UNIT 5

STOCHASTIC PROCESSES AND MARKOV CHAINS

PART-A

SHORT QUESTIONS WITH SOLUTIONS

Q1. Define stochastic process.

Answer :

A stochastic process with state space S is the set of all possible random variables, indexed by time variable 't' where $t \geq 0$ and is represented by $(X_t, t \geq 0)$ from the etymology of Greek word stochastic, the stochastic process is also known as random process. It defines the random variables whose outcomes continuously changes with time.

Q2. What are the two different types of stochastic process?

Model Paper-I, Q1(i)

Answer :

Stochastic process are of two types. They are,

(i) Continuous stochastic process

(ii) Discrete stochastic process.

(i) **Continuous Stochastic Process**

If the process is defined for all instants of time (i.e., $t \geq 0$), then it is called continuous stochastic process and is represented by $(X_t, t \geq 0)$.

(ii) **Discrete Stochastic Process**

If the process is defined for discrete time instants (i.e., $n = 1, 2, 3, ...$), then it is called discrete stochastic process. It is represented by $(X_n, n = 1, 2, 3.....)$.

Q3. Define random process.

Answer :

Random process is defined as a random variable that is a function of time. It is also know as stochastic process.

It is denoted by $X(t, s)$

Where,

X is a random variable

s represents the possible outcomes of an experiment.

Q4. Classify the random process.

Model Paper-II, Q1(i)

Answer :

Random processes are classified into four types according to the characteristics of t and the random variable $X = X(t)$ at time t. They are,

(a) Continuous random process

(b) Discrete random process

(c) Continuous random sequence

(d) Discrete random sequence.

Q5. Define continuous and discrete random process.

Answer : Model Paper-III, Q1(i)

Continuous Random Process

If X is continuous and 't' can have any of a continuum of values, then $X(t)$ is called a continuous random process.

Discrete Random Process

If X is having only discrete values while 't' is continuous, then $X(t)$ is called a discrete random process.

Q6. Define continuous random sequence and discrete random sequence.

Answer :

Continuous Random Sequence

A random process fro which X is continuous but time has only discrete value is called a continuous random sequence.

Discrete Random Sequence

If both the time and random variable X are having only discrete values, then $X(t)$ is called a discrete random sequence.

Q7. Define stationary random process and classify them.

Answer : Model Paper-I, Q1(j)

A random or stochastic process $X(t)$ is called strict-sense stationary or simply stationary if its statistical properties are invariant to a shift of origin i.e., statistical properties do not change with time. This means that $X(t)$ and $X(t+\tau)$ has the same statistics for any value of τ.

The various types of stationary random process are,

(i) First order stationary process

(ii) Second order or wide sense stationary process

(iii) Nth order or strict- sense stationary process.

Q8. Define non-stationary random process along with its properties.

Answer :

Non-Stationary Random Process

A random process $X(t)$ is called non stationary if its statistical properties are variant to a shift of origin i.e., statistical properties change with time.

Properties of Non Stationary Random Process

1. $f(x_1,... x_N; t_1,......t_N) \neq f(x_1,... x_N; t_1+\tau,......t_N+\tau)$
2. Means is not constant
3. Its auto correlation function is time dependent.

Q9. Define Markov process.

Answer : Model Paper-II, Q1(j)

Markov process is a time varying random phenomenon for which a specific property holds.

Markov processes arise in probability and statistics in one of two ways. A stochastic process, defines via a separate argument, may be shown to have the Markov property and as a consequence to have the properties that can be deduced from this for all Markov processes.

Q10. List the classification states of random process.

Answer : Model Paper-III, Q1(j)

The three classification states of random process are,

(i) Transient

(ii) Periodic

(iii) Ergodic.

(i) Transient

A state is said to be transient, if it is possible to leave the state and never return back.

(ii) Periodic

A state is said to be periodic, if it is not transient and that state is returned to only one multiples of some positive integer which are greater than 1. This integer is know as the period of the state.

(iii) Ergodic

A state is said to be ergodic if it is neither transient nor periodic.

Q11. Define Markov chains.

Answer :

A stochastic process ($X(n)$, where n =0,1,....) is said to be a Markov chain if,

$$P(X(n+1) = K_{n+1} \mid X(n) = K_n, X(n-1) = K_{n-1}, ..., X(1) = K_1, X(0) = K_0) = P(X(n+1) = K_{n+1} \mid X(n) = K_n)$$

For all $K_0, K_1, ..., K_{n+1} \in S$ and all $n \in N$

PART-B

ESSAY QUESTIONS WITH SOLUTIONS

5.1 INTRODUCTION TO STOCHASTIC PROCESSES

Q12. Give a brief introduction on stochastic process.

Answer : Model Paper-I, Q10(a)

Stochastic Process

A stochastic process with state space S is the set of all possible random variables, indexed by time variable 't' where $t \geq 0$ and is represented by $(X_t, t \geq 0)$ from the etymology of Greek word stochastic, the stochastic process is also known as random process. It defines the random variables whose outcomes continuously changes with time.

Based on the characteristics of source, the generated stochastic process is of two types. They are,

(i) Continuous stochastic process

(ii) Discrete stochastic process.

1. **Continuous Stochastic Process**

If the process is defined for all instants of time (i.e., $t \geq 0$), then it is called continuous stochastic process and is represented by $(X_t, t \geq 0)$.

2. **Discrete Stochastic Process**

If the process is defined for discrete time instants (i.e., $n = 1, 2, 3, ...$), then it is called discrete stochastic process. It is represented by $(X_n, n = 1, 2, 3,....)$.

Consider an example for a discrete stochastic process, "A Random Walk".

Let, Y_n be an IID random sequence (Independent Identically Distributed Random sequence) that takes only two values $1, -1$.

For the 'Random Walk' process, let '1' represents the event, when walker takes a step to the right.

'-1' represents the event, when walker takes a step to the left and 'P' be the probability of occurrence of event '1'.

i.e., $P(Y(n) = 1) = P$

'q' be the probability of occurence of event '-1'.

i.e., $P(Y(n) = -1) = q$.

Then, the stochastic process, 'A random walk' is characterised as,

$$(X_n, n = 0, 1, 2,....)$$

Such that X(0) = 0 and

$$X(n) = Y(0) + Y(1) + ... + Y(n)$$

Based on the probabilities of occurrence of events '1' and '-1', random walk may be symmetric or asymmetric.

Symmetric random walk is the random walk with probabilities p and q equal to $\frac{1}{2}$.

$p \neq q \neq \frac{1}{2}$ is for asymmetric random walk.

The sample of path simulations from a random walk with $p = q = \frac{1}{2}$ (i.e., symmetric random walk) is as shown in figure (1).

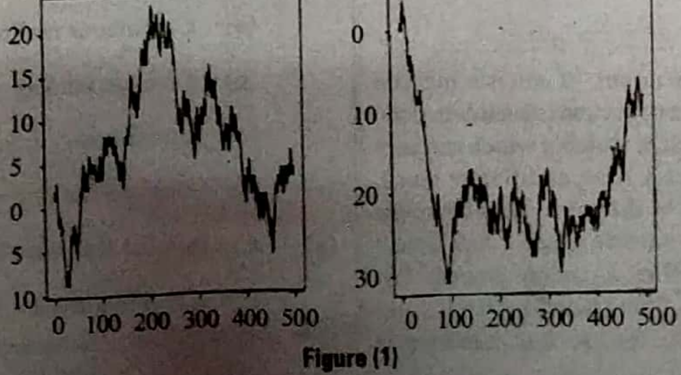


Figure (1)

The sample of path simulations from a random walk with unequal p and q is shown in figure (2).

Consider $p = 0.6$ and $q = 0.4$ for asymmetric random walk shown in figure (2).

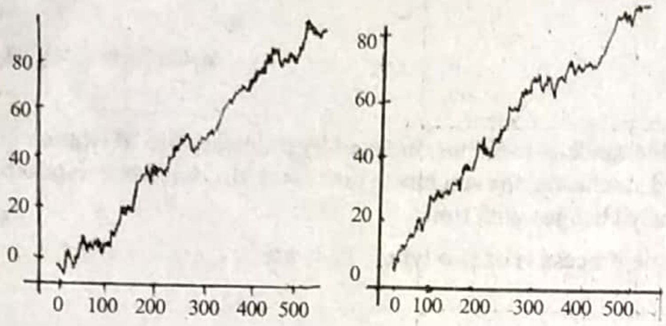


Figure (2): Samples Showing Simulation of Asymmetric Random Walk

The S-plus instructions used in simulation of random walk are given below.

