A solutions manual for Algebra by Thomas W. Hungerford

Chapter I: Groups - 2. Homomorphisms and Subgroups

Exercises

1. If $f : G \to H$ is a homomorphism of groups, then $f(e_G) = e_H$ and $f(a^{-1}) = f(a)^{-1}$ for all $a \in G$. Show by example that the first conclusion may be false if G, H are monoids that are note groups.

2. A group G is abelian if and only if the map $G \to G$ given by $x \to x^{-1}$ is an automorphism.

3. Let Q_8 be the group (under ordinary matrix multiplication) generated by the complex matrices $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, where $i^2 = -1$. Show that Q_8 is a non-abelian group of order 8. Q_8 is called the **quaternion group**. [*Hint:* Observe that $BA = A^3B$, whence every element of Q_8 is of the form A^iB^j. Note also that $A^4 = B^4 = I$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity element of Q_8.]

4. Let H be the group (under matrix multiplication) of real matrices generated by $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Show H is a non-abelian group of order 8 which is *not* isomorphic to the quaternion group of Exercise 3, but is isomorphic to the group D_4^*.

5. Let S be a nonempty subset of a group G and define a relation on G by $a \sim b$ if and only if $ab^{-1} \in S$. Show that $\sim$ is an equivalence relation if and only if S is a subgroup of G.

6. A nonempty finite subset of a group is a subgroup if and only if it is closed under the product in G.

7. If n is a fixed integer, then $\{kn \mid k \in \mathbb{Z}\} \subseteq \mathbb{Z}$ is an additive subgroup of $\mathbb{Z}$, which is isomorphic to $\mathbb{Z}$.

8. The set $\{\sigma \in S_n \mid \sigma(n) = n\}$ is a subgroup of S_n, which is isomorphic to S_{n-1}.

9. Let $f : G \to H$ be a homomorphism of groups, A a subgroup of G, and B a subgroup of H.
   (a) $\mathrm{Ker} f$ and $f^{-1}(B)$ are subgroups of G.
   (b) $f(A)$ is a subgroup of H.

10. List all subgroups of $Z_2 \bigoplus Z_2$. Is $Z_2 \bigoplus Z_2$ isomorphic to Z_4?

11. If G is a group, then $C = \{a \in G \mid ax = xa \text{ for all } x \in G\}$ is an abelian subgroup of G. C is called the center of G.

12. The group D_4^* is not cyclic, but can be generated by two elements. The same is true of S_n (nontrivial). What is the minimal number of generators of the additive group $\mathbb{Z} \bigoplus \mathbb{Z}$?

13. If $G = \langle a \rangle$ is a cyclic group and H is any group, then every homomorphism $f : G \to H$ is completely determined by the element $f(a) \in H$.

14. The following cyclic subgroups are all isomorphic: the multiplicative group $\langle i \rangle$ in C, the additive group Z_4 and the subgroup $\left\langle \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \right\rangle$ of S_4.

15. Let G be a group and Aut G the set of all automorphisms of G.

(a) Aut G is a group with composition of functions as a binary operation. [*Hint:* $1_G \in$ Aut G is an identity; inverses exist by Theorem 2.3.]

(b) Aut $\mathbb{Z} \cong Z_2$ and Aut $Z_6 \cong Z_2$; Aut $Z_8 \cong Z_2 \bigoplus Z_2$; Aut $Z_p \cong Z_{p-1}$ (p prime).

(c) What is the Aut Z_n for arbitrary $n \in \mathbb{N}^*$?

16. For each prime p the additive subgroup $\mathbb{Z}(p^\infty)$ of $\mathbb{Q}/\mathbb{Z}$ (Exercise 1.10) is generated by the set $\{\overline{1/p^n} \mid n \in \mathbb{N}^*\}$.

17. Let G be an abelian group and let H, K be subgroups of G. Show that the join $H \vee K$ is the set $\{ab \mid a \in H, b \in K\}$. Extend this result to any finite number of subgroups of G.

18. (a) Let G be a group and $\{H_i \mid i \in I\}$ a family of subgroups. State and prove a condition that will imply that H is a subgroup, that is, that $\bigcup_{i \in I} H_i = \langle \bigcup_{i \in I} H_i \rangle$.

(b) Give an example of a group G and a family of subgroups $\{H_i \mid i \in I\}$ such that $\bigcup_{i \in I} H_i \neq \langle \bigcup_{i \in I} H_i \rangle$.

19. (a) The set of all subgroups of a group G, partially ordered by set theoretic inclusion, forms a complete lattice (Introduction, Exercise 7.1 and 7.2) in which the g.l.b. of $\{H_i \mid i \in I\}$ is $\bigcap_{i \in I} H_i$ and the l.u.b. is $\bigcup_{i \in I} H_i$.

(b) Exhibit the lattice of subgroups of the groups S_3, D_4^*, Z_6, Z_{27}, and Z_{36}.