5.8 Groups of Permutations

Definition

A permutation of set A is a bijective f-n from A to A

Ex

For $A = \{1, 2, 3\}$

The following are permutations of A

$\delta_1 : A \to A \qquad \delta_1(1) = 1 \wedge \delta_1(2) = 2 \wedge \delta_1(3) = 3$

$\delta_2 : A \to A \qquad \delta_2(1) = 2 \wedge \delta_2(2) = 3 \wedge \delta_2(3) = 1$

$\delta_3 : A \to A \qquad \delta_3(1) = 3 \wedge \delta_3(2) = 2 \wedge \delta_3(3) = 1$

denoted

$$\delta_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\delta_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\delta_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Consider collection of all permutations of set A, denoted S_A wrt f-n composition

Notice: $\forall\, \delta, \sigma \in S_A \quad \exists!\, \delta \circ \sigma \in S_A$

(composition of bij. f-ns is bij)

hence f-n composition is everywhere well-defined in S_A

$\therefore$ f-n composition is a binary operation on S_A, called permutation multiplication denoted "$\cdot$"

Theorem

$(S_A, \cdot)$ is a non-abelian group

Proof:

1) Since f-n comp. is associative, $\cdot$ is associative in S_A

2) Identity of S_A is id_A (where $id_A(a)=a$)

$$\forall \delta \in S_A \quad \delta \cdot e = e \cdot \delta = \delta$$

3) Let $\delta \in S_A$, the inverse

$$\delta^{-1}(b) = a \quad \text{iff} \quad \delta(a) = b$$

bcs $\quad \delta \cdot \delta^{-1} = \delta^{-1} \cdot \delta = e$

$\therefore$ $(S_A, \cdot)$ has group structure

Notation: $A = I_n = \{1,2,3,\dots,n\}$ then $S_A = S_{I_n} = S_n$ is the

symmetric group on n-letters

Ex

$A = \{1,2,3,4,5\} = I_5$

consider elems of S_5, find their products

$$\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \in S_5$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix} \in S_5$$

$$\delta \cdot \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

$$\neq$$

$$\sigma \cdot \delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 2 & 3 \end{pmatrix}$$

so S_{I_5} is non-abelian

In general,

D_n is the nth dihedral group of symmetries of regular n-gon

Example

Consider 4th dim dihedral group D_4 of symmetries of a square w/ elems

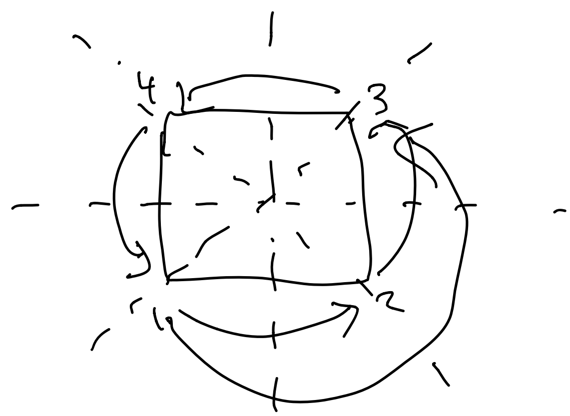
rotations

$$\delta_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$\delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$\delta_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$



mirror images

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\mu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

diagonal flips

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

$\cdot$	δ_0	δ_1	δ_2	δ_3	μ_1	μ_2	σ_1	σ_2
δ_0	δ_0	δ_1	δ_2	δ_3	μ_1	μ_2	σ_1	σ_2
δ_1	δ_1	δ_2		δ_0				
δ_2	δ_2	δ_3						
δ_3	δ_3	δ_0						
μ_1	μ_1	μ_2						
μ_2	μ_2							
σ_1	σ_1							
σ_2	σ_2							

From the table,

D_4 is not abelian

δ_0 - identity

$\delta_1^{-1} = \delta_3 \wedge \delta_3^{-1} = \delta_1^{-1}$

$\delta_2^{-1} = \delta_2$

$\mu_1^{-1} = \mu_1$

$\mu_2^{-1} = \mu_2$

$\sigma_1^{-1} = \sigma_1$

$\sigma_2^{-1} = \sigma_2$

Inverses

Subgroups :