$t = 500$

$z =$ Sample ($C(-1, 1)$, size $= t$, replace $= T$, prob $= C(q, p)$)

$x =$ cumsum (z)

t splot(x).

Q13. Explain the concept of random process.

Answer :

The concept of a random process is an extension of a random variable concept. Since a random variable X is by its definition a function of all possible outcomes s of an experiment, it now becomes a function of both s and time.

Definition

Random process is defined as a random variable that is a function of time. It is also known as stochastic process.

It is denoted by $X(t, s)$

Where,

X is a random variable and,

s represents the possible outcomes of an experiment.

Example

Let us consider an experiment of measuring the temperature of a room. Let there be a collection of thermometers. Each thermometer reading is a random variable which can take on any value from the sample space s. Also, at different times, the readings of thermometers may be different. Thus the room temperature is a function of both sample space s and time t which is denoted by $X(t, s)$. Therefore, a random process $X(t, s)$ represents a family or ensemble of time functions when t and s are variables. The following figure shows a few members of the ensemble.

$X_1(t)$ is the reading of first thermometer.

$X_2(t)$ is the reading of second thermometer and so on.

Each member is also known as sample function or ensemble function or realization of the process. A random process represents a single time function when t is variable and s is fixed. $X_1(t)$ and $X_2(t)$ are examples of single time functions.

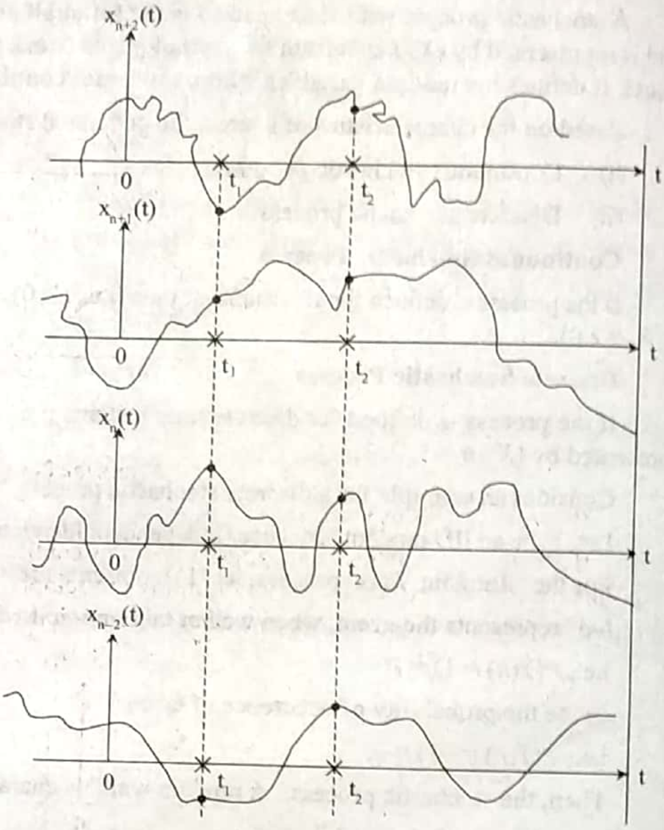


Figure: A Few Members of Esemble

Q14. Explain the classification of random process with neat sketches.

Answer : Model Paper-II, Q10(a)

Random processes are classified into four types according to the characteristics of t and the random variable $X = X(t)$ at time t. They are,

(a) Continuous random process

(b) Discrete random process

(c) Continuous random sequence

(d) Discrete random sequence.

(a) Continuous Random Process

If X is continuous and 't' also have continuous values, then $X(t)$ is called a continuous random process as shown in figure (a).

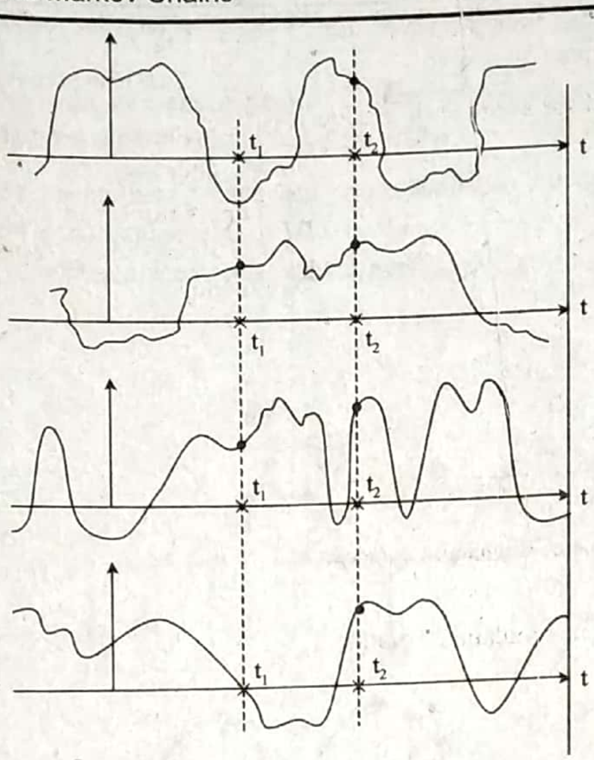


Figure (a): Continuous Random Process

(b) Discrete Random Process

If X is having only discrete values while 't' is continuous, then $X(t)$ is called a discrete random process as shown in figure (b). This process is derived by heavily limiting the sample functions. The sample functions have only two discrete values, the positive level is generated whenever a sample function is positive and negative level occurs for other times.

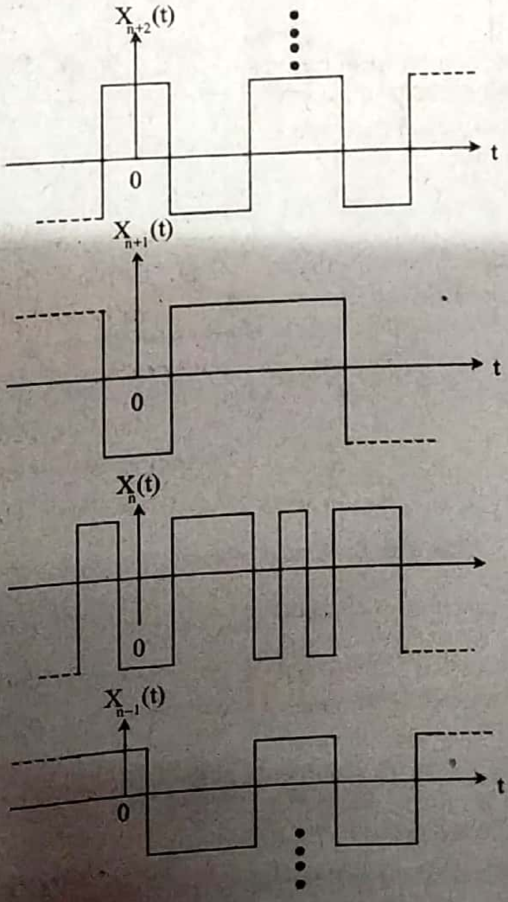


Figure (b): Discrete Random Process

(c) **Continuous Random Sequence**

A random process for which X is continuous but time has only discrete values is called as continuous random sequence. The following figure shows a sequence that is formed by periodically sampling the ensemble members of figure (c).

Since a continuous random sequence is defined at only discrete or sample times, it is also known as Discrete Time (DT) random process, its sample functions are often referred to as a *DT* random signal. These types of processes are important in the analysis of various Digital Signal Processing (DSP) systems where the sampling interval T_s is not important. So, this process is also called as a discrete-time sequence.

