A solutions manual for Algebra by Thomas W. Hungerford

Introduction: Prerequisites and Preliminaries - 7. The Axiom of Choice, Order and Zorn's Lemma

Exercises

1. Let $(A, \leq)$ be a partially ordered set and B a nonempty subset. A lower bound of B is an element $d \in A$ such that $d \leq b$ for every $b \in B$. A **greatest lower bound (g.l.b.)** of B is a lower bound d_0 of B such that $d \leq d_0$ for every other lower bound d of B. A **least upper bound (l.u.b.)** of B is an upper bound t_0 of B such that $t_0 \leq t$ for every other upper bound t of B. $(A, \leq)$ is a **lattice** if for all $a, b \in A$ the set $\{a, b\}$ has both a greatest lower bound and a least upper bound.

(a) If $S \neq \emptyset$, then the power set $P(S)$ ordered by set-theoretic inclusion is a lattice, which has a unique maximal element.

(b) Give an example of a partially ordered set which is not a lattice.

(c) Give an example of a lattice with no maximal element and an example of a partially ordered set with two maximal elements.

Proof. (a) Let $X, Y \in P(S)$. From $X \subset T, Y \subset T \iff X \cup Y \subset T$, and from $T \subset X, T \subset Y \iff T \subset X \cap Y$, $X \cap Y$ is the greatest lower bound and $X \cup Y$ is the least upper bound of $\{X, Y\}$. Therefore, $P(S)$ is a lattice. Given any set $A \in P(S), A \cup S = S$, so S is a maximal element of $P(S)$. Furthermore, any set $M \in P(S)$ that is maximal still has the property that $M \cup S = S$. Therefore, $P(S)$ has a unique maximal element, S. $\square$

Examples. (b) The set $\{\emptyset, \{0, 2\}, \{1, 3\}\}$ ordered by inclusion.

(c) The natural numbers ordered in the traditional way, and the set $\{\emptyset, \{0, 2\}, \{1, 3\}\}$ ordered by inclusion.

2. A lattice $(A, \leq)$ (see Exercise 1) is said to be **complete** if every nonempty subset of A has both a least upper bound and a greatest lower bound. A map of partially ordered sets $f : A \to B$ is said to preserve order if $a \leq a'$ in A implies $f(a) \leq f(a')$ in B. Prove that an order-preserving map f of a complete lattice A onto itself has at least one fixed element (that is, an $a \in A$ such that $f(a) = a$).

Proof. Let $P = \{x \in A : f(x) \geq x\}$. Note that P is nonempty as the g.l.b of $A \in P$. Let a be the l.u.b of P. Since $a \geq x$ for all $x \in P$, $f(a) \geq f(x)$ for all $x \in P$, and so $f(a) \geq x$ for all $x \in P$; thus $f(a) \geq a$, and so $a \in P$.

But then $a \leq f(a)$ implies $f(a) \leq f(f(a))$; thus $f(a) \in A$, and so $f(a) \leq a$. Therefore, $f(a) = a$. $\square$

3. Exhibit a well-ordering of the set $\mathbb{Q}$ of rational numbers.

Example. $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \frac{1}{5}, \frac{5}{1}, \frac{1}{6}, \ldots, -\frac{1}{1}, -\frac{1}{2}, -\frac{2}{1}, -\frac{1}{3}, -\frac{3}{1}, -\frac{1}{4}, -\frac{2}{3}, -\frac{3}{2}, -\frac{4}{1}, \ldots$

4. Let S be a set. A **choice function** for S is a function f from the set of all nonempty subsets of S to S such that $f(A) \in A$ for all $A \neq \emptyset, A \subset S$. Show that the Axiom of Choice is equivalent to the statement that every set S has a choice function.

Proof. When $S = \emptyset$, there does not exist nonempty subset of S; vacuously true, so we suppose S is nonempty. Let $X = \{X_i \mid i \in I\}$ be the family of all nonempty subsets of S. Suppose that the Axiom of Choice is true, then we have $\prod_{i \in I} X_i$ which is nonempty; from each element of the procuct, $\langle x_i \mid i \in I \rangle$ which is a sequence of x_i such that $x_i \in X_i$, we have a function $f(i) = x_i$. So every set S has a choice function. Conversely, If there is a choice function f, then $\langle f(i) \mid i \in I \rangle$ is an element of the product $\prod_{i \in I} X_i$. So the product is nonempty. $\square$

5. Let S be the set of all points (x, y) in the plane with $y \leq 0$. Define an ordering by $(x_1, y_1) \leq (x_2, y_2) \iff x_1 = x_2$ and $y_1 \leq y_2$. Show that this is a partial ordering of S, and that S has infinitely many maximal elements.

Sketch of proof. It's easy to show that the relation on S is reflexive, antisymmetric, and transitive. The elements $(x, 0)$ are all maximal elements.

6. Prove that if all the sets in the family $\{A_i \mid i \in I \neq \emptyset\}$ are nonempty, then each of the projections $\pi_k : \prod_{i \in I} A_i \to A_k$ is surjective.

Proof. By definition, the product $\prod_{i \in I} A_i$ is the collection of all functions $f : I \to \bigcup_{i \in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$. Then each of the projections $\pi_k : \prod_{i \in I} A_i \to A_k$ is given by $f \mapsto f(k)$. It is obvious that each of the projections is well-defined, and dom $\pi_k = A_k$, and so injective. $\square$ *FIXME: maybe not rigorous.*

7. Let $(A, \leq)$ be a linearly ordered set. The **immediate successor** of $a \in A$ (if it exists) is the least element in the set $\{x \in A \mid a < x\}$. Prove that if A is well-ordered by $\leq$, then at most one element of A has no immediate successor. Give an example of a linearly ordered set in which precisely two elements have no immediate successor.

Proof. Given any $a \in A$. If a is not a maximal element of A, then $X_a = \{x \in A \mid x > a\}$ is nonempty; by well-orderedness, X_a has a least element m, which is the immediate successor of a. Otherwise, a has no immediate successor; given any two maximal elements M and N, either $M < N, M = N$, or $N < M$; thus $M = N$. $\square$

Example. Consider a subset of $\mathbb{R}$, $[0, 1] \cup (1.1, 2]$. $1, 2$ have no immediate successor.