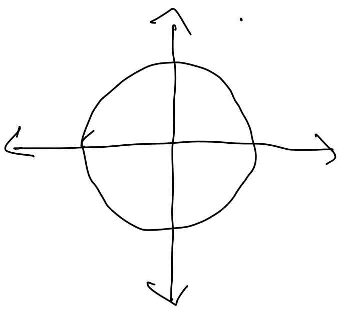
Groups + Subgroups

Consider $U = \{ z \in \mathbb{C} \mid |z| = 1 \} \subseteq \mathbb{C}$



Note each elem of U is defined by $\theta \in [0, 2\pi) = \mathbb{R}_{2\pi}$

Every angle $\theta \in \mathbb{R}_{2\pi}$ given by

$f : U \to \mathbb{R}_{2\pi}$, $f(z) = \theta$ for $z = e^{i\theta}$

+

$f^{-1} : \mathbb{R}_{2\pi} \to U$, $f^{-1}(\theta) = e^{i\theta}$

Consider U w/ multiplication

Let $z_1 = e^{i\theta_1}$, $z_2 = e^{i\theta_2} \in U$

then $z_1 \cdot z_2 = e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$

where $\theta_1, \theta_2 \in \mathbb{R}_{2\pi}$, $\theta_1 +_{2\pi} \theta_2 \in \mathbb{R}_{2\pi}$

Just U is closed under multiplication

$\forall z_1, z_2 \in U \quad z_1 \cdot z_2 \in U$

Definition

Let S be a set

Let $*$ be a binary operation on S

$* : S \times S \to S$

$(S, *)$ is an **algebraic structure**

So $(U, \cdot)$ + $(\mathbb{R}_{2\pi}, +_{2\pi})$ are alg. structs. s.t.

"1-1" btwn U + $\mathbb{R}_{2\pi}$ and

operations are preserved

$(U, \cdot) \simeq (\mathbb{R}_{2\pi}, +_{2\pi})$

or

$U \simeq \mathbb{R}_{2\pi}$

<u>Isomorphism</u>

To show 2 alg. structs are isomorphic,
find a bijective f-n that preserves the operations

Isomorphic structures share the same algebraic properties
(associativity, commutative, distributive, ...)

To show 2 alg structs are not isomorphic, find a property that they don't share
(the easiest is to check the cardinality)

Examples

$(\mathbb{N}, +) \not\cong (\mathbb{Z}, +)$ bcs $\mathbb{N}$ has no additive identity

$(\mathbb{Z}^*, \cdot) \not\cong (\mathbb{Q}^*, \cdot)$ bcs not all elems of $\mathbb{Z}^*$ have inverses

$(n/m)^{-1} = m/n$

$(\mathbb{R}, +) \not\cong (\mathbb{R}, \cdot)$ bcs $x * x + 1 = 0$

is solvable in $(\mathbb{R},+)$: $2x+1=0$

but unsolvable in $(\mathbb{R},\cdot)$: $x^2+1=0$