Recall

let $f: A \to B$

The set $f(C) = \{f(x) \mid x \in C\} \subseteq B$ is the image of C in B

Lemma

If $\phi: G \to G'$ is an injective group homomorphism, then

$$\phi(G) \leq G'$$

and

$$\phi: G \to \phi(G) \text{ is an isomorphism}$$

Proof

$\phi(G) = \{\phi(g) \mid g \in G\} \subseteq G'$ s.t.

1) let $x', y' \in \phi(G)$ since ϕ is injective,

$\exists! \, x, y \in G, \quad \phi(x) = x' \wedge \phi(y) = y'$

(so ϕ is closed)

Consider $x' \cdot y' = \phi(x) \cdot \phi(y) = \phi(x \cdot y)$
where $x \cdot y \in G$ thus $x' \cdot y' \in \phi(G)$

2) Let e be identity of G
e' " " G'

$\phi(e) \in \phi(G) \subseteq G'$

Consider $e' \cdot \phi(e) = \phi(e) = \phi(e \cdot e)$
$= \phi(e) \cdot \phi(e)$

by cancellation of G

$e' \cdot \phi(e) = \phi(e) \cdot \phi(e) \Rightarrow e' = \phi(e)$

3) Let $x' \in \phi(G)$, then

$\exists! \, x \in G, \ \phi(x) = x'$

Consider $e' = \phi(e) = \phi(x x^{-1}) \overset{\text{by homomorphism}}{=} \phi(x) \cdot \phi(x^{-1})$

hence $(x')^{-1} = \phi(x^{-1})$, so $(x')^{-1} \in \phi(G)$

$\therefore \ \phi(G) \leq G'$

since

$\phi: G \to G'$ is an injective homomorphism
$\phi: G \to \phi(G)$ is a bijective homomorphism

so $G \cong \phi(G)$ ~ $\phi(G) \leq G'$

Cayley's Theorem

Every group is isomorphic to a group of permutations

Proof

Let G be a group

Define $\phi: G \to S_G$ by $\phi(x) = \lambda_x$ for $x \in G$

where $\lambda_x : G \to G$ is defined by

$$\lambda_x(g) = x \cdot g$$

Notice λ_x is injective bcs

$\forall g_1, g_2 \in G$ $\lambda_x(g_1) = \lambda_x(g_2) \iff x \cdot g_1 = x \cdot g_2$

$\implies g_1 = g_2$

λ_x is surjective bcs

$\forall h \in G$, $\exists g \in G$ $\lambda_x(g) = x \cdot (x' \cdot h)$

$= (x \cdot x') h$

$= eh = h$

Therefore λ_x is a permutation of G
bcs it is bijective f-n from G onto G

Notice ϕ is also injective
$\forall x, y \in G \iff \lambda_x = \lambda_y \iff \forall g \in G, x \cdot g = x \cdot y \Rightarrow x = y$

And ϕ is homomorphic
$\forall x, y \in G \quad \phi(x) \cdot \phi(y) \iff \lambda_x \cdot \lambda_y$
$$= \lambda_x \circ \lambda_y$$
$$= \lambda_{xy} = \phi(xy)$$

By previous lemma, since
$\phi : G \to S_G$ is an injective homomorphism
$\phi(G) \leq S_G \quad \wedge \quad G \simeq \phi(G)$

Example

$(\mathbb{Z}_3, +_3)$ Find left regular representation

$+_3$	0	1	2
0	0	1	2

1	1	2	0
2	2	0	1

Let $\lambda_0 : \mathbb{Z}_3 \to \mathbb{Z}_3$ $\quad \lambda_0(n) = 0 +_3 n \quad \forall n \in \mathbb{Z}_3$

$\lambda_1 : \mathbb{Z}_3 \to \mathbb{Z}_3 \quad \lambda_1(n) = 1 +_3 n$

$\lambda_2 : \mathbb{Z}_3 \to \mathbb{Z}_3 \quad \lambda_2(n) = 2 +_3 n$

Then op on $\phi(\mathbb{Z}_3) = \{\lambda_0, \lambda_1, \lambda_2\} \subseteq S_{\mathbb{Z}_3}$

$\circ$	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	λ_2	λ_0
λ_2	λ_2	λ_0	λ_1