COSM

Unit - 1

I

① probability : definations.
multiplication Theorem.
addition Theorem.
Bayers Theorem.

② Random variables & probability distribution.
: Discrete prob distribution
: Continuous prob distribution.

II

Unit - 2

Mathematical Expectations & probability distribution

① Mathematical Expectations

Disc → mean, variance.
→ PMF
→ CDF.

cont → mean, variance
→ PDF
→ CDF.

② Discrete probability distribution

binomial → mean variance
poson

Unit - III

(1) Cont... prob distribution & sampling distr

Normal dist.
cal Area, mean, variance.

with replacement
without replacement
F - Test.
Chi - square.

(1)

Unit - IV

→ Estimation & Test Hypothesis
1. Z test for single mean.
2. diff mean.
3. single proportion.
4. Diff proportion.

(1)

→ Statistical hypothesis.
1. T test for single mean.
2. T test for diff mean.

(1)

paired sample test.

Unit V

Stocastic process markov process.

(1)

→ ergodic
→ regular
→ irreducible
→ steady state vector.

UNIT - I

Random Experiment :-
eg Tambola ; Shuffle in Spotify, eg Head Tails.
→ We know the outcome but we dont know the actual

Sample Space :-
no of possible ways
eg :- $S = \{HH, HT, TH, TT\}$.

$$\text{Probability} = \frac{\text{no of favourable events}}{\text{Total no of cases}}$$

$$P(E) = \frac{n(E)}{n(S)}$$

(1). $S = \{HH, HT, TH, TT\}$.

E_1 = getting '2' heads.
$\therefore P(E_1) = \frac{n(E_1)}{n(S)} = \frac{1}{4}$.

E_2 = getting '2' tails
$\therefore P(E_2) = \frac{n(E_2)}{n(S)} = \frac{1}{4}$.

E_3 = getting '1' head
$\therefore P(E_3) = \frac{n(E_3)}{n(S)} = \frac{3}{4}$.

E_4 = getting '1' tail
$P(E_4) = \frac{n(E_4)}{n(S)} = \frac{3}{4}$

Here, $\boxed{\text{not}}$ (because elements should be

E_1 & E_2 are EQUALLY LIKED EVENTS.

E_3 & E_4 are EQUALLY LIKED EVENTS

E_1 & E_4 are MUTUALLY EXCLOSIVE EVENTS (OR) DISJOINT EVENT.

$$\boxed{\therefore E_1 \cap E_4 = \phi}$$

E_3 & E_4 are MUTUALLY EXHAUSTIVE EVENTS

$$\boxed{E_3 \cup E_4 = S}$$

② Throwing a die.

E_1 = getting no 4.
E_2 = getting even no.
E_3 = getting odd no.
E_4 = getting prime no.
E_5 = getting no 7.
E_6 = getting natural no.

Sol $S = \{1, 2, 3, 4, 5, 6\}$ $n(S) = 6$.

$P(E_1) = \frac{1}{6}$ $\{4\}$

$P(E_2) = \frac{3}{6} = \frac{1}{2}$ $\{2, 4, 6\}$

$P(E_3) = \frac{3}{6} = \frac{1}{2}$ $\{1, 3, 5\}$

$P(E_4) = \frac{3}{6} = \frac{1}{2}$ $\{2, 3, 5\}$

$P(E_5) = \frac{0}{6} = 0$. $\{\}$

$P(E_6) = \frac{6}{6} = 1$. $\{1, 2, 3, 4, 5, 6\}$

E_2, E_3, E_4 are ~~mutually~~ equally liked events.

$E_2 \cap E_3 = \phi$ are mutually exclusive or disjoint events.

$E_2 \cup E_3 = S$ are mutually exhaustible events.

$E_4 \cup E_6 = S$ " " " "

$E_5 = 0$ is an impossible event.

$E_6 = 1$ is a sure event

Axioms of Probability

1. $P(S) = 1$.
2. $P(\phi) = 1$.
3. $0 \leq P(E) \leq 1$.
4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
5. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$
6. $P(E) + P(\bar{E}) = 1$.
7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
8. $(A \cup B)^C = A^C \cap B^C$.
9. $(A \cap B)^C = A^C \cup B^C$.
10. $P(A \cap \bar{B}) = P(A) - P(A \cap B)$.
11. $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

Permutations

* Arrangements.

Permutations :- out of 'n' diff positions, 'r' elements can be arranged in.

^nP_r ways : without repetations

r^n ways : with repeatations.

* Ex :- $n=3$, $r=2$.

AB _	BA _
A _ B	B _ A
_ AB	_ BA

} 6 ways.

$^nP_r = {}^3P_2$.

$= \frac{n!}{n-r!} = \frac{3!}{3-2!} = 6$

* Eg :-

A A A.
A A B.
A B A.
A B B.
B A A.
B A B.
B B A.
B B B.

} 8 ways.

$r^n = 2^3 = 8$.

Combinations

* Selections.

* Out of 'n' different elements, 'r' elements can be selected in ^nC_r ways: without repetation.

$(n+r-1)C_r$ ways : with repetation.

* Eg:- AB, BC, CA } 3 ways $\rightarrow {}^nC_r = {}^3C_2 = \frac{n!}{(n-r)!\,r!}$

* Eg2:- AB, BB, CC / AB, AC, BC } $\rightarrow (n+r-1)C_r$

$= (3+2-1)C_2 = 4C_2$

$= \frac{4\times3}{2\times1} = \frac{12}{2} = 6$

__/__/__

① What is the probability for a leap year to have 52 mondays & 53 sundays

→ Tot no of days in a leap year = 366.
→ Tot no of weeks in a leap year.

```
7)366(52
  -35
   16
  -14
    2
```

52 weeks & 2 days.

→ The 2 days would be arranged as.

sun & mon
mon & tue
tue & wed
wed & thur
thur & fri
fri & sat
sat & sun

→ The 52 mondays are already covered. as in 365, days, each week has mon
→ Now 52 sundays are also covered.
→ To get 53 sundays, check the criteria where there is a sun
→ we got sun & mon.
sat & sun.
→ we want sun & mon cuz, it becomes 53 sundays & 54 mondays
→ So take sat & sun.

∴ we get 52 mondays & 53 Sundays.

$$\therefore P(E) = \frac{\text{no. of valuable cases}}{\text{Tot no of events}}$$

$$= \frac{1}{7}$$

2) A card is drawn at random from a pack of 52 playing cards find the probability that the card drawn is.
① King ② neither a king nor a Jack.

① $\frac{{}^4C_1}{{}^{52}C_1} = \frac{n!}{n-r!\,r!} = \frac{4!}{4-1!\,1!} = \frac{\frac{(4\times 3!)}{3!\,1!}}{\frac{52\times 51!}{51!\times 1!}} = \frac{4}{52}$

$= \frac{1}{13}$

② 8 Remove

∴ $52-8 = 44$

∴ $\frac{44}{52}$

__/__/__

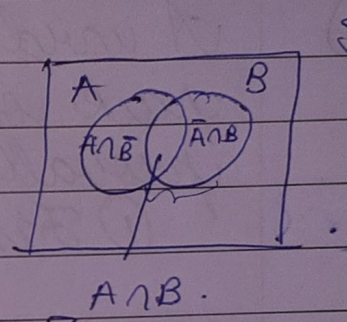
Addition Theorem on probability:

Stat: If A & B are any 2 events subset of sample space 'S', and are not disjoint

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Pf Let $A \cup B = A \cup (\bar{A} \cap B)$

$\boxed{\text{Apply } P \text{ on B.S}}$



$$P(A \cup B) = P[A \cup (\bar{A} \cap B)]$$

$$= P(A \cup B) = P(A) + P(\bar{A} \cap B)$$

[∵ Axiom : $P(A \cup B) = P(A) + P(B)$].

$\boxed{\text{Add \& sub } A \cap B \text{ on B.S}}$.

$$\therefore P(A \cup B) = P(A) + \underline{P(\bar{A} \cap B) + P(A \cap B)} - P(A \cap B)$$

$$= P(A) + P(B) - P(A \cap B)$$

[∵ from fig]

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Hence proved

case 1 :- If A & B are mutually exclusive or disjoint.

→ Inorder to be disjoint i.e $P(A \cap B) = 0$.

→ As to addition theorum :-

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\therefore P(A \cup B) = P(A) + P(B) - 0$$

→ $$\therefore P(A \cup B) = P(A) + P(B)$$

case 2 :- If A & B & C are 3 events subset of sample space 'S'. Then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$P(A \cup B \cup C) = P[(A \cup B) \cup C]$$

$$= P(A \cup B) + P(C) - P[(A \cap B) \cap C]$$

[∵ Addition Theorum]

$$= P(A) + P(B) - P(A \cap B) + P(C) - [P(A \cap C) \cup P(B \cap C)]$$

[∵ distributive Law]

$$= P(A) + P(B) + P(C) - P(A \cap B) - [P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)]$$

[∵ addition Theorum]

$$\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

OR = addition theorem
union

Problems on Addition Theorem

① One card is drawn from regular deck of 52 cards. What is the probability either red (or) king.

Let A be an event of red.
Let B be an event of king

52
red 26 | king 4 | 2

$$P(A) = \frac{\text{no of fav cases}}{\text{Tot no of cases in sample space}}$$

$\therefore \boxed{P(A) = \frac{26}{52}}$, $\boxed{P(B) = \frac{4}{52}}$, $\boxed{P(A \cap B) = \frac{2}{52}}$

$$\therefore \boxed{P(A \cup B)} = P(A) + P(B) - P(A \cap B)$$
$$= \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52}$$

$$\therefore \boxed{P(A \cup B) = \frac{28}{52}}$$

② A card is drawn from a well shuffled pack of cards. What is the prob either a spade (or) ace.

52
spade 13 | ace 4

$P(A) = \frac{13}{52}$, $P(B) = \frac{4}{52}$, $P(A \cap B) = \frac{1}{52}$

$$P(A \cup B) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52}$$

$\cap$ / /

(3) From a city '3' news papers A, B, C are published.

A is read by 20%.
B " " " 16%.
C " " " 14%
A & B 8%
A & C " " 5%.
B & C " " 4%.
all 3 2%.

What is the percentage that read at least one paper.

As they gave %'s, sample space = 100.

$\therefore \boxed{P(A) = \frac{20}{100}}$ $\boxed{P(B) = \frac{16}{100}}$ $\boxed{P(C) = \frac{14}{100}}$

$\boxed{P(A \cap B) = \frac{8}{100}}$ $\boxed{P(A \cap C) = \frac{5}{100}}$ $\boxed{P(B \cap C) = \frac{4}{100}}$

$$\boxed{P(A \cap B \cap C) = \frac{2}{100}}$$

w.k.t $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

$$= \frac{20}{100} + \frac{16}{100} + \frac{14}{100} - \frac{8}{100} - \frac{5}{100} - \frac{4}{100} + \frac{2}{100} = \frac{31}{100}.$$

To get % i.e $\frac{31}{100} \times 100 = 31\%$.

$\therefore \boxed{P(A \cup B \cup C) = 35\%}$

Conditional Probability

Based on conditional probability we can solve Multipliation theorem and Bayers thorum.

