Let $H \leq G$

Define "left" relation on G by $a \sim_L b$ iff

$a^{-1} \cdot b \in H$

Similarly we can define the "right" relation: $a \sim_R b$ iff

$ab^{-1} \in H$

The left relation is reflexive

$\forall a \in G \quad a \sim_L a$ bcs

$a^{-1} a = e \in H$ by ident. prop

and symmetric

$\forall a, b \in G \quad a \sim_L b \Leftrightarrow a^{-1} b \in H$

$b \sim_L a \Leftrightarrow b^{-1} a \in H$

by inverse prop

and transitive

$\forall a, b, c \in G \quad a \sim_L b \wedge b \sim_L c \Leftrightarrow a^{-1} b \in H \wedge b^{-1} c \in H$

$a \sim_L c \Leftarrow a^{-1}c = (a^{-1}b)(b^{-1}c) \in H$

by closure prop

$\therefore$ $\sim_L$ is an equivalence relation

(the right relation is also an equiv. relation)

$\therefore$ The left relation partitions G into nonempty, disjoint subsets called equiv. classes of form

$$
\begin{aligned}
E_a &= \{ b \in G \mid a \sim_L b \} \\
&= \{ b \in G \mid a^{-1} b \in H \} \\
&= \{ b \in G \mid a^{-1} b = h, \ h \in H \} \\
&= \{ b \in G \mid b = ah, \ h \in H \} \\
&= \{ ah \mid h \in H \} \\
&= aH \text{ is the left coset of } H \text{ in } G
\end{aligned}
$$

(for some $a \in G$)

Also $Ha = \{h \cdot a \mid h \in H\}$
is the right coset

<u>Fact</u> If G is an abelian group & $H \leq G$, then
the left & right cosets for H in G coincide

<u>Proof</u>
let $a \in G$ so $aH = \{ah \mid h \in H\}$
$= \{ha \mid h \in H\}$
$= Ha$

<u>Examples</u>
Find all left & right cosets for the subgroups

1) $\langle 3 \rangle \leq \mathbb{Z}_6$ under addition

$\langle 3 \rangle = \{3, 0\}$ $\quad$ $\mathbb{Z}_6 = \{\cancel{0}, 1, 2, \cancel{3}, 4, 5\}$

$0 + \langle 3 \rangle = \langle 3 \rangle + 0 = \{0, 3\} = 3 + \langle 3 \rangle$

$1 + \langle 3 \rangle = \langle 3 \rangle + 1 = \{1, 4\}$

$2 + \langle 3 \rangle = \langle 3 \rangle + 2 = \{2, 5\}$

Since $\langle 3 \rangle \cup \langle 3 \rangle + 1 \cup \langle 3 \rangle + 2 = \mathbb{Z}_6$

there are 3 left & right cosets

& they coincide bcs $\mathbb{Z}_6$ is abelian

$\mathbb{Z}_6$

$+_6$	0	3	1	4	2	5
0	0	3	1	4	2	5
3	3	0	4	1	5	2
1	1	4	2	5	3	0
4	4	1	5	2	0	3
2	2	5	3	0	4	1
5	5	2	0	3	1	4

$+_3$	$\langle 3\rangle+0$	$\langle 3\rangle+1$	$\langle 3\rangle+2$
$\langle 3\rangle+0$	$\langle 3\rangle+0$	$\langle 3\rangle+1$	$\langle 3\rangle+2$
$\langle 3\rangle+1$	$\langle 3\rangle+1$	$\langle 3\rangle+2$	$\langle 3\rangle+0$
$\langle 3\rangle+2$	$\langle 3\rangle+2$	$\langle 3\rangle+0$	$\langle 3\rangle+1$

has group structure

$$\{\langle 3\rangle+0,\ \langle 3\rangle+1,\ \langle 3\rangle+2\} \simeq (\mathbb{Z}_3, +_3)$$