A solutions manual for Set Theory by Thomas Jech

1. Axioms of Set Theory

Exercises

1.1. Verify (1.1) $\quad (a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Proof. If $a = c$ and $b = d$, then $(a, b) = \{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} = (c, d)$. Assume that $(a, b) = (c, d)$ and $a = b$. Then $\{\{c\}, \{c, d\}\} = \{\{a\}\}$; thus $\{c, d\} \in \{\{a\}\}$ and $\{c\} \in \{\{a\}\}$, so $\{c, d\} = \{a\} = \{c\}$. Hence $c = d = a$. Since it was assumed that $a = b$, $a = c$ and $b = d$. Assume that $(a, b) = (c, d)$ and $a \neq b$. Since $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ and $\{a\} \neq \{a, b\}$, $\{c\} = \{a\}$ and $\{a, b\} = \{c, d\}$; thus $c = a$ and $d = b$. $\square$

1.2. There is no set X such that $P(X) \subset X$.

Proof. Suppose $P(X) \subset X$, then we have a function f from X onto $P(X)$. But the set $Y = \{x \in X : x \notin f(x)\}$ is not in the range of f: If $z \in X$ were such that $f(z) = Y$, then $z \in Y$ if and only if $z \notin Y$, a contradiction. Thus f is not a surjective function; also a contradiction. $\square$

Let
$$\mathbb{N} = \bigcap \{X : X \text{ is inductive}\}.$$
$\mathbb{N}$ is the smallest inductive set. Let us use the following notation:
$$0 = \emptyset, \quad 1 = \{0\}, \quad 2 = \{0, 1\}, \quad 3 = \{0, 1, 2\}, \quad ....$$
If $n \in \mathbb{N}$, let $n + 1 = n \cup \{n\}$. Let us define $<$ (on $\mathbb{N}$) by $n < m$ if and only if $n \in m$.

A set T is transitive if $x \in T$ implies $x \subset T$.

1.3. If X is inductive, then the set $\{x \in X : x \subset X\}$ is inductive. Hence $\mathbb{N}$ is transitive, and for each $n, n = \{m \in \mathbb{N} : m < n\}$.

Proof. Let $Y = \{x \in X : x \subset X\}$. Since $\emptyset \subset X$, and $\emptyset \in X$, $\emptyset \in Y$. Now let $y \in Y$. Since $Y \subset X$, and X is inductive, $y \in X$, i.e., $\{y\} \subset X$, and $y \cup \{y\} \in X$, and since $y \subset X$, $y \cup \{y\} \subset X$; thus $y \cup \{y\} \in Y$. Therefore, Y is inductive.

Let $Y_{\mathbb{N}} = \{x \in \mathbb{N} : x \subset \mathbb{N}\}$, then $Y_{\mathbb{N}} \subset \mathbb{N}$, and since $Y_{\mathbb{N}}$ is inductive, $\mathbb{N} \subset Y_{\mathbb{N}}$; thus $\mathbb{N} = Y_{\mathbb{N}}$, and so we have that $x \in \mathbb{N}$ implies $x \subset \mathbb{N}$. Therefore, $\mathbb{N}$ is transitive.

It's obvious that $k \in n \cup \{n\}$ if and only if $k \in n$ or $k = n$. So it follows that for all $k, n \in \mathbb{N}, k < n + 1$ if and only if $k < n$ or $k = n$. Now we show that for each $n, n = \{m \in \mathbb{N} : m < n\}$ by induction. Let $P(x)$ be the property "$x = \{m \in \mathbb{N} : m < x\}$". $P(0)$ holds, and assume that $P(n)$ holds. $n + 1 = n \cup \{n\}$ $= \{m \in \mathbb{N} : m < n\} \cup \{n\} = \{m \in \mathbb{N} : m < n \text{ or } m = n\} = \{m \in \mathbb{N} : m < n + 1\} =$ $P(n + 1)$ holds. Therefore, for each $n, n = \{m \in \mathbb{N} : m < n\}$. $\square$

1.4. If X is inductive, then the set $\{x \in X : x \text{ is transitive}\}$ is inductive. Hence every $n \in \mathbb{N}$ is transitive.

Proof. Let $Y = \{x \in X : x \text{ is transitive}\}$. Since $\emptyset \in X$, and $\emptyset$ is transitive, $\emptyset \in Y$. Now let $y \in Y$. Since $Y \subset X$, and X is inductive, $y \in X$, and $y \cup \{y\} \in X$. Let $z \in y \cup \{y\}$, then either $z \in y$ or $z = y$; since y is transitive, in any case, $z \subset y \cup \{y\}$. Thus $y \cup \{y\}$ is transitive, and so $y \cup \{y\} \in Y$. Therefore, Y is inductive.

