Recall

If $G_1, G_2, .. G_n$ - groups, their direct product

$$G_1 \times G_2 \times .. \times G_n$$

is a group

If all groups are abelian, their direct sum is too. Denoted

$$G_1 \oplus G_2 \oplus \ldots \oplus G_n$$

<u>Fact 1</u>

Let $(g_1, \ldots, g_n) \in \prod_{i=1}^{n} G_i$ then $(g_1, \ldots, g_n)$ is of order

$$k = \text{lcm}(r_1, \ldots, r_n) \quad \text{where}$$

r_i is the order of g_i in G_i

<u>Proof</u>

Let g_i be of order r_i in G_i

Then $g_i^{r_i} = e_i \wedge \forall k < r_i, \; g_i^{r_i} \neq e_i$

To find least pos int k s.t.

$$(g_1, \dots, g_n)^k = (e_1, \dots, e_n)$$

take lcm of order of g_i's

$$k = \mathrm{lcm}(r_1, \dots, r_n)$$

<u>Example</u>

Find order of $(2,3,4)$ in $\mathbb{Z}_{12} \times \mathbb{Z}_8 \times \mathbb{Z}_{24}$

Since 2 is order 6 in $\mathbb{Z}_{12}$

3 is order 8 in $\mathbb{Z}_8$

4 " " 6 in $\mathbb{Z}_{24}$

$(2,3,4)$ is order $k = \mathrm{lcm}(6,8,6) = 24$

<u>Fact 2</u>

The group $\mathbb{Z}_n \times \mathbb{Z}_m$ is cyclic & isomorphic to $\mathbb{Z}_{n \cdot m}$ iff n & m are relatively prime

$$\gcd(n,m) = 1$$

Proof

If n, m are rel. prime, then

$k = \text{lcm}(n, m) = n \cdot m$, so

$|\langle (1,1) \rangle| = k \cong \mathbb{Z}_n \times \mathbb{Z}_m$

$\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_{n \cdot m}$

bcs $\mathbb{Z}_n \times \mathbb{Z}_m$ is cyclic

In general,

Group $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_i}$

is cyclic + isomorphic to

$\mathbb{Z}_{n_1 \cdot n_2 \cdot \ldots \cdot n_i}$ if

$\mathbb{Z}_{n_1}, \mathbb{Z}_{n_2}, \ldots, \mathbb{Z}_{n_i}$ are rel. prime

Fundamental Thm of Finitely Generated Abelian Groups

Every finitely generated abelian group can be presented as a direct product of cyclic groups of form:

$$\mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \mathbb{Z}_{p_k^{n_3}} \times \mathbb{Z} \times \dots \times \mathbb{Z}$$

where p_i's are prime #s (not necessarily distinct) and

$$n_i \in \mathbb{Z}^+ \quad i = 1, \dots, k$$

<u>Corollary</u>

Every finitely generated abelian group can be presented as

$\mathbb{Z}_{p_1^{n_1}} \times \dots \times \mathbb{Z}_{p_k^{n_k}}$ where

p_i's are primes,

n_k's are pos. ints

Example

Find all f.g.a. groups of order 180

Since

180	2
90	3
30	3
10	2
5	5
1	1

$180 = 2^2 \cdot 3^2 \cdot 5$, we have the non-isomorphic f.g.a. groups order 180

$$\mathbb{Z}_{180} \simeq \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5$$
$$\simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$$
$$\simeq \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$$
$$\simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$$

Example

Determine if groups are isomorphic

$$\mathbb{Z}_{15} \times \mathbb{Z}_8 \times \mathbb{Z}_2 \simeq \mathbb{Z}_6 \times \mathbb{Z}_{40}$$

$$\begin{array}{ccc} \text{12} & & \text{12} \end{array}$$

$$\mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_8 \times \mathbb{Z}_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8 \times \mathbb{Z}_5$$