→ Conditional probability :- probability of one event after happing of another event.

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = A \text{ given } B.$$

$$P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)} = B \text{ given } A.$$

→ Multipliation Theorem of probability :-
A, B are two events in a sample space and $P(A) \neq 0$; $P(B) \neq 0$. Then

$$P(A \cap B) = P(B) \cdot P\left(\frac{A}{B}\right) = P(A) \cdot P\left(\frac{B}{A}\right)$$

Proof: w.k.t A, B are any two events in a sample space.

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}, P(B) \neq 0 \quad -(1)$$

$$P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)}, P(A) \neq 0 \quad -(2)$$

CROSS PRODUCT.

$$(1) \Rightarrow \boxed{P\left(\frac{A}{B}\right) P(B) = P(A \cap B)} \quad -(3)$$

$$(2) \Rightarrow \boxed{P\left(\frac{B}{A}\right) P(A) = P(B \cap A)} \quad -(4)$$

$\therefore$ from (3) & (4).

$$P\left(\frac{A}{B}\right) P(B) = P\left(\frac{B}{A}\right) \cdot P(A) = P(A \cap B).$$

Hence proved

Problems on Conditional probability

(1). A couple has two children, find probability (1) both are boys
(1) one of the child is boy.
(2) older child is boy.

Let 'A' the ran event where both are boys? 1/4.
Let 'B' the ran event where one is boy 3/4.
Let 'C' the ran event where older child is boy 2/4.

$\therefore S = \{B_1, B_2, B_1, G_2, G_1, B_2, G_1, G_2\}$.

$$\therefore P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{4} \times \frac{4}{3} = \frac{1}{3}$$

$$\therefore P\left(\frac{A}{C}\right) = \frac{P(A \cap C)}{P(C)} = \frac{1/4}{2/4} = \frac{1}{4} \times \frac{4}{2} = \frac{1}{2}$$

(2). What is the probability of getting a total of 7 (or) 11 when a pair of dice is thrown.
union

Sol: $S = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6)$
$(2,1)\ (2,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6)$
$(3,1)\ (3,2)\ (3,3)\ (3,4)\ (3,5)\ (3,6)$

(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)
(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)
(6,1) (6,2) (6,3) (6,4) (6,5) (6,6)

$$\boxed{\text{If it is an independent event, } P(A \cup B) = P(A) + P(B)}$$

getting a total of 7 means add die 1 + die 2 and it should give 7.

Let $P(A)$ = prob of getting 7.

$$P(A) = \frac{6}{36}$$

Let $P(B)$ be prob of getting 11.

$$P(B) = \frac{2}{36}$$

$$\therefore P(A \cup B) = P(A) + P(B)$$

$$= \frac{6}{36} + \frac{2}{36}$$

$$= \frac{8}{36} = \frac{2}{9}$$

BAYER'S THEOREM:-

$E_1, E_2 \ldots E_n$ are 'n' mutually Exclusive & Exhaustive events

Proof → $E_1, E_2 \ldots E_n$ are mutually exclusive
$E_1 \cap E_2 \ldots E_n = \phi \quad (i \neq j)$.

→ $E_1, E_2 \ldots E_n$ are mutually exhaustive
$E_1 \cup E_2 \ldots E_n = S$.

$$A = A \cap S$$
$$= A \cap (E_1 \cup E_2 \cup \ldots E_n)$$
$$= (A \cap E_1) \cup (A \cap E_2) \cup \ldots \cup (A \cap E_n)$$

POBS.

$$P(A) = P(A \cap E_1) \cup P(A \cap E_2) \cup \ldots P(A \cap E_n)$$
$$P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots P(A \cap E_n)$$

(from axiom ③ def)

$$\boxed{P(A) = P(E_1) \cdot P\left(\frac{A}{E_1}\right) + P(E_2) \cdot P\left(\frac{A}{E_2}\right) \ldots P(E_n) \cdot P\left(\frac{A}{E_n}\right) \quad \text{—①}}$$

Tot probability

(from multiplicative theorem)

$$\cancel{P\left(\frac{E_i}{A}\right) = P(E_i) \cdot P\left(\frac{A}{E_i}\right)}$$

let $P\left(\frac{E_i}{A}\right) = \frac{P(E_i \cap A)}{P(A)} = \frac{P(E_i) \cdot P\left(\frac{A}{E_i}\right)}{P(A)}$

$$\therefore P\left(\frac{E_i}{A}\right) = \frac{P(E_i) \cdot P\left(\frac{A}{E_i}\right)}{P(E_1) \cdot P\left(\frac{A}{E_1}\right) + P(E_2) \cdot P\left(\frac{A}{E_2}\right) + \ldots P(E_n) \cdot P\left(\frac{A}{E_n}\right)}$$

Problems on Bayes Theorem

(1) In a bolt factory machines A, B, C manufacture 20%, 30% 50% of the total of their input and 6%, 3% & 2% are defective. A bolt is drawn at random & found to be defective. Find the probabilities it is manufactured from
(1) Machine A (2) Machine B (3) Machine C

(Sol). $P(A) = \frac{20}{100} = \frac{1}{5}$

$P(B) = \frac{30}{100} = \frac{3}{10}$

$P(C) = \frac{50}{100} = \frac{1}{2}$

D = defective bolts

$P(D/A) = 6/100$
$P(D/B) = 3/100$
$P(D/C) = 2/100$

i) Machine A

$$P(A/D) = \frac{P(A) \cdot P(D/A)}{P(D/A) \cdot P(A) + P(D/B) \cdot P(B) + P(D/C) \cdot P(C)}$$

$$= \frac{6/100 \times 1/5}{6/100 \times 1/5 + 3/100 \cdot 3/10 + 3/100 \cdot 1/2} = \frac{12}{31}$$

(ii) $P(B/D) = \dfrac{P(B) \cdot P(D/B)}{P(A) \cdot P\left(\frac{D}{A}\right) + P(B) \cdot P\left(\frac{D}{B}\right) + P(C) \cdot P\left(\frac{D}{C}\right)}$

$= 9/31.$

(iii) $P(C/D) = 10/31.$

(2). A businessman goes to hotel x, y, z & 20%, 50%, 30% of the time. 5%, 4%, 8% of rooms are faulty. Prob of that business having faulty is assigned to z?

Sol Let the probability of businessman going to x, y, z respectively be $P(x), P(y), P(z)$

$$P(x) = \frac{20}{100}, \quad P(y) = \frac{50}{100} \quad P(z) = \frac{30}{100}.$$

Let E be the event that the hotel room has faulty plumbing. The prob that hotel x, y, z have faulty plumbing

$$P\left(\frac{E}{x}\right) = \frac{5}{100} = \frac{1}{20}, \quad P\left(\frac{E}{y}\right) = \frac{4}{100} = \frac{1}{25}. \quad P\left(\frac{E}{z}\right) = \frac{2}{25}$$

$$P\left(\frac{z}{E}\right) = \frac{P(z) \cdot P(E/z)}{P(x) \cdot P\left(\frac{E}{x}\right) \cdot P(y) \cdot P\left(\frac{E}{y}\right) \cdot P(z) \cdot P\left(\frac{E}{z}\right)}$$

$$= \frac{30/100 \cdot 2/25}{20/100 \cdot 1/20} = \ldots$$

Random Variables

→ Random variables 'X' on sample space 'S' is function from 'S' into real no set 'R'

$$X : S \to R$$

Eg:- Toss 2 coins

$S = \{HH, HT, TH, TT\}$

no of tails = $X(HH) = 0$ ✓
$X(HT) = 1$ ✓
$X(TH) = 1$ ✓
$X(TT) = 2$

$\therefore X : S \to R$

→ Random Variables

Discrete RV (finite)
$X : S \to R$
→ Binomial
→ poison
$P(x) > 0$
$\Sigma P(x) = 1$
$E(x) = \Sigma x P(x)$

Continuous RV (unfinite)
$X : S \to R$
→ Normal, Uniform
→ Exponential
$f(x) \geq 0$
$E(x) = \int_{-\infty}^{+\infty} x f(x)\, dx$

Discrete Probability Distribution
(Probability Mass Function)

Let x be a discrete RV such that $P(x = x_i) = P_i$, then P_i said to be PMF. It satisfies

$$\boxed{\begin{array}{l} ① \ P(x) \geq 0 \\ ② \ \sum_{i=0}^{n} P(x_i) = 1. \end{array}}$$

Ex

① $P(x) = \frac{x-2}{2}, \ \forall x = 1, 2, 3, 4.$

Sub $x = 1$ in $P(x)$.

$$\frac{1-2}{2} = \frac{-1}{2} < 0.$$

condition ① fails so no need to check ② cond

Eg ② $f(x) = \frac{1}{6}, \ x = 1, 2, 3, 4, 5, 6.$

Sub x values in func.

$1 > \frac{1}{6}, \ 2 > \frac{1}{6}, \ 3 > \frac{1}{6}, \ 4 > \frac{1}{6}, \ 5 > \frac{1}{6}.$

$\therefore$ condition ① satisfies.

$$\sum_{n=1}^{6} f(x_i) = 1/6 + 1/6 + 1/6 + 1/6 + 1/6 + 1/6.$$
$$= 6/6 = 1.$$

$\therefore \sum_{i=0}^{n} P(x_i) = 1.$

$\therefore$ condition ① & ② satisfies.

① A discrete RV : 'x' has foll distrib.

Val of x	0	1	2	3	4	5	6	7	8
$P(x=x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$	15	$17k$

i) find the value of 'k'
ii) find $P(x \geq 5)$.
iii) find $P(x<4)$
iv) find $(0<x<4)$

$$\sum_{i=0}^{n} f(x) = 1.$$

i). $f(0)+f(1)+f(2)+f(3)+f(4)+f(5)+f(6)+f(7)+f(8)$
$= k+3k+5k+7k+9k+11k+13k+15k+17k$.
$= 8k = 1$.

$$\boxed{k = \frac{1}{81}}$$

ii). $P(x \geq 5) = P(x=5)+P(x=6)+P(x=7)+P(x=8)$
$= 11k+13k+15k+17k$.
$= 56k$.
$= 56 \times \frac{1}{81} = \frac{56}{81}$

$$\boxed{\therefore P(x \geq 5) = \frac{56}{81}}$$

(iii) $P(x<4) = P(x=0)+P(x=1)+P(x=2)+P(x=3)$.
$= k+3k+5k+7k = 16k$.

$$\boxed{P(x<4) = \frac{16}{81}}$$

$P(0<x<4) = P(x=1) + P(x=2) + P(x=3)$

$= 3k + 5k + 7k = 15k$

$$\boxed{P(0<x<4) = \frac{15}{81}}$$

(2) A RV 'X' has the foll func.

