

# A solutions manual for Set Theory by Thomas Jech

## 1. Axioms of Set Theory

### Exercises

**1.1.** Verify (1.1)  $(a, b) = (c, d)$  if and only if  $a = c$  and  $b = d$ .

**Proof.** If  $a = c$  and  $b = d$ , then  $(a, b) = \{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} = (c, d)$ . Assume that  $(a, b) = (c, d)$  and  $a = b$ . Then  $\{\{c\}, \{c, d\}\} = \{\{a\}\}$ ; thus  $\{c, d\} \in \{\{a\}\}$  and  $\{c\} \in \{\{a\}\}$ , so  $\{c, d\} = \{a\} = \{c\}$ . Hence  $c = d = a$ . Since it was assumed that  $a = b$ ,  $a = c$  and  $b = d$ . Assume that  $(a, b) = (c, d)$  and  $a \neq b$ . Since  $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$  and  $\{a\} \neq \{a, b\}$ ,  $\{c\} = \{a\}$  and  $\{a, b\} = \{c, d\}$ ; thus  $c = a$  and  $d = b$ .  $\square$

**1.2.** There is no set  $X$  such that  $P(X) \subset X$ .

**Proof.** Suppose  $P(X) \subset X$ , then we have a surjective function  $f : X \rightarrow P(X)$ . But the set  $Y = \{x \in X : x \notin f(x)\}$  is not in the range of  $f$ . Suppose not. If  $z \in X$  were such that  $f(z) = Y$ , then  $z \in Y$  if and only if  $z \notin Y$ , a contradiction; hence  $f$  is not a surjective function.  $\square$

Let

$$\mathbb{N} = \bigcap \{X : X \text{ is inductive}\}.$$

$\mathbb{N}$  is the smallest inductive set. Let us use the following notation:

$$0 = \emptyset, \quad 1 = \{0\}, \quad 2 = \{0, 1\}, \quad 3 = \{0, 1, 2\}, \quad \dots$$

If  $n \in \mathbb{N}$ , let  $n + 1 = n \cup \{n\}$ . Let us define  $<$  (on  $\mathbb{N}$ ) by  $n < m$  if and only if  $n \in m$ .

A set  $T$  is transitive if  $x \in T$  implies  $x \subset T$ .

**1.3.** If  $X$  is inductive, then the set  $\{x \in X : x \subset X\}$  is inductive. Hence  $\mathbb{N}$  is transitive, and for each  $n$ ,  $n = \{m \in \mathbb{N} : m < n\}$ .

**Proof.** Let  $Y = \{x \in X : x \subset X\}$ . Since  $\emptyset \subset X$ , and  $\emptyset \in X$ ,  $\emptyset \in Y$ . Now let  $y \in Y$ . Since  $Y \subset X$ , and  $X$  is inductive,  $y \in X$ , i.e.,  $\{y\} \subset X$ , and  $y \cup \{y\} \in X$ , and since  $y \subset X$ ,  $y \cup \{y\} \subset X$ ; thus  $y \cup \{y\} \in Y$ . Therefore,  $Y$  is inductive.

Let  $Y_{\mathbb{N}} = \{x \in \mathbb{N} : x \subset \mathbb{N}\}$ , then  $Y_{\mathbb{N}} \subset \mathbb{N}$ , and since  $Y_{\mathbb{N}}$  is inductive,  $\mathbb{N} \subset Y_{\mathbb{N}}$ ; thus  $\mathbb{N} = Y_{\mathbb{N}}$ , and so we have that  $x \in \mathbb{N}$  implies  $x \subset \mathbb{N}$ . Therefore,  $\mathbb{N}$  is transitive.

