Recall

$(G, *)$ - group iff

1) Associative $*$ on G
2) Unique identity e in G
3) $\forall g \in G$ have unique inverse

Definition

Let $(G, *)$ be a group

let $S \subseteq G$

Then S is a **subgroup** of G

$$S \leq G \text{ iff}$$

S has group structure under $*$ induced from G

If $S \subset G$, then S is a **proper subgroup** of G

$$S < G$$

Example

$$(\mathbb{Z}, +) < (\mathbb{Q}, +) < (\mathbb{R}, +) < (\mathbb{C}, +)$$

$(U, \cdot) < (\mathbb{C}^{\times}, \cdot)$

$\{z \in \mathbb{C} \mid |z| = 1\}$

<u>Fact</u>

Every nontrivial group has at least 2 subgroups

1) trivial / identity subgroup ($\{e\}$)
2) improper subgroup G

other subgroups are called the nontrivial subgroups of G

$$\forall G, \quad \underset{\text{trivial}}{\{e\} \leq G}, \quad \underset{\text{improper}}{G \leq G}$$

If $|G| = n$ $n \in \mathbb{Z}^{+}$, then G is a **finite** group

(i.e. finitely countable)

If G is denumerable (that is, $|G| = \aleph_0$) or uncountable, then G is an **infinite** group

Recall

Up to order 3, there is only one group structure

$$|G| = 1 \Rightarrow G = \{e\}$$
$$|G| = 2 \Rightarrow G \cong \mathbb{Z}_2$$
$$|G| = 3 \Rightarrow G \cong \mathbb{Z}_3$$

consider set $S = \{e, a, b, c\}$ wrt $*$

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

There are 2 group structures (abelian) of order 4, namely

$(\mathbb{Z}_4, +_4) \qquad (\mathbb{Z}_2 \times \mathbb{Z}_2, +_2)$

Subgroup diagram illustrater sub struct. for finite groups

$\mathbb{Z}_4$

$\{0\}$

$\{0, 2\}$

$$\begin{tikzcd}
 & \mathbb{Z}_2 \times \mathbb{Z}_2 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\{(0,0), (1,1)\} \arrow[dr, dash] & \{(0,0), (0,1)\} \arrow[d, dash] & \{(0,0), (1,0)\} \arrow[dl, dash] \\
 & \{(0,0)\} &
\end{tikzcd}$$

$\Rightarrow \mathbb{Z}_4 \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$

Note: for finite group order n, any possible subgroup has to have an order that is a divisor on n

Fact

Let $(G, *)$ be a group

let $S \subseteq G$

Then $S \leq G$ under $*$ iff

1) S is closed under $*$ induced from G

$\forall s_1, s_2 \in S \quad s_1 * s_2 \in S \implies$ associativity

2) Identity of G wrt $*$ is in S

3) All elems of S have inverses in S

$\forall s \in S \quad s' \in S \quad s * s' = s' * s = s$

Example

$GL(n, \mathbb{R}) = GL_n(\mathbb{R})$ wrt matrix mult

$= \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\}$

consider $S = \{A \in M_n(\mathbb{R}) \mid \det A = 1\} \subset GL_n(\mathbb{R})$

1) $\forall A, B \in S \quad A \cdot B \in S$ bcs

$\det(A\cdot B) = \det A \cdot \det B = 1\cdot 1 = 1$

2) $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \in S$ bcs $\det I_n = 1$

3) $\forall A \in S$, $A^{-1} \in S$ bcs

$1 = \det I_n = \det A \cdot \det A^{-1} = 1 \cdot \det A^{-1} = \det A^{-1}$

example

$(F = \{ f : A \to \mathbb{R} \mid A \subseteq \mathbb{R} \}, +)$ - group (abelian)

let $C = \{ f \in F \mid f \text{ - continuous} \} \subset F$

1) $\forall f, g \in C$, $f+g \in C$

2) The zero f-n $(\forall x \in A, f(x) = 0)$ is in C

bcs it is constant

3) $f \in C \Rightarrow -f \in C$, hence $C < F$

consider $D = \{ f \in F \mid f \text{-differentiable} \} \subset C$

Then $D < C < F$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$+_2$	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)
(1,0)	(1,0)	(1,1)	(0,0)	(0,1)
(1,1)	(1,1)	(1,0)	(0,1)	(0,0)