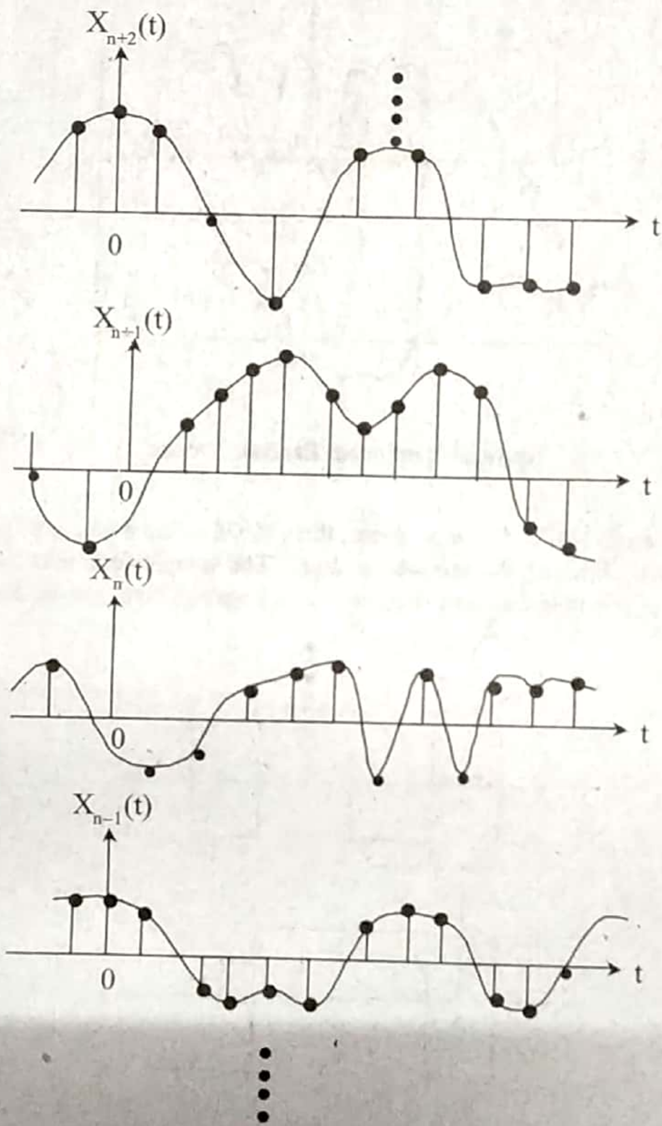


Figure (c): Continuous Random Sequence

(d) **Discrete Random Sequence**

If both the time and random variable X are having only discrete values, then $X(t)$ is called as discrete random sequence. It is developed by sampling the sample functions of figure (d) or it can derive from rounding off samples of a DT random process (continuous random process).

This operation is exactly what happens in DSP systems. The rounding consists of choosing a discrete amplitude from a finite set of discrete amplitudes, that most closely equals each sample value of DT random process.

The operation is called quantization and is necessary to convert the process to a form suitable for use in a digital computer.

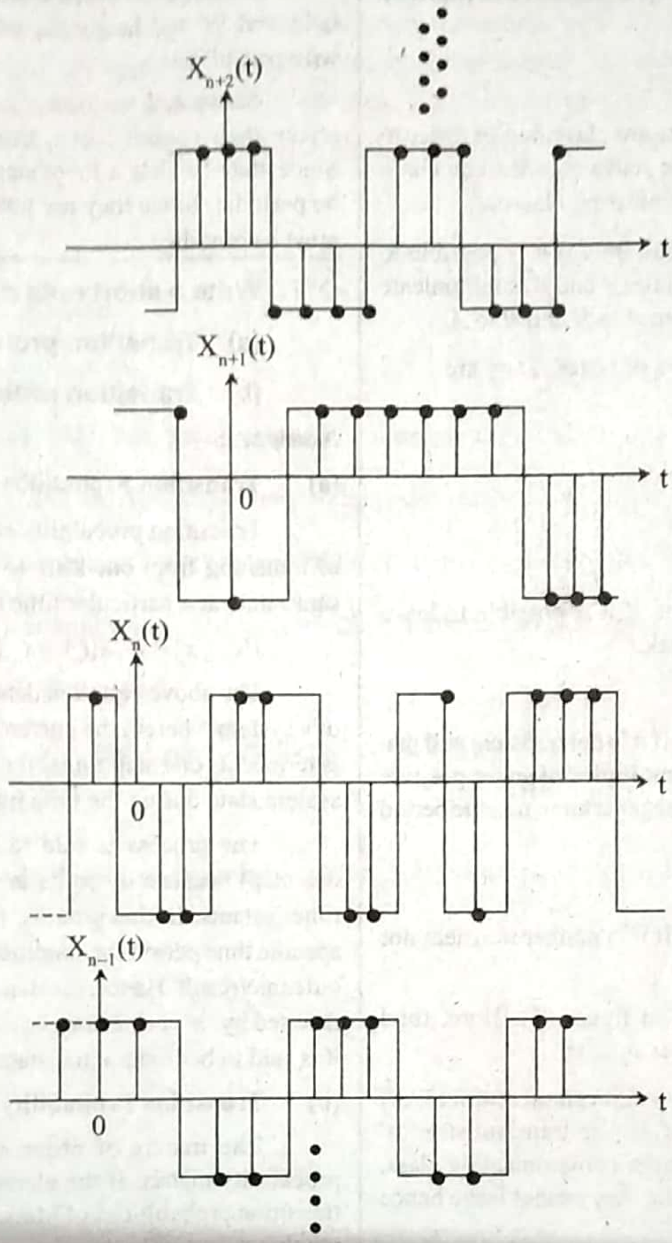


Figure (d): Discrete Random Sequence

5.2 MARKOV PROCESS, TRANSITION PROBABILITY, TRANSITION PROBABILITY MATRIX

Q15. Discuss in brief about Markov process.

Answer : Model Paper-III, Q11(a)

Markov process is a stochastic (random) process wherein the occurrence of future state of the process depends only on the probability of the outcome of its former state. A process is said to be Markov process if it satisfies the markovian property wherein $t_0 < t_1 < \ldots < t_n$ denotes the points in time scale then the family of random variables $(X(t_n))$ is given by,

$$P\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}, \ldots X(t_0) = x_0\}$$

$$P\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}\}$$

It is a process consisting of sequence of 'n' experiments wherein every individual experiments has 'n' resultant outcomes (i.e., $x_1, x_2, \ldots x_n$). Here, each resultant outcome is termed as state. The probability of outcome occurrence relies on the probability of outcome occurring in its preceeding experiment. A simple markov process is discrete and constant over time. It is preferred when the entire experiment is defined with respect to the states or possible outcomes. If a system is examined at regular intervals of time then it is said to be discrete. However, a process can only be in one state at a particular time instant.

Q16. Explain the classification of states in the Markov process.

Answer :

In Markov process, the states are classified in order to find the communicating classes. The states of a Markov chain can be partitioned into these communicating classes.

Two states communicate if and only if it is possible to go from one state to other state i.e., states A and B communicate if and only if it is possible to go from A to B and B to A.

There are three classifications of states. They are,

(i) Transient

(ii) Periodic

(iii) Ergodic

1. **Transient**

A state is said to be transient, if it is possible to leave the state and never return back.

2. **Periodic**

A state is said to be periodic, if it is not transient and that state is returned to only on multiples of some positive integer greater than 1. This integer is known as the period of the state.

3. **Ergodic**

A state is said to be ergodic if it is neither transient nor periodic.

Consider the graph shown in figure (1). Here, total communicating classes are 13, i.e., $a, b, \ldots, m$.

Once the chain goes from a to d, it cannot return to 'a', hence states a, b, c are transient. 'd' is also transient state 'e' is not transient states f, g, h, i, j form a communicating class, once they are in their part of the chain, they cannot leave hence they are not transient.

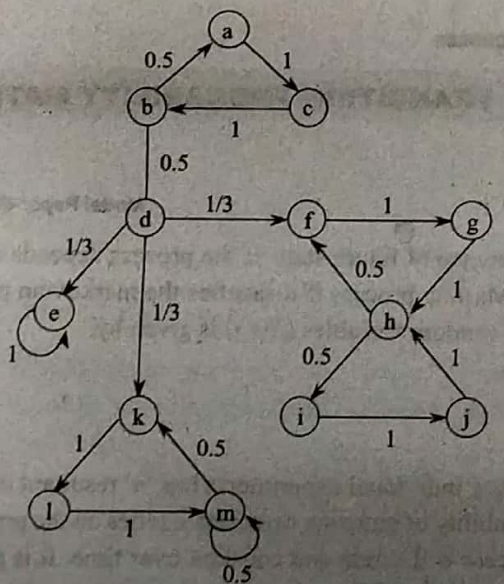


Figure (1): Graph

It can be observed that, if the state 'h' then will always is leaved 'h' and hence the whole class f-g-h-i-j are periodic with period 3.

