## A solutions manual for Algebra by Thomas W. Hungerford

## Chapter I: Groups - 1. Semigroups, Monoids, and Groups

## Exercises

- 1. Give examples other than those in the text of semigroups and monoids that are not groups.
- 2. Let G be a group (written additively), S a nonempty set, and M(S,G) the set of all functions  $f: S \to G$ . Define additions in M(S,G) as follows:  $(f+g): S \to G$  is given by  $s \mapsto f(s) + g(s) \in G$ . Prove that M(S,G) is a group, which is abelian if G is.
- 3. Is it true that a semigroup which has a *left* identity element and in which every element has a *right* inverse (see Proposition 1.3) is a group?
- 4. Write out a multiplication table for  $D_4^*$ .
- 5. Prove that the symmetric group on n letters,  $S_n$ , has order n!.
- 6. Write out an addition table for  $Z_2 \oplus Z_2$ .  $Z_2 \oplus Z_2$  is called the **Klein Four Group**.
- 7. If p is prime, then the nonzero elements of  $\mathbb{Z}_p$  form a group of order p-1 under multiplication. [Hint:  $\overline{a} \neq \overline{0} \Rightarrow (a,p) = 1$ ; use Introduction, Theorem 6.5.] Show that this statement is false if p is not prime.
- 8. (a) The relation given by  $a \sim b \Leftrightarrow a b \in Z$  is a congruence relation on the additive group  $\mathbb{Q}$  [see Theorem 1.5].
  - (b) The set  $\mathbb{Q}/\mathbb{Z}$  of equivalence classes is an infinite abelian group.
- 9. Let p be a fixed prime. Let  $R_p$  be the set of all those rational numbers whose denominator is relatively prime to p. Let  $R^p$  be the set of rationals whose denominator is a power of p ( $p^i, i \ge 0$ ). Prove that both  $R_p$  and  $R^p$  are abelian groups under ordinary addition of rationals.
- 10. Let p be a prime and let  $Z(p^{\infty})$  be the following subset of the group  $\mathbb{Q}/\mathbb{Z}$  (see pg. 27):

$$Z(p^{\infty}) = \{\overline{a/b} \in Q/Z \mid a, b \in Z \text{ and } b = p^i \text{ for some } i \ge 0\}.$$

Show that  $\mathbb{Z}(p^{\infty})$  is an infinite group under the addition operation of  $\mathbb{Q}/\mathbb{Z}$ .

- 11. The following conditions on a group G are equivalent: (i) G is abelian; (ii)  $(ab)^2 = a^2b^2$  for all  $a, b \in G$ ; (iii)  $(ab)^{-1} = a^{-1}b^{-1}$  for all  $a, b \in G$ ; (iv)  $(ab)^n = a^nb^n$  for all  $n \in \mathbb{Z}$  and all  $a, b \in G$ ; (v)  $(ab)^n = a^nb^n$  for three consecutive integers n and all  $a, b \in G$ . Show (v)  $\Rightarrow$  (i) is false if "three" is replaced by "two."
- 12. If G is a group,  $a, b \in G$  and bab = a for some  $r \in \mathbb{N}$ , then bab = a for all  $i \in \mathbb{N}$ .
- 13. If  $a^2 = e$  for all elements a of a group G, then G is abelian.
- 14. If G is a finite group of even order, then G contains an element  $a \neq e$  such that  $a^2 = e$ .
- 15. Let G be a nonempty finite set with an associative binary operation such that for all  $a, b, c \in G$   $ab = ac \Rightarrow b = c$  and  $ba = ca \Rightarrow b = c$ . Then G is a group. Show that this conclusion may be false if G is infinite.
- 16. Let  $a_1, a_2, ...$  be a sequence of elements in a semigroup G. Then there exists a unique function  $\phi: \mathbb{N}^* \to G$  such that  $\phi(1) = a_1, \phi(2) = a_1 a_2, \phi(3) = (a_1 a_2) a_3$  and for  $n \geq 1, \phi(n+1) = (\phi(n)) a_{n+1}$ . Note that  $\phi(n)$  is precisely the standard n product  $\prod_{i=1}^n a_i$  [Hint: Applying the Recursion Theorem 6.2 of the Introduction with  $a = a_1, S = G$  and  $f_n: G \to G$  given by  $x \to x a_{n+2}$  yields a function  $\varphi: N \to G$ . Let  $\phi = \varphi \theta$ , where  $\theta: \mathbb{N}^* \to \mathbb{N}$  is given by  $k \mapsto k-1$ .]