Let $Y_{\mathbb{N}} = \{x \in \mathbb{N} : x \text{ is transitive}\}$, then $Y_{\mathbb{N}} \subset \mathbb{N}$, and since $Y_{\mathbb{N}}$ is inductive, $\mathbb{N} \subset Y_{\mathbb{N}}$; thus $\mathbb{N} = Y_{\mathbb{N}}$. Threfore, every $n \in \mathbb{N}$ is transitive. $\square$

1.5. If X is inductive, then the set $\{x \in X : x \text{ is transitive and } x \notin x\}$ is inductive. Hence $n \notin n$ and $n \neq n + 1$ for each $n \in \mathbb{N}$.

Proof. Let $Y = \{x \in X : x \text{ is transitive and } x \notin x\}$. Since $\emptyset \in X$, and $\emptyset$ is transitive and $\emptyset \notin \emptyset$, $\emptyset \in Y$. Now let $y \in Y$. Since $Y \subset X$, and X is inductive, $y \in X$, and $y \cup \{y\} \in X$. We already have that $y \cup \{y\}$ is transitive. Suppose $y \cup \{y\} \in y \cup \{y\}$, then $y \cup \{y\} \in y$, i.e., $y \cup \{y\} \subset y$ or $y \cup \{y\} = y$; in any case, $\{y\} \subset y$, i.e., $y \in y$; a contradiction. Thus $y \cup \{y\} \notin y \cup \{y\}$, and so $y \cup \{y\} \in Y$. Therefore, Y is inductive.

Let $Y_{\mathbb{N}} = \{x \in \mathbb{N} : x \text{ is transitive and } x \notin x\}$, then $Y_{\mathbb{N}} \subset \mathbb{N}$, and since $Y_{\mathbb{N}}$ is inductive, $\mathbb{N} \subset Y_{\mathbb{N}}$; thus $\mathbb{N} = Y_{\mathbb{N}}$, and so $n \notin n$. Suppose $n + 1 = n$, i.e., $n \cup \{n\} = n$, then $\{n\} \subset n$, i.e., $n \in n$; a contradiction. Therefore, $n \notin n$ and $n \neq n + 1$ for each $n \in \mathbb{N}$. $\square$

1.6. If X is inductive, then $\{x \in X : x \text{ is transitive and every nonempty } z \subset x \text{ has an } \epsilon\text{-minimal element}\}$ is inductive (t is ϵ-*minimal* in z if there is no $s \in z$ such that $s \in t$).

Proof. Let $Y = \{x \in X : x \text{ is transitive and every nonempty } z \subset x \text{ has an } \epsilon\text{-minimal element}\}$. Since $\emptyset \in X$, and $\emptyset$ is transitive and has no nonempty set, $\emptyset \in Y$. Now let $y \in Y$. Since $Y \subset X$, and X is inductive, $y \in X$, and $y \cup \{y\} \in X$. We already have that $y \cup \{y\}$ is transitive. Now suppose that there exists $a \in y$ such that $y \in a$, then $y \in a \in y$, and by transitivity of y, $y \in a \subset y$, i.e., $y \in y$, but then $\{y\} \subset y$ and $\{a, y\} \subset y$ do not have ϵ-minimal element ($\cdots y \in y \in y \cdots$, $\cdots y \in a \in y \in a \cdots$); a contradiction. It follows that y is ϵ-*maximal* in y; thus every nonempty $z \subset y \cup \{y\}$ has an ϵ-minimal element, and so $y \cup \{y\} \in Y$. Therefore, Y is inductive. $\square$

1.7. Every nonempty $X \subset \mathbb{N}$ has an ϵ-minimal element.
[Pick $n \in X$ and look at $X \cap n$.]

Proof. Since $\mathbb{N}$ is the smallest inductive set, from 1.6, we have that every $n \in \mathbb{N}$ has an ϵ-minimal element. Let $n \in X$. If $n \cap X = \emptyset$, then n is an ϵ-minimal element. Suppose not. There exists $m \in X \setminus n$ such that $m \in n$, but then since $n = \{m \in \mathbb{N} : m < n\}$, $n \cap X \neq \emptyset$; a contradiction. If $n \cap X \neq \emptyset$, then $n \cap X \subset n$ has an ϵ-minimal element, and it's an ϵ-minimal element of X; otherwise similarly to the previous, a contradiction. $\square$

1.8. If X is inductive then so is $\{x \in X : x = \emptyset \text{ or } x = y \cup \{y\} \text{ for some } y\}$. Hence each $n \neq 0$ is $m + 1$ for some m.

Proof. Let $A = \{x \in X : x = \emptyset \text{ or } x = y \cup \{y\} \text{ for some } y\}$; let $a \neq \emptyset \in A$. Since $a = y \cup \{y\}$ for some y, so is $a \cup \{a\}$ for a; thus $a \cup \{a\} \in A$. Therefore, A is inductive, and each $n \neq 0$ is $m + 1$ for some m. $\square$

1.9 (Induction). Let A be a subset of $\mathbb{N}$ such that $0 \in A$, and if $n \in A$ then $n+1 \in A$. Then $A = \mathbb{N}$.

