Definition

A nontrivial group is **simple** iff
it doesn't contain any
nontrivial proper normal subgroups

Any group contains 2 normal subgroups

- Trivial subgroup
- Improper subgroup

Note:

$\mathbb{Z}_p$ - simple group for all prime p
bcs it contains 2 subgroups
$\{0\}$ ^ $\mathbb{Z}_p$

<u>Lemma</u>

Let $\phi : G \to G'$ be a group homomorphism

1) If $N \trianglelefteq G$, then $\phi(N) \trianglelefteq \phi(G)$

2) If $N' \trianglelefteq \phi(G)$, then $\phi^{-1}(N') \trianglelefteq G$

Definition

A normal subgroup M of G is a **maximal** normal subgroup iff

G does not contain a proper normal subgroup that properly contains M.

Theorem

Let $M \trianglelefteq G$

Then M is a maximal normal subgroup of G iff

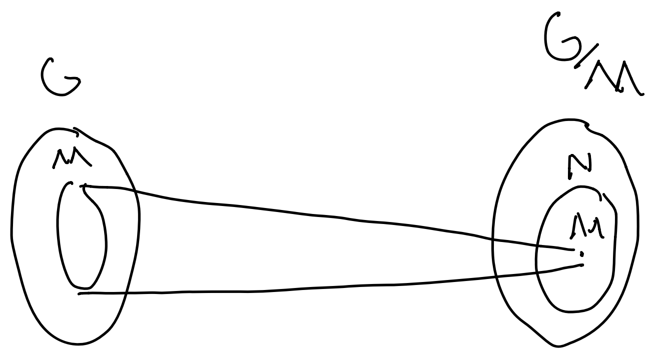
G/M is simple

Proof

$\Rightarrow$ "Assume M is maximal & G/M is not simple"

$\gamma : G \to G/M$ be a canonical homomorphism

$\gamma(g) = gM$



Since G/M is not simple, there is a nontrivial proper normal subgroup N

$N \ntrianglelefteq G/M$

By lemma 2

$\gamma^{-1}(N) \ntrianglelefteq G$ s.t. $N \subsetneq \gamma^{-1}(N)$

This is a contradiction, M is a max. norm. subgroup of G

$\Leftarrow$ "Assume G/M is simple + M is not a max. norm. subgroup"

let $\gamma : G \to G/M$ be a canonical homomorphism

Since M is not a max. normal subgroup, there is a proper normal subgroup N of G s.t. $M \subsetneq N$

By lemma 1
$\gamma(N) \trianglelefteq G/M \wedge \gamma(N) \neq \{M\}$

A contradiction, bcs G/M is simple

Definition

Let G be a group

The **center** of G is

$$Z(G) = \{ z \in G \mid \forall g \in G,\ g \cdot z = z \cdot g \}$$

Note:

$Z(G) \subseteq G$ s.t. $Z(G) \neq \emptyset$ bcs identity of G is in $Z(G)$

Moreover, $Z(G) \trianglelefteq G$

($Z(G)$ is a normal subgroup of G)