States k, l, m form a communicating class, once they arrive then cannot leave, therefore, they are not transient. Since state 'm' has a loop, state m and its whole class cannot be periodic. Since they are neither transient nor periodic they must be ergodic.

Q17. Write a short note on,

(a) Transition probability

(b) Transition probability matrix.

Answer : Model Paper-I, Q11

(a) Transition Probability

Transition probability can be defined as the probability of transiting from one state to other state or containing in its same state at a particular time instant. It is given by,

$$Px_{n-1}, x_n = P\{X(t_n) = x_n \{X(t_{n-1}) = x_{n-1}\}\}$$

The above equation denotes the conditional probability of a system wherein the current state is x_{n-1} at time t_{n-1}. Thus, it is termed as one step transition probability since it depicts the system state during the time interval (t_{n-1}, t_n).

The process is said to have stepped (or incremented one step) because it results in different outcomes at different time instants. In this process, every individual step denotes a specific time period (or condition) which leads to other possible outcome/result. Hence, the number of steps (or increments) are denoted by 'n'. For instance, if the value of the 'n' is zero then it is said to be in the intial state.

(b) Transition Probability Matrix

The matrix of order $n \times n$ is said to be transition probability matrix, if the elements of the matrix are one-step transition probabilities of Markov chain. Here, n represents the number of possible states.

The two conditions to be satisfied by the elements of transition probability/stochastic matrix are,

(i) $P_{ij} \geq 0$

(ii) $\sum_{j=0}^{\infty} P_{ij} = 1$

However, condition (i) is satisfied from the basic theory of probabilities.

The transition probability matrix or stochastic matrix is exhibited as square matrix.

$$P = \begin{bmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ P_{i0} & P_{i1} & P_{i2} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}_{n \times n}$$

Where each element of P is defined as, $P_{ij} = P\{X(t_n) = j \mid X(t_{n-1}) = i\}$ i.e., one step transition probability.

In other words it is the probability of transitioning from i to j in one step. Hence, it is also known as stochastic matrix. The stochastic matrix is also known as Markov matrix, as it describes the transitions of Markov chain. The other names stochastic matrix are transition matrix of probability matrix or transition probability matrix. The elements of the stochastic matrix does not depend on the state n and are said to be homogenous with time. Each element of transition probability matrix is also known as probability vector. Specifically, a probability vector is said to be stationary state vector, if it is a eigen vector of transition probability matrix.

i.e., $\pi P = \pi$

Where, π is stationary state vector.

Furthermore, it can be observed that,

- If stochastic matrix is reducible, the corresponding homogenous Markov chain is also reducible.
- If stochastic matrix is irreducible, the corresponding homogenous Markov chain is also irreducible.
- If stochastic matrix is primitive, the corresponding homogenous Markov chain is cyclic.
- If stochastic matrix is imprimitive, the corresponding homogenous Markov chain is acyclic.

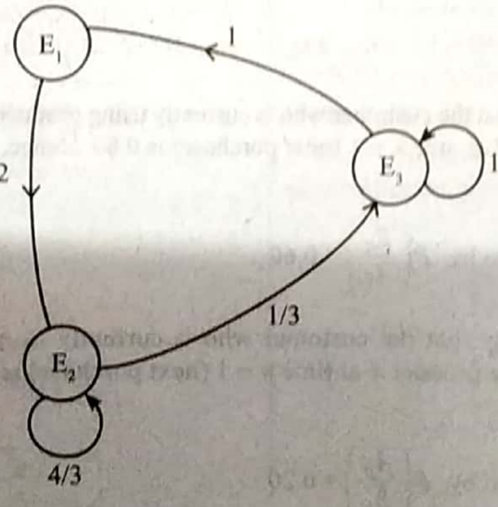
Representation of Transition Probabilities

Transition probabilities can be represented with the help of following diagrams,

(a) Transition Diagram

This diagram demonstrates the transition probabilities or shifts occurring in a specific situation.

Example



$$P = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{4}{3} & \frac{1}{4} \\ 1 & 0 & \frac{1}{4} \end{bmatrix} \end{array}$$

Here, zero in the matrix denotes that there is no transition from one state to another state.

(b) Probability Tree Diagram

This diagram focuses on the probabilities and their transitions (i.e., movement from one step to another). Besides this, it also covers all the possible paths or branches that links the outcomes over a particular time.

PROBLEM

Q18. Two manufacturers A and B are competing with each other in a restricted market. Over the years, A's customers have exhibited a high degree of loyalty as measured by the fact that customers using A's product 80% of time. Also, former customers purchasing the product from B have switched back to A's 60% of time.

(a) Construct and interpret the state transition matrix in terms of,

(i) Retention and loss

(ii) Retention and gain.

(b) Calculate the probability of a customer purchasing A's product at the end of the second period.

Solution : **Model Paper-II, Q11**

(a) The transition probabilities in the matrix form is given by,

$$P_{(\text{presnet})} = \text{Purchase } (n=0) \begin{array}{c} A \\ B \end{array}\begin{bmatrix} 0.80 & 0.20 \\ 0.60 & 0.40 \end{bmatrix} \downarrow \text{retention and gains}$$

$$\rightarrow \text{retention and losses}$$

The probability of a customer's pruchase at the next step '$n-1$' rely on the product which a customer posses at prent ($n=0$). The above matrix denotes the conditional probability for passing from one state to other state. In mathematical concepts, the conditional probabilities in the above matrix are stated as follows,

(i) The conditional probability is given by,

$$P\left(\frac{A_0}{A_1}\right) = P_{11} = 0.80$$

This implies that, the probability that the customer who is currently using product 'A' at time n = 0 (present purcahse) will again purchase the same product A at time $n = 1$ (next purchase) is 0.80. Hence, the product 'A' is retained (or retention to product A).

(ii) The conditional probability is given by, $P\left(\frac{B_0}{A_1}\right) = 0.60$

This implies that, the probability that the customer who is currently using product 'B' at time $n = 0$ (present purchase) purchase will purchase product A at time $n = 1$ (next purchase) is 0.60. Hence, the product B is loss (or loss to product B).

(iii) The conditional probability is given by, $P\left(\frac{A_0}{B_1}\right) = 0.20$

This implies that, the probability that the customer who is current using product 'A' at time $n = 0$ (present purcahse) will purchase product 'B' at time $n = 1$ (next purchase) is 0.20. Hence, the product A is loss (or loss to product A).

(iv) The conditional probability is given by, $P\left(\frac{B_0}{B_1}\right) = 0.40$

This implies that, the probability that the customer who is currently using product 'B' at time $n = 0$ (present purchase) will again purchase the same product B at time $n = 1$ (next purchase) is 0.40.

Hence, the product B is retained (or retention to product B).

(b) Transition probabilities can be diagrammatically represented by either the transition diagram or probability tree diagram.

Transition Probability Diagram

The transition probability diagram is as shown below,

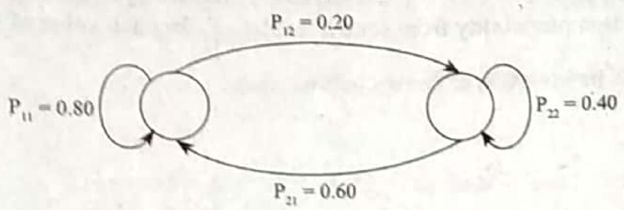


Figure: Transition Probability Diagram

In the above figure, nodes denotes states and arrows denotes the transition probabilities among states.

