Remember a <u>bin op</u>

$$\forall a,b \in S \quad \exists! \, a*b, \quad a*b \in S$$

$$(a,b) \in S \times S$$

Ex

$(U, \cdot)$ is algebraically the same as $(\mathbb{R}_{2\pi}, +_{2\pi})$ where

$$U = \{z \in \mathbb{C} \mid |z| = 1\}$$

$$\mathbb{R}_{2\pi} = [0, 2\pi)$$

bcs there is a "1-1" correspondence btwn U & $\mathbb{R}_{2\pi}$

$|U| = |\mathbb{R}_{2\pi}|$ bcs a bijective fn exists

$$\phi : U \to \mathbb{R}_{2\pi}, \quad \phi(z) = \phi(e^{i\theta}) = \theta$$

that preserves operations (homomorphism)

Definition

Let $(S, *)$, $(S', *')$ be bi

Let $(S,*)$, $(S',*')$ be binary structures.

Then $(S,*)$ & $(S',*')$ are **isomorphic** (algebraically the same), denoted

$$S \simeq S'$$

iff

$\exists \phi : S \to S'$, ϕ is bijective

ϕ is a homomorphism (satisfies a homomorphic property)

function ϕ is **homomorphic** iff

$$\forall s_1, s_2 \in S \quad \phi(s_1 * s_2) = \phi(s_1) *' \phi(s_2)$$

Ex

a) $\phi : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$, $\phi(n) = n+1$

? ϕ-bij - ϕ-injective : $\forall n, m \in \mathbb{Z}$,

$\phi(n) = \phi(m) \Leftrightarrow n+1 = m+1 \Rightarrow n = m$

ϕ - surjective: $\forall k \in \mathbb{Z} \ \exists n \in \mathbb{Z},\ \phi(n) = k$

? ϕ - homomorphic:

$\forall n, m \in \mathbb{Z},\ \phi(n+m) \neq \phi(n) + \phi(m)$ bcs

$n+m+1 \neq n+1+m+1 = n+m+2$

(i.e. $n=m=1$)

ϕ is not an isomorphism

$\Downarrow$

ϕ - not an isomorphism

b) $\phi : (M_2(\mathbb{R}), \cdot) \to (\mathbb{R}, \cdot),\ \phi(A) = \det A$

? ϕ - bij

injective: $\forall A, B \in M_2(\mathbb{R}),\ \overbrace{\phi(A) = \phi(B)}^{\det A = \det B} \Rightarrow A = B$

ϕ - not bij bcs not injective

ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

$\det A = \det B = 1$

$\Rightarrow \phi$ - not isomorphic

however, ϕ-surjective

$$\forall r \in \mathbb{R} \quad \exists A \in M_2(\mathbb{R}) \quad \phi(A) = r$$

$$\begin{matrix} & & & \| \\ & & & \begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix} \end{matrix}$$

ϕ-homomorphism bcs

$$\forall A, B \in M_2(\mathbb{R}), \ \phi(A \cdot B) = \phi(A) \cdot \phi(B)$$
$$= \det A \cdot \det B$$

Recall

let $(S, *)$, $(S', *')$ be alg. structs

Then $S \simeq S'$ iff

$\exists\, \phi : S \to S'$, ϕ - bijective

(that is $|S| = |S'|$)

and ϕ is a homomorphism

Finding a f-n for isomorphism

Assuming that $|S| = |S'|$, in order to show 2 bin. structs, $(S, *)$, $(S', *')$ are not isomorphic, find a property that is not shared ex/ × solvability of equations

$(\mathbb{C}, \cdot) \not\cong (\mathbb{R}, \cdot)$ bcs

equation $x \cdot x + 1 = 0$ is not solvable in $\mathbb{R}$ but has 2 solns in $\mathbb{C}$

ex/ $(\mathbb{Q}, +) \not\cong (\mathbb{Z}, +)$

bcs eq $x + x = 1$ has no solns in $\mathbb{Z}$ but has soln in $\mathbb{Q}$

ex $(\mathbb{Z}, \cdot) \not\cong (\mathbb{N}, \cdot)$

bcs eq. $x \cdot x = x$ has 2 solns in $\mathbb{Z}$ but 1 in $\mathbb{N}$

can also check for

× commutative/associative

× existence of identity

× existence of inverse elements

(i.e. $(\mathbb{Z}^*, \cdot) \not\cong (\mathbb{Q}^*, \cdot)$)

Recall

Let $(S, *)$ be an alg bin. structure

Then $e \in S$ is the **identity element** of S w.r.t. $*$ iff

$$(\forall s \in S)\ s * e = e * s = s$$

* Note

for addition, $e = 0$ is the additive ident.

for mult, $e = 1$ is the mult. identity in $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$

Prove

fact 1: Identity of an alg struct $(S, *)$ is **unique**, if it exists

Assume identity is not unique

Let $e \neq e'$ be identities

$$(\forall s \in S)\ e * s = s * e \ \wedge\ e' * s = s * e' = s$$

$$e * e' = e' \ \wedge\ e * e' = e$$

$$\Rightarrow e' = e$$

fact 2:

Let $(S, *), (S', *')$ be isomorphic structs

$\phi : S \to S'$ is an isomorphism

&

e is the unique identity of S

then $\phi(e)$ is the identity of S'

Proof

since e is ident. of S,

$(\forall s \in S)\ e * s = s * e = s$

consider $\phi(e*s) = \phi(s)$

since ϕ is a homomorphism

$\phi(e*s) = \phi(e) *' \phi(s)$, $\phi(s*e) = \phi(s) *' \phi(e)$

$\phi(e) *' \phi(s) = \phi(s) *' \phi(e) = \phi(s)$

since ϕ is a bijection

$\forall s \in S\ \exists ! s' \in S',\ \phi(s) = s'$

&

$\forall s' \in S'\ \exists s \in S,\ \phi(s) = s'$

$\therefore\ \phi(e) *' s' = s' *' \phi(e) = s'$, where $s' \in S'$

which means $\phi(e)$ is identity of S'

Ex $(U, \cdot) \simeq (\mathbb{R}_{2\pi}, +_{2\pi})$