INDIAN INSTITUTE OF INFORMATION TECHNOLOGY, SRI CITY

End Sem-EXAMINATION-Monsoon 2025

Real Analysis, Numerical Analysis, and Calculus

CSE & ECE: UG2 **Date: 28-11-2025**
Duration: 180 Minutes (02:30 PM-05:30 PM) **Max. Marks: 45**

Instructions: **Roll No:** ______________

- Marks are indicated in [ ] after each question.
- Rough work should be done separately, not with the answers.
- Answers should be reasoned and derived clearly, not a single-word answer.
- Be short and precise. Provide the answer to exactly what is asked for.
- The exam is not open book, and students are not allowed to bring textbooks, photocopies, hand-written notes, or laptops.
- Follow the instructions mentioned in the questions. Answer all the questions.
- Return the question paper and answer sheet before leaving the exam hall.
- Calculator is allowed.

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1. (a) Let $x_1 = 8$ and $x_{n+1} = \frac{1}{2}x_n + 2$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone. Find the limit.

Solution: Recurrence: $x_{n+1} = \dfrac{1}{2}x_n + 2,\ x_1 = 8.$

Step 1. Find fixed point L. Assume $x_n \to L$. Then

$$L = \frac{1}{2}L + 2 \quad \Rightarrow \quad \frac{1}{2}L = 2 \quad \Rightarrow \quad L = 4.$$

[1 Mark]

Step 2. Monotonicity. Compute

$$x_{n+1} - x_n = -\frac{1}{2}(x_n - 4).$$

If $x_n > 4$ then $x_{n+1} - x_n < 0$. Since $x_1 = 8 > 4$ and $x_2 = \frac{1}{2} \cdot 8 + 2 = 6 > 4$, by induction $x_n > 4$ for all n and hence the sequence is strictly decreasing.

Step 3. Boundedness. Each $x_n > 4$, so sequence is bounded below by 4 and above by $x_1 = 8$.

Conclusion. Sequence is monotone decreasing and bounded, so it converges. Limit:

$$\boxed{\lim_{n\to\infty} x_n = 4.}$$

[1 Mark]

(b) Test the convergence of the series

$$\left(\frac{1}{3}\right)^2+\left(\frac{1\cdot 2}{3\cdot 5}\right)^2+\left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2+\left(\frac{1\cdot 2\cdot 3\cdot 4}{3\cdot 5\cdot 7\cdot 9}\right)^2+\cdots.$$

Solution: Series:

$$\left(\frac{1}{3}\right)^2+\left(\frac{1\cdot 2}{3\cdot 5}\right)^2+\left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2+\cdots$$

Let

$$t_n=\prod_{k=1}^{n}\frac{k}{2k+1}, \qquad a_n=t_n^2.$$

Step 1. Ratio for t_n:

$$\frac{t_{n+1}}{t_n}=\frac{n+1}{2n+3}\xrightarrow{n\to\infty}\frac{1}{2}.$$

[1 Mark]

Step 2. Ratio for a_n:

$$\frac{a_{n+1}}{a_n}=\left(\frac{t_{n+1}}{t_n}\right)^2\to\left(\frac{1}{2}\right)^2=\frac{1}{4}<1.$$

Conclusion. By ratio test (terms decay like a geometric sequence with ratio about $1/4$), the series $\sum a_n$ converges.

$$\boxed{\text{The series converges.}}$$

[1 Mark]

(c) Prove that $\mu=\dfrac{2+\Delta}{2\sqrt{1+\Delta}}$, where Δ denotes the forward difference operator and μ denotes the average operator.

Solution: Let E be forward shift: $(Ef)(n)=f(n+1)$. Then $\Delta=E-I$ so $1+\Delta=E$. Also $2+\Delta=I+E$. Hence

$$\frac{2+\Delta}{2\sqrt{1+\Delta}}=\frac{I+E}{2E^{1/2}}=\frac{E^{-1/2}+E^{1/2}}{2},$$

which equals the average operator μ (since $\mu f=\frac{1}{2}(f(n)+f(n+1))$). Thus

$$\boxed{\mu=\frac{2+\Delta}{2\sqrt{1+\Delta}}.}$$

[1 Mark]

2. (a) Let us find an approximate solution to the nonlinear equation $\sin x+x^2-1=0$ using regula–falsi method. Exact solution is approximately 0.637.

Solution:

Step 1. Define $f(x)=\sin x+x^2-1$. Note $f(0)=-1<0$, $f(1)=\sin 1>0$. So a root lies in $[0,1]$.

Step 2. Regula–falsi iteration:

$$x_{\text{new}}=b-\frac{f(b)(b-a)}{f(b)-f(a)}.$$

Start with $a_0 = 0,\ b_0 = 1$.

