Recall

$\phi: G \to G'$ group homomorphism

$\phi(a)^{-1} = \phi(a^{-1})$ for $a \in G$

$H \leq G \Rightarrow \phi(H) \leq G'$

$\text{Ker}\,\phi = \phi^{-1}(\{e'\}) = \{x \in G \mid \phi(x) = \phi(e)\}$

Theorem

Let $\phi: G \to G'$ be a g.h. w/ kernel H & $a \in G$

Then $aH = Ha = \phi^{-1}(\{\phi(a)\})$

$= \{x \in G \mid \phi(x) = \phi(a)\}$

Proof

Need to show $aH = \phi^{-1}(\phi(a))$

Let $x \in aH$, then $x = ah$ $h \in H$

Consider $\phi(x) = \phi(ah) \underset{\text{g.h.}}{=} \phi(a) \cdot \phi(h)$

$= \phi(a) \cdot e'$

$h \in H = \ker\phi$

$= \phi(a)$

hence $x \in \phi^{-1}(\phi(a))$

$aH \subseteq \phi^{-1}(\phi(a))$

Let $x \in \phi^{-1}(\phi(a))$ then $\phi(x) = \phi(a)$

Since $\phi(a) \in G'$, $\exists\, \phi(a)^{-1}$ s.t. $\phi(a)\phi(a)^{-1} = e'$

So $\phi(a)^{-1} \cdot \phi(x) = e'$

$\phi(a^{-1}) \cdot \phi(x) = e'$

$\phi(a^{-1} \cdot x) = e'$ for $a^{-1} \cdot x \in G$

then $a^{-1}x \in H = \ker\phi$

$a^{-1}x = h$ $h \in H$ so $x = ah \in aH$

Thus $\phi^{-1}(\phi(a)) \subseteq aH$

$\therefore \phi^{-1}(\phi(a)) = aH$ and $\phi^{-1}(\phi(a)) = Ha$

Examples

1) $\phi : S_n \to \mathbb{Z}_2$ $\quad \phi(\sigma) = \begin{cases} 0 & \text{if even} \\ 1 & \text{if odd} \end{cases}$

g.h. w/ Kernel $H = A_n \leq S_n$

w/ cosets $H = id H = H\, id$ $\quad id = (12)(21)$

2) $\phi : GL_n(\mathbb{R}) \to \mathbb{R}^*$ $\quad \phi(A) = \det A$

g.h. w/ Kernel

$$H = \{ A \in GL_n(\mathbb{R}) \mid \det A = 1 \}$$

$$= I_n \cdot H = H \cdot I_n \qquad I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

other cosets of form

$$B \cdot H = H \cdot B = \{ A \in GL_n(\mathbb{R}) \mid \phi(A) = \phi(B) \}$$

for $B \in GL_n(\mathbb{R}) \setminus H$ $\quad (\det B \neq 0 \wedge \det B \neq 1)$

3) $\phi : \mathbb{C}^* \to \mathbb{R}^+ \qquad \phi(z) = |z|$

g.h. w/ kernel $H = \{ z \in \mathbb{C}^* \mid |z| = 1 \}$

$= U \leq \mathbb{C}^*$

cosets

$(1+0i) \cdot H = H \cdot (1+0i)$

$z' \cdot H = H \cdot z' = \{ z \in \mathbb{C}^* \mid |z| = |z'| \}$

4) $\phi : \overset{G_1 \times \ldots \times G_n}{\prod_{i=1}^{n}} G_i \to G_k \ , \ \phi((g_1, \ldots, g_n)) = g_k \quad k = 1, \ldots, n$

g.h. w/ kernel

$H = \{ (g_1, \ldots, g_n) \in \prod_{i=1}^{n} G_i \mid g_k = e_k \}$
$k = 1 \ldots n$

$\shortparallel$

$(e_1 \ldots e_n) \cdot H = H \cdot (e_1 \ldots e_n)$

other cosets

$(g_1' \ldots g_n') \cdot H = H \cdot (g_1' \ldots g_n')$

$= \{ (g_1 \ldots g_n) \in \prod_{i=1}^{n} G_i \mid \phi((g_1 \ldots g_n)) = \phi((g_1' \ldots g_n')) \}$

5) $\phi : F \to \mathbb{R}$ $\quad \phi(f) = f(c) \quad c \in D \subseteq \mathbb{R}$

g.h. w/ $\ker \phi = \{ f \in F \mid f(c) = 0 \}$

$= \bar{0} + H = H + \bar{0}$

other cosets of form

$g + H = H + g = \{ f \in F \mid \phi(f) = \phi(g) \}$

$= \{ f \in F \mid f(c) = f(g) \}$

<u>Fact</u>

Let: $\phi : G \to G'$ be a group homomorphism w/ Kernel H
Then ϕ is injective iff the kernel of ϕ is trivial

<u>Proof</u>

$\Rightarrow$ Assume ϕ is injective

Since $\phi(e) = e'$ $\quad \neg(\exists a \in G \setminus \{e\},\ \phi(a) = e')$

$\Rightarrow$ Assume $\ker \phi$ is trivial

$H = \ker \phi = \{e\}$

since $aH = Ha = \{ae\} = \{a\}$ $a \in G$

ϕ - injective

Note: To show $\phi: G \to G'$ is a group isomorphism, show:

1) ϕ - group homomorphism
2) $\ker \phi$ is trivial $\Leftrightarrow$ ϕ - injective
3) $\phi(G) = G' \Leftrightarrow \phi$ - surjective