Midterm 2019

Special thanks to BI8 has given us this :*

Problem 1.1:

Prove that: For $n \in \mathbb{N}$,

$$\sum_{j=1}^{n} j^3 = \frac{1}{4}n^2(n+1)^2 \tag{1}$$

Base case: When $n = 1, LHS = 1^3 = 1, RHS = \frac{1}{4}1^2(1+1)^2 = 1$. Thus, (1) holds for $n = 1$.
Induction step: Let $k \in \mathbb{N}$ be given and suppose that (1) is true for $n = k$. Then:

$$\begin{aligned}
\sum_{j=1}^{k+1} j^3 &= \frac{1}{4}(k+1)^2(k+2)^2 \\
\sum_{j=1}^{k} j^3 + (k+1)^3 &= \frac{1}{4}(k+1)^2(k+2)^2 \\
\frac{1}{4}k^2(k+1)^2 + (k+1)^3 &= \frac{1}{4}(k+1)^2(k+2)^2 \\
(k+1)^2(k^2+4k+4) &= (k+1)^2(k+2)^2
\end{aligned}$$

Thus, (1) holds for $n = k+1$, the proof of induction step is complete.
Conclusion: By principle of induction, (1) is true for all $n \in \mathbb{N}$.

Problem 2.1:

$$\begin{cases} 3x + 5y \equiv 14 \pmod{17} \\ 7x + 3y \equiv 6 \pmod{17} \end{cases} \iff \begin{cases} x + 13y \equiv 16 \pmod{17} \\ 7x + 3y \equiv 6 \pmod{17} \end{cases}$$

$$\iff \begin{cases} x + 13y \equiv 16 \pmod{17} \\ 3y \equiv 4 \pmod{17} \end{cases} \iff \begin{cases} x \equiv 10 \pmod{17} \\ y \equiv 7 \pmod{17} \end{cases}$$

Problem 3.1:

$$f(x) = \sqrt{x^3 - 7}$$

$f(x)$ has domain $D = [\sqrt[3]{7}, +\infty)$, range $R = [0, +\infty)$
Let $y = f(x)$, we can re-write the function as:

$$y = \sqrt{x^3 - 7} \iff x = \sqrt[3]{y^2 + 7}$$

Thus, the inverse function of $f(x)$ is $f^{-1} = \sqrt[3]{x^2 + 7}$

$$f \circ f^{-1} = f(f^{-1}(x)) = \sqrt{(\sqrt[3]{x^2 + 7})^3 - 7} = |x| = x \tag{2}$$

$$f^{-1} \circ f = f^{-1}(f(x)) = \sqrt[3]{(\sqrt{x^3 - 7})^2 + 7} = x \tag{3}$$

Conclusion: Since (2) = (3) with $x \in D$, $f \circ f^{-1} = f^{-1} \circ f\ \forall x \in D$.

Problem 1.2:

Prove that: For $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

Since i'm sleepy and this is such trivial that I proved it in 5^{th} grade, reader can prove it himself/herself.

Problem 2.2:

$$\begin{cases} 5x - 6y \equiv 9 \pmod{22} \\ 8x + y \equiv 12 \pmod{22} \end{cases} \iff \begin{cases} x - 10y \equiv 15 \pmod{22} \\ 8x + y \equiv 12 \pmod{22} \end{cases}$$

$$\iff \begin{cases} x - 10y \equiv 15 \pmod{22} \\ 15y \equiv 2 \pmod{22} \end{cases} \iff \begin{cases} x \equiv 9 \pmod{22} \\ y \equiv 6 \pmod{22} \end{cases}$$

Problem 3.2:

$$f(x) = x^2 - 3x + 2$$

$f(x)$ has domain $D = \mathbb{R}$, range $R = (-\frac{1}{4}, +\infty)$.
Let $f(x) = y$, we can re-write the function as:

$$y = x^2 - 3x + 2 \iff 0 = x^2 - 3x + (2 - y) \iff x = \frac{3 \pm \sqrt{3^2 - (2 - y)}}{2}$$

Thus, the inverse function of $f(x)$ is $f^{-1} = \frac{3 \pm \sqrt{y+7}}{2}$.
We can simple prove that $f \circ f^{-1} = f^{-1} \circ f = x$ by using Linear Algebra which is already learned in high school ;).