It's obvious that  $k \in n \cup \{n\}$  if and only if  $k \in n$  or  $k = n$ . So it follows that for all  $k, n \in \mathbb{N}$ ,  $k < n + 1$  if and only if  $k < n$  or  $k = n$ . Now let  $P(x)$  be the property " $x = \{m \in \mathbb{N} : m < x\}$ ".  $P(0)$  holds, and assume that  $P(n)$  holds.  $n + 1 = n \cup \{n\} = \{m \in \mathbb{N} : m < n\} \cup \{n\} = \{m \in \mathbb{N} : m < n \text{ or } m = n\} = \{m \in \mathbb{N} : m < n + 1\} = P(n + 1)$  holds. Therefore, for each  $n$ ,  $n = \{m \in \mathbb{N} : m < n\}$ .  $\square$

**1.4.** If  $X$  is inductive, then the set  $\{x \in X : x \text{ is transitive}\}$  is inductive. Hence every  $n \in \mathbb{N}$  is transitive.

**Proof.** Let  $Y = \{x \in X : x \text{ is transitive}\}$ . Since  $\emptyset \in X$ , and  $\emptyset$  is transitive,  $\emptyset \in Y$ . Now let  $y \in Y$ . Since  $Y \subset X$ , and  $X$  is inductive,  $y \in X$ , and  $y \cup \{y\} \in X$ . Let  $z \in y \cup \{y\}$ , then either  $z \in y$  or  $z = y$ ; since  $y$  is transitive, in any case,  $z \subset y \cup \{y\}$ . Thus  $y \cup \{y\}$  is transitive, and so  $y \cup \{y\} \in Y$ . Therefore,  $Y$  is inductive.

Let  $Y_{\mathbb{N}} = \{x \in \mathbb{N} : x \text{ is transitive}\}$ , then  $Y_{\mathbb{N}} \subset \mathbb{N}$ , and since  $Y_{\mathbb{N}}$  is inductive,  $\mathbb{N} \subset Y_{\mathbb{N}}$ ; thus  $\mathbb{N} = Y_{\mathbb{N}}$ . Therefore, every  $n \in \mathbb{N}$  is transitive.  $\square$

**1.5.** If  $X$  is inductive, then the set  $\{x \in X : x \text{ is transitive and } x \notin x\}$  is inductive. Hence  $n \notin n$  and  $n \neq n + 1$  for each  $n \in \mathbb{N}$ .

**Proof.** Let  $Y = \{x \in X : x \text{ is transitive and } x \notin x\}$ . Since  $\emptyset \in X$ , and  $\emptyset$  is transitive and  $\emptyset \notin \emptyset$ ,  $\emptyset \in Y$ . Now let  $y \in Y$ . Since  $Y \subset X$ , and  $X$  is inductive,  $y \in X$ , and  $y \cup \{y\} \in X$ . We already have that  $y \cup \{y\}$  is transitive. Suppose  $y \cup \{y\} \in y \cup \{y\}$ , then  $y \cup \{y\} \in y$ , i.e.,  $y \cup \{y\} \subset y$  or  $y \cup \{y\} = y$ ; in any case,  $\{y\} \subset y$ , i.e.,  $y \in y$ ; a contradiction. Thus  $y \cup \{y\} \notin y \cup \{y\}$ , and so  $y \cup \{y\} \in Y$ . Therefore,  $Y$  is inductive.

Let  $Y_{\mathbb{N}} = \{x \in \mathbb{N} : x \text{ is transitive and } x \notin x\}$ , then  $Y_{\mathbb{N}} \subset \mathbb{N}$ , and since  $Y_{\mathbb{N}}$  is inductive,  $\mathbb{N} \subset Y_{\mathbb{N}}$ ; thus  $\mathbb{N} = Y_{\mathbb{N}}$ , and so  $n \notin n$ . Suppose  $n + 1 = n$ , i.e.,  $n \cup \{n\} = n$ , then  $\{n\} \subset n$ , i.e.,  $n \in n$ ; a contradiction. Therefore,  $n \notin n$  and  $n \neq n + 1$  for each  $n \in \mathbb{N}$ .  $\square$

**1.6.** If  $X$  is inductive, then  $\{x \in X : x \text{ is transitive and every nonempty } z \subset x \text{ has an } \epsilon\text{-minimal element}\}$  is inductive ( $t$  is  $\epsilon\text{-minimal}$  in  $z$  if there is no  $s \in z$  such that  $s \in t$ ).

