Definition

Let δ be a permutation on S_A

Define a relation on A by

$$a \sim b \text{ iff } b = \delta^n(a) \text{ for some } n \in \mathbb{Z}$$

Notice : this relation is

- reflexive $a \sim a$ bcs $a = \delta^0(a) = id(a)$
- symmetric $\forall a, b \in A$

$$a \sim b \Leftrightarrow \exists n \in \mathbb{Z},\ b = \delta^n(a)$$
$$\Rightarrow \exists n' = -n \in \mathbb{Z},\ a = \delta^{n'}(b) \Leftrightarrow b \sim a$$

- transitive $\forall a, b \in A$

$$a \sim b \wedge b \sim c \Leftrightarrow \exists n, n' \in \mathbb{Z},$$
$$b = \delta^n(a) \wedge c = \delta^{n'}(b)$$
$$a \sim c \Leftrightarrow \exists \underset{n+n'}{\overset{\shortparallel}{n''}} \in \mathbb{Z},\ c = \delta^{n'}(\delta^n(a))$$
$$= \delta^{n'+n}(a)$$

$\therefore$ it is an equivalence relation on A + partitions A into nonempty, disjoint subsets called equivalence classes, of form

$$E_{a,\sim} = E_a = \{ b \in A \mid a \sim b \}$$

$$\quad \| \qquad \| = \{ b \in A \mid b = \delta^n(a) \text{ for } n \in \mathbb{Z} \}$$

$\mathcal{O}_{a,\delta,} = \mathcal{O}_a$ is the **orbit** of elements

Example

$$\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 1 & 4 & 2 & 9 & 3 & 8 \end{pmatrix} \in S_9$$

Find all orbits of δ

$$\mathcal{O}_{1,\delta} = \{ k \in I_9 \mid k = \delta^n(1) \text{ for } n \in \mathbb{Z} \}$$

$$= \{ 1, 4, 5 \}$$

$$= \mathcal{O}_4 = \mathcal{O}_5$$

$$\mathcal{O}_{2,\delta} = \{ k \in I_9 \mid k = \delta^n(2) \text{ for } n \in \mathbb{Z} \}$$

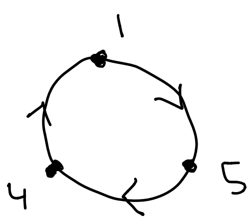
$$= \{6, 2\}$$
$$= \mathcal{O}_6$$

$$\mathcal{O}_{3,8} = \{3, 7, 9, 8\}$$
$$= \mathcal{O}_9, \mathcal{O}_8$$

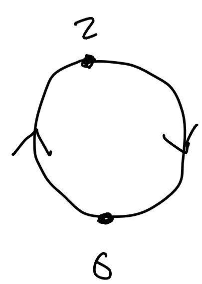
Since $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 = I_9$

there are exactly 3 orbits

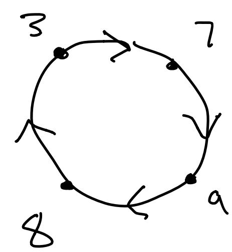
$\mathcal{O}_1$



$\mathcal{O}_2$



$\mathcal{O}_3$



Cycles:

$(1, 5, 4)$ $\qquad$ $(2, 6)$ $\qquad$ $(3, 7, 9, 8)$

$$(1,5,4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 2 & 3 & 1 & 4 & 6 & 7 & 8 & 9 \end{pmatrix}$$

Definition

A cycle is a permutation w/ at most 1 orbit w/ more than 1 element.

The length of a cycle is the number of elements in the orbit.

Cycles are disjoint if they do not move any of the same elements.

Notice:

Any permutation can be presented as a product of disjoint cycles by finding its orbits + corresponding cycles

Can use permutation multiplication to multiply cycles, however, a product of cycles is not necessarily a cycle

Permutation multiplication of <u>disjoint</u> cycles is commutative

$$\delta = (1,5,4)\cdot(2,6)\cdot(3,7,9,8)$$
$$= (2,6)\cdot(1,5,4)\cdot(3,7,9,8)$$

Example

Let $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 5 & 7 & 1 & 8 & 4 & 2 \end{pmatrix} \in S_8$

$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 4 & 5 & 3 & 7 & 6 & 1 & 2 \end{pmatrix} \in S_8$

Present δ, σ as products of disjoint cycles + find their products

$$\delta = \underset{\overset{\shortparallel}{\theta_1}}{(1,3,5)}\,\underset{\overset{\shortparallel}{\theta_2}}{(2,6,8)}\,\underset{\overset{\shortparallel}{\theta_4}}{(4,7)}$$

$$\begin{array}{ccc} \shortparallel & \shortparallel & \shortparallel \\ O_3 & O_6 & O_7 \\ \shortparallel & \shortparallel & \\ O_8 & O_5 & \end{array}$$

$$\sigma = (1,8,2,4,3,5,7)(6)$$

$$O_1 \cup O_6 = I_8$$

$$\begin{aligned} \delta \cdot \sigma &= (1,3,5)(2,6,8)(4,7) \\ &\quad \cdot (1,8,2,4,3,5,7) \\ &= (1,2,7,3) \cdot (4,5) \cdot (6,8) \end{aligned}$$

$$\begin{aligned} \sigma \cdot \delta &= (1,8,2,4,3,5,7) \\ &\quad \cdot (1,3,5)(2,6,8)(4,7) \\ &= (1,5,8,4) \cdot (2,6) \cdot (3,7) \end{aligned}$$

Note: every permutation can be presented as a product of disjoint cycles uniquely up to the rearangement of cycles in the product.

Definition

A cycle of length 2 is called a **transposition**, that is,

$$(a,b) \in S_A \quad \text{where } a,b \in A$$

Any cycle can be presented as a product of transposition as follows

$$(a_1, a_2, \dots, a_{n-1}, a_n) = (a_1, a_n) \cdot (a_1, a_{n-1}) \cdot \dots (a_1, a_2)$$

$$\text{(OR)} \quad = (a_1, a_2) \cdot (a_2, a_3) \cdot \dots (a_{n-1}, a_n)$$

for a total of $n-1$ transpositions

Note:

Any permutation can be presented as a product of transpositions by first presenting it as a product of disjoint cycles

Definition

A permutation is **even** iff it can be presented as a product of an even number of transpositions

Otherwise it is **odd**

Example

Let $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 2 & 8 & 9 & 4 & 1 & 3 & 7 \end{pmatrix} \in S_9$

Decide whether δ is odd or even

$$\delta = (1,5,9,7) \cdot (2,6,4,8,3)$$

now as a product of transpositions

$$= (1,7)(1,9)(1,5) \cdot (2,6)(6,4)(4,8)(8,3)$$

is an <u>odd</u> permutation

Note:

Identity of S_n is

$$id = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix} = (k, m)(m, k) = (k, m)(k, m)$$
$$= (k, m)^2$$

for $m, k \in I_n$

$\therefore$ The rule is that the identity of S_n for $n \geq 2$ is

$$(1, 2)(2, 1) = (1, 2)^2$$

hence, it is an <u>even</u> permutation

Let A_n be the set of all even permutations of I_n

let B_n " " odd permutations of I_n

$$A_n, B_n \subseteq S_n$$

$|A_n| = |B_n|$ bcs $\phi : A_n \to B_n$

$\phi(\delta) = \sigma \cdot \delta$ where σ is a fixed transposition in S_n

ϕ is a bijection

ϕ is injective bcs