A solutions manual for Set Theory by Thomas Jech

1. Axioms of Set Theory

Exercises

1.1. Verify (1.1) $\quad (a,b)=(c,d)$ if and only if $a=c$ and $b=d$.

Proof. If $a=c$ and $b=d$, then $(a,b)=\{\{a\},\{a,b\}\}=\{\{c\},\{c,d\}\}=(c,d)$. Assume that $(a,b)=(c,d)$ and $a=b$. Then $\{\{c\},\{c,d\}\}=\{\{a\}\}$; thus $\{c,d\}\in\{\{a\}\}$ and $\{c\}\in\{\{a\}\}$, so $\{c,d\}=\{a\}=\{c\}$. Hence $c=d=a$. Since it was assumed that $a=b$, $a=c$ and $b=d$. Assume that $(a,b)=(c,d)$ and $a\neq b$. Since $\{\{a\},\{a,b\}\}=\{\{c\},\{c,d\}\}$ and $\{a\}\neq\{a,b\}$, $\{c\}=\{a\}$ and $\{a,b\}=\{c,d\}$; thus $c=a$ and $d=b$. $\square$

1.2. There is no set X such that $P(X)\subset X$.

Proof. Suppose $P(X)\subset X$, then we have a surjective function $f:X\to P(X)$. But the set $Y=\{x\in X : x\notin f(x)\}$ is not in the range of f. Suppose not. If $z\in X$ were such that $f(z)=Y$, then $z\in Y$ if and only if $z\notin Y$, a contradiction; hence f is not a surjective function. Therefore, there is no set X such that $P(X)\subset X$. $\square$

Let
$$\mathbb{N}=\bigcap\{X : X \text{ is inductive}\}.$$
$\mathbb{N}$ is the smallest inductive set. Let us use the following notation:
$$0=\emptyset,\quad 1=\{0\},\quad 2=\{0,1\},\quad 3=\{0,1,2\},\quad ....$$
If $n\in\mathbb{N}$, let $n+1=n\cup\{n\}$. Let us define $<$ (on $\mathbb{N}$) by $n<m$ if and only if $n\in m$.

A set T is transitive if $x\in T$ implies $x\subset T$.

1.3. If X is inductive, then the set $\{x\in X : x\subset X\}$ is inductive. Hence $\mathbb{N}$ is transitive, and for each $n, n=\{m\in\mathbb{N} : m<n\}$.

Proof. Let $Y=\{x\in X : x\subset X\}$. Since $\emptyset\subset X$, and $\emptyset\in X$, $\emptyset\in Y$. Now let $y\in Y$. Since $Y\subset X$, and X is inductive, $y\in X$, i.e., $\{y\}\subset X$, and $y\cup\{y\}\in X$, and since $y\subset X$, $y\cup\{y\}\subset X$; thus $y\cup\{y\}\in Y$. Therefore, we have that Y is inductive.

Let $Y_{\mathbb{N}}=\{x\in\mathbb{N} : x\subset\mathbb{N}\}$, then $Y_{\mathbb{N}}\subset\mathbb{N}$, and since $Y_{\mathbb{N}}$ is inductive, $\mathbb{N}\subset Y_{\mathbb{N}}$; thus $\mathbb{N}=Y_{\mathbb{N}}$, and so we have that $x\in\mathbb{N}$ implies $x\subset\mathbb{N}$. Therefore, $\mathbb{N}$ is transitive.

It's obvious that $k\in n\cup\{n\}$ if and only if $k\in n$ or $k=n$. So it follows that for all $k,n\in\mathbb{N}, k<n+1$ if and only if $k<n$ or $k=n$. Now let $P(x)$ be the property "$x=\{m\in\mathbb{N} : m<x\}$". $P(0)$ holds, and assume that $P(n)$ holds. $n+1=n\cup\{n\}=\{m\in\mathbb{N} : m<n\}\cup\{n\}=\{m\in\mathbb{N} : m<n \text{ or } m=n\}=$

$\{m \in \mathbb{N} : m < n+1\} = P(n+1)$ holds. Therefore, for each $n, n = \{m \in \mathbb{N} : m < n\}$. $\square$

1.4. If X is inductive, then the set $\{x \in X : x \text{ is transitive}\}$ is inductive. Hence every $n \in \mathbb{N}$ is transitive.