Probability Tree Diagram

The probability tree diagram is as shown below,

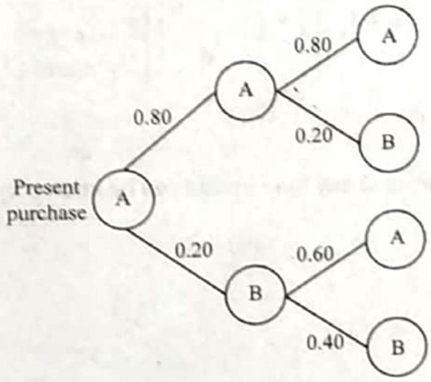


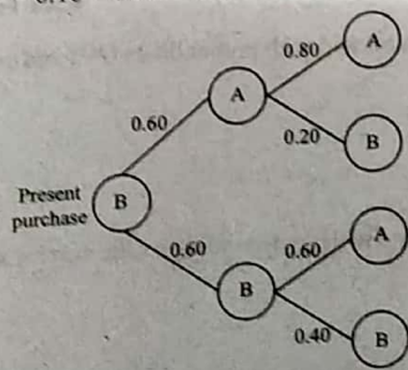
Figure: Probability Tree Diagram

$P_{11} = (0.80)(0.80) + (0.20)(0.60)$

$= 0.64 + 0.12 = 0.76$

$P_{12} = (0.80)(0.20) + (0.20)(0.40)$

$= 0.16 + 0.80 = 0.24$



$P_{21} = (0.60)(0.80) + (0.40)(0.60)$

$= 0.48 + 0.24 = 0.72$

$P_{22} = (0.60)(0.20) + (0.40)(0.40)$

$= 0.12 + 0.16 = 0.28$

5.3 FIRST ORDER AND HIGHER ORDER MARKOV PROCESS, N-STEP TRANSITION PROBABILITIES

Q19. Write a short note on,

(i) First order Markov process

(ii) Second order Markov process

(iii) n-step transition probabilities.

Model Paper-III, Q10

Answer :

(i) First Order Markov Process

The first order Markov process is build by considering three assumptions. They are,

1. The group of probable result(outcome) is finite.
2. The next outcome (result) probability rely on the immediately preceeding outcome.
3. The transition probabilities are uniform (constant) throughout the time.

(ii) Second Order Markov Process

The second order Markov process predicts that the next outcome's probability relies on the two preceeding outcomes.

Similarly, the third order markov process is computed by considering the three preceeding outcomes.

(iii) n-Step Transition Probabilities

Consider a system that requires to determine the probability of moving from a state E_i at time $t = 0$ to a state E_j at time $t = n$. So, these time periods are known as number of steps.

If $P_{ij}^{(n)}$ represent the n-step transition then the transition probabilities in the matrix form is as shown below,

$$P^{(n)} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ \vdots \\ E_m \end{array} \begin{array}{c} \begin{array}{ccccc} E_1 & E_2 & E_3 & \cdots & E_m \end{array} \\ \begin{bmatrix} P_{11}^{(n)} & P_{12}^{(n)} & P_{13}^{(n)} & \cdots & P_{1m}^{(n)} \\ P_{21}^{(n)} & P_{22}^{(n)} & P_{23}^{(n)} & \cdots & P_{2m}^{(n)} \\ P_{31}^{(n)} & P_{32}^{(n)} & P_{33}^{(n)} & \cdots & P_{3m}^{(n)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ P_{m1}^{(n)} & P_{m2}^{(n)} & P_{m3}^{(n)} & \cdots & P_{mm}^{(n)} \end{bmatrix} \end{array}$$

Where,

$P_{31}^{(n)}$ indicates the probability that the system which occupied by state E_3 will move to state E_1 after performing n – steps.

Assume $P_i(n)$ denotes the probability of moving from state E_i to state E_j in one-step transition. Here, the transition probability is time independent but the absolute probability is time dependent.

If 'm' is the probable state then, $\sum_{i=1}^{m} P_i(n) = 1$ and $\sum_{j=1}^{m} P_{ij} = 1 \forall i$

When the state probabilities are known at time $t = n$, the state probabilities at time $t = n + 1$ can be computed by using the below equation.

$$P_j(n+1) = \sum_{i=1}^{m} P_i(n) P_{ij} \text{ ; where n} = 0, 1, 2, 3, \ldots$$

Alternatively, the probability of finding the system in the state E_j at time $t = n + 1$ is equal to the probability of finding the system in the state E_i at time $t = n$ which is multiplied by the transition probability from state E_i to state E_j for each value of i.

The equations for each state probability at time $t = n + 1$ can be rewritten as shown below,

$$P_1(n+1) = P_1(n)P_{11} + P_2(n)P_{21} + P_3(n)P_{31} + \ldots + P_m(n)\, P_{m1}$$

$$P_2(n+1) = P_1(n)P_{12} + P_2(n)P_{22} + P_3(n)P_{31} + \ldots + P_m(n)\, P_{m2}$$

$$P_3(n+1) = P_1(n)P_{13} + P_2(n)P_{23} + P_3(n)P_{33} + \ldots + P_m(n)\, P_{mm}$$

The above equations can be written in the form of matrix as shown below,

$$R(n+1) + R(n)P$$

Where,

$R(n + 1)$ denotes the row vector of state probabilities at time, $t = n + 1$,

$R(n)$ denotes the row vector of state probability at time, $t = n$.

P denotes the matrix of transition probabilities.

If the state probability at time t = 0 are known then the state probabilities at any time instant can be known by computing the following matrix equation.

$$\begin{bmatrix} R(1) = R(0)P \\ R(2) = R(1)P^2 \\ R(3) = R(2)P = R(0)P^3 \\ \vdots \\ R(n) = R(n-1)P = R(0)P^n \end{bmatrix}$$

5.4 MARKOV CHAIN, STEADY STATE CONDITION

Q20. Discuss in brief about Markov chain.

Answer :

Model Paper-III, Q11(b)

Markov chain is completely defiend by a transition matrix which is associated with probabilities (P_j^0) and states $E_j(j = 0, 1, 2, \ldots)$.

The Markov chain is classified into two type and they are,

(i) Ergodic Markov Chain

It contains a property where it can move form one state to the other state without depending on the current state within a finite number of steps.

(ii) Regular Markov Chain

It is a chain that contains transition matrix P, so that in any hour of P it contains positive non zero probability values only.

However, a regular Markov chain is considered to be a special type of ergoidc Markov chain.

Stochastic Matrix

For answer refer Unit-V, Q17(b).

Q21. Discuss the procedure for determining the steady state condition of a Markov chain.

Answer :

Steady State Condition in Markov Chain

In a regular ergodic Markov chain, the steady state condition is determined by evaluating P^n in case of large values of n. In this case, the limits are applied to the Markov process to identify the probability of number of transitions of the system for reaching the steady state.

Therefore, the equation $R(n+1) = R(n)P$ in case of larger values of n can be written as,

$$\underset{n\to\infty}{Lt}\ R(n+1) = \underset{n\to\infty}{Lt}\ R(n)P$$

When $n \to \infty$ the value of $R(n)$ becomes independent with respect to time and becomes constant i.e., $R(n+1) = R(n) = R$. At this point, the system is said to be at a steady state.

PROBLEMS

Q22. Determine if the following transition matrix is ergodic Markov chain.

Future states

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \textbf{Present states} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 0 & 1/2 & 1/4 & 1/4 \\ 1/4 & 0 & 1/2 & 1/4 \\ 0 & 0 & 1/3 & 2/3 \end{bmatrix} \end{array}$$

Solution :

Given that,

Future states

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \text{Present states} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 0 & 1/2 & 1/4 & 1/4 \\ 1/4 & 0 & 1/2 & 1/4 \\ 0 & 0 & 1/3 & 2/3 \end{bmatrix} \end{array}$$

The transition diagram for the given transition matrix is as shown below,

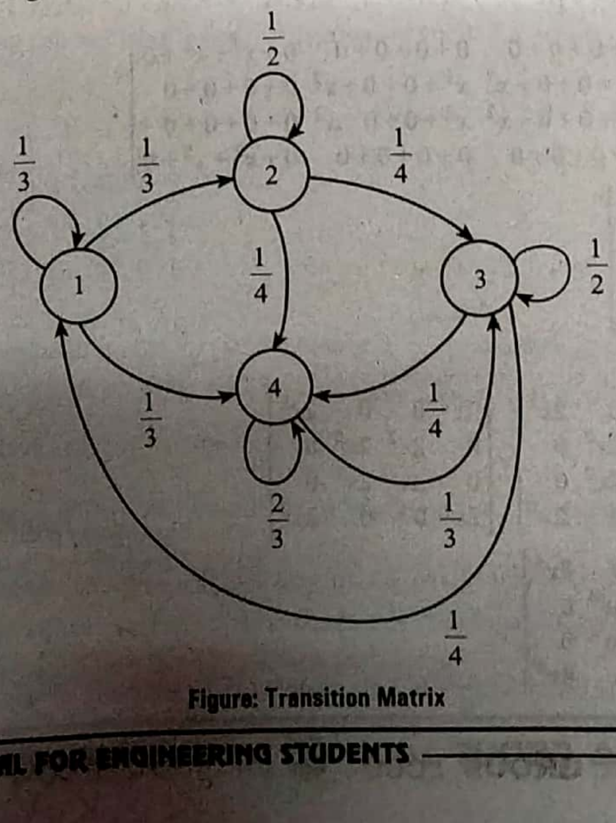


Figure: Transition Matrix

In the above transition diagram, state 1 can go to any other state directly except to state 3. To go to state 3, one must go from state 1 to state 2 and then to state 3.

