

# A solutions manual for Algebra by Thomas W. Hungerford

## Chapter I: Groups - 1. Semigroups, Monoids, and Groups

### Exercises

1. Give examples other than those in the text of semigroups and monoids that are not groups.
2. Let  $G$  be a group (written additively),  $S$  a nonempty set, and  $M(S, G)$  the set of all functions  $f : S \rightarrow G$ . Define additions in  $M(S, G)$  as follows:  $(f + g) : S \rightarrow G$  is given by  $s \mapsto f(s) + g(s) \in G$ . Prove that  $M(S, G)$  is a group, which is abelian if  $G$  is.
3. Is it true that a semigroup which has a *left* identity element and in which every element has a *right* inverse (see Proposition 1.3) is a group?
4. Write out a multiplication table for  $D_4^*$ .
5. Prove that the symmetric group on  $n$  letters,  $S_n$ , has order  $n!$ .
6. Write out an addition table for  $Z_2 \oplus Z_2$ .  $Z_2 \oplus Z_2$  is called the **Klein Four Group**.
7. If  $p$  is prime, then the nonzero elements of  $\mathbb{Z}_p$  form a group of order  $p - 1$  under multiplication. [*Hint:*  $\bar{a} \neq \bar{0} \Rightarrow (a, p) = 1$ ; use Introduction, Theorem 6.5.] Show that this statement is false if  $p$  is not prime.
8. (a) The relation given by  $a \sim b \Leftrightarrow a - b \in Z$  is a congruence relation on the additive group  $\mathbb{Q}$  [see Theorem 1.5].  
(b) The set  $\mathbb{Q}/Z$  of equivalence classes is an infinite abelian group.
9. Let  $p$  be a fixed prime. Let  $R_p$  be the set of all those rational numbers whose denominator is relatively prime to  $p$ . Let  $R^p$  be the set of rationals whose denominator is a power of  $p$  ( $p^i, i \geq 0$ ). Prove that both  $R_p$  and  $R^p$  are abelian groups under ordinary addition of rationals.
10. Let  $p$  be a prime and let  $Z(p^\infty)$  be the following subset of the group  $\mathbb{Q}/Z$  (see pg. 27):

$$Z(p^\infty) = \{\overline{a/b} \in \mathbb{Q}/Z \mid a, b \in \mathbb{Z} \text{ and } b = p^i \text{ for some } i \geq 0\}.$$

Show that  $Z(p^\infty)$  is an infinite group under the addition operation of  $\mathbb{Q}/Z$ .

11. The following conditions on a group  $G$  are equivalent: (i)  $G$  is abelian; (ii)  $(ab)^2 = a^2b^2$  for all  $a, b \in G$ ; (iii)  $(ab)^{-1} = a^{-1}b^{-1}$  for all  $a, b \in G$ ; (iv)  $(ab)^n = a^n b^n$  for all  $n \in \mathbb{Z}$  and all  $a, b \in G$ ; (v)  $(ab)^n = a^n b^n$  for three consecutive integers  $n$  and all  $a, b \in G$ . Show (v)  $\Rightarrow$  (i) is false if “three” is replaced by “two.”

12. If  $G$  is a group,  $a, b \in G$  and  $bab = a$  for some  $r \in \mathbb{N}$ , then  $bab = a$  for all  $i \in \mathbb{N}$ .
13. If  $a^2 = e$  for all elements  $a$  of a group  $G$ , then  $G$  is abelian.
14. If  $G$  is a finite group of even order, then  $G$  contains an element  $a \neq e$  such that  $a^2 = e$ .
15. Let  $G$  be a nonempty finite set with an associative binary operation such that for all  $a, b, c \in G$   $ab = ac \Rightarrow b = c$  and  $ba = ca \Rightarrow b = c$ . Then  $G$  is a group. Show that this conclusion may be false if  $G$  is infinite.
16. Let  $a_1, a_2, \dots$  be a sequence of elements in a semigroup  $G$ . Then there exists a unique function  $\phi : \mathbb{N}^* \rightarrow G$  such that  $\phi(1) = a_1, \phi(2) = a_1a_2, \phi(3) = (a_1a_2)a_3$  and for  $n \geq 1, \phi(n+1) = (\phi(n))a_{n+1}$ . Note that  $\phi(n)$  is precisely the standard  $n$  product  $\prod_{i=1}^n a_i$  [Hint: Applying the Recursion Theorem 6.2 of the Introduction with  $a = a_1, S = G$  and  $f_n : G \rightarrow G$  given by  $x \rightarrow xa_{n+2}$  yields a function  $\varphi : \mathbb{N} \rightarrow G$ . Let  $\phi = \varphi\theta$ , where  $\theta : \mathbb{N}^* \rightarrow \mathbb{N}$  is given by  $k \mapsto k - 1$ .]