X	0	1	2	3	4	5	6	7
P(x)	0	k	2k	2k	2k	3k	$2k^2$	$7k^2+k$

i) find the value of k

ii) Evaluate $P(X \leq 6)$, $P(X \leq 6)$

iii) Evaluate $P(0<x<5)$

i) $f(0)+f(1)+f(2)+f(3)+f(4)+f(5)+f(6)+f(7)=1$

$0+k+2k+2k+2k+3k+k^2+7k^2+k=1$

$= 8k+8k^2 = 1$

$8k^2+8k = 1$

$8k(k+1) = 0$

$8k = 0$ | $k+1 = 0$

$k = 0$ | $k = -1$

$= 8k^2+9k = 1$

$= 8k^2+9k-1 = 0$

$k(8k+9k) = 0$

$= 8k^2+9k-1 = 0$

1	8	9	-1
	8	8	
	8		
2	8	9	-1
	0	16	
	8		
-1	8	9	-1
	0	-8	-1
	8	1	-2
1	8	9	-1
	0	8	
	8	17	

$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 2k^2 + k = 1$

$10k^2 + 9k - 1 = 0$

10

$-10 \quad 1$

$\cancel{10k^2 - 10k + 1k - 1 = 1}$

$\cancel{10k(k-1) 1(k-1) = 1}$

$\cancel{(k-1)(10k-1) = 0}$

$10k^2 + 9k - 1 = 0$

$(10k-1)(k+1) = 0$

$k = \frac{1}{10}, -1$

$\therefore \boxed{k = \frac{1}{10}}$

i) $k = \frac{1}{10}$

ii) $P(x<6) = P(x=0) + P(x=1) + P(x=2)$
$+ P(x=3) + P(x=4) + P(x=5)$

$= 0 + k + 2k + 2k + 3k + k2$

$= k^2 + 8k$

$= \left(\frac{1}{10}\right)^2 + 8\left(\frac{1}{10}\right) =$

$= 0.01 + 0.8 = \boxed{\frac{81}{100}}$

iii) $P(x \leq 6) =$

$0 + k + 2k + 2k + 3k + k^2 + 2k^2$

$= 8k + 3k^2$

$= 8\left(\frac{1}{10}\right) + 3\left(\frac{1}{10}\right)^2$

$0.8 + 0.03 = 0.83 = \boxed{\frac{83}{100}}$

iv) $P(0 < x < 5)$

$= k + 2k + 2k + 3k$

$= 8k$

$= \frac{8}{10} = 0.8$

* Probability Distribution of Discrete Random Variable:

[PMF, cdf, mean, variance].

→ **Probability Mass Function**

Let 'x' be a Discrete Random Variable such that $P(x = x_i) = P_i$ then P_i is said to the PMF.

→ ① $P(x_i) > 0$
$0 < P_i \leq 1$.

② $\Sigma P(x_i) = 1$.
$i = 1, 2 \ldots$

→ Eg: Toss 2 coins, $\boxed{\text{no of heads}}$

$S = \{HH, HT, TH, TT\}$.

→

no of heads $(x = x_i)$	0	1	2
$P(x = x_i)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

→ $X: S \to R$.

$X(HH) = 2$
$X(HT) = 1$.
$X(TH) = 1$
$X(TT) = 0$.

→ Checking conditions

$1/4 + 2/4 + 1/4 = 4/4 = 1$

cond ② ✓

cond ① ✓

∴ It satisfies PMF

→ DISTRIBUTION FUNCTION :- (CDF)

Let 'x' be a discrete Random variable. Then its PDF or CDF is defined as

→ $\boxed{F(x) = \Sigma P_i = P(x \leq x_i)}$

→ Taking the previous table for Example

$$F(x) = \begin{cases} 1/4, & x < 0. \\ 1/4 + 2/4 = 3/4, & x \leq 1. \\ 1/4 + 2/4 + 1/4 = 1, & x < 2. \end{cases}$$

//

→ MEAN OF DRV.

Expected value of Random Variable.

$E(X) = x_1 p_1 + x_2 p_2 + x_3 p_3 \cdots x_n p_n.$

$$\mu = \boxed{E(x) = \sum_{i=1}^{n} P_i x_i}$$

→ Taking the prev table for Example.

$E(\text{no of heads}) = x_1 p_1 + x_2 p_2 + x_3 p_3.$

$= 0 \times \frac{1}{4} + 1 \times \frac{2}{4} + 2 \times \frac{1}{4} = 1.$

$\therefore \boxed{E(X) = 1.}$

Expectation of x is 1.

__/__/____

→ **VARIANCE** of DRV

$$\boxed{Var(X) = E(X)^2 - [E(X)]^2}$$

$$\boxed{Var(X) = E(X)^2 - \mu^2}$$

→ Example taking table & Mean value:

$$E(X)^2 = x_1^2 p_1 + x_2^2 p_2 + x_3^2 p_3 + \cdots x_n^2 p_n$$

$$= 0 \times \frac{1}{4} + 1^2 \times \frac{2}{4} + 2^2 \times \frac{1}{4}$$

$$= \frac{0}{4} + \frac{2}{4} + \frac{4}{4} = \frac{\cancel{6}\,3}{\cancel{4}\,2}$$

$$\therefore E(X)^2 = \frac{3}{2}$$

$$\therefore [E(X)]^2 = \mu = (1)^2$$

$$\therefore Var(X) = \frac{3}{2} - 1 = \frac{3-2}{2} = \frac{1}{2}$$

→ **STANDARD DEVIATION** of DRV

$$\boxed{\sigma = \sqrt{E(X)^2 - [E(X)]^2}}$$

$$\boxed{\sigma = \sqrt{E(X)^2 - \mu^2}}$$

→ Example taking table & Variance

$$= \boxed{SD = \sqrt{\frac{1}{2}}}$$

PROBLEMS.

① A random sample with replacement of size '2' is taken from $S = \{1,2,3\}$. Let RV 'x' denotes the sum of two nos taken

① write the PDF of 'x'
② find mean
③ find variance

→ $S = \{(1,1)(1,2)(1,3)$
$(2,1)(2,2)(2,3)$
$(3,1)(3,2)(3,3)\}$

$X(S)$ = sum of 2 nos

→ $X(S) = \{2,3,4,5,6\}$.

⇒ $P(2) = P(X=2) = P(1,1) = 1/9$.
$P(3) = P(X=3) = P[(1,2),(2,1)] = 2/9$.
$P(4) = P(X=4) = P[(1,3)(2,2)(3,1)] = 3/9$.
$P(5) = P(X=5) = P[(2,3)(3,2)] = 2/9$.
$P(6) = P(X=6) = P[(3,3)] = 1/9$.

i) PP of x.

x	2	3	4	5	6
$P(x)$	1/9	2/9	3/9	2/9	1/9

ii) Mean $= \Sigma P_i x_i = E(X)$.

$= 2 \times \frac{1}{9} + 3 \times \frac{2}{9} + 4 \times \frac{3}{9} + 5 \times \frac{2}{9} + 6 \times \frac{1}{9}$

$= \boxed{4}$

iii) Variance $= E(x)^2 - \mu^2$

$P_i x_i^2 - \mu^2$

$\frac{1}{9} \times (2^2) + \frac{2}{9} \times (3)^2 + \frac{3}{9} \times (4)^2 + \frac{2}{9} \times (5)^2 + \frac{1}{9} \times (6)^2$

$= 17.33$

$Var(x) = E(x)^2 - \mu^2$

$= 17.33 - 16.6$

$= 1.33$

② Calculate Expectation & variance of x, If the probability distribution of the random variable is

x	-1	0	1	2	3
f	0.3	0.1	0.1	0.3	0.2

Expectation =

$\mu = E(x) = \sum_{i=0}^{n} 0.3 \times (-1) + 0.1 \times 0 + 0.1 \times 1 + 0.3 \times 2 + 0.2 \times 3$

$= 1.$

$\boxed{E(x) = 1.} = \mu$

$Var(x) = E(x)^2 - [E(x)]^2$

$E(x)^2 - \mu^2$

$= (-1)^2 \times 0.3 + \cdots$

$E(x)^2 = 3.4$

$3.4 - 1 = 2.4$

cont of RV ②

A random variable 'X' has the foll prob function

x	0	1	2	3	4	5	6	7
$P(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

(iii) If $P(X \le k) > \frac{1}{2}$ find a minimum value of 'k' ie, iv) distribution fun of X.
(v) determine Mean
(vi) Variance.

(iii) $P(X \le k) > \frac{1}{2}$

$P(X \le 1) = P(X=0) + P(X=1) = 0 + k = 0.1$ ×

$P(X \le 2) = P(X=0) + P(X=1) + P(X=2) = 0 + k + 2k = 3k$
$= 3 \times 0.1 = 0.3$

$P(X \le 3) = 0 + k + 2k + 2k = 5k = 5 \times 0.1 = 0.5$ ×

$P(X \le 4) = 0 + k + 2k + 2k + 3k + 8k^2$
$= k(k+8)$
$= 0.1(0.1+8)$
$= 0.8$

$\therefore 0.8 > \frac{1}{2}$

$\therefore$ min val of k is '4'

(iv) Distribution function of X.

X	$F(X) = P(X \leq x)$
0	0
1	$K = 1/10$
2	$0 + K + 2K = 3K = 3/10$
3	$0 + K + 2K + 2K = 5K = 5/10$
4	$8K = 8/10$
5	$8K + K^2 = 81/10$
6	$8K + 3K^2 = 83/10$
7	$9K + 10K^2 = 1$

v) Mean $= \mu = \sum_{i=0}^{n} p_i x_i = E(X)$

$= 0 \times 0 + 1 \times K + 2 \times 2K + 3 \times 2K \ldots\ldots$

$= 66K^2 + 30K$

$= 66\left(\frac{1}{10}\right)^2 + 30\left(\frac{1}{10}\right)$

$= 0.66 + 3 = 3.66$

$\therefore \mu = 3.66$

vi) $Var(X) = E(X)^2 - [E(X)^2]$

$= K + 8K + 18K + 48K + 25K^2 + \ldots$

$= 440K^2 + 124 - (3.66)^2$

$= \frac{440}{100} + \frac{124}{10} - (3.66)^2$

$= 3.4044$

(4). Two dies are thrown, Let X assign each point (a,b) in S. The maximum of its no's i.e $X(a,b) = \max(a,b)$. Find the probability distribution 'X' is a random variable with $X(S) = \{1,2,3,4,5,6\}$.
Also find the mean & variance

(OR)

A RV has a foll dist.

x	1	2	3	4	5	6
$p(x)$	1/36	3/36	5/36	7/36	9/36	11/36

→ Total no. of cases $= 6 \times 6 = 36$.
→ The max no would be 1,2,3,4,5,6
i.e $X(S) = X(a,b) = \max(a,b)$

→ now take 1, it will appear as
(1,1) (1,2) (1,3) ....
so one will appear (1,1) only once $= 1/36$.
so two will appear (1,2) (2,1) (2,2) $= 3/36$.
and ....

i) → $\mu = \sum_{i=1}^{6} P_i x_i = E(X)$.

$$\frac{1\times1}{36} + \frac{2\times3}{36} + \frac{3\times5}{36} + \frac{4\times7}{36} \ldots$$

$$= \frac{166}{36} = \boxed{4.47} = \mu$$

ii). → $Var X = \sum_{i=1}^{6} P_i x_i^2 - \mu = \left[E(x)^2 - [E[X]^2]\right]^2$

$$\frac{1}{36}\times 1+\frac{3}{36}\times 4+\frac{5}{36}\times 9+\frac{7}{36}+16+\frac{9}{36}+25+\frac{11}{36}\times 36$$

$Var\,X = 1.9912$.