It is possible for state 2 to directly go to all other states except to state 1. Now, from state 2 one can reach state 1 in two ways. One way is from state 3 to state 1 and the other way is from state 4 to state 3 and then to state 1.

Also, it is possible to directly go to all states except to state 2. To go to state 2, one must go to state 1 and then to state 2.

Finally, from state 4 it is possible to go to state 3 and state 4 except to state 1 and state 2.

From state 4, go to state 3 and then to state 1. Also, from state 4 goto state 3 and then to state 1 and next to state 2. Thus, in this way one can reach state 1 and state 2 from state 4.

Hence, the given matrix is an ergodic Markov chain.

Q23. Test the following transition matrix to see if the Markov chain is ergodic where x is some positive P_{ij} value.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & x & x & 0 \\ x & 0 & 0 & x \\ x & 0 & 0 & x \\ 0 & x & x & 0 \end{bmatrix} \end{array}$$

Solution :

Given that,

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & x & x & 0 \\ x & 0 & 0 & x \\ x & 0 & 0 & x \\ 0 & x & x & 0 \end{bmatrix} \end{array}$$

Now, compute P^2

$$P^2 = \begin{bmatrix} 0 & x & x & 0 \\ x & 0 & 0 & x \\ x & 0 & 0 & x \\ 0 & x & x & 0 \end{bmatrix}\begin{bmatrix} 0 & x & x & 0 \\ x & 0 & 0 & x \\ x & 0 & 0 & x \\ 0 & x & x & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0+x^2+x^2+0 & 0+0+0+0 & 0+0+0+0 & 0+x^2+x^2+0 \\ 0+0+0+0 & x^2+0+0+x^2 & x^2+0+0+x^2 & 0+0+0+0 \\ 0+0+0+0 & x^2+0+0+x^2 & x^2+0+0+x^2 & 0+0+0+0 \\ 0+x^2+x^2+0 & 0+0+0+0 & 0+0+0+0 & 0+x^2+x^2+0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 2x^2 & 0 & 0 & 2x^2 \\ 0 & 2x^2 & 2x^2 & 0 \\ 0 & 2x^2 & 2x^2 & 0 \\ 2x^2 & 0 & 0 & 2x^2 \end{bmatrix}$$

Now, compute P^4

$$P^4 = P^2 \times P^2 = \begin{bmatrix} 2x^2 & 0 & 0 & 2x^2 \\ 0 & 2x^2 & 2x^2 & 0 \\ 0 & 2x^2 & 2x^2 & 0 \\ 2x^2 & 0 & 0 & 2x^2 \end{bmatrix} \times \begin{bmatrix} 2x^2 & 0 & 0 & 2x^2 \\ 0 & 2x^2 & 2x^2 & 0 \\ 0 & 2x^2 & 2x^2 & 0 \\ 2x^2 & 0 & 0 & 2x^2 \end{bmatrix}$$

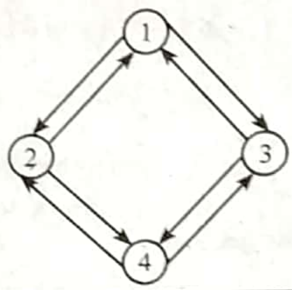
$$P^4 = \begin{bmatrix} 8x^4 & 0 & 0 & 8x^4 \\ 0 & 8x^4 & 8x^4 & 0 \\ 0 & 8x^4 & 8x^4 & 0 \\ 8x^4 & 0 & 0 & 8x^4 \end{bmatrix}$$

Next, compute P^8.

$$P^8 = P^4 \times P^4 = \begin{bmatrix} 8x^4 & 0 & 0 & 8x^4 \\ 0 & 8x^4 & 8x^4 & 0 \\ 0 & 8x^4 & 8x^4 & 0 \\ 8x^4 & 0 & 0 & 8x^4 \end{bmatrix} \times \begin{bmatrix} 8x^4 & 0 & 0 & 8x^4 \\ 0 & 8x^4 & 8x^4 & 0 \\ 0 & 8x^4 & 8x^4 & 0 \\ 8x^4 & 0 & 0 & 8x^4 \end{bmatrix}$$

$$P^8 = \begin{bmatrix} 128x^8 & 0 & 0 & 128x^8 \\ 0 & 128x^8 & 128x^8 & 0 \\ 0 & 128x^8 & 128x^8 & 0 \\ 128x^8 & 0 & 0 & 128x^8 \end{bmatrix}$$

The given matrix is not a regular matrix because the elements of the matrix are not non-zero positive elements. However, the given matrix is ergodic because it is possible to travel from one state to all other states. As shown in the below figure, it is possible to travel from one state to all other states in one or more steps.



Q24. A manufacturing company has a certain piece of equipment that is inspected at the end of each day and classified as just overhauled good, pair or inoperative. If the item is inoperative, it is overhauled, a procedure that takes one day. Let us denote the four classifications as state 1, 2, 3 and 4 respectively. Assume that the working condition of the equipment follows a Markov processes the following transition matrix:. If it costs Rs. 125 to overhaul a machine (including lost time), on the average and Rs. 75 in production? a machine is found inoperative. Using steady state probabilities, compute the expected per day cost of maintenance.

Solution :

Given that,

A machine used by the company can be classified as just overhauled good, fair or inoperative after using it for one day.

Now, consider four states i.e., 1, 2, 3, 4.

$$\text{Transition matrix, } P = \text{Today} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3/4 & 1/4 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Here, the machine can be in fair condition for $\frac{1}{4}^{th}$ of the time and $\frac{3}{4}^{th}$ of the time in good condition after it for one day.

However, the machine that is in good condition can be in the same condition (i.e., good condition) or in fair condition after using it for one day. Since, the chances are equal, the time will be $\frac{1}{2}$.

Next, the machine that is in fair condition can be in the same condition (i.e., fair condition) or inoperative after using it for one day. Since, the chances are equal, the time will be $\frac{1}{2}$.

Finally, an inoperative machine will be overhauled the other days, so the time will be 1 day.

Now, consider P_1, P_2, P_3, P_4 as steady state probabilities and 1, 2, 3, 4 are the states.

Let, 'P' is an ergodic regular Markov process. The steady state equation is given by,

$R = RP$

$$(P_1\ P_2\ P_3\ P_4) = (P_1\ P_2\ P_3\ P_4)\begin{bmatrix} 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

From the above equation, the following equations are obtained,

$$P_1 = P_4 \quad \text{... (1)}$$

$$P_2 = \frac{3}{4}P_1 + \frac{1}{2}P_2 \quad \text{... (2)}$$

$$P_3 = \frac{1}{4}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3 \quad \text{... (3)}$$

$$P_4 = \frac{1}{2}P_3 \quad \text{... (4)}$$

$$P_1 + P_2 + P_3 + P_4 = 1 \quad \text{... (5)}$$

Now, substituting $P_1 = P_4$ in equation (2), we get,

$$P_2 = \frac{3}{4}P_4 + \frac{1}{2}P_2$$

$$P_2 = \frac{3}{4}(P_4) + \frac{1}{4}(P_2)$$

$$\frac{1}{2}P_2 - \frac{3}{4}P_4 = 0$$

$$\frac{1}{2}P_2 = \frac{3}{4}P_4$$

$$\boxed{P_2 = \frac{3P_4}{2}} \quad \text{... (6)}$$

Substitute equations (1), (4), (6) in equation (5), we get,

$$P_1 + P_2 + P_3 + P_4 = 1$$

$$P_4 + \frac{3P_4}{2} + 2P_4 + P_4 = 1$$

$$4P_4 + \frac{3P_4}{2} = 1$$

$$\frac{8P_4 + 3P_4}{2} = 1$$

$$\frac{11}{2}P_4 = 1$$

$$\boxed{P_4 = \frac{2}{11}} \quad \text{... (7)}$$

Substitute equation (7) in equations (1), (4), (6), we get,

$$P = P_4 \Rightarrow P_1 = \frac{2}{11}$$

$$\boxed{\therefore P_1 = \frac{2}{11}}$$

$$P_3 = \frac{3P_4}{2}$$

$$P_4 = 2\left(\frac{2}{11}\right)$$

$$\boxed{\therefore P_3 = \frac{4}{11}}$$

$$P_2 = \frac{3P_4}{2}$$

$$P_2 = \frac{3\left(\frac{2}{11}\right)}{2}$$

$$P_2 = \frac{6}{22}$$

$$\boxed{\therefore P_2 = \frac{3}{11}}$$

Hence, $P_1 = \frac{2}{11}, P_2 = \frac{3}{11}, P_3 = \frac{4}{11}, P_4 = \frac{2}{11}$

Thus, the machine all will be overhauled for 2 out of every 11 days, $P_1 = \frac{2}{11}$.

