A solutions manual for Algebra by Thomas W. Hungerford

Introduction: Prerequisites and Preliminaries - 8. Cardinal Numbers

Exercises

1. Let $I_0 \neq \emptyset$ and for each $n \in \mathbb{N}^*$ let $I_n = \{1, 2, 3, ..., n\}$.
(a) I_n is not equipollent to any of its proper subsets [*Hint:* induction.]
(b) I_m and I_n are equipollent if and only if $m = n$.
(c) I_m is equipollent to a subset of I_n but I_n is not equipollent to any subset of I_m if and only if $m < n$.

2. (a) Every infinite set is equipollent to one of its proper subsets.
(b) A set is finite if and only if it is not equipollent to one of its proper subsets [see Exercise 1].

3. (a) $\mathbb{Z}$ is a denumerable set.
(b) The set $\mathbb{Q}$ of rational numbers is denumerable. [*Hint:* show that $|\mathbb{Z}| \leq |\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{Z}| = |\mathbb{Z}|$.]

4. If A, A', B, B' are sets such that $|A| = |A'|$ and $|B| = |B'|$, then $|A \times B| = |A' \times B'|$. If in addition $A \cap B = \emptyset = A' \cap B'$, then $|A \cup B| = |A' \cup B'|$. Therefore multiplication and addition of cardinals is well-defined.

5. For all cardinal numbers α, β, γ
(a) $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ (commutative laws).
(b) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ (associative laws).
(c) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ and $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$ (distributivelaws).
(d) $\alpha + 0 = \alpha$ and $\alpha 1 = \alpha$.
(e) If $\alpha \neq 0$, then there is no β such that $\alpha + \beta = 0$ and if $\alpha \neq 1$, then there is no β such that $\alpha\beta = 1$. Therefore subtraction and division of cardinal numbers cannot be defined.

6. Let I_n be as in Exercise 1. If $A \sim I_m$ and $B \sim I_n$ and $A \cap B = \emptyset$, then $(A \cup B) \sim I_{m+n}$ and $A \times B \sim I_{mn}$. Thus if we identify $|A|$ with m and $|B|$ with n, then $|A| + |B| = m + n$ and $|A||B| = mn$.

7. If $A \sim A', B \sim B'$, and $f : A \to B$ is injective, then there is an injective map $A' \to B'$. Therefore the relation $\leq$ on cardinal numbers is well defined.

8. An infinite subset of a denumerable set is denumerable.

9. The infinite set of real numbers R is not denumerable (that is $\aleph_0 < |R|$). [*Hint:* It suffices to show that the open interval $(0, 1)$ is not denumerable by Exercise 8. You may assume each real number can be written as an infinite decimal. If $(0, 1)$ is

denumerable there is a bijection $f : \mathbb{N}^* \to (0, 1)$. Construct an infinite decimal (real number) $.a_1 a_2 \cdots$ in $(0, 1)$ such that a_n is not the n th digit in the decimal expansion of $f(n)$. This number cannot be in $Im\ f$.]

10. If α, β are cardinals, define $\alpha\beta$ to be the cardinal number of the set of all functions $B \to A$, where A, B are sets such that $|A| = \alpha, |B| = \beta$.

(a) $\alpha\beta$ is independent of the choice of A, B.

(b) $\alpha\beta + \gamma = (\alpha\beta)(\alpha\gamma)$; $(\alpha\beta)\gamma = (\alpha\gamma)(\beta\gamma)$; $\alpha\beta\gamma = (\alpha\beta)\gamma$.

(c) If $\alpha \leq \beta$, then $\alpha\gamma \leq \beta\gamma$.

(d) If α, β are finite with $\alpha > 1, \beta > 1$ and γ is infinite, then $\alpha\gamma = \beta\gamma$.

(e) For every finite cardinal n, $\alpha^n = \alpha\alpha \cdots \alpha$ (n factors). Hence $\alpha^n = \alpha$ if α is infinite.

(f) If $P(A)$ is the power set of a set A, then $|P(A)| = 2^{|A|}$.

11. If I is an infinite set, and for each $i \in I A_i$ is a finite set. Then $|\bigcup_{i \in I} A_i| \leq |I|\alpha$.