$\langle \delta_0 \rangle = \{\delta_0\}$

D_4

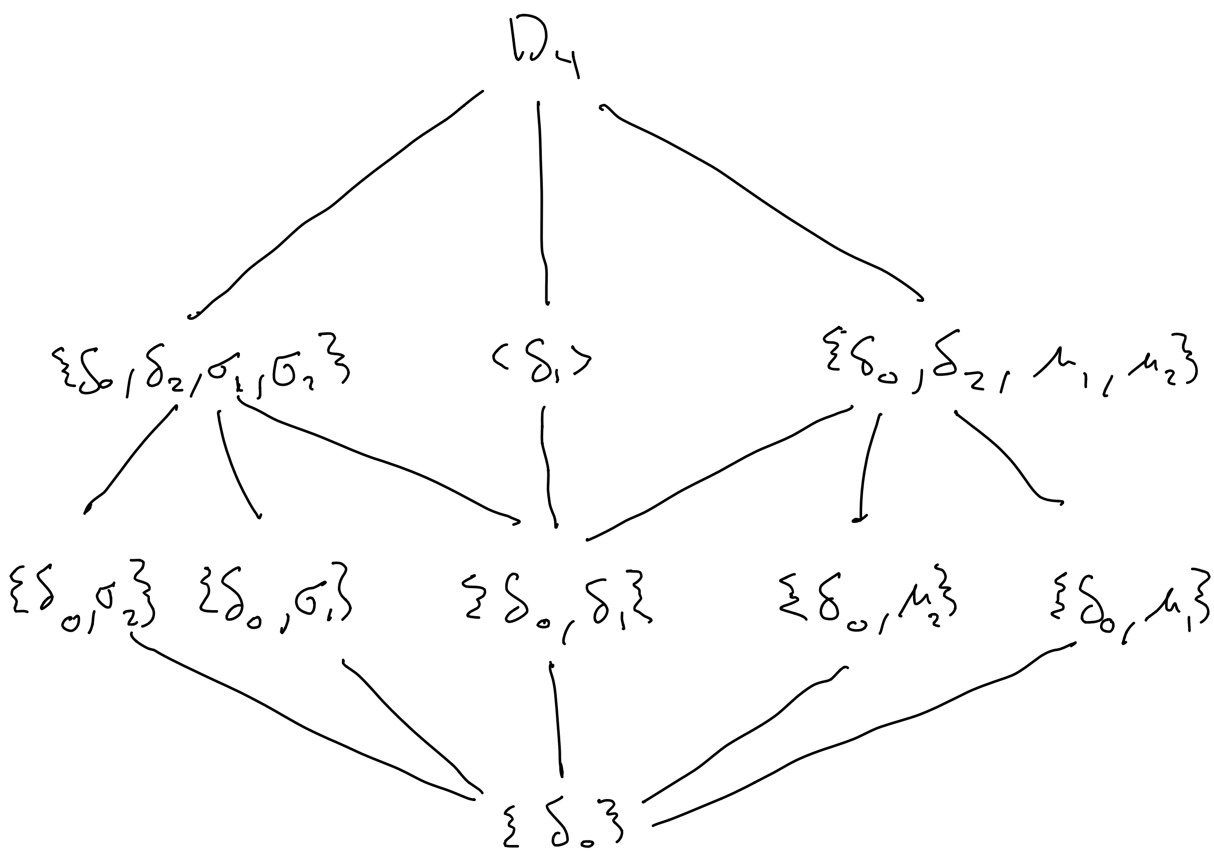
$\langle \delta_2 \rangle = \{\delta_0, \delta_2\}$

$\langle \mu_1 \rangle = \{\delta_0, \mu_1\}$

$\langle \sigma_1 \rangle = \{\delta_0, \sigma_1\}$

$\langle \sigma_2 \rangle = \{\delta_0, \sigma_2\}$

$\langle \delta_3 \rangle = \langle \delta_1 \rangle = \{\delta_0, \delta_1, \delta_2, \delta_3\}$



$D_4 < S_4$ bcs $D_4 \subset S_4$

$|D_4| = 8$ $\qquad$ $|S_4| = 4! = 24$

In general, for $n \geq 4$, $D_n < S_n$