**Proof.** Let  $Y = \{x \in X : x \text{ is transitive and every nonempty } z \subset x \text{ has an } \epsilon\text{-minimal element}\}$ . Since  $\emptyset \in X$ , and  $\emptyset$  is transitive and has no nonempty set,  $\emptyset \in Y$ . Now let  $y \in Y$ . Since  $Y \subset X$ , and  $X$  is inductive,  $y \in X$ , and  $y \cup \{y\} \in X$ . We already have that  $y \cup \{y\}$  is transitive. Now suppose that there exists  $a \in y$  such that  $y \in a$ , then  $y \in a \in y$ , and by transitivity of  $y$ ,  $y \in a \subset y$ , i.e.,  $y \in y$ , but then  $\{y\} \subset y$  and  $\{a, y\} \subset y$  do not have  $\epsilon\text{-minimal element}$  ( $\cdots y \in y \in y \cdots$ ,  $\cdots y \in a \in y \in a \cdots$ ); a contradiction. It follows that  $y$  is  $\epsilon\text{-maximal}$  in  $y$ ; thus every nonempty  $z \subset y \cup \{y\}$  has an  $\epsilon\text{-minimal element}$ , and so  $y \cup \{y\} \in Y$ . Therefore,  $Y$  is inductive.  $\square$

**1.7.** Every nonempty  $X \subset \mathbb{N}$  has an  $\epsilon\text{-minimal element}$ .

[Pick  $n \in X$  and look at  $X \cap n$ .]

**Proof.** Since  $\mathbb{N}$  is the smallest inductive set, from **1.6**, we have that every  $n \in \mathbb{N}$  has an  $\epsilon\text{-minimal element}$ . Let  $n \in X$ . If  $n \cap X = \emptyset$ , then  $n$  is an  $\epsilon\text{-minimal element}$ . Suppose not. There exists  $m \in X \setminus n$  such that  $m \in n$ , but then since  $n = \{m \in \mathbb{N} : m < n\}$ ,  $n \cap X \neq \emptyset$ ; a contradiction. If  $n \cap X \neq \emptyset$ , then  $n \cap X \subset n$  has an  $\epsilon\text{-minimal element}$ , and it's an  $\epsilon\text{-minimal element of } X$ ; otherwise similarly to the previous, a contradiction.  $\square$

**1.8.** If  $X$  is inductive then so is  $\{x \in X : x = \emptyset \text{ or } x = y \cup \{y\} \text{ for some } y\}$ . Hence each  $n \neq 0$  is  $m + 1$  for some  $m$ .

**Proof.** Let  $A = \{x \in X : x = \emptyset \text{ or } x = y \cup \{y\} \text{ for some } y\}$ ; let  $a \neq \emptyset \in A$ . Since  $a = y \cup \{y\}$  for some  $y$ , so is  $a \cup \{a\}$  for  $a$ , i.e.,  $y \cup \{y\}$ ; thus  $a \cup \{a\} \in A$ . Therefore,  $A$  is inductive, and each  $n \neq 0$  is  $m + 1$  for some  $m$ .  $\square$

**1.9 (Induction).** Let  $A$  be a subset of  $\mathbb{N}$  such that  $0 \in A$ , and if  $n \in A$  then  $n + 1 \in A$ . Then  $A = \mathbb{N}$ .

**Proof.** By definition,  $A$  is an inductive subset of  $\mathbb{N}$ . Therefore,  $A = \mathbb{N}$ .  $\square$

A set  $X$  has  $n$  elements (where  $n \in \mathbb{N}$ ) if there is a one-to-one mapping of  $n$  onto  $X$ . A set is *finite* if it has  $n$  elements for some  $n \in \mathbb{N}$ , and *infinite* if it is not finite.