Proof. Let $Y = \{x \in X : x \text{ is transitive}\}$. Since $\emptyset \in X$, and $\emptyset$ is transitive, $\emptyset \in Y$. Now let $y \in Y$. Since $Y \subset X$, and X is inductive, $y \in X$, and $y \cup \{y\} \in X$. Let $z \in y \cup \{y\}$, then either $z \in y$ or $z = y$; since y is transitive, in any case, $z \subset y \cup \{y\}$. Thus $y \cup \{y\}$ is transitive, and so $y \cup \{y\} \in Y$. Therefore, we have that Y is inductive.

Let $Y_{\mathbb{N}} = \{x \in \mathbb{N} : x \text{ is transitive}\}$, then $Y_{\mathbb{N}} \subset \mathbb{N}$, and since $Y_{\mathbb{N}}$ is inductive, $\mathbb{N} \subset Y_{\mathbb{N}}$; thus $\mathbb{N} = Y_{\mathbb{N}}$, and so we have that every $n \in \mathbb{N}$ is transitive. $\square$

1.5. If X is inductive, then the set $\{x \in X : x \text{ is transitive and } x \notin x\}$ is inductive. Hence $n \notin n$ and $n \neq n+1$ for each $n \in \mathbb{N}$.

Proof. Let $Y = \{x \in X : x \text{ is transitive and } x \notin x\}$. Since $\emptyset \in X$, and $\emptyset$ is transitive and $\emptyset \notin \emptyset$, $\emptyset \in Y$. Now let $y \in Y$. Since $Y \subset X$, and X is inductive, $y \in X$, and $y \cup \{y\} \in X$. We already have that $y \cup \{y\}$ is transitive. Suppose $y \cup \{y\} \in y \cup \{y\}$, then $y \cup \{y\} \in y$, i.e., $y \cup \{y\} \subset y$ or $y \cup \{y\} = y$; in any case, $\{y\} \subset y$, i.e., $y \in y$; a contradiction. Thus $y \cup \{y\} \notin y \cup \{y\}$, and so $y \cup \{y\} \in Y$. Therefore, we have that Y is inductive.

Let $Y_{\mathbb{N}} = \{x \in \mathbb{N} : x \text{ is transitive and } x \notin x\}$, then $Y_{\mathbb{N}} \subset \mathbb{N}$, and since $Y_{\mathbb{N}}$ is inductive, $\mathbb{N} \subset Y_{\mathbb{N}}$; thus $\mathbb{N} = Y_{\mathbb{N}}$, and so $n \notin n$. Suppose $n+1 = n$, i.e., $n \cup \{n\} = n$, then $\{n\} \subset n$, i.e., $n \in n$; a contradiction. Therefore, $n \notin n$ and $n \neq n+1$ for each $n \in \mathbb{N}$. $\square$

1.6. If X is inductive, then $\{x \in X : x \text{ is transitive and every nonempty } z \subset x$ has an ϵ-minimal element$\}$ is inductive (t is ϵ-*minimal* in z if there is no $s \in z$ such that $s \in t$).

Proof. Let $Y = \{x \in X : x \text{ is transitive and every nonempty } z \subset x$ has an ϵ-minimal element$\}$. Since $\emptyset \in X$, and $\emptyset$ is transitive and has no nonempty set, $\emptyset \in Y$. Now let $y \in Y$. Since $Y \subset X$, and X is inductive, $y \in X$, and $y \cup \{y\} \in X$. We already have that $y \cup \{y\}$ is transitive. Now suppose that there exists $a \in y$ such that $y \in a$, then $y \in a \in y$, and by transitivity of y, $y \in a \subset y$, i.e., $y \in y$, and then $\{y\} \subset y$ and $\{a, y\} \subset y$ do not have ϵ-minimal element (...$y \in y \in y$..., ...$y \in a \in y \in a$...); a contradiction. It follows that y is ϵ-*maximal* in y; thus every nonempty $z \subset y \cup \{y\}$ has an ϵ-minimal element, and so $y \cup \{y\} \in Y$. Therefore, we have that Y is inductive. $\square$

