= 17+ < 8 > \\
2+ < 8 > &= \{2, 10, 18\} = 10+ < 8 >= 18+ < 8 > \\
3+ < 8 > &= \{3, 11, 19\} = 11+ < 8 >= 19+ < 8 > \\
4+ < 8 > &= \{4, 12, 20\} = 12+ < 8 >= 20+ < 8 > \\
5+ < 8 > &= \{5, 13, 21\} = 13+ < 8 >= 21+ < 8 > \\
6+ < 8 > &= \{6, 14, 22\} = 14+ < 8 >= 22+ < 8 > \\
7+ < 8 > &= \{7, 15, 23\} = 15+ < 8 >= 23+ < 8 >
\end{aligned}$$

  It's trivial that $(\mathbb{Z}_{24}, +)$ is an abelian groanan. Thus $(\mathbb{Z}_{24}, +)$ is commutative and the right cosets is the same as above.

- $< 3 >$ in $(U(8), .)$
  Since $< 3 >= \{1, 3\}, (U(8), .) = \{1, 3, 5, 7\}$, the cosets of $< 3 >$ in $(U(8), .)$ is

$$\begin{aligned}
1* < 3 > &= \{1, 3\} = 3* < 3 > \\
5* < 3 > &= \{5, 7\} = 7* < 3 >
\end{aligned}$$

  It's trivial that $(U(8), .)$ is an abelian group. Thus $(U(8), .)$ is commutative and the right cosets is the same as above.

Exercise 2:

Let k be the order of group G. Assume that the order k does not divides n. That we have: $n = kd + r$ with $d, r \in \mathbb{N}, 0 < r < k$. Since $g^n = e \iff g^{kd+r} = e \iff g^{kd}g^r = e \iff eg^r = e \iff g^r = e$, this contradicts with the property of order of the group.

Conclusion: The order of group k must divides n.

Exercise 3:

$H = \{3k, k \in \mathbb{N}\}$, the left cosets of $(H, +)$ in $\mathbb{Z}$ are:

$$\begin{aligned} 0 + H &= \{3k, k \in \mathbb{N}\} && = 3d + H && \forall d \in \mathbb{Z} \\ 1 + H &= \{3k + 1, k \in \mathbb{N}\} && = 3d + 1 + H && \forall d \in \mathbb{Z} \\ 2 + H &= \{3k + 2, k \in \mathbb{N}\} && = 3d + 2 + H && \forall d \in \mathbb{Z} \end{aligned}$$

1. 11 + H and 17 + H
   Since 11 and 17 $\in \{3d + 1, d \in \mathbb{Z}\}$, these two cosets are equivalent.
2. -1 + H and 5 + H
   Since -1 and 5 $\in \{3d + 2, d \in \mathbb{Z}\}$, these two cosets are equivalent.
3. 7 + H and 23 + H
   Since 7 + H $\in \{3d + 1, d \in \mathbb{Z}\}$, 23 + H $\in \{3d + 2, d \in \mathbb{Z}\}$, these two cosets are not equivalent.

Exercise 4:

G is a group order by 15 and is generated by a. Thus, $G = \{a^1, a^2, ...a^{14}, e\}$.
Left cosets of $< a^5 >$ in $< a >$ are:

$$
\begin{aligned}
e* < a^5 > &= \{e, a^5, a^{10}\} \\
a* < a^5 > &= \{a, a^6, a^{11}\} \\
a^2* < a^5 > &= \{a^2, a^7, a^{12}\} \\
a^3* < a^5 > &= \{a^3, a^8, a^{13}\} \\
a^4* < a^5 > &= \{a^4, a^9, a^{14}\}
\end{aligned}
$$

Exercise 5:

Since G is a group of order 60, let $a \in G$, we have $a^{60} = e$ following the property of order of a group.
Let k be the order of the subgroup of G. Thus, $a^k = e,\ k \in \mathbb{N}$.
Since $e^n = e\ \forall n \in \mathbb{N}$ and $a^k = e, a^{60} = e$, there exist such $n \in \mathbb{N}$ satisfies that $kn = 60$. Hence, k must divides n.
Conclusion: $k \in \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$