A solutions manual for Algebra by Thomas W. Hungerford

Chapter I: Groups - 1. Semigroups, Monoids, and Groups

Exercises

1. Give examples other than those in the text of semigroups and monoids that are not groups.

2. Let G be a group (written additively), S a nonempty set, and $M(S,G)$ the set of all functions $f : S \to G$. Define additions in $M(S,G)$ as follows: $(f+g) : S \to G$ is given by $s \mapsto f(s)+g(s) \in G$. Prove that $M(S,G)$ is a group, which is abelian if G is.

3. Is it true that a semigroup which has a *left* identity element and in which every element has a *right* inverse (see Proposition 1.3) is a group?

4. Write out a multiplication table for D_4^*.

5. Prove that the symmetric group on n letters, S_n, has order $n!$.

6. Write out an addition table for $Z_2 \bigoplus Z_2$. $Z_2 \bigoplus Z_2$ is called the **Klein Four Group**.

7. If p is prime, then the nonzero elements of $\mathbb{Z}_p$ form a group of order $p-1$ under multiplication. [*Hint:* $\overline{a} \neq \overline{0} \Rightarrow (a,p) = 1$; use Introduction, Theorem 6.5.] Show that this statement is false if p is not prime.

8. (a) The relation given by $a \sim b \Leftrightarrow a - b \in Z$ is a congruence relation on the additive group $\mathbb{Q}$ [see Theorem 1.5].
(b) The set $\mathbb{Q}/\mathbb{Z}$ of equivalence classes is an infinite abelian group.

9. Let p be a fixed prime. Let R_p be the set of all those rational numbers whose denominator is relatively prime to p. Let R^p be the set of rationals whose denominator is a power of p ($p^i, i \geq 0$). Prove that both R_p and R^p are abelian groups under ordinary addition of rationals.

10. Let p be a prime and let $Z(p^\infty)$ be the following subset of the group $\mathbb{Q}/\mathbb{Z}$ (see pg. 27):

$$Z(p^\infty) = \{\overline{a/b} \in Q/Z \mid a, b \in Z \text{ and } b = p^i \text{ for some } i \geq 0\}.$$

Show that $\mathbb{Z}(p^\infty)$ is an infinite group under the addition operation of $\mathbb{Q}/\mathbb{Z}$.

11. The following conditions on a group G are equivalent: (i) G is abelian; (ii) $(ab)^2 = a^2b^2$ for all $a, b \in G$; (iii) $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$; (iv) $(ab)^n = a^nb^n$ for all $n \in \mathbb{Z}$ and all $a, b \in G$; (v) $(ab)^n = a^nb^n$ for three consecutive integers n and all $a, b \in G$. Show (v) $\Rightarrow$ (i) is false if "three" is replaced by "two."

12. If G is a group, $a, b \in G$ and $bab = a$ for some $r \in \mathbb{N}$, then $bab = a$ for all $i \in \mathbb{N}$.

13. If $a^2 = e$ for all elements a of a group G, then G is abelian.

14. If G is a finite group of even order, then G contains an element $a \neq e$ such that $a^2 = e$.

15. Let G be a nonempty finite set with an associative binary operation such that for all $a, b, c \in G$ $ab = ac \Rightarrow b = c$ and $ba = ca \Rightarrow b = c$. Then G is a group. Show that this conclusion may be false if G is infinite.

16. Let $a_1, a_2, ...$ be a sequence of elements in a semigroup G. Then there exists a unique function $\phi : \mathbb{N}^* \to G$ such that $\phi(1) = a_1, \phi(2) = a_1 a_2, \phi(3) = (a_1 a_2) a_3$ and for $n \geq 1, \phi(n+1) = (\phi(n)) a_{n+1}$. Note that $\phi(n)$ is precisely the standard n product $\prod_{i=1}^{n} a_i$ [*Hint:* Applying the Recursion Theorem 6.2 of the Introduction with $a = a_1, S = G$ and $f_n : G \to G$ given by $x \to x a_{n+2}$ yields a function $\varphi : N \to G$. Let $\phi = \varphi\theta$, where $\theta : \mathbb{N}^* \to \mathbb{N}$ is given by $k \mapsto k - 1$.]