Arbitrage Model for AMM Liquidity Pool structure.

Due to the simplicity of the AMM pricing model, cyclic arbitrage opportunities can be derived from the following model:

$$Given\ an\ arbitrage\ route\ of\ n-Tokens\ \{T_1, T_2, T_3, .., T_n\}$$

$$Where\ T_n = T_1\ for\ route\ to\ be\ cyclic$$

$$And\ let\ Lp_n\ be\ the\ liquidity\ pool\ for\ Pair\ T_n\ /\ T_{n-1}\ (or\ vice\ versa)$$

$$And\ let\ Ta_n\ be\ Token\ A\ for\ Lp_n\ \&\ Tb_n\ be\ Token\ B\ for\ Lp_n$$

$$And\ let\ Ra_n\ be\ the\ reserves\ of\ Ta_n\ for\ Lp_n\ \&\ Rb_n\ be\ the\ reserves\ of\ Tb_n\ for\ Lp_n$$

$$And\ let\ \mu_n\ be\ a\ "reverse"\ flag\ where:$$

$$\mu_n = 0\ if\ Ta_n\ = T_{n-1}$$

$$\mu_n = 1\ if\ Ta_n\ = T_n$$

$$And\ let\ \ \gamma_n\ to\ be\ the\ fee\ associated\ with\ Lp_n\ \ and\ the\ product\ of\ fees\ to\ be:$$

$$\omega(k) = \prod_{i=1}^{k} \gamma_i$$

$$(i.e. \gamma_0 = 0.997, \gamma_1\ = 0.998, \omega(1) = 0.997 * 0.998)$$

$$The\ output\ \Delta n\ of\ the\ route, given\ input\ \Delta a$$
$$considering\ all\ the\ fees\ of\ varying\ CEXs\ involved\ in\ the\ pool, will\ be:$$

$$\Delta n = \frac{Eb_n \omega(n) \Delta a}{Ea_n + (\Delta a\ \gamma_0)} \quad (eq. 5)$$

$$Where\ Ea_n = \frac{Rj_n Ea_{n-1}}{Rj_n + (Eb_{n-1}(\frac{\omega(n)}{\gamma_0}))} \quad for\ n > 1\ and\ Ea_n = Rj_n\ for\ n = 1\ (eq. 6)$$

$$And\ Eb_n = \frac{Rk_n Eb_{n-1}}{Rj_n + (Eb_{n-1}(\frac{\omega(n)}{\gamma_0}))} \quad for\ n > 1\ and\ Eb_n = Rk_n\ for\ n = 1\ (eq. 7)$$

$$Where\ Rj_n = \begin{cases} Ra_n\ when\ \mu_n = 0 \\ Rb_n\ when\ \mu_n = 1 \end{cases} \quad (eq. 8)$$

$$And\ Rk_n = \begin{cases} Rb_n\ when\ \mu_n = 0 \\ Ra_n\ when\ \mu_n = 1 \end{cases} \quad (eq. 9)$$

$$The\ Optimal\ Input\ for\ the\ Arbitrage\ Route\ can\ be\ determined\ when\ the\ derivative\ of$$
$$the\ return\ function\ f(n), where\ f(n) = \Delta n - \ \Delta a, equals\ to\ 0$$

$$\frac{df}{da} = \frac{Ea_n * Eb_n * \omega(n)}{(Ea_n + (\Delta a *\ \gamma_0))^2} - 1 = 0$$

$$Hence\ optimal\ input\ amount\ \Delta a_{optimal} = \frac{\sqrt{Ea_n Eb_n \omega(n)} -\ Ea_n}{\gamma_0} \quad (eq. 10)$$

Therefore maximum ouptut amount, without knowing the Optimal input amount:

$$\Delta n_{maximum} = \frac{Eb_n\omega(n)(\sqrt{Ea_nEb_n\omega(n)} - Ea_n)}{\gamma_0\sqrt{Ea_nEb_n\omega(n)}} \quad (eq. 11)$$

Hence, difference between Optimal Input & Maximum Ouptut amount (λ),
without knowing the Optimal input amount :

$$\lambda = \frac{\Delta n_{maximum}}{\Delta a_{optimal}} = \frac{Eb_n\omega(n)(\sqrt{Ea_nEb_n\omega(n)} - Ea_n)}{\gamma_0\sqrt{Ea_nEb_n\omega(n)}} * \frac{\gamma_0}{\sqrt{Ea_nEb_n\omega(n)} - Ea_n}$$

$$\lambda = \frac{\Delta n_{maximum}}{\Delta a_{optimal}} = \frac{Eb_n\omega(n)}{\sqrt{Ea_nEb_n\omega(n)}} = \sqrt{\frac{Eb_n\omega(n)}{Ea_n}} \, (eq. 12)$$

Knowing the Optimal Input Amount we can then simply calculate the maximum output amount:

$$\Delta n_{maximum} = \Delta a_{optimal} * \lambda = \Delta a_{optimal} * \sqrt{\frac{Eb_n\omega(n)}{Ea_n}} \quad (eq. 13)$$

Finally, we can intrepret the unit difference between input amount and output amount as :

$$\beth = \Delta n_{maximum} - \Delta a_{optimal} = \Delta a_{optimal} * (\lambda - 1)$$