Iterations (numerical).

$$
\begin{aligned}
x_1 &\approx 0.5430441252, & f(x_1) &\approx -0.1883585,\\
x_2 &\approx 0.6266225471, & f(x_2) &\approx -0.0209319,\\
x_3 &\approx 0.6356849974, & f(x_3) &\approx -0.0021757,\\
x_4 &\approx 0.6366245537, & f(x_4) &\approx -0.0002246,\\
x_5 &\approx 0.6367215023, & f(x_5) &\approx -2.316 \times 10^{-5}.
\end{aligned}
\qquad [1Mark]
$$

[1 Mark]

Conclusion. Root ≈ 0.63672. Rounded to 3 decimals:

$$\boxed{x \approx 0.637.}$$

(b) Evaluate using Simpson's $\dfrac{1}{3}$rd rule $\displaystyle\int_0^1 \frac{dx}{x^3+x+1}$, choose step length 0.25.

Simpson's rule: $\displaystyle\int_0^1 \frac{dx}{x^3+x+1}$ with $h = 0.25$.

Solution: Step 1. Nodes: $x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1$.

Step 2. Function values:

$$
\begin{aligned}
f(0) &= 1,\\
f(0.25) &\approx 0.7901234568,\\
f(0.5) &\approx 0.6153846154,\\
f(0.75) &\approx 0.4604316547,\\
f(1) &= \tfrac{1}{3}.
\end{aligned}
$$

[1 Mark]

Step 3. Composite Simpson's rule:

$$I \approx \frac{h}{3}\big(f_0 + 4f_1 + 2f_2 + 4f_3 + f_4\big) \approx \frac{0.25}{3} \cdot 7.5661 \approx 0.6305269.$$

Conclusion.

$$\boxed{\int_0^1 \frac{dx}{x^3+x+1} \approx 0.63053.}$$

[1 Mark]

3. (a) Evaluate the integral

$$\iint_\Omega xy\,dx\,dy,$$

where Ω is the interior of the circle $x^2 + y^2 \le 1$ bounded by the straight line $2x + y = 1$.

Solution: Step 1. Find intersection points of line and circle. Solve

$$x^2 + (1-2x)^2 = 1 \ \Rightarrow\ 5x^2 - 4x = 0 \ \Rightarrow\ x = 0,\ x = \tfrac{4}{5}.$$

Corresponding points: $(0, 1)$ and $(\frac{4}{5}, -\frac{3}{5})$.

Step 2. Set up integral using vertical slices for the (smaller) segment:

$$I = \int_{x=0}^{4/5} \int_{y=1-2x}^{\sqrt{1-x^2}} xy\, dy\, dx.$$

[1 Mark]

Step 3. Inner integral:

$$\int_{1-2x}^{\sqrt{1-x^2}} xy\, dy = x \left[\frac{y^2}{2} \right]_{1-2x}^{\sqrt{1-x^2}} = \frac{x}{2}\big(1 - x^2 - (1-2x)^2\big) = \frac{x^2(4-5x)}{2}.$$

Step 4. Outer integral:

$$I = \frac{1}{2} \int_0^{4/5} (4x^2 - 5x^3)\, dx.$$

Compute:

$$\int_0^{4/5} 4x^2\, dx = \frac{256}{375}, \qquad \int_0^{4/5} 5x^3\, dx = \frac{256}{500},$$

so

$$\int_0^{4/5} (4x^2 - 5x^3)\, dx = \frac{64}{375},$$

hence

$$I = \frac{1}{2} \cdot \frac{64}{375} = \boxed{\frac{32}{375}}.$$

[1 Mark]

(b) Let $R = [0,1] \times [0,1]$ and $f : R \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0, & \text{when } x \notin \mathbb{Q}, \\ 0, & \text{when } x \in \mathbb{Q},\ y \notin \mathbb{Q}, \\ \frac{1}{q}, & \text{when } x \in \mathbb{Q},\ y = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

Calculate $\int_0^1 \int_0^1 f(x,y) dx dy$, where $\mathbb{Q}$ is the set of rational numbers.
Solution: The set where $f \neq 0$ is contained in $\{x \in \mathbb{Q}\} \times \{y \in \mathbb{Q}\}$, which is countable, measure zero in $\mathbb{R}^2$. Hence $f = 0$ almost everywhere and

$$\boxed{\iint_{[0,1]^2} f(x,y)\, dx\, dy = 0.}$$

[2 Mark]

(c) Find the volume of the region Ω in $\mathbb{R}^3$ bounded by the surfaces of the paraboloids

$$z = x^2 + y^2, \qquad \text{and} \qquad 2z = 12 - x^2 - y^2.$$

Volume between $z = x^2 + y^2$ and $2z = 12 - x^2 - y^2$.

