

# A solutions manual for Algebra by Thomas W. Hungerford

## Introduction: Prerequisites and Preliminaries - 8. Cardinal Numbers

### Exercises

1. Let  $I_0 \neq \emptyset$  and for each  $n \in \mathbb{N}^*$  let  $I_n = \{1, 2, 3, \dots, n\}$ .
  - (a)  $I_n$  is not equipollent to any of its proper subsets [*Hint*: induction.]
  - (b)  $I_m$  and  $I_n$  are equipollent if and only if  $m = n$ .
  - (c)  $I_m$  is equipollent to a subset of  $I_n$  but  $I_n$  is not equipollent to any subset of  $I_m$  if and only if  $m < n$ .
2.
  - (a) Every infinite set is equipollent to one of its proper subsets.
  - (b) A set is finite if and only if it is not equipollent to one of its proper subsets [see Exercise 1].
3.
  - (a)  $\mathbb{Z}$  is a denumerable set.
  - (b) The set  $\mathbb{Q}$  of rational numbers is denumerable. [*Hint*: show that  $|\mathbb{Z}| \leq |\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{Z}| = |\mathbb{Z}|$ .]
4. If  $A, A', B, B'$  are sets such that  $|A| = |A'|$  and  $|B| = |B'|$ , then  $|A \times B| = |A' \times B'|$ . If in addition  $A \cap B = \emptyset = A' \cap B'$ , then  $|A \cup B| = |A' \cup B'|$ . Therefore multiplication and addition of cardinals is well-defined.
5. For all cardinal numbers  $\alpha, \beta, \gamma$ 
  - (a)  $\alpha + \beta = \beta + \alpha$  and  $\alpha\beta = \beta\alpha$  (commutative laws).
  - (b)  $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$  and  $(\alpha\beta)\gamma = \alpha(\beta\gamma)$  (associative laws).
  - (c)  $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$  and  $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$  (distributive laws).
  - (d)  $\alpha + 0 = \alpha$  and  $\alpha 1 = \alpha$ .
  - (e) If  $\alpha \neq 0$ , then there is no  $\beta$  such that  $\alpha + \beta = 0$  and if  $\alpha \neq 1$ , then there is no  $\beta$  such that  $\alpha\beta = 1$ . Therefore subtraction and division of cardinal numbers cannot be defined.
6. Let  $I_n$  be as in Exercise 1. If  $A \sim I_m$  and  $B \sim I_n$  and  $A \cap B = \emptyset$ , then  $(A \cup B) \sim I_{m+n}$  and  $A \times B \sim I_{mn}$ . Thus if we identify  $|A|$  with  $m$  and  $|B|$  with  $n$ , then  $|A| + |B| = m + n$  and  $|A||B| = mn$ .
7. If  $A \sim A', B \sim B'$ , and  $f : A \rightarrow B$  is injective, then there is an injective map  $A' \rightarrow B'$ . Therefore the relation  $\leq$  on cardinal numbers is well defined.
8. An infinite subset of a denumerable set is denumerable.
9. The infinite set of real numbers  $R$  is not denumerable (that is  $\aleph_0 < |R|$ ). [*Hint*: It suffices to show that the open interval  $(0, 1)$  is not denumerable by Exercise 8. You may assume each real number can be written as an infinite decimal. If  $(0, 1)$  is

denumerable there is a bijection  $f : \mathbb{N}^* \rightarrow (0, 1)$ . Construct an infinite decimal (real number)  $.a_1a_2\cdots$  in  $(0, 1)$  such that  $a_n$  is not the  $n$ th digit in the decimal expansion of  $f(n)$ . This number cannot be in  $Im f$ .]

10. If  $\alpha, \beta$  are cardinals, define  $\alpha\beta$  to be the cardinal number of the set of all functions  $B \rightarrow A$ , where  $A, B$  are sets such that  $|A| = \alpha, |B| = \beta$ .

(a)  $\alpha\beta$  is independent of the choice of  $A, B$ .

(b)  $\alpha\beta + \gamma = (\alpha\beta)(\alpha\gamma)$ ;  $(\alpha\beta)\gamma = (\alpha\gamma)(\beta\gamma)$ ;  $\alpha\beta\gamma = (\alpha\beta)\gamma$ .

(c) If  $\alpha \leq \beta$ , then  $\alpha\gamma \leq \beta\gamma$ .

(d) If  $\alpha, \beta$  are finite with  $\alpha > 1, \beta > 1$  and  $\gamma$  is infinite, then  $\alpha\gamma = \beta\gamma$ .

(e) For every finite cardinal  $n$ ,  $\alpha^n = \alpha\alpha \cdots \alpha$  ( $n$  factors). Hence  $\alpha^n = \alpha$  if  $\alpha$  is infinite.

(f) If  $P(A)$  is the power set of a set  $A$ , then  $|P(A)| = 2^{|A|}$ .

11. If  $I$  is an infinite set, and for each  $i \in I$   $A_i$  is a finite set. Then  $|\bigcup_{i \in I} A_i| \leq |I|\alpha$ .