

# A solutions manual for Algebra by Thomas W. Hungerford

## Introduction: Prerequisites and Preliminaries - 7. The Axiom of Choice, Order and Zorn's Lemma

### Exercises

1. Let  $(A, \leq)$  be a partially ordered set and  $B$  a nonempty subset. A lower bound of  $B$  is an element  $d \in A$  such that  $d \leq b$  for every  $b \in B$ . A **greatest lower bound (g.l.b.)** of  $B$  is a lower bound  $d_0$  of  $B$  such that  $d \leq d_0$  for every other lower bound  $d$  of  $B$ . A **least upper bound (l.u.b.)** of  $B$  is an upper bound  $t_0$  of  $B$  such that  $t_0 \leq t$  for every other upper bound  $t$  of  $B$ .  $(A, \leq)$  is a **lattice** if for all  $a, b \in A$  the set  $\{a, b\}$  has both a greatest lower bound and a least upper bound.

(a) If  $S \neq \emptyset$ , then the power set  $P(S)$  ordered by set-theoretic inclusion is a lattice, which has a unique maximal element.

(b) Give an example of a partially ordered set which is not a lattice.

(c) Give an example of a lattice with no maximal element and an example of a partially ordered set with two maximal elements.

**Proof.** (a) Let  $X, Y \in P(S)$ . From  $X \subset T, Y \subset T \iff X \cup Y \subset T$ , and from  $T \subset X, T \subset Y \iff T \subset X \cap Y$ ,  $X \cap Y$  is the greatest lower bound and  $X \cup Y$  is the least upper bound of  $\{X, Y\}$ . Therefore,  $P(S)$  is a lattice. Given any set  $A \in P(S)$ ,  $A \cup S = S$ , so  $S$  is a maximal element of  $P(S)$ . Furthermore, any set  $M \in P(S)$  that is maximal still has the property that  $M \cup S = S$ . Therefore,  $P(S)$  has a unique maximal element,  $S$ .  $\square$

**Examples.** (b) The set  $\{\emptyset, \{0, 2\}, \{1, 3\}\}$  ordered by inclusion.

(c) The natural numbers ordered in the traditional way, and the set  $\{\emptyset, \{0, 2\}, \{1, 3\}\}$  ordered by inclusion.

2. A lattice  $(A, \leq)$  (see Exercise 1) is said to be **complete** if every nonempty subset of  $A$  has both a least upper bound and a greatest lower bound. A map of partially ordered sets  $f : A \rightarrow B$  is said to preserve order if  $a \leq a'$  in  $A$  implies  $f(a) \leq f(a')$  in  $B$ . Prove that an order-preserving map  $f$  of a complete lattice  $A$  onto itself has at least one fixed element (that is, an  $a \in A$  such that  $f(a) = a$ ).

**Proof.** Let  $P = \{x \in A : f(x) \geq x\}$ . Note that  $P$  is nonempty as the g.l.b of  $A \in P$ . Let  $a$  be the l.u.b of  $P$ . Since  $a \geq x$  for all  $x \in P$ ,  $f(a) \geq f(x)$  for all  $x \in P$ , and so  $f(a) \geq x$  for all  $x \in P$ ; thus  $f(a) \geq a$ , and so  $a \in P$ .

But then  $a \leq f(a)$  implies  $f(a) \leq f(f(a))$ ; thus  $f(a) \in A$ , and so  $f(a) \leq a$ . Therefore,  $f(a) = a$ .  $\square$

3. Exhibit a well-ordering of the set  $\mathbb{Q}$  of rational numbers.

**Example.**  $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{1}, \frac{2}{4}, \frac{3}{2}, \frac{4}{1}, \frac{5}{1}, \frac{1}{6}, \dots, -\frac{1}{1}, -\frac{1}{2}, -\frac{2}{1}, -\frac{1}{3}, -\frac{3}{1}, -\frac{1}{4}, -\frac{2}{3}, -\frac{3}{2}, -\frac{4}{1}, \dots$

4. Let  $S$  be a set. A **choice function** for  $S$  is a function  $f$  from the set of all nonempty subsets of  $S$  to  $S$  such that  $f(A) \in A$  for all  $A \neq \emptyset, A \subset S$ . Show that the Axiom of Choice is equivalent to the statement that every set  $S$  has a choice function.

**Proof.** When  $S = \emptyset$ , there does not exist nonempty subset of  $S$ ; vacuously true, so we suppose  $S$  is nonempty. Let  $X = \{X_i \mid i \in I\}$  be the family of all nonempty subsets of  $S$ . Suppose that the Axiom of Choice is true, then we have  $\prod_{i \in I} X_i$  which is nonempty; from each element of the product,  $\langle x_i \mid i \in I \rangle$  which is a sequence of  $x_i$  such that  $x_i \in X_i$ , we have a function  $f(i) = x_i$ . So every set  $S$  has a choice function. Conversely, If there is a choice function  $f$ , then  $\langle f(i) \mid i \in I \rangle$  is an element of the product  $\prod_{i \in I} X_i$ . So the product is nonempty.  $\square$

5. Let  $S$  be the set of all points  $(x, y)$  in the plane with  $y \leq 0$ . Define an ordering by  $(x_1, y_1) \leq (x_2, y_2) \iff x_1 = x_2$  and  $y_1 \leq y_2$ . Show that this is a partial ordering of  $S$ , and that  $S$  has infinitely many maximal elements.

**Sketch of proof.** It's easy to show that the relation on  $S$  is reflexive, antisymmetric, and transitive. The elements  $(x, 0)$  are all maximal elements.

6. Prove that if all the sets in the family  $\{A_i \mid i \in I \neq \emptyset\}$  are nonempty, then each of the projections  $\pi_k : \prod_{i \in I} A_i \rightarrow A_k$  is surjective.

**Proof.** By definition, the product  $\prod_{i \in I} A_i$  is the collection of all functions  $f : I \rightarrow \bigcup_{i \in I} A_i$  such that  $f(i) \in A_i$  for all  $i \in I$ . Then each of the projections  $\pi_k : \prod_{i \in I} A_i \rightarrow A_k$  is given by  $f \mapsto f(k)$ . It is obvious that each of the projections is well-defined, and  $\text{dom } \pi_k = A_k$ , and so injective.  $\square$  *FIXME: maybe not rigorous.*

7. Let  $(A, \leq)$  be a linearly ordered set. The **immediate successor** of  $a \in A$  (if it exists) is the least element in the set  $\{x \in A \mid a < x\}$ . Prove that if  $A$  is well-ordered by  $\leq$ , then at most one element of  $A$  has no immediate successor. Give an example of a linearly ordered set in which precisely two elements have no immediate successor.

**Proof.** Given any  $a \in A$ . If  $a$  is not a maximal element of  $A$ , then  $X_a = \{x \in A \mid x > a\}$  is nonempty; by well-orderedness,  $X_a$  has a least element  $m$ , which is the immediate successor of  $a$ . Otherwise,  $a$  has no immediate

successor; given any two maximal elements  $M$  and  $N$ , either  $M < N$ ,  $M = N$ , or  $N < M$ ; thus  $M = N$ .  $\square$

**Example.** Consider a subset of  $\mathbb{R}$ ,  $[0, 1] \cup (1.1, 2]$ .  $1, 2$  have no immediate successor.