<u>Recall</u>

$H \trianglelefteq G$ iff $gH = Hg \quad \forall g \in G$

$gHg^{-1} = H$

H is invariant under conjugation by element of G

$\forall g \in G \quad \forall h \in H \quad \exists h' \in H \quad ghg^{-1} = h'$

Definition

An **automorphism** of a group G is an isomorphism from G onto G

Let $g \in G$ Define $i_g : G \to G$

$$i_g(x) = gxg^{-1} \quad x \in G$$

$\forall x_1, x_2 \in G$

$$\begin{aligned} i_g(x_1 \cdot x_2) &= g(x_1 \cdot x_2)g^{-1} \\ &= (gx_1) \cdot e \cdot (x_2 g^{-1}) \\ &= (gx_1)(gg^{-1})(x_2 g^{-1}) \\ &= (gx_1 g^{-1})(g x_2 g^{-1}) \end{aligned}$$

$= i_g(x_1) \cdot i_g(x_2)$

so i_g is a homomorphism w/ kernel

$$\ker i_g = \{x \in G \mid i_g(x) = e\}$$
$$= \{x \in G \mid gxg^{-1} = e\}$$
$$= \{x \in G \mid gx = g\}$$
$$= \{x \in G \mid x = e\}$$

So the kernel is trivial, meaning i_g is injective. Also, i_g is surjective

$i_g(G) = G$ bcs

$\forall y \in G \quad \exists x \in G \quad i_g(x) = y$

$$i_g(x) = y = g(g^{-1}yg)g^{-1}$$

$\therefore$ i_g is an isomorphism G to G, more specifically, an inner automorphism

5.15 Factor Group Computations

If $N \trianglelefteq G$, then we can define the factor group G/N whose elems are (left/right) cosets $gN = Ng$ $g \in G$

+ identity of G/N is

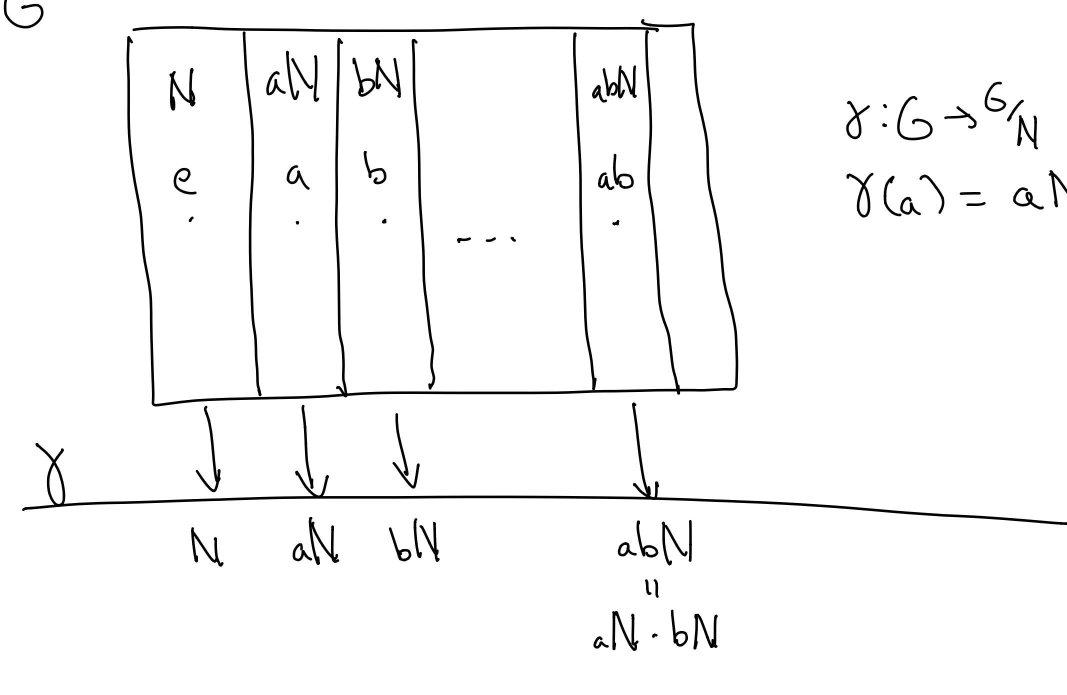
$$N = eN = Ne$$

+ inverse of gN is $g^{-1}N = Ng^{-1}$

G

$\gamma : G \to G/N$

$\gamma(a) = aN$



If $N = \{e\}$, then $G/N \simeq G$

$\gamma(a) = a\{e\} = a$

$\gamma(ab) = ab\{e\} = ab$

If $N = G$, then $G/G \simeq \{e\}$

$\gamma(a) = aG = G$ by closure prop

If $N \neq \{e\}$ and $N \trianglelefteq G$, the structure of G/N is "simpler" than G + may give some info about G.