The machine will be in good condition for 3 out of every 11 days, $P_2 = \frac{3}{11}$.

The machine will be in fair condition for 4 out of every 11 days, $P_3 = \frac{4}{11}$.

The machine will be inoperative for 2 out of every 11 days, $P_4 = \frac{2}{11}$.

Therefore, the expected maintenance cost per day is,

$$= \frac{2}{11}(125) + \frac{2}{11}(75)$$

$$= \frac{250}{11} + \frac{150}{11}$$

$$= \frac{400}{11}$$

= Rs. 36.36.

5.5 MARKOV ANALYSIS

Q25. Discuss in detail about Markov Analysis.

Answer : **Model Paper-II, Q10(b)**

Markov Analysis

Markov Analysis is used to evaluate various decision situations so as to take an appropriate decision and it also provides the probabilities information with respect to the decision made.

Markov Analysis is applicable on those systems in which probabilities movement is observed from time to time in between the states. It is demonstrated by using the following example,

Brand Switching

Consider two brands A and B. The consumer samples are distributed over these brands represents the entire group in terms of their brand loyalty, switching patterns etc.

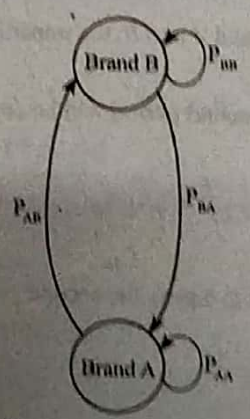


Figure: Brand Switching

The transition matrix represents the probabilities.

$$\text{From}\quad \begin{array}{c c} & \text{To} \\ & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} P_{AA} & P_{AB} \\ P_{BA} & P_{BB} \end{bmatrix} \end{array}$$

Generally, the customers buy similar class of products but they switch the branch from time to time.

Suppose if there are 'n' brands, i.e., $A_1, A_2, \ldots, A_n$ with probability 'P_{ij}' where P_{ij} is based on the customer choice of brands i or j from time to time, then the transition matrix is represented as follows,

$$\text{From}\quad \begin{array}{c c} & \begin{array}{cccc} A_1 & A_2 & \cdots & A_n \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_n \end{array} & \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \end{array}$$

Here, $\sum_{j=1}^{n} P_{ij} = 1 \; ; \; i = 1, 2, \ldots, n$

Thus, the brand switching problem analyzes the customer behaviour that changes from time to time.

PROBLEMS

Q26. Suppose there are two market products of brand A and B respectively. Let each of these two brands have exactly 50% of the total market in same period and let the market be of a fixed size. The transition matrix is given below,

If the initial market share breakdown is 50% for each brand, then determine their market shares in the steady state.

$$\text{From}\quad \begin{array}{c c} & \text{To} \\ & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \boxed{\begin{array}{cc} 0.9 & 0.1 \\ 0.5 & 0.5 \end{array}} \end{array}$$

Solution : Model Paper-I, Q10(b)

Given that,

The initial state for two market products of brand 'A' and 'B' are 50% of total market in same period.

$$\text{Transition matrix} = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \boxed{\begin{array}{cc} 0.9 & 0.1 \\ 0.5 & 0.5 \end{array}} \end{array}$$

After the promotional efforts made to the brand A and B, the transition matrix depicts that the brand A will retain 90% of its customers and confine 50% of brand B.

The market share form brand A during the second period will be computed as shown below,

$(50\%)(0.9) + (50\%)(0.5)$

$= 45\% + 25\%$

$= 70\%$

Now, computing the market share for brand B during the second period will be computed as shown below,

$(50\%)(0.1) + (50\%)(0.5)$

$= 5\% + 25\%$

$= 30\%$

During the period '1', the transition matrix is given by,

$$(50\% \quad 50\%)\begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix} = (70\% \quad 30\%)$$

Similarly, compute the market share of two brands (A and B) for different periods by assuming the fixed market share.

Period	Market Share (Brand A)	Market Share (Brand B)
0	(50%)	(50%)
1	(50%)(0.9) + (50%)(0.5) = 70%	(50%)(0.1) + (50%)(0.5) = 30%
2	(70%)(0.9) + (30%)(0.5) = 78%	(70%)(0.1) + (30%)(0.5) = 22%
3	(78%)(0.9) + (22%)(0.5) = 81.20%	(78%)(0.1) + (22%)(0.5) = 18.80%
4	(81.20%)(0.9) + (18.80%)(0.5) = 82.48%	(81.20%)(0.1) + (18.80%)(0.5) = 17.52%
5	(82.48%)(0.9) + (17.52%)(0.5) = 82.99%	(82.48%)(0.1) + (17.52%)(0.8) = 17.01%
6	(82.99%)(0.9) + (17.01%)(0.5) = 83.20%	(82.99%)(0.1) + (17.01%)(0.5) = 16.80%

By observing the above table, it can be inferred that, the market share starts with 50%, 50% and after completion of 6th period. The market shares are nearly 83%, 17% respectively. Thus, the steady state (equilibrium) market share position of brand A and B will be $\frac{5}{6}$ and $\frac{1}{6}$ of the total market.

Now, obtaining the steady state position through the matrix equation.

$$(x \quad y)\begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix} = (x \quad y)$$

$$0.9x + 0.5y = x$$

$$0.1x + 0.5y = y$$

Hence, it can be concluded that, if the transition matrix doesnot change through out the period of time then it is said to be independent of initial market share.

Q27. Suppose there are three dairies in a town, say A, B and C. They supply all the milk consumed in the two. It is known by all the dairies that consumers switch from dairy to dairy overtime because of advertising, dissatisfaction with service and other reasons. All these dairies maintain records of the number of their customers and the dairy from which the obtained each new customer. Following table illustrates the flow of customers over an observation period of one month, say June.

Dairy	June 1 (Customers)	Gains from			Losses to			July 1 (Customers)
		A	B	C	A	B	C	
A	200	0	35	25	0	20	20	220
B	500	20	0	20	35	0	15	490
C	300	20	15	0	25	20	0	290

We assume that the matrix of transition probabilities remain fairy stable and that the July market shares are,

A = 22%, B = 49% and C = 29%

Managers of these dairies are willing to know,

(i) Market share of their dairies on Ist August and Ist September,

(ii) Their market shares in steady state.

Solution :

Given that,

$A = 22\%, B = 49\%, C = 29\%$

From the given tabular data, the matrix of the transition probabilities are as shown below,

	A	B	C
A	$\frac{220-35-25}{20}=\frac{160}{200}=0.8$	$\frac{20}{200}=0.1$	$\frac{20}{200}=0.1$
B	$\frac{35}{500}=0.07$	$\frac{490-20-20}{500}=\frac{450}{500}=0.90$	$\frac{15}{500}=0.03$
C	$\frac{25}{300}=0.083$	$\frac{20}{300}=0.067$	$\frac{290-20-15}{300}=\frac{255}{300}=0.85$

$$= \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.800 & 0.100 & 0.100 \\ 0.070 & 0.900 & 0.030 \\ 0.083 & 0.067 & 0.850 \end{bmatrix} \end{array}$$

