Recall

$(R, +, \cdot)$ - ring iff

$(R, +)$ - abelian group
$(R, \cdot)$ - semigroup
DLs hold

"multiplicative" - wrt 2^{nd} op.

Definition

1) Let $(R, +, \cdot)$ be a ring
If "multiplication" is commutative, then R is a **commutative ring**.

$(R, +, \cdot)$ - commutative ring iff
$(R, +)$ - abelian group
$(R, \cdot)$ - commutative semigroup
DLs hold

2) If R contains the "multiplicative" identity, then R is a ring with **unity**

$(R, +, \cdot)$ - ring w/ unity iff
$(R, +)$ - abelian group
$(R, \cdot)$ - monoid

DLs hold

Examples

$(\mathbb{R}, +, \cdot)$ - commutative ring w/ unity

$(\mathbb{Q}, +, \cdot)$ - " " " " " "

$(\mathbb{Z}, +, \cdot)$ - " " " " " "

$(n\mathbb{Z}, +, \cdot)$ - " " " " w/o unity

$(M_n(\mathbb{R}), +, \cdot)$ - non-commutative ring w/ unity

$(F, +, \cdot)$ - commutative ring w/ unity

set of real func's of real variable

Facts

Let $R_1 .. R_n$ be rings

1. If $R_1, .., R_n$ are commutative rings,

   $R_1 \times .. \times R_n$ is a commutative ring

2. If $R_1, \ldots, R_n$ are rings w/ unity
$R_1 \times \ldots \times R_n$ is a ring w/ unity

Theorem

If R is a ring
(w/ additive inverse 0)
then for all $a, b \in R$, we have

1) $a \cdot 0 = 0 \cdot a = 0$
2) $(-a) \cdot b = a \cdot (-b) = -a \cdot b$
3) $(-a) \cdot (-b) = a \cdot b$

<u>Proof</u>

1) Consider $0 \cdot a + 0a \underset{\text{DL}}{=} (0+0) \cdot a = 0 \cdot a$

$$= 0 \cdot a + 0$$

$$\Rightarrow 0 \cdot a = 0$$

left additive cancel law

2) Consider

$$a \cdot (-b) + ab \underset{DL}{=} a(-b+b) = a \cdot 0 \underset{①}{=} 0$$

hence $a(-b) = -a \cdot b$

3) Consider

$$(-a)(-b) \underset{②}{=} -(a \cdot (-b)) \underset{②}{=} -(-a \cdot b) = a \cdot b$$

Defintion

Let R + R' be rings

$\phi : R \to R'$ is a **ring homomorphism** iff

$$\forall a, b \in R \quad \begin{cases} \phi(a+b) = \phi(a) + \phi(b) \\ \phi(a \cdot b) = \phi(a) \cdot \phi(b) \end{cases}$$

In a ring $(R, +, \cdot)$, the set R w.r.t. $+$ is an abelian group called the additive group of a ring

Thus, a ring 'homomorphism'

$\phi:(R,+,\cdot) \to (R',+',\cdot')$ is the additive group homomorphism

Therefore, all "group homomorphism" results hold for rings.

ex

$\mathrm{Ker}\,\phi = \{ r \in R \mid \phi(r) = 0' \} \trianglelefteq R$

$\wedge$ $\dfrac{R}{\mathrm{Ker}\,\phi}$ is a factor group

Moreover, if $\mathrm{Ker}\,\phi = \{0\}$, then ϕ is an injective homomorphism

To show $\phi: R \to R'$ is a ring isomorphism, need to show:

1. ϕ is a ring homomorphism
2. $\mathrm{Ker}\,\phi$ is trivial (ϕ-injective)
3. $\mathrm{rng}\,\phi = \phi(R) = R'$, that is, ϕ-surjective

Examples

1) Let $\phi : F \to R$ be an evaluation (group) homomorphism defined by

$\forall f, g \in F, \ \phi(f+g) = \phi(f) + \phi(g)$

$$\phi(f) = f(c) \quad \text{for} \quad c \in D \subseteq R$$

Then ϕ is a ring homom. bcs.

$$\forall f, g \in F \quad \phi(f \cdot g) = (f \cdot g)(c) = f(c) \cdot g(c) = \phi(f) \cdot \phi(g)$$

2) let $\phi : \mathbb{Z} \to \mathbb{Z}_n$ def by

$\phi(m) = m \bmod n = r$, where $m = n \cdot k + r$, $k \in \mathbb{Z}$, $0 \leq r < n$

Then we showed ϕ is the additive group homom.

$$\forall m_1, m_2 \in \mathbb{Z} \quad \phi(m_1 + m_2) = \phi(m_1) +_n \phi(m_2)$$

which is a ring homom. bcs

$\forall m_1, m_2 \in \mathbb{Z} \quad \phi(m_1 \cdot m_2) = \phi((k_1 n + r_1)(k_2 n + r_2))$