Example, consider $A_n \trianglelefteq S_n$ bcs

$$(S_n : A_n) = \frac{n!}{\frac{n!}{2}} = 2$$

Then S_n/A_n has 2 elements

- A_n
- $\delta A_n = B_n$ $\quad \delta \in S_n/A_n$

the operation is described by

$\cdot$	A_n	B_n
A	A	B

A_n	A_n	B_n
B_n	B_n	A_n

Notice $A_n^2 = A_n$

$B_n^2 = A_n$

hence $\forall \delta \in S_n \quad \delta^2 \in A_n$

In general, if $N \ntrianglelefteq G$ s.t. $(G:N) = 2$

G/N has 2 elements N + gN $g \in G/N$

w/ table

$\cdot$	N	gN
N	N	gN
gN	gN	N

Notation:

$N \ntrianglelefteq G$ - Proper non-trivial subgroup

Computing Factor Groups

Example Compute $\mathbb{Z}_4 \times \mathbb{Z}_6 / \langle (0,1) \rangle$

Since $\left| \mathbb{Z}_4 \times \mathbb{Z}_6 / \langle (0,1) \rangle \right| = \dfrac{|\mathbb{Z}_4 \times \mathbb{Z}_6|}{|\langle (0,1) \rangle|} = \dfrac{4 \cdot 6}{\operatorname{lcm}(1,6)}$

$$= \frac{4 \cdot 6}{6} = 4$$

Order of the group is 4

$\dfrac{\mathbb{Z}_4 \times \mathbb{Z}_6}{\langle (0,1) \rangle}$ is isomorphic to either

$\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$

To check which, find at least one element in $\dfrac{\mathbb{Z}_4 \times \mathbb{Z}_6}{\langle (0,1) \rangle}$ of order 4.
If one exists, it must be isomorphic to $\mathbb{Z}_4$

Elements of $\dfrac{\mathbb{Z}_4 \times \mathbb{Z}_6}{\langle (0,1) \rangle}$

Recall $gN \in G/N$ is order n iff n is least pos int s.t. $(gN)^n = g^n N = N$ iff $g^n \in N$

$\langle(0,1)\rangle = \langle(0,1)\rangle + (0,0)$

$= \{(0,0), (0,1), (0,2), (0,3), (0,4), (0,5)\}$

$\langle(0,1)\rangle + (1,0) =$

$\{(1,0), (1,1), (1,2), (1,3), (1,4), (1,5)\}$

is order $n=4$ bcs

4 is least pos int s.t. $\underset{\underset{(0,0)}{\shortparallel}}{4\cdot(1,0)} \in \langle(0,1)\rangle$

$\langle(0,1)\rangle + (2,0) =$

$\{(2,0), (2,1), (2,2), (2,3), (2,4), (2,5)\}$

$\langle(0,1)\rangle + (3,0) =$

$\{(3,0), (3,1), (3,2), (3,3), (3,4), (3,5)\}$

$\therefore \langle(0,1)\rangle + (1,0)$ is a generator of $\dfrac{\mathbb{Z}_4 \times \mathbb{Z}_6}{\langle(0,1)\rangle}$ hence the group

is cyclic of order 4,

hence isomorphic to $\mathbb{Z}_4$

Theorem

Let $G = H \times K$ be a direct product of groups

1. $\bar{H} = \{(h,e) \mid h \in H\} \trianglelefteq G$

$G/\bar{H} \cong K$

2. $\bar{K} = \{(e,k) \mid k \in K\} \trianglelefteq G$

$G/\bar{K} \cong H$

<u>Proof</u>

$\phi : G = H \times K \to H$

$\phi((h,k)) = h$

is a projection homomorphism w/ Kernel

$\ker \phi = \{(h,k) \in G \mid \phi((h,k)) = h = e\}$

$$= \{(e,k) \mid k \in K\}$$

$$= \overline{K} \trianglelefteq G$$

$$G/\overline{K} \simeq \phi(G) = H$$