(5) A random variable 'X' is foll distribution.

Val of X	0	1	2	3	4	5	6	7	8
P(x)	a	3a	5a	7a	9a	11a	13a	15a	17a

(1) a. (2) $P(X<3)$ (3) $P(X\geq 3)$ (3) Find $DF = F(x)$

$\sum_{i=1}^{8} P_i = 1$

(1) $1+3a+5a+7a+9a+11a+13a+15a+17a = 1$

$81a = 1$

$$\boxed{a = \frac{1}{81}}$$

ii) $P(X<3) = P(X=0)+P(X=1)+P(X=2)$

$= a+3a+5a$

$= 9a = \frac{9}{81} = \boxed{\frac{1}{9}}$

(iii) $P(X\geq 3) = \cancel{P(X=3)+(X=4)+P(X=5)+P(X=6)+7+8}$

$= \cancel{7a+9a+11a+13+15a+17a}$

$=$

$= 1 - P(X<3)$

$= 1 - \frac{1}{9}$

$= \frac{9-1}{9} = \boxed{\frac{8}{9}}$

(3) Finding $DF = F(x)$.

PDF of $F(x)$ of DRV.

$$F(x) = P(X \le x) = \sum_{i=1}^{x} P(x_i)$$

X.	$F(x) = P(X \le x)$
0	$1 = 1 = 1$.
1	$1+3a = 4a = 4/81$.
2	$1+3a+5a = 9a = 4/81$
3	$1+3a+5a+7a = 16/81$
4	$1+3a+5a+7a+9a = 25/81$.
5	$36/81$
6	$49/81$.
7	$64/81$
8	$81/81 = 1$.

Continuous RV:- A random variable which can take infinite no. of values in an interval is k.a. CRV.

Probability density Function (PMF):- A function $F(x)$ is said to be in PDF if

① $\boxed{f(x) > 0}$; $-\infty < x < \infty$

② $\boxed{\int_{-\infty}^{\infty} (x)\,dx = 1}$; It is known as density func

Cummulative Distribution function (CDF):-

Let X be a continuous random variable f(x), Then $F_x(X)$ will be CDF of X, If

$$\boxed{F_x(X) = P(X \le x) = \int_{-\infty}^{\infty} f(x)\,dx}$$ CDF.

Mean:-

$$\mu = \boxed{E(X) = \int_{-\infty}^{\infty} x\, f(x) \quad \text{(or)} \quad = E\phi(x) = \int_{-\infty}^{\infty} \phi(x)\, f(x)\,dx}$$

Median = Median is a point which divides the entire distribution into 2 equal parts

$$\boxed{\int_a^M f(x)\,dx = \int_M^b f(x)\,dx = 1/2}$$

Mode =

$$\boxed{f'(x) = 0} \;\&\; \boxed{f''(x) < 0 \quad \forall\, a < x < b.}$$

Variance:

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2.$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 E(X) - [E(X)]^2$$

Mean Deviation:

$$\int_{-\infty}^{\infty} |x-\mu| f(x)\,dx$$

PROBLEMS:

1. If the probability density of a random variable is given by.

$$f(x) = \begin{cases} k(1+x^2), & \text{for } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the value of 'k' & the probability that RV having this probality will take on a value

(i) b/w 0.1 & 0.2

(ii) b/w greater than 0.5

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx = 1$$

$$0 + \int_{0}^{1} K(1-x^2)\,dx + 0 = 1.$$

$$= \int_{0}^{1} (K - Kx^2)\,dx = 1.$$

$$= K\int_{0}^{1} dx - K\int_{0}^{1} x^2\,dx = 1.$$

$$K.[x]_0^1 - K\left[\frac{x^3}{3}\right]_0^1 = 1.$$

$$K[1-0] - K[1/3 - 0] = 1.$$

$$= K - \frac{K}{3} = 1.$$

$$= K\left[1 - \frac{1}{3}\right] = 1.$$

$$= K\left[\frac{2}{3}\right] = 1.$$

$$\boxed{K = 3/2.}$$

$K \times \frac{2}{3} = 1$

$2K = 3$

$K = \frac{3}{2}$.

$$P(0.1 < x < 0.2) = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_{0.1}^{0.2} k(1-x^2)\,dx$$

$$\left[kx - \frac{kx^3}{3}\right]_{0.1}^{0.2}$$

$$= k\left[x - \frac{x^3}{3}\right]_{0.1}^{0.2}$$

$$= \frac{3}{2}\left[0.2 - \frac{(0.2)^3}{3} - 0.1 - \frac{(0.1)^3}{3}\right]$$

$$= \frac{3}{2}\left[0.2 - \frac{0.008}{3} - 0.1 - \frac{0.001}{3}\right]$$

$P(0.1 < x < 0.2) = 0.2965$ //

(ii) greater than 0.5.

$$= \int_{0.5}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= \int_{0.5}^{1} k(1-x^2)\,dx + 0$$

$$= K\left[x - \frac{x^3}{3}\right]_{0.5}^{1}$$

$$= \frac{3}{2}\left[\left(1 - \frac{1^3}{3}\right) - \left(0.5 - \frac{(0.5)^3}{3}\right)\right]$$

$$= \frac{3}{2}\left(\frac{2}{3} - 0.4583\right.$$

$$\therefore \boxed{> 0.5 \quad = 0.3122}$$

* ② A Cont RV has PDF

→ $f(x) = \begin{cases} Kxe^{-\lambda x}, & \text{for } x \geq 0, \lambda \geq 0 \\ 0, & \text{otherwise} \end{cases}$

→ Determine :- i) K (ii) Mean (iii) Variance.

Sol. Given $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 1.$$

$$= 0 + \int_{0}^{\infty} (Kxe^{-\lambda x})\,dx = 1.$$

$$= K\left[x\int e^{-\lambda x}dx - \int\left[\frac{d}{dx}x\int e^{-\lambda x}dx\right]dx\right].$$

Range

$\int_{-\infty}^{0} \quad \int_{0}^{\infty}$

K = const

x = Algebr

e = expo.

LIATE

$$\therefore \int f(x)g(x)\,dx =$$

$$\therefore \left[f(x)\int g(x)\,dx - \int \left[\frac{d}{dx} f(x) \int g(x)\,dx \right] dx \right]$$

$$= K\left[\frac{x e^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right]_0^{\infty} = 1$$

$$= K\left[\left(\frac{\infty e^{-\lambda \infty}}{-\lambda} - \frac{e^{-\lambda \infty}}{\lambda^2} \right) - \left(\frac{0 e^{-\lambda 0}}{-\lambda} - \frac{e^{\lambda 0}}{\lambda^2} \right) \right] = 1.$$

$$= K\left[(0-0) - \left(0 - \frac{1}{\lambda^2}\right) \right] = 1$$

$$= \frac{K}{\lambda^2} = 1.$$

$$\therefore \boxed{K = \lambda^2.}$$

(b) Now $f(x) = \begin{cases} \lambda^2 x e^{-\lambda x}, & \text{for } x > 0. \\ 0, & \text{otherwise.} \end{cases}$

ii) Mean $= \mu = \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \int_0^{\infty} x \lambda^2 x e^{-\lambda x}\,dx.$$

$$\lambda^2 \int_0^\infty \overset{u}{x^2} \overset{v}{e^{-\lambda x}} dx \qquad \text{[Bernoulis Rule]}$$

$$\boxed{\int uv\,dx = uv_1 - u'v_2 + u''v_3 + \cdots}$$

$$= \lambda^2 \left[x^2 \left[\frac{e^{-\lambda x}}{-\lambda}\right] - 2x \left[\frac{e^{-\lambda x}}{\lambda^2}\right] + 2 \left[\frac{e^{-\lambda x}}{-\lambda^3}\right] \right]_0^\infty$$

$$= \lambda^2 \left[(0-0-0) - \left(\frac{-2}{\lambda^3}\right) \right].$$

$$\boxed{\mu = \frac{2}{\lambda}}$$

(iii) Variance : $\sigma^2 = \int_0^\infty x^2 f(x)\,dx - \mu^2$

$$= \lambda^2 \int_0^\infty x^3 e^{-\lambda x} dx - \left(\frac{2}{\lambda}\right)^3$$

$$= \lambda^2 \int_0^\infty x^3 e^{-\lambda x} dx - \frac{8}{\lambda^3}$$

$$= \lambda^2 \left[\frac{x^3 e^{-\lambda x}}{\lambda} - 3x^2 \left[\frac{e^{-\lambda x}}{\lambda^2}\right] + 6x \left[\frac{e^{-\lambda x}}{\lambda^3}\right] + 6 \left[\frac{e^{-\lambda x}}{\lambda^4}\right] \right]_0^\infty - \frac{4}{\lambda^2}$$

$$= \lambda^2 \left[(0-0-0+0) - \left[\frac{-6}{\lambda^4} \right] \right] - \frac{4}{\lambda^2}$$

$$= \frac{6}{\lambda^2} - \frac{4}{\lambda^2}$$

$$= \frac{2}{\lambda^2}$$ is the variance //

* (2) PDF of a RV 'x' is

$$f(x) = \begin{cases} 1/2 \sin x & \text{for } 0 \le x < \pi \\ 0, & \text{elsewhere} \end{cases}$$

Find (1) Mean (2) Median (3) Mode.
(4) prob b/w 0 & $\pi/2$.

i) $\boxed{\text{Mean} = \int_{-\infty}^{\infty} x f(x)\, dx}$

Range

$$\int_{-\infty}^{0} \quad \int_{0}^{\pi} \quad \int_{\pi}^{\infty}$$

$$0 + \boxed{\quad} + 0.$$

Mean

$$= \int_0^{\pi} \frac{1}{2} x \sin x \, dx.$$

$$= \frac{1}{2} \left[x(\sin x) dx \right]_0^{\pi}$$

$$= \frac{1}{2} \left[x \int \sin x\, dx - \int \left[\frac{d}{dx} x \int \sin x\, dx \right] \right]$$

$\cos\pi = -1$
$\sin\pi = 0$

$$\boxed{\int f(x)g(x)\,dx = f(x)\int g(x)\,dx - \int\left[\frac{d}{dx}f(x)\int g(x)\,dx\right]dx}$$

$$= \frac{1}{2}\left[-x\cos x + \int \cos x\,dx\right]_0^{\pi}$$

$$= \frac{1}{2}\left[-x\cos x + \sin x\right]_0^{\pi}$$

$$= \frac{1}{2}\left[(-\pi\cos\pi + \sin\pi) - (-0\cos 0 + \sin 0)\right]$$

$$= \frac{1}{2}\left\{-\pi(-1) + 0 - 0\right\} = \boxed{\frac{\pi}{2}}$$

$$\boxed{\mu = \pi/2}$$

Mode: $f'(x) = 0$ & $f''(x) < 0$ $\forall\, a < x < b$.
$0 < x < \pi$.

$f(x) = \frac{1}{2}\sin x$

$f'(x) = \frac{1}{2}\cos x$.

$\therefore \frac{1}{2}\cos x = 0$.