1.7. Every nonempty $X \subset \mathbb{N}$ has an ϵ-minimal element.
[Pick $n \in X$ and look at $X \cap n$.]

Proof. Since $\mathbb{N}$ is the smallest inductive set, from **1.6**, we have that every $n \in \mathbb{N}$ has an ϵ-minimal element. Let $n \in X$. If $n \cap X = \emptyset$, then n is an ϵ-minimal element. Suppose not. There exists $m \in X \setminus n$ such that $m \in n$, but since $n = \{m \in \mathbb{N} : m < n\}$, a contradiction. If $n \cap X \neq \emptyset$, then $n \cap X \subset n$ has an ϵ-minimal element, and it's an ϵ-minimal element of X; otherwise similarly to the previous, a contradiction. $\square$

1.8. If X is inductive then so is $\{x \in X : x = \emptyset \text{ or } x = y \cup \{y\} \text{ for some } y\}$. Hence each $n \neq 0$ is $m + 1$ for some m.

Proof. Let $A = \{x \in X : x = \emptyset \text{ or } x = y \cup \{y\} \text{ for some } y\}$; let $a \neq \emptyset \in A$. Since $a = y \cup \{y\}$ for some y, so is $a \cup \{a\}$ for $y \cup \{y\}$; thus $a \cup \{a\} \in A$. Therefore, A is inductive, and each $n \neq 0$ is $m + 1$ for some m. $\square$

1.9 (Induction). Let A be a subset of $\mathbb{N}$ such that $0 \in A$, and if $n \in A$ then $n + 1 \in A$. Then $A = \mathbb{N}$.

Proof. By definition, A is a inductive subset of $\mathbb{N}$. Therefore, $A = \mathbb{N}$. $\square$

A set X *has n elements* (where $n \in \mathbb{N}$) if there is a one-to-one mapping of n onto X. A set is *finite* if it has n elements for some $n \in \mathbb{N}$, and *infinite* if it is not finite.

A set S is *T-finite* if every nonempty $X \subset P(S)$ has a $\subset$-maximal element, i.e., $u \in X$ such that there is no $v \in X$ with $u \subset v$ and $u \neq v$. S is T-*infinite* if it is not T-finite. (T is for Tarski.)

1.10. Each $n \in \mathbb{N}$ is T-finite.

Proof. Let $A = \{n \in \mathbb{N} : n \text{ is T-finite}\}$. We show that $A = \mathbb{N}$ by induction.

Since $P(\emptyset) = \{\emptyset\}$ has the only subset $\{\emptyset\}$, and it's T-finite, $\emptyset \in A$.

Let $n \in A$; let $X \subset P(n+1)$. For some $Y \subset P(n)$, X is either Y or $\{n\} \cup Y$. From **1.6** and by transitivity of n, n is a $\subset$-maximal element of $\{n\} \cup Y$; thus $n + 1 \in A$. Therefore, $A = \mathbb{N}$ $\square$

1.11. $\mathbb{N}$ is T-infinite; the set $\mathbb{N} \subset P\ (\mathbb{N})$ has no $\subset$-maximal element.

1.12. Every finite set is T-finite.

1.13. Every infinite set is T-infinite.

[If S is infinite, consider $X = \{u \subset S : u \text{ is finite}\}$.]

1.14. The Separation Axioms follow from the Replacement Schema.

[Given ϕ, let $F = \{(x, x) : \phi(x)\}$. Then $\{x \in X : \phi(x)\} = F(X)$, for every X.]

1.15. Instead of Union, Power Set, and Replacement Axioms consider the following weaker versions:

(1.8) $\forall X \exists Y \bigcup X \subset Y$, i.e., $\forall X \exists Y (\forall x \in X)(\forall u \in x) u \in Y$,
(1.9) $\forall X \exists Y P(X) \subset Y$, i.e., $\forall X \exists Y \forall u (u \subset X \rightarrow u \in Y)$,
(1.10) If a class F is a function, then $\forall X \exists Y F(X) \subset Y$.

Then axioms 1.4, 1.5, and 1.7 can be proved from (1.8), (1.9), and (1.10), using the Separation Schema (1.3).