Proof. By definition, A is a inductive subset of $\mathbb{N}$. Therefore, $A = \mathbb{N}$. $\square$

A set X *has n elements* (where $n \in \mathbb{N}$) if there is a one-to-one mapping of n onto X. A set is *finite* if it has n elements for some $n \in \mathbb{N}$, and *infinite* if it is not finite.

A set S is *T-finite* if every nonempty $X \subset P(S)$ has a $\subset$-maximal element, i.e., $u \in X$ such that there is no $v \in X$ with $u \subset v$ and $u \neq v$. S is T-*infinite* if it is not T-finite. (T is for Tarski.)

1.10. Each $n \in \mathbb{N}$ is T-finite.

Proof. Let $A = \{n \in \mathbb{N} : n \text{ is T-finite}\}$. We show that $A = \mathbb{N}$ by induction.

$P(\emptyset) = \{\emptyset\}$ has the only nonempty subset $\{\emptyset\}$ which has a $\subset$-maximal element $\emptyset$.

Let $n \in A$; let $X \subset P(n+1)$. For some $Y \subset P(n)$, X is either Y or $Z = \{x \cup \{n\} : x \in Y\}$. For the latter case, let a be a $\subset$-maximal element of Y. Then it's obvious that $a \cup \{n\}$ is a $\subset$-maximal element of Z; thus X is T-finite. $\square$

1.11. $\mathbb{N}$ is T-infinite; the set $\mathbb{N} \subset P\ (\mathbb{N})$ has no $\subset$-maximal element.

Proof. For any $n \in \mathbb{N}$, there exists $n + 1$ such that $n \subsetneq n + 1$; thus $\mathbb{N} \subset P(\mathbb{N})$ has no $\subset$-maximal element. $\square$

Note that $\mathbb{N} \in P(\mathbb{N})$, $\mathbb{N} \subset P(\mathbb{N})$, *and* $\bigcup \mathbb{N} = \mathbb{N}$.

1.12. Every finite set is T-finite.

Proof. Let F be a finite set, then there is a one-to-one mapping f of F onto $n \in \mathbb{N}$. Let $A \subset P(F)$ be a nonempty set. Then $B = \{f(X) \subset P(n) : X \in A\}$ is nonempty, and has a $\subset$-maximal element. It's obvious that $\forall X, Y \in A (X \subset Y \iff f(X) \subset f(Y))$; A has a $\subset$-maximal element. $\square$

1.13. Every infinite set is T-infinite.

[If S is infinite, consider $X = \{u \subset S : u \text{ is finite}\}$.]

Proof. Since $\emptyset \in X$, X is nonempty. Suppose X has a $\subset$-maximal element m. Then $S \setminus m \neq \emptyset$; otherwise S is a subset of a finite set; a contradiction, and so there exists $x \in S \setminus m$. Then $m \subsetneq m \cup \{x\} \in X$; a contradiction. $\square$

1.14. The Separation Axioms follow from the Replacement Schema.

[Given ϕ, let $F = \{(x, x) : \phi(x)\}$. Then $\{x \in X : \phi(x)\} = F(X)$, for every X.]

Proof. Let $\varphi(x, y)$ be $x = y \wedge \phi(x)$. Then $F = \{(x, x) : \phi(x)\} = \{(x, y) : \varphi(x, y)\}$. Since $\forall x \forall y \forall z (\varphi(x, y) \wedge \varphi(x, z) \rightarrow y = z)$ holds, $\varphi(x, y)$ is a functional formula. Therefore, we have that The Separation Axioms follow from the Replacement Schema.

$F(X) = \{y : (\exists x \in X)\varphi(x,y)\} = \{y : (\exists x \in X)x = y \wedge \phi(x)\} = \{x : (\exists x \in X)\phi(x)\} = \{x \in X : \phi(x)\}$. $\square$

1.15. Instead of Union, Power Set, and Replacement Axioms consider the following weaker versions:

(1.8) $\forall X \exists Y \bigcup X \subset Y$, i.e., $\forall X \exists Y (\forall x \in X)(\forall u \in x) u \in Y$,
(1.9) $\forall X \exists Y P(X) \subset Y$, i.e., $\forall X \exists Y \forall u (u \subset X \to u \in Y)$,
(1.10) If a class F is a function, then $\forall X \exists Y F(X) \subset Y$.

Then axioms 1.4, 1.5, and 1.7 can be proved from (1.8), (1.9), and (1.10), using the Separation Schema (1.3).

Proof. Using the Separation Schema,
(1.8) $\implies \{x \in Y : (\exists a \in X) x \in a\} = \bigcup X$,
(1.9) $\implies \{x \in Y : x \subset X\} = P(X)$,
(1.10) $\implies \{y \in Y : (\exists x \in X)\varphi(x,y,p)\} = F(X)$. $\square$