$\cos x = 0$

$\boxed{x = \frac{\pi}{2}}$

$f''(x) = -\frac{1}{2}\sin x$.

when $x = \frac{\pi}{2}$, then $f''(x) = \frac{-1}{2} < 0$

mode $= \boxed{x = \frac{\pi}{2}}$

$$\boxed{\text{Median} = \int_a^M f(x)\,dx = \int_M^b f(x)\,dx = \frac{1}{2}}$$

$$= \int_0^M f(x)\,dx = \int_M^\pi f(x)\,dx = \frac{1}{2}$$

$$\int_0^M \frac{1}{2}\sin x\,dx = \int_M^\pi \frac{1}{2}\sin x\,dx = 1/2.$$

$$\int_0^M 1/2 \sin x\,dx = \int_M^\pi \frac{1}{2}\sin x\,dx = 1/2$$

$$= \frac{1}{2}\int^M \sin x\,dx = 1/2.$$

$$= \frac{1}{2}(-\cos x)_0^M = 1/2.$$

$$= (\cos M + \cos 0)\,1/2 = 1/2.$$

$$= (1 - \cos M) = 1.$$

$$1 - 1 = \cos M.$$

$$\cos M = 0.$$

$$\therefore \boxed{M = \frac{\pi}{2}}$$

$\therefore$ Mean = Mode = Median = $\pi/2$.

iv) Prob b/w $\left(0 < x < \frac{\pi}{2}\right)$.

$$P(0<x<\pi/2) = \int_{-\infty}^{\infty} f(x)\,dx.$$

$$= \int_0^{\pi} f(x)\,dx = \int_0^{\pi/2} \frac{1}{2}\sin x\,dx.$$

$$= \left(-\frac{1}{2}\cos x\right)_0^{\pi/2} = -1/2\left(\cos\frac{\pi}{2} - \cos 0\right).$$

$$= 1/2\,(0-1)$$

$$= 1/2.$$

(2) If 'X' is a continuous RV & $Y = aX+b$ prove that $E(Y) = a\,E(X)+b$ & $V(Y) = a^2 V(X)$, where V stands for variance & a, b are constants

Sol. i) By defination

$$E(Y) = E(aX+b) = \int_{-\infty}^{\infty} (ax+b)\,f(x)\,dx.$$

$$\left[\because E\phi(x) = \int_{-\infty}^{\infty} \phi(x) f(x)\,dx\right]$$

$$= a\int_{-\infty}^{\infty} x\,f(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx$$

$$= a\,E(X) + b(1)$$

[∴ Tot porob unity]

$$\boxed{E(Y) = a\,E(X)+b.}$$

(ii). From (i) we have

$$E(y) = aE(x) + b \quad -(1)$$

where $y = ax + b \quad -(2)$

(2) − (1).

$$y - E(y) = a[x - E(x)]$$

S.O.B.S

$$[y - E(y)]^2 = a^2[x - E(x)^2]$$

E.O.B.S

$$E\{[y - E(y)]^2\} = a^2 E\{[x - E(x)]^2\}$$

$$V(y) = a^2 V(x)$$

proved

(2) If 'x' is CRV and 'k' is constant. Then prove that.

i) $Var(x+k) = Var(x)$

ii) $Var(kx) = k^2 Var(x)$

Sol. By defination

$$Var(x) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} x f(x)\,dx\right]^2$$

i) $$Var(x+k) = \int_{-\infty}^{\infty} (x+k^2) f(x)\,dx - \left[\int_{-\infty}^{\infty} (x+k) f(x)\,dx\right]^2$$

$$\int_{-\infty}^{\infty} (x^2+2kx+k^2) f(x)\,dx - \left[\int_{-\infty}^{\infty} x f(x)\,dx + k\int_{-\infty}^{\infty} f(x)\,dx\right]^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx + 2k\int_{-\infty}^{\infty} x f(x)\,dx + k^2 - \left[\int_{-\infty}^{\infty} x f(x)\,dx + k\right]^2$$

$$= E(X^2) + 2k\,E(X) + k^2 - [E(X)+k]^2$$

$$= E(X^2) + 2k\,E(X) + k^2 - [E(X)]^2 - 2k\,E(X) - k^2$$

$$= E(X^2) - [E(X)]^2.$$

$$Var(X+k) = Var(X).$$

(ii) $$Var(kx) = \int_{-\infty}^{\infty} k^2 x^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} kx f(x)\,dx\right]^2$$

$$= k^2\int_{-\infty}^{\infty} x^2 f(x)\,dx - k^2\left[\int_{-\infty}^{\infty} f(x)\,dx\right]^2$$

$$= k^2\left[E(X)^2 - \{E(X)\}^2\right]$$

$$= k^2\,Var(X).$$

② Z test for diff of means
(2 samples with mean & SD)

① Null hypothesis $H_0 : \mu_1 = \mu_2$ [H_0 std]

② Alternate hypothesis $H_1 : \mu_1 \neq \mu_2$ [Two tail]
$\mu_1 > \mu_2$ (OR) $\mu_1 < \mu_2$ [one tail]

③ Level of significance (α) = 5%, (OR) 1%
(OR) 2% (OR) 10%.

④ Test significance

$$\boxed{Z = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}}$$

⑤ Conclusion $\therefore$ $|Z| \leq Z$ Table Value
$\therefore$ Accept H_0.

Table values.

α	1 tail	2 tail
5.	1.96	1.645
1%	2.58	2.33
2%	2.33	—
10%.	1.645	—

① Roughs

Given

$n_1 = 1000$

$n_2 = 2000$

$SD = 2.5$

$\bar{x}_1 = 67.5$

$\bar{x}_2 = 68.0$

Let:
μ_1 & μ_2 be mean of 2 populations

① Null hypothesis H_0: The samples have been drawn from same pop

$\therefore \mu_1 = \mu_2$ & $\sigma = 2.5$

② Alternate hyp H_1 = $\mu_1 \neq \mu_2$.

③ Test statistic = $Z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}} = \dfrac{67.5 - 68}{\sqrt{(2.5)^2\left(\dfrac{1}{1000} + \dfrac{1}{2000}\right)}}$

$= -5.16$

$|Z| = 5.16 > 1.96$

Hence the hypothesis is rejected.

⑤

$SD_1 = 15$	$SD_2 = 20$	use $\alpha = 0.01$
$n_1 = 150$	$\bar{x}_1 = 75$	sig = 1%
$n_2 = 250$	$x_2 = 70$	

$\therefore Z =$

① Null hyp = $\mu_1 = \mu_2$

② Alt = $\mu_1 \neq \mu_2$

③ Test = $Z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$

$= \dfrac{75 - 70}{\sqrt{\dfrac{(15)^2}{150} + \dfrac{(20)^2}{250}}} = 27$

$|Z| \leq$ table value

$2.7 > 2.33$

Hence we reject the null hypothesis at 1% level of significance & conclude that the groups have not been taken from same population.

③ Z test for Single proportion :-
[no mean, no SD, one sample]

① H_0 = Null hypothesis $P = 0.5$
(There is no proportion)

② H_1 = Alternative hyp $P \neq 0.5$ (2 tail)
$P > 0.5$ (1 tail)
(OR)
$P < 0.5$.

③ α = level of significance = 5% or 2%, 1%, 10%

④ Test Statistic formula.

P = population proportion
p = Sample proportion.

$$\boxed{Z = \frac{p - P}{\sqrt{\frac{PQ}{n}}}}, \quad p = x/n, \quad Q = 1 - P$$

⑤ Conclusion :

Z value ≤ Z table

Accept H_0

① Null hyp = $P = 0.5$ (Both are equally pop

② Alt hyp = $P \neq 0.5$. (2 tail).

③ $\alpha = 0.01$.

④ Test stats = $n = 1000$, $\bar{x} = 540$, $\boxed{p = \frac{x}{n}}$

$= \frac{540}{1000}$ $\therefore \boxed{p = 0.54}$

$\boxed{P = 1/2}$

$\therefore$ sample $p = 0.54$. $\boxed{Q = 1 - P}$

$Q = 1 - 0.54 = 0.5$.

$\boxed{Q = 0.5}$

$$z = \frac{p - P}{\sqrt{\frac{PQ}{n}}} = \frac{0.54 - 0.5}{\sqrt{\frac{0.5 \times 0.5}{1000}}} = 2.532.$$

$|z| <$ table

$2.53 < 2.58$.

we accept H_0.

$\frac{}{n}$ $\sqrt{\frac{}{100}}$

$= \frac{-0.16}{0.0876}$

$= -18.26$

Given $= x = 100$

$n = 500$

$p = \frac{x}{n} = 0.2$

$P = 0.04$

$Q = 1 - P = 0.96$

$|Z| <$ table

$18.26 < 1.625$

but x

$18.26 > 1.6245$

we reject

(4) Z test for difference of proportions.

$H_0 = P_1 = P_2$.

$H_1 = P_1 \neq P_2$. (2 tail).
$P_1 > P_2$ (OR) $P_1 < P_2$ (1 tail).

$\alpha = 5\%$ or 1%, 2% 10%.

$$Z = \frac{P_1 - P_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$p = \frac{x_1 + x_2}{n_1 + n_2}$ $\quad q = 1 - p$.

$P_1 = \frac{x_1}{n_1}$ $\quad P_2 = \frac{x_2}{n_2}$.

$|Z| < Z_{Table}$ (Accept H_0).

(1) $n_1 = 180$. $\quad x_1 = 45$ $\quad \alpha = 5\%$.
$n_2 = 120$. $\quad x_2 = 34$.

Here $P = \frac{x}{n} = P_1 = \frac{45}{180}$ $\quad P_2 = \frac{34}{120}$.

$\boxed{P_1 = 0.25}$ $\boxed{P_2 = 0.283}$

$$P = \frac{n_1 P_1 + n_2 P_2}{n_1 + n_2} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{45 + 34}{300}$$

$\boxed{P = 0.268}$ $\boxed{Q = 1 - 0.268} = 0.732$

① $P_1 = P_2$.

② $P_1 \neq P_2$ (2 tail).

③ level of signi = $\alpha = 0.05 = 5\%$.

④ $$\boxed{Z = \frac{P_1 - P_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}}$$

$$= \frac{0.25 - 0.283}{(0.26)(0.737)\left(\frac{1}{180} + \frac{1}{20}\right)}$$

$$= \boxed{-0.6358}$$

$|Z| < |Z \text{ table}|$

$= 0.635 < 1.9$.

We accept

//

② $x_1 = 20$ | $n_1 = 400$ | $P_1 = \frac{20}{400} = 1/20 = 0.05$

$x_2 = 10$ | $n_2 = 300$ | $P_2 = \frac{10}{300} = 1/300 = 0.03$

$$P = \frac{x_1 + x_2}{n_1 + n_2} = \frac{20 + 10}{700} = \boxed{\frac{30}{700}}$$

$$Q = 1 - \frac{30}{700} = \frac{67}{70} \quad \boxed{P = \frac{3}{70}}, \boxed{Q = \frac{67}{70}}$$

① $P_1 = P_2$

② $P_1 > P_2$ (1 tail).

③ $\alpha = 5\%$.

④ $$Z = \frac{P_1 - P_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{\frac{1}{20} - \frac{1}{30}}{\left(\frac{3}{20}\right)\left(\frac{67}{70}\right)\left[\frac{1}{400} + \frac{1}{300}\right]}$$

$= 1.0844$.

