Tutorial #4: Isomorphism

Exercise 1:

1. $f : (\mathbb{R}^*, .) \rightarrow (\mathbb{R}^{2\times 2}, .)$; $f(x) = \begin{bmatrix} x & 0 \\ 1 & 1 \end{bmatrix}$
   Let $a, b \in \mathbb{R}^*$,
   $$f(a) = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, f(b) = \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix}, f(a)f(b) = \begin{bmatrix} ab & 0 \\ 0 & 1 \end{bmatrix}$$
   Since $a, b \in (\mathbb{R}^*, .)$, $ab \in (\mathbb{R}^*, .)$ as the closure property of a group,
   $$f(ab) = \begin{bmatrix} ab & 0 \\ 0 & 1 \end{bmatrix}$$
   Since $f(ab) = f(a)f(b)$, f is homomorphism.
   Kernel of $f = \{x : f(x) = e_H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\}$. Thus, $ker(f) = \{1\}$.

2. $f : (R, +) \rightarrow (R^{2\times 2}, .)$; $f(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$
   Let $a, b \in \mathbb{R}$,
   $$f(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, f(b) = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, f(a)f(b) = \begin{bmatrix} 1 & b + a \\ 0 & 1 \end{bmatrix}$$
   Since $a, b \in (\mathbb{R}, +)$, $(a + b) \in (\mathbb{R}, +)$ as the closure property of the group,
   $$f(a + b) = \begin{bmatrix} 1 & a + b \\ 0 & 1 \end{bmatrix}$$
   Since $f(a + b) = f(a)f(b)$, f is homomorphism.
   Kernel of $f = \{x : f(x) = e_H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\}$. Thus, $ker(f) = \{0\}$.

Exercise 2:

Prove that: $G = (\mathbb{R}\backslash\{-1\}, *)$ is an abelian group with $a*b = a+b+ab$.

i) Let a, b, c be arbitrary in $\mathbb{R} \setminus \{-1\}$, we have:

$$\begin{aligned}(a * b) * c = (a + b + ab) * c &= (a + b + ab) + c + (a + b + ab)c \\ &= a + b + c + ab + bc + ac + abc \quad (1)\end{aligned}$$

$$\begin{aligned}a * (b * c) = a * (b + c + bc) &= a + (b + c + bc) + a(b + c + bc) \\ &= a + b + c + ab + bc + ac + abc \quad (2)\end{aligned}$$

With (1) = (2), we conclude that G is associative.

ii) There exists $e = 0$ such that $a * e = (a + 0 + a * 0) = a$. Thus, G has identity element.

iii) With arbitrary element $a \in \mathbb{R} \setminus \{-1\}$, there exists a^{-1} is an inverse element of a. Indeed:

$$\begin{aligned}a * a^{-1} = e &\iff a + a^{-1} + aa^{-1} = 0 \\ &\iff a^{-1} = \frac{-a}{a+1}\end{aligned}$$

iv) Since $a * b = a + b + ab = b + a + ba = b * a$, G is commutative.

Hence, G is an abelian group.
Prove that: $f : (G, *) \to (\mathbb{R} \setminus \{0\}, .); f(x) = x + 1$ is homomorphism.
Since $f(a) = a+1$, $f(b) = b+1$, $f(a)f(b) = (a+1)(b+1) = ab+a+b+1$.
On the other hand, $f(a * b) = f(a + b + ab) = a + b + ab + 1$.
Conclusion: f is homomorphism.

Exercise 3:

Prove that: $\phi(g^n) = (\phi(g))^n$. Given that $g \in G$, $f : (G, *) \to (H, \circ)$.
Base case: When $n = 2$, $\phi(g * g) = \phi(g) \circ \phi(g) \iff \phi(g^2) = (\phi(g))^2$.

Thus, $\phi(g^n) = (\phi(g))^n$ holds for $n = 2$.
Induction step: Let $k \in \mathbb{N}$ be given and suppose that $\phi(g^n) = (\phi(g))^n$ for $n = k$,

$$\phi(g^{k+1}) = \phi(g^k * g) = (\phi(g))^k \circ \phi(g) = (\phi(g))^{k+1}$$

Hence, $\phi(g^n) = (\phi(g))^n$ and the proof of induction step is complete.
Conclusion: By the principle of induction, $\phi(g^n) = (\phi(g))^n, \forall n \in \mathbb{N}$

Exercise 4:

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