Recall

$\phi: G \to G'$ - group homomorphism

$H' \leq G' \Rightarrow \phi^{-1}(H') \leq G'$

$H = \ker\phi = \phi^{-1}(\{e'\}) \leq G$

Examples

Find kernel & value of ϕ at given points for group homomorphism ϕ

1) $\phi: \mathbb{Z} \to \mathbb{Z}_7 \qquad \phi(1) = 4$

$$\begin{aligned}\ker\phi &= \{n \in \mathbb{Z} \mid \phi(n) = 0 \text{ in } \mathbb{Z}_7\} \\ &= \{n \in \mathbb{Z} \mid n \cdot 4 = 0 \text{ in } \mathbb{Z}_7\} \\ &= \{n \in \mathbb{Z} \mid n = 7 \cdot k\} = \langle 7 \rangle = 7\mathbb{Z}\end{aligned}$$

4 is of order 7 in $\mathbb{Z}_7$ bcs $n = 7$ is least pos. int. s.t. $n \cdot 4 = 0$

$$\begin{aligned}\phi(25) = \phi(21+4) = \phi(21) +_7 \phi(4) &= \phi(4) \\ &= 4\phi(1) \\ &= 4 \cdot_4 4\end{aligned}$$

$= 2$

2) $\phi: \mathbb{Z} \to \mathbb{Z}_{10}$ $\qquad \phi(1) = 6$

$\phi(n) = n\,\phi(1)$

$\text{Ker}\,\phi = \{ n \in \mathbb{Z} \mid \phi(n) = 0 \}$

$= \{ n \in \mathbb{Z} \mid n \cdot 6 = 0 \text{ in } \mathbb{Z}_{10} \}$

6 is order 5 in $\mathbb{Z}_{10}$

$= \{ n = 5k \mid k \in \mathbb{Z} \}$

find point @ $\phi(18)$

$\phi(18) = \phi(15+3) = \phi(15) +_{10} \phi(3) = \phi(3) = 3\phi(1)$

$= 3 \cdot_{10} 6$

$= 8$

$\phi(15) = 0$ bcs $15 = 3 \cdot 5 \in \text{Ker}\,\phi$

3) $\phi: \mathbb{Z} \to S_8$

$\phi(1) = (1, 4, 2, 5, 7, 6)$ order 6

$\text{Ker}\,\phi = \{ n \in \mathbb{Z} \mid \phi(n) = id \}$

$\phi(n) = \phi(\underbrace{1 + \dots + 1}_{n})$

$= \{ n \in \mathbb{Z} \mid (\phi(1))^n = id \}$

$= \{ n \in \mathbb{Z} \mid (1,4,2,5,7,6)^n = id$

$= \{ n = 6k \mid k \in \mathbb{Z} \}$

$$= \langle 6 \rangle = 6\mathbb{Z}$$

4) $\phi : \mathbb{Z} \times \mathbb{Z} \to S_{10}$

$$\phi((1,0)) = \underbrace{(3,5)(2,4)}_{\text{disjoint}}$$

$$\underset{4}{\phi}((0,1)) = \underbrace{(1,7)(6,10,8,9)}_{\text{disjoint}}$$

$$\ker\phi = \{(n,m) \in \mathbb{Z} \times \mathbb{Z} \mid \phi((n,m)) = \underset{(1)(2)\dots(10)}{\underset{\shortparallel}{id}}\}$$

$$= \{(n,m) \in \mathbb{Z} \times \mathbb{Z} \mid ((3,5)(2,4))^n \cdot ((1,7)(6,10,8,9))^m = id\}$$

$$= \{(n,m) \in \mathbb{Z} \times \mathbb{Z} \mid n = \operatorname{lcm}(2,2) \cdot k = 2k \ \wedge \ m = \operatorname{lcm}(2,4) \cdot k' = 4k'\}$$

$$= \{(2k, 4k') \mid k, k' \in \mathbb{Z}\}$$

$$= \langle 2 \rangle \times \langle 4 \rangle$$

$$= 2\mathbb{Z} \times 4\mathbb{Z}$$

$$\phi((3,10)) = \phi((2+1, 8+2)) = \phi((2,8) + (1,2))$$

$$= \phi((2,8)) \cdot \phi((1,2))$$

$\parallel$

id bcs $(2,8) = (2, 2\cdot 4) \in 2\mathbb{Z} \times 4\mathbb{Z} = \ker\phi$

$= \phi((1,2))$

$= \phi((1,0) + (0,2))$

$= \phi((1,0)) \cdot \phi((0,1))^2$

plug in beginning params