$|Z| \leq Z$ table

$1.0844 < 1.645$.

Accepted.

when ~~percentages~~ given

③ $P_1 = 0.30$ $n_1 = 1200$

$P_2 = \cancel{0.25}$ 0.25 $n_2 = 900$.

$\frac{x}{n} = p$ $\frac{30}{100}$

$x = pn$. $\frac{3}{10}$.

0.3.

$\frac{25}{10}$.

$\frac{30}{10}$.

$\frac{03}{10}$

$\frac{100}{1}$

$\frac{03}{1} \times \frac{10}{100}$

① H_0 $P_1 = P_2$.

② H_1 $P_1 \neq P_2$.

$Q_1 = 1 - P_1 = 1 - 0.3$.

$Q_2 = 1 - P_2 = 1 - 0.25$.

③ $$Z = \frac{P_1 - P_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{P_1 - P_2}{\left(\sqrt{\frac{P_1 q_1}{n_1} + \frac{P_2 q_2}{n_2}}\right)}$$

$Z = 2.56 > 1.96$

$\neq$ Reject

Normal Distribution

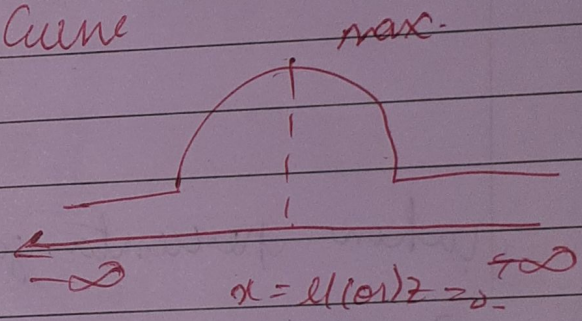
It is defined on CRV.

$$f(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

(OR)

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Normal Distribution Curve



1. bell shaped, symetric $x = \mu$.
2. ext from $-\infty$ to ∞.
3. Tot area is 1.
4. nue enters into x-axis.
5. mean, median, mode coinside at $x = \mu$.

So normal curve is unimodal.

Mean of Normal Distribution

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$f(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right]$$

$$\mu = \int_{-\infty}^{\infty} x\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \quad \text{--- (1)}$$

let $z = \frac{x-\mu}{\sigma}$

$$\boxed{x = z\sigma\mu}$$

$$dz = \frac{dx}{\sigma}$$

$$dx = \sigma dz$$

$$\cancel{\mu = \int_{-\infty}^{\infty}}$$

$$\mu = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (z\sigma + \mu) e^{-\frac{1}{2}\left(\frac{\sigma z}{\sigma}\right)^2}$$

$$\sigma = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (z\sigma + \mu)^{-\frac{z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z\sigma$$

1. If X is normal distribution with mean is 1 & SD = 3. Find the probabilities that

(i) $3.43 \le x \le 6.19$ x_2

(ii) $-1.43 \le 6.19$

Given mean $\boxed{\mu = 1}$

SD = $\boxed{\sigma = 3}$

$$\boxed{z = \frac{x-\mu}{\sigma}}$$

(i) $P(3.43 \le x \le 6.19)$

$x_1 = 3.43$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{3.43-1}{3} = \frac{2.43}{3}$$

$$\boxed{z_1 = 0.81}$$

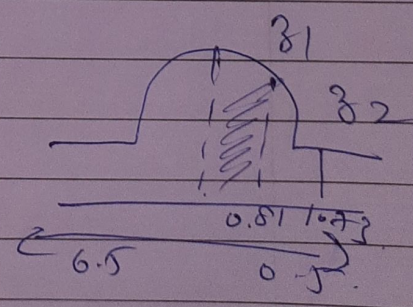
$x_2 = 6.19$

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{6.19-1}{3} =$$

$$\boxed{z_2 = 1.73}$$

$P(3.43 \le x \le 6.19)$

$P(0.81 \le z \le 1.73)$



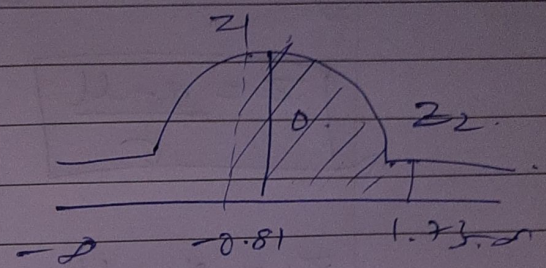
$A(z_2) - A(z_1)$

$A(1.73) - A(0.81)$

$= 0.4582 - 0.2910$

$= 0.$

(ii) $-1.43 \le x \le 6.19$.



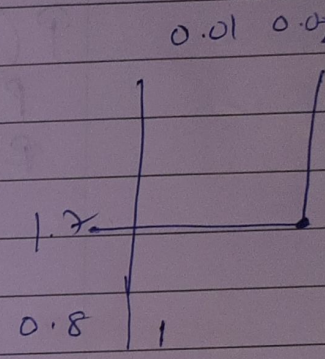
$$z = \frac{x-\mu}{\sigma}$$

$x_1 = -1.43$.

$x_2 = 6.19$.

$$z_1 = \frac{x_1-\mu}{\sigma} = \frac{-1.43-1}{3} = \boxed{-0.81}$$

$$z_2 = \frac{x_2-\mu}{\sigma} = \frac{6.19-1}{3} = \boxed{1.73}$$



$\therefore P(-1.43 < x < 6.19)$.

$= P(-0.81 < x < 1.73)$.

$A(z_2) + A(z_1)$

$= A(1.73) + A(0.81)$.

$= 0.45 + 0.29$.

$= 0.74$.

Q. Suppose the wts of 800 male students are normally distributed with mean $\mu = 140$ pounds & SD $= 10$ pounds. Find the no of students whose wts are

(i) b/w 138 & 148 pounds.

ii) more than 152 pounds.

$n = 800$.

$\mu = 140$.

$\sigma = 10$.

i) $P(138 \le x \le 148)$.

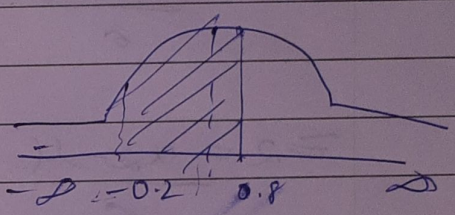
$$\boxed{Z = \frac{x-\mu}{\sigma}}$$

$$Z_1 = \frac{x_1-\mu}{\sigma} = \frac{136-140}{10} = -0.2$$

$$Z_2 = \frac{x_2-\mu}{\sigma} = \frac{148-140}{10} = 0.8.$$

$P(138 \le x < 148)$

$P(-0.2 \le z \le 0.8)$.



~~P(Z2~~

$A(Z_2) + A(Z_1)$

$= A(0.8) + A(-0.2)$.

$= 0.2881 + (0.0793)$.

$0.2881 + 0.0793$

$= 0.3674$.

Hence whose mts r b/w these 2 i.e.

138 & 148.

$= 0.3674 \times 800$.

≈ 294

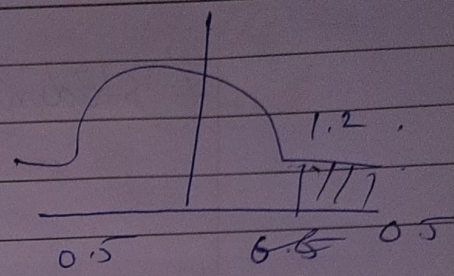
ii) $P(x \ge 152)$.

$$Z = \frac{x-\mu}{\sigma}.$$

$$Z = \frac{152-140}{10} = 1.2.$$

$P(z > 1.2)$

check positive side A1.



$P(z > 1.2)$

$= 0.5 - A(z_1)$

$= 0.5 - 152$

$A_2(0.5) - A(1.2)$

$0.5 - 0.3849$

$= 0.1151$

$\therefore P(x > 152) = 0.1151$

$\therefore 0.1151 \times 800 = 92$

*

If the masses of 30 students are ~~manually~~ normally distributed with mean 68 kgs & std deviation 3kgs. how many due students have masses

(i) greater than 72 kgs

(ii) Less than or equal to 64 kgs.

iii) b/w 65 & 71 kgs

$\mu = 68$

$SD = 3$

i) $P(x > 72)$

$$z_1 = \frac{x - \mu}{\sigma} = \frac{72 - 68}{3} = 1.33$$

$A_1(0.5) - A_2(1.33)$

$0.5 - 0.4082 = (0.0918$ *30)

$\therefore$ Students with more than 72.

$= 0.0918 \times 30 = 28$.

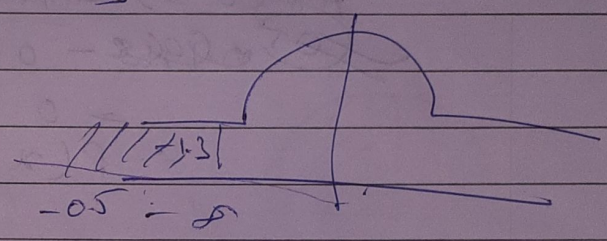
ii) $x = 64$.

$$z = \frac{x - \mu}{\sigma} = \frac{68 - 64}{3} = 1.33$$

$P(x \leq 64)$

$P(z \leq 64)$

1.33.



$0.5 - (-1.33)$

$0.5 - 0.4082$

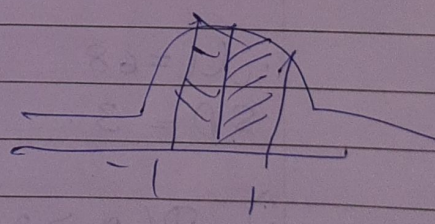
$= 0.0918$

iii) $P(65 \leq x \leq 71)$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{65 - 68}{3} = -1$$

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{71 - 68}{3} = 1.$$

$P(-1 < z < 1)$.



$A(-1) + A(1)$

$A(1) + A(1)$

$2A(1)$

$= 0.3413$

$= 0.6826$.

Req no of students $= 0.6826 \times 30$

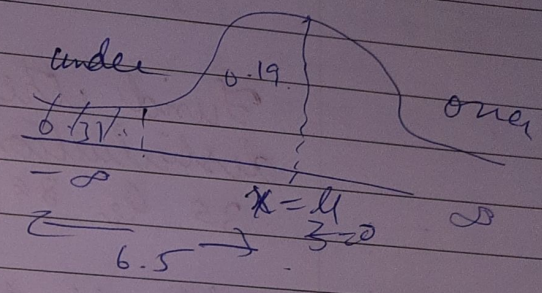
In a normal Distribution, 31% of the items under 45 & 8% are over 64. Find mean & varience of distribution

Sol

Let x be CRV.

let μ be mean

σ be SD



$P(x<45) = 31$

$= 0.31$

$Z_1 = \frac{x-\mu}{\sigma}$

$= Z_1 = \frac{45-\mu}{\sigma}$

$Z_2 = \frac{45-\mu}{\sigma}$

$Z_1 = -\frac{45-\mu}{\sigma}$

$0.5 = \frac{45-\mu}{\sigma}$

$0.5\sigma + 45 = \mu$

$\frac{45-\mu}{\sigma} = -0.5$

$0.5\sigma + \mu = 45$ — (1)

$P(x>64) = 8 = 0.08$

$Z_2 = \frac{x_2-\mu}{\sigma}$

$= \frac{64-\mu}{\sigma}$

$64 = Z_2\sigma + \mu$

$64 = 1.4\sigma + \mu$ — (2)

(1) − (2)

$(45-\mu)(64-\mu) = -0.5\sigma - 1.4\sigma$

$= -\frac{19}{\sigma} = -1.9\sigma$

$\sigma = \frac{49}{1.9} = 10$

$\sigma = 18$

08/. In a Normal Distribution, 7% of the items under 35 & 89% of items under 63. Determine the mean & variance of dist

(OR)

Find the mean & SD of normal distribution 7% of the items under 35 & 89% are under 63

Sol Let μ be the mean (at $z=0$) & σ be the SD of NDC. 7% of the items are under 35 means the area to the left of the ordinate $x=35$.