(i) Market Share of the Diaries on 1[st] August and 1[st] September

1[st] Agugust

The market share of 1[st] August will be,

$$(0.22 \quad 0.49 \quad 0.20)\begin{bmatrix} 0.800 & 0.100 & 0.100 \\ 0.070 & 0.900 & 0.030 \\ 0.083 & 0.067 & 0.850 \end{bmatrix}$$

$$= (0.176 + 0.034 + 0.024 \quad 0.022 + 0.441 + 0.019 \quad 0.022 + 0.015 + 0.247)$$

$$= (0.235 \quad 0.482 \quad 0.283)$$

1[st] September

The market share of 1st September will be,

$$(0.234 \quad 0.482 \quad 0.284)\begin{bmatrix} 0.800 & 0.100 & 0.100 \\ 0.070 & 0.900 & 0.030 \\ 0.083 & 0.067 & 0.850 \end{bmatrix}$$

$$= (0.187 + 0.034 + 0.024 \quad 0.023 + 0.434 + 0.019 \quad 0.023 + 0.014 + 0.241)$$

$$= (0.245 \quad 0.476 \quad 0.279)$$

(ii) Market Shares in Steady State

$$(x \quad y \quad z)\begin{bmatrix} 0.800 & 0.100 & 0.100 \\ 0.070 & 0.900 & 0.030 \\ 0.083 & 0.067 & 0.850 \end{bmatrix} = (x \quad y \quad z)$$

$$0.800x + 0.070y + 0.083z = x \quad \text{... (1)}$$

$$0.100x + 0.900y + 0.067z = y \quad \text{... (2)}$$

$$0.100x + 0.030y + 0.850z = z \quad \text{... (3)}$$

$$x + y + z = 1 \quad \text{... (4)}$$

Here, $z = 1 - x - y$

Substitute z in equation (1), (2) and (3)

$$0.800x + 0.070y + 0.083(1 - x - y) = x$$

$$0.800x + 0.070y + 0.083 - 0.083x - 0.083y - x = 0$$

$$-0.283x - 0.013y + 0.083 = 0 \quad \text{... (5)}$$

$0.100x + 0.900y + 0.067(1 - x - y) = y$

$0.100x + 0.900y + 0.067 - 0.067x - 0.067y - y = 0$

$0.033x - 0.167y + 0.067 = 0$ ... (6)

$0.100x + 0.030y + 0.850\,(1 - x - y) = 1 - x - y$

$0.100x + 0.030y + 0.850 - 0.850x - 0.850y - 1 + x + y = 0$

$0.25x + 0.180y - 0.15 = 0$ ... (7)

Solving equations (6) and (7), we get,

$x = 0.272$ and

$y = 0.455$

Now, substitute x, y values in equation (4).

$x + y + z = 1$

$0.272 + 0.455 + z = 1$

$z = 1 - 0.272 - 0.455$

$z = 0.273$

$\boxed{\therefore X = 0.272, Y = 0.455, Z = 0.273}$

Q28. The number of units of an item that are withdrawn from inventroy on a day-to-day basis is a Markov chain process in which requirements for tomorrow depend on today's requirements. A one day transition matrix is given below,

(i) Construct a tree diagram showing inventory requirements on two consective days.

(ii) Develop a two-day transition matrix.

(iii) Comment on "how a two day transition matrix might be helpful to a manager who is responsible for inventory management."

Number of units withdrown from Inventory

$$\begin{array}{cc} & \text{Tomorrow} \\ & \begin{array}{ccc} 5 & 10 & 12 \end{array} \\ \text{Today} \begin{array}{c} 5 \\ 10 \\ 12 \end{array} & \begin{bmatrix} 0.6 & 0.4 & 0.0 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{bmatrix} \end{array}$$

Solution :

(i) The inventory requirements on two consecutive days is represented by a tree diagram as shown below,

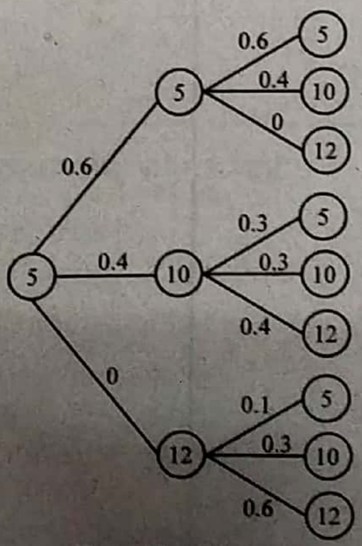


Figure: Tree Diagram

$P_{11}^{2} = (0.6)(0.6) + (0.4)(0.3) + (0)(0.1)$

$= 0.36 + 0.12 + 0$

$\therefore \quad P_{11}^{2} = 0.48$

$P_{12}^{2} = (0.6)(0.4) + (0.4)(0.3) + (0)(0.3)$

$= 0.24 + 0.12 + 0$

$\therefore \quad P_{12}^{2} = 0.36$

$P_{13}^{2} = (0.6)(0) + (0.4)(0.4) + (0)(0.6)$

$= 0 + 0.16 + 0$

$\therefore \quad P_{13}^{2} = 0.16$

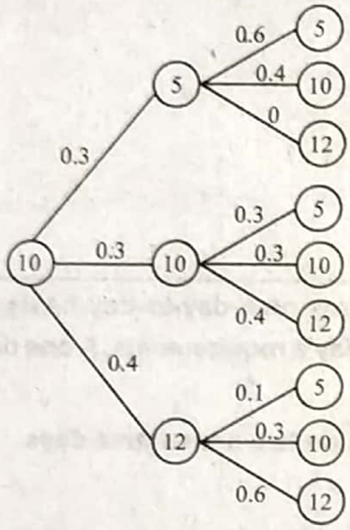


Figure: Tree Diagram

$P_{21}^{(2)} = (0.3)(0.6) + (0.3)(0.3) + (0.4)(0.1)$

$= 0.18 + 0.09 + 0.04$

$P_{21}^{(2)} = 0.31$

$P_{22}^{(2)} = (0.3)(0.4) + (0.3)(0.3) + (0.4)(0.3)$

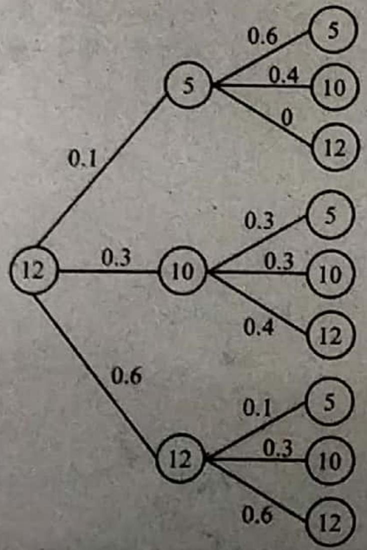
$= 0.12 + 0.09 + 0.12$

$P_{22}^{(2)} = 0.33$

$P_{23}^{(2)} = (0.3)(0) + (0.3)(0.4) + (0.4)(0.6)$

$= 0 + 0.12 + 0.24$

$P_{23}^{(2)} = 0.36$



$P_{31}^{(2)} = (0.1)(0.6) + (0.3)(0.3) + (0.6)(0.1)$

$= 0.06 + 0.09 + 0.06$

$P_{31}^{(2)} = 0.21$

$P_{32}^{(2)} = (0.1)(0.4) + (0.3)(0.3) + (0.6)(0.3)$

$= 0.04 + 0.09 + 0.18$

$P_{32}^{(2)} = 0.31$

$P_{33}^{(2)} = (0.1)(0) + (0.3)(0.4) + (0.6)(0.6)$

$= 0 + 0.12 + 0.36$

$P_{33}^{(2)} = 0.48$

(ii) Let 'P' be a transition matrix.

The two day transition matrix is,

$$P^2 = \begin{array}{c} \\ 5 \\ 10 \\ 12 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 5 & 10 & 12 \end{array} \\ \begin{bmatrix} 0.6 & 0.4 & 0.0 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{bmatrix} \end{array} \begin{array}{c} \\ 5 \\ 10 \\ 12 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 5 & 10 & 12 \end{array} \\ \begin{bmatrix} 0.6 & 0.4 & 0.0 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{bmatrix} \end{array}$$

$$P^2 = \begin{array}{c} \\ 5 \\ 10 \\ 12 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 5 & 10 & 12 \end{array} \\ \begin{bmatrix} 0.48 & 0.36 & 0.16 \\ 0.31 & 0.33 & 0.36 \\ 0.21 & 0.31 & 0.48 \end{bmatrix} \end{array}$$

(iii) Consider that a manager should place an order for inventory management every morning. Due to the time requirement for product delivery, the order placed today is delivered after 2 days. Thus, the two day transition matrix is used for making ordering decisions.

Example

Suppose if a manager notices an increase in demand for 5 units, then after two days the probability of requiring 5 units will be 0.48, 10 units will be 0.36 and 12 units will be 0.16.