A set  $S$  is *T-finite* if every nonempty  $X \subset P(S)$  has a  $\subset$ -maximal element, i.e.,  $u \in X$  such that there is no  $v \in X$  with  $u \subset v$  and  $u \neq v$ .  $S$  is *T-infinite* if it is not T-finite. (T is for Tarski.)

**1.10.** Each  $n \in \mathbb{N}$  is T-finite.

**Proof.** Let  $A = \{n \in \mathbb{N} : n \text{ is T-finite}\}$ . We show that  $A = \mathbb{N}$  by induction.

$P(\emptyset) = \{\emptyset\}$  has the only nonempty subset  $\{\emptyset\}$  which has a  $\subset$ -maximal element  $\emptyset$ .

Let  $n \in A$ ; let  $X \subset P(n+1)$ . For some  $Y \subset P(n)$ ,  $X$  is either  $Y$  or  $Z = \{x \cup \{n\} : x \in Y\}$ . For the latter case, let  $a$  be a  $\subset$ -maximal element of  $Y$ . Then it's obvious that  $a \cup \{n\}$  is a  $\subset$ -maximal element of  $Z$ ; thus  $X$  is T-finite.  $\square$

**1.11.**  $\mathbb{N}$  is T-infinite; the set  $\mathbb{N} \subset P(\mathbb{N})$  has no  $\subset$ -maximal element.

**Proof.** For any  $n \in \mathbb{N}$ , there exists  $n + 1$  such that  $n \subsetneq n + 1$ ; thus  $\mathbb{N} \subset P(\mathbb{N})$  has no  $\subset$ -maximal element.  $\square$

**1.12.** Every finite set is T-finite.

**Proof.** Let  $F$  be a finite set. By definition, we have a bijective function  $f : F \rightarrow n \in \mathbb{N}$ . Let  $A \subset P(F)$  be a nonempty set. Then  $B = \{f(X) \subset P(n) : X \in A\}$  is nonempty, and has a  $\subset$ -maximal element. Since it's obvious that  $\forall X, Y \in A (X \subset Y \iff f(X) \subset f(Y))$ ,  $A$  has a  $\subset$ -maximal element.  $\square$

**1.13.** Every infinite set is T-infinite.

[If  $S$  is infinite, consider  $X = \{u \subset S : u \text{ is finite}\}$ .]

**Proof.** Since  $\emptyset \in X$ ,  $X$  is nonempty. Suppose  $X$  has a  $\subset$ -maximal element  $m$ . Then  $S \setminus m \neq \emptyset$ , and so there exists  $x \in S \setminus m$ ; otherwise  $S$  is a subset of a finite set, a contradiction. Then  $m \subsetneq m \cup \{x\} \in X$ ; a contradiction.  $\square$

**1.14.** The Separation Axioms follow from the Replacement Schema.

[Given  $\phi$ , let  $F = \{(x, x) : \phi(x)\}$ . Then  $\{x \in X : \phi(x)\} = F(X)$ , for every  $X$ .]

**1.15.** Instead of Union, Power Set, and Replacement Axioms consider the following weaker versions:

(1.8)  $\forall X \exists Y \cup X \subset Y$ , i.e.,  $\forall X \exists Y (\forall x \in X)(\forall u \in x) u \in Y$ ,

(1.9)  $\forall X \exists Y P(X) \subset Y$ , i.e.,  $\forall X \exists Y \forall u (u \subset X \rightarrow u \in Y)$ ,

(1.10) If a class  $F$  is a function, then  $\forall X \exists Y F(X) \subset Y$ .

Then axioms 1.4, 1.5, and 1.7 can be proved from (1.8), (1.9), and (1.10), using the Separation Schema (1.3).