Given CIP $P(X<35) = 7\% = 0.07$.

$P(X<63) = 89\% = 0.89$.

$\therefore P(X>63) = 1 - P(X<63) = 1-0.89 = 0.11$.

when $X=35$, $z = \frac{x-\mu}{\sigma} = \frac{35-\mu}{\sigma} = -z_1$ — (1).

when $X=63$, $z = \frac{x-\mu}{\sigma} = \frac{63-\mu}{\sigma} = z_2$ — (2).

From the fig, we have:

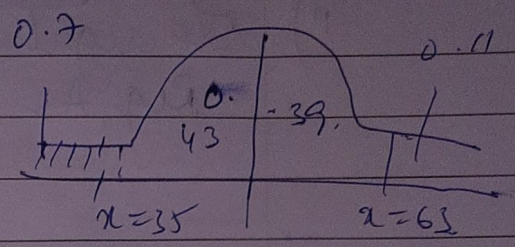
$P(0<z<z_1) = 0.43$.

$\Rightarrow z_1 = 1.48$ {from NP table}.

$P(0<z<z_2) = 0.39$.

$z_2 = 1.23$.

① $\Rightarrow \frac{35-\mu}{\sigma} = -1.48$



$z=21$ $x=\mu$

$z=0$ $z=2$

$= 35-\mu = -1.48\sigma$

$= 1.48\sigma + \mu = 35$ — ③

② $\Rightarrow \frac{63-\mu}{\sigma} = 1.23$

$63-\mu = 1.23\sigma$

$= 1.23\sigma + \mu = 63$ — ④

④ − ③

$\frac{28}{\sigma} = 2.71$

$\sigma = \frac{-28}{2.71} = 10.332$

put 'σ' in ③

$35-\mu = -1$

$35-\mu = -1.48$

1) If 'X' is normal variable, find area
(i) to left of $z=-1.78$.

to left of $z=-1.78$.

Required Area

$= 0.5 - Ar(-1.78)$

$0.5 - 1.78$

$0.5 - 0.4625$

$= 0.0375$

Right = +0.5
left = −0.5

ii) To right of $Z = \cancel{1.28}\ -1.45$.

Req Ar $= 0.5 + (-1.45)$

$= 0.5 + 1.45$

$= 0.5 + 0.426$

$= 0.926$

iii) corresponding to $-0.8 \le Z < 1.53$.

Req Ar $= A(Z_2) + A(Z_1)$.

$A(1.53) + A(-0.8)$

$= 0.437 + 0.2881$.

$= 0.7251$.

iv). To the left of $Z = -2.52$ & Right of $Z = 1.8$.

$[0.5 - Ar(2.52)] + [0.5 + Ar(1.8)]$.

$= (0.5 - 0.4941) + (0.5 - 0.4664)$.

$= 0.0059 + 0.0336$.

$= 0.0395$

Sampling Distribution

large sample $(n > 30)$

Small $(n < 30)$.

Sampling distribution

population $- N$

Sample : n.

Sample with replacement :- = $\boxed{N^n}$ ways.

Sample without replacement = $\boxed{{}^N C_n}$ ways

	Mean	S.D.
Sample	$\bar{x}$	s
population	μ	σ

Q). A population consists of five no's 2, 3, 6, 5 & 11. Consider all possible samples of size two which can be drawn with replacement from this population.
Find
① The mean of population
② The stand Deviation of pop
③ The mean of the sampling dist of mean.
④ The SD of sampling deviation of mean

① $\mu = \frac{\Sigma x_i}{n} = \bar{x}$

$= \frac{2+3+6+8+11}{5}$

$\boxed{\mu = 6}$

② $\sigma = \sigma^2 = \Sigma \left(\frac{x_i - \bar{x}}{n}\right)^2$

$= \frac{(2-6)^2 + (3-6)^2 + (6-6)^2 + (8-6)^2 + (11-6)^2}{5}$

$$\frac{16+9+0+4+25}{5} = 10.8$$

Variance $\sigma^2 = \sqrt{10.8} = 3.29$

(5) → Sampling with replacement (Infinite population).

$N^n = 5^2$

$= 25$. (Sample of size 2)

∴ all possible samples of size '2' from population 2, 3, 4, 5, 6, 8 we get 25 samples.

→ Sampling without replacement

2, 3, 6, 8, 11

(2,2)	(2,3)	(2,6)	(2,8)	(2,11)
(3,3)	(3,2)	(3,6)	(3,8)	(3,11)
(6,2)	(6,3)	(6,6)	(6,8)	(6,11)
(8,2)	(8,3)	(8,6)	(8,8)	(8,11)
(11,2)	(11,2)	(11,6)	(11,8)	(11,11)

→ Mean for each sample $= \frac{2+2}{2} = \frac{4}{2}$

2	2.5	4	5	6.5
3	2.5	4.5	5.5	7
4	4.5	6	7.0	8.5
5	5.5	7	8	9.5
6.5	7	8.5	9.5	11

— (1)

$$\mu = \frac{\text{sum of sample means}}{25}$$

$$= \frac{150}{25} = 6.$$

$$\therefore \mu_{\bar{x}} = \mu$$
$$= 6 = 6.$$

(4). The variance σx^2 of SD of mean is obtained by subtracting the mean '6' from each no in (1).
& Squaring the result,
Adding all 25 numbers & dividing by 25.

$$\sigma x^2 = \frac{(2-6)^2 + (2.5-6)^2 + (4-6)^2 + (5-6)^2 + \ldots (11-6)^2}{25}.$$

$$\sigma x^2 = \frac{135}{25} = 5.40$$

$$\sigma x = \sqrt{5.40} = 2.32.$$

UNIT-5 8

→ **Stochastic process:-** It is a set of random variables depends on some real parameters like time 't'.

→ **Markov process:-** A random process in which the occurence of future state depends on the immediately preceding state & only on it is called Markov chain.

Uses:- brand check
machine check.
state is a condition
(a) location.

Assumptions:- ① finite no of state
② states are mutually exclusive
③ states are collectively exclusive (both known)
④ prob of moving from one state to another is const

Transition probability:- Moving from one state to another state or remaining in the same state during a single time period is called transition probability

Mathematically : $P_{ij} = P\left(\frac{\text{next state } S_j \text{ at } t=1}{\text{Initial state } S_i \text{ at } t=0}\right)$

Transition probability Matrix :-

$$P = \text{Initial state } (i) \; (n=0) \quad i \quad \begin{array}{c|ccc} & S_1 & S_2 & S_3 \\ \hline S_1 & P_{11} & P_{12} & P_{13} \\ S_2 & P_{21} & P_{22} & P_{23} \\ S_3 & P_{31} & P_{32} & P_{33} \end{array} \quad \text{next state } j$$

Eg: $P_{11} = P\left[\frac{S_1 \text{ at time } t=1}{S_1 \text{ at time } t=0}\right]$

$\therefore$ $\boxed{\text{One step Transition}}$

TPM with 2 square.

$$P^2 = \begin{array}{c|ccc} & S_1 & S_2 & S_3 \\ \hline S_1 & P_{11}^2 & P_{12}^2 & P_{13}^2 \\ S_2 & " & " & " \\ S_3 & & & \end{array}$$

Eg: $P_{11}^n = P\left[\frac{S_1 \text{ at time } t=n}{S_1 \text{ at time } t=0}\right]$

$\boxed{\text{two step transition}}$

nth step probability

$$P_{11}^n \quad P_{12}^n \quad P_{13}^n$$

→ properties of TPM:-

1. It is a square matrix
2. All the entries are b/w 0 & 1.
3. The sum of entries in any row must be 1.
4. entries should be non negative

→ Eg) Which of the foll are Stochastic Matrix

① $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ ✗

② $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

③ $\begin{bmatrix} 0 & 1 \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}$ ✗

④ $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ ✓

⑤ $\begin{bmatrix} 0 & 2 \\ 1/4 & 1/4 \end{bmatrix}$ ✗

→ Regular (OR) not regular

* A Stochastic matrix P is said to be regular if all the entries of some power P^m are positive.

* A Stochastic matrix 'P' is not regular if '1' occurs in the main principle diagnol

Q). $A = \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0 & 0.5 & 0.5 \\ 0.6 & 0.4 & 0 \end{bmatrix}$ is regular?

$$\boxed{A^2} = \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0 & 0.5 & 0.5 \\ 0.6 & 0.4 & 0 \end{bmatrix} \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0 & 0.5 & 0.5 \\ 0.6 & 0.4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.56 & 0.31 & 0.12 \\ 0.3 & 0.45 & 0.25 \\ 0.45 & 0.35 & 0.2 \end{bmatrix}$$

$\therefore$ no 0, no 1, A^2 having all positive +ve entries

$\therefore$ It is regular.

Q). $B = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$B^2 = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} (0.5)(0.5) + 0(0) & 0.5(0) \\ & \\ & \end{bmatrix} = \begin{bmatrix} 0.25 & 0 & 0.25 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we get 0's & 1's

$$B^3 = \begin{bmatrix} 0.125 & 0 & 0.875 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \theta$$

$$B^4 = \begin{bmatrix} 0.625 & 0 & 0.937 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ A is not regular

Q) Find the value of x, y, z if

$$\begin{bmatrix} 0 & x & 1/3 \\ 0 & 0 & y \\ 1/3 & 1/4 & z \end{bmatrix}$$ is a TPM

Sol w.k.t $0 + x + \frac{1}{3} = 1$ — (1)

$0 + 0 + y = 1$ — (2)

$1/3 + 1/4 + z = 1$ — (3)

$\rightarrow x = 1 - \frac{1}{3} = 2/3$

$\rightarrow y = 1 = 1$

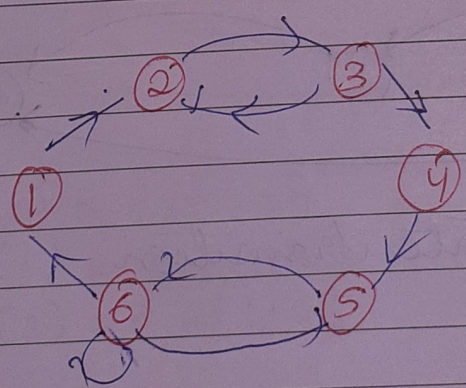
$\rightarrow z = 1 - = 5/12$

Period:- A period is a state is the GCD of n such that $P_{ij}^{n} \neq 0$. denoted by $d(i)$.

① If $\boxed{d(i) > 1}$, its $\boxed{\text{periodic}}$
② If $\boxed{d(i) = 1}$, its $\boxed{\text{aperiodic}}$

Recurrent:- If at any time we leave that state & we will return to that state with prob 1.

Transistent:- If the probability of returning is less than 1 (<1) then the state is called Transistent.

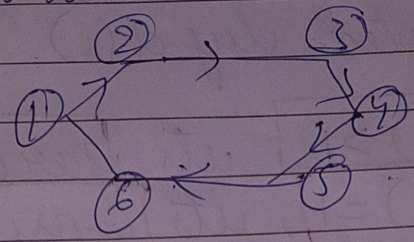


$d(1) = 1$
$d(2) = 2$
$d(3) = 3$
$d(4) = 1$
$d(5) = \{2, 3\} = 1$
$d(6) = \{2, 3\} = 1$

$\therefore$ 2, 3 are periodic

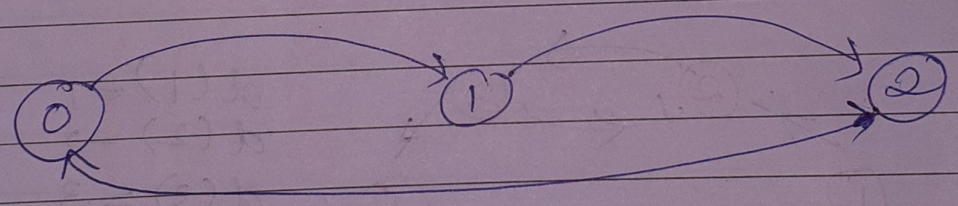
$\therefore$ Transistant = not come back
Recurrent = come back to the same state

with each other.



Q) $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is it irreducible?

$$= \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \end{array}$$



State transition

Class of state

$C(0) = \{1, 2, 0\}$
$C(1) = \{1, 0, 2\}$
$C(2) = \{1, 0, 2\}$

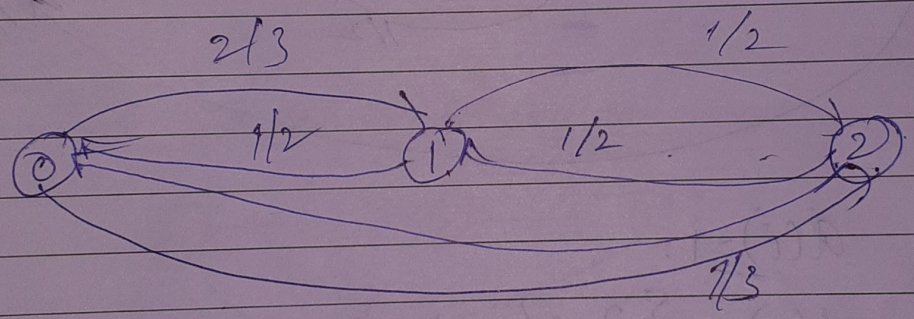
$\therefore$ each state communicate with

all other states ∴ it is irreducible

∴ Markov chain is irreducible

Q) $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$.

$$= \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 2/3 & 1/3 \\ 1 & 1/2 & 0 & 1/2 \\ 2 & 1/2 & 1/2 & 0 \end{array}$$



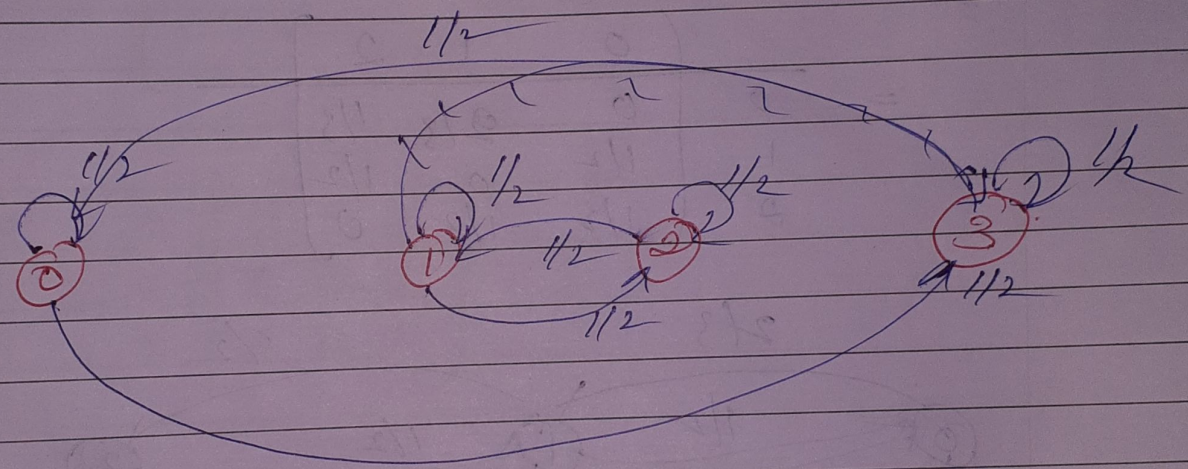
$C(0) = \{0, 1, 2\}$
$C(1) = \{1, 0, 2\}$
$C(2) = \{2, 1, 0\}$

∴ It is irreducible

Ergodic :- * Consists of '1' equivalence class.
* All elements of this class r recurrent
* All states are aperiodic

Eg:-

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1/2 & 0 & 0 & 1/2 \\ 1 & 0 & 1/2 & 1/2 & 0 \\ 2 & 0 & 1/2 & 1/2 & 0 \\ 3 & 1/2 & 0 & 0 & 1/2 \end{array}$$



GCD's $d(i) = 1$.

$d(0) = \{2, 3\} = 1$.

$d(1) = 1$.

$d(2) = 1$.

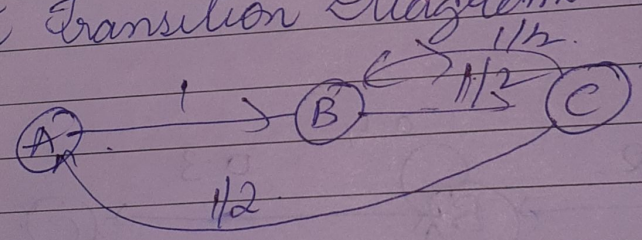
$d(3) = 1$.

$\therefore$ It is ergodic.

Sol

$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 0 & 0 & 1 \\ C & 1/2 & 1/2 & 0 \end{array}$$

State Transition Diagram



$d(i) = 1$, $d(A) = \{3, 5\} = 1$

$d(B) = \{3, 5\} = 1.$

$d(C) = \{2, 3, 5\} = 1.$

The given matrix is aperiodic & recurrent Hence it is ergodic

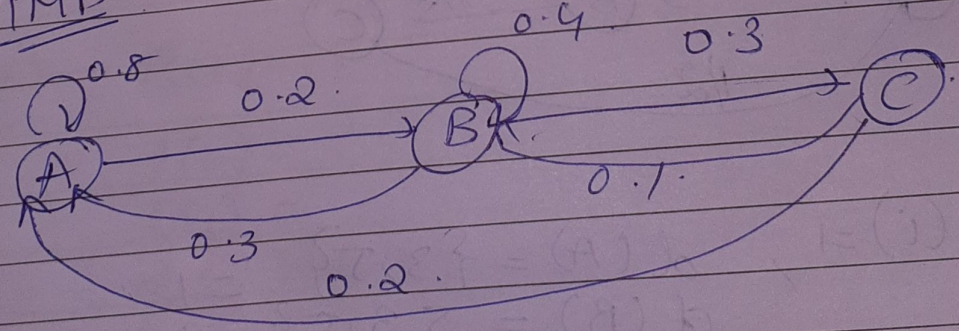
Q) 3 universities A, B, C are admitting students. If it is given that 80 percent of the children went to A & rest went to B. 40 % of children of B went to B & rest split evenly b/w A & C of children of C 70% went to C, 20% went to A & 10% went to B. Form a TM.

Sol

TM

$$\begin{array}{c|ccc} & A & B & C \\ \hline A & 0.8 & 0.2 & 0 \\ B & 0.3 & 0.4 & 0.3 \\ C & 0.2 & 0.1 & 0.7 \end{array}$$

TMD



Markov chain

Steady State Vector :-

(OR) fixed vector (OR) Equilibrium vector of Markov chain (OR) long range trend of the markov chain.

→ If a markov chain with Transition matrix P is regular, then there is a unique vector v such that for any probability vector 'v' i.e for large value of n

$$\boxed{vP^n = v}$$

Vector v is called steady state vector.

Q) Find the long range trend (OR) steady state vector for markov chain with TM

$$\begin{bmatrix} 0.55 & 0.28 & 0.07 \\ 0.15 & 0.67 & 0.18 \\ 0.12 & 0.36 & 0.52 \end{bmatrix}$$

Sol All the entries in the matrix are positive. So that given matrix is regular

$$P = \begin{bmatrix} 0.65 & 0.28 & 0.07 \\ 0.15 & 0.67 & 0.18 \\ 0.12 & 0.36 & 0.52 \end{bmatrix}$$

Let 'V' the apcolutii vector

$[v_1\ v_2\ v_3] \Rightarrow \boxed{vp = v}$

$$[v_1\ v_2\ v_3]\begin{bmatrix} 0.65 & 0.28 & 0.07 \\ 0.15 & 0.67 & 0.18 \\ 0.12 & 0.36 & 0.52 \end{bmatrix}$$

$$\left[(0.65v_1 + 0.15v_2 + 0.12v_3)\ (0.28v_1 + 0.67v_2 + 0.36v_3)\ (0.07v_1 + 0.18v_2 + 0.52v_3)\right] = [v_1\ v_2\ v_3]$$

equating corresponding entries

$0.65v_1 + 0.15v_2 + 0.12v_3 = v_1$

$= -0.35v_1 + 0.15v_2 + 0.12v_3 = 0$ — (1)

$= 0.28v_1 - 0.33v_2 + 0.36v_3 = 0$ — (2)

$= 0.07v_3 + 0.18v_2 \cancel{+} 0.48v_3 = 0$ — (3)

(3) is the sum of (2) & (1) multiply by -1

V is the probability vector

$\boxed{v_1 + v_2 + v_3 = 1}$ — (4)

$\boxed{Ax = 0}$

$$\begin{bmatrix} -0.35 & 0.15 & 0.12 \\ 0.28 & -0.33 & 0.36 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By using gauss jordan

$R_1 \leftrightarrow R_3$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0.28 & -0.33 & 0.36 \\ -0.35 & 0.15 & 0.13 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$R_2 \to R_2 - 0.28R_1$
$R_3 \to R_3 + 0.35R_1$.

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 104/363 \\ 532/1089 \\ 245/1089 \end{bmatrix}$$

$v_1 = 0.28$
$v_2 = 0.4$
$v_3 = 0.22$.

$[v_1 \; v_2 \; v_3] = [0.281 \quad 0.48 \quad 0.22]$