Solution: Step 1. Intersection: solve

$$x^2 + y^2 = \frac{12 - x^2 - y^2}{2} \ \Rightarrow\ x^2 + y^2 = 4.$$

So region in xy-plane is disk of radius 2. [1 Mark]

Step 2. Volume:

$$V = \iint_{r \le 2} \left(\frac{12 - r^2}{2} - r^2 \right) dA = \int_0^{2\pi} \int_0^2 \frac{12 - 3r^2}{2}\, r \, dr \, d\theta.$$

Inner integral evaluates to 6, so

$$V = 6 \cdot 2\pi = \boxed{12\pi}.$$

[1 Mark]

4. (a) Compute the integral

$$\iint_\Omega e^{\frac{y-x}{y+x}} \, dx \, dy,$$

where

$$\Omega = \{(x, y) : x \ge 0,\ y \ge 0,\ x + y \le 1\}.$$

Solution: Change of variables:

$$u = y - x, \quad v = x + y.$$

Then $x = (v - u)/2,\ y = (v + u)/2$.

Step 2. Jacobian:

$$\det \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}.$$

Hence $dx\,dy = \frac{1}{2}\, du\, dv$.

Step 3. Integrand:

$$\frac{y - x}{y + x} = \frac{u}{v}, \qquad e^{(y-x)/(y+x)} = e^{u/v}.$$

Step 4. Region in (u, v)-plane:

$$x \ge 0 \Rightarrow v - u \ge 0 \Rightarrow u \le v, \quad y \ge 0 \Rightarrow v + u \ge 0 \Rightarrow u \ge -v,$$

and $0 \le v \le 1$. So $0 \le v \le 1,\ -v \le u \le v$. [1 Mark]

Step 5. Integral:

$$I = \int_{v=0}^1 \int_{u=-v}^v e^{u/v} \frac{1}{2} \, du \, dv.$$

Evaluate inner integral: put $t = u/v,\ du = v\,dt,\ t \in [-1, 1]$:

$$\int_{-v}^v e^{u/v} \, du = v \int_{-1}^1 e^t \, dt = v(e - e^{-1}).$$

Thus

$$I = \int_0^1 \frac{1}{2} v(e - e^{-1}) \, dv = \frac{e - e^{-1}}{2} \cdot \frac{1}{2} = \boxed{\frac{e - e^{-1}}{4}}.$$

[1 Mark]

(b) $\int_0^2 \int_{y/2}^1 e^{-x^2}\,dx\,dy$.
Solution: Region: $0 \le y \le 2,\ y/2 \le x \le 1$.
Swapping order: $0 \le x \le 1,\ 0 \le y \le 2x$.

$$J = \int_0^1 \int_0^{2x} e^{-x^2}\,dy\,dx = \int_0^1 2xe^{-x^2}\,dx.$$

[1 Mark]

Let $u = x^2$, $du = 2x\,dx$:

$$J = \int_0^1 e^{-u}\,du = 1 - e^{-1}.$$

$$\boxed{J = 1 - e^{-1}}.$$

[1 Mark]

5. (a) If $\mathbf{F}(x, y, z) = xyz\mathbf{i} - e^z \cos x\,\mathbf{j} + xy^2z^3\,\mathbf{k}$, then calculate $\nabla \cdot (\nabla \times \mathbf{F})$.
Solution. Given $\mathbf{F} = (P, Q, R) = (xyz,\ -e^z \cos x,\ xy^2z^3)$.
First compute $\nabla \times \mathbf{F}$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ xyz & -e^z \cos x & xy^2z^3 \end{vmatrix}.$$

Compute each component:

$$(\nabla \times \mathbf{F})_1 = \frac{\partial}{\partial y}(xy^2z^3) - \frac{\partial}{\partial z}(-e^z \cos x) = 2xyz^3 - (-e^z \cos x) = 2xyz^3 + e^z \cos x,$$

$$(\nabla \times \mathbf{F})_2 = \frac{\partial}{\partial z}(xyz) - \frac{\partial}{\partial x}(xy^2z^3) = xy - y^2z^3,$$

$$(\nabla \times \mathbf{F})_3 = \frac{\partial}{\partial x}(-e^z \cos x) - \frac{\partial}{\partial y}(xyz) = e^z \sin x - xz.$$

Thus

$$\nabla \times \mathbf{F} = \big(2xyz^3 + e^z \cos x,\ \ xy - y^2z^3,\ \ e^z \sin x - xz\big).$$

Now compute the divergence:

$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial}{\partial x}(2xyz^3 + e^z \cos x) + \frac{\partial}{\partial y}(xy - y^2z^3) + \frac{\partial}{\partial z}(e^z \sin x - xz).$$

[1 Mark]

Evaluate each derivative:

$$\frac{\partial}{\partial x}(2xyz^3 + e^z \cos x) = 2yz^3 - e^z \sin x,$$

$$\frac{\partial}{\partial y}(xy - y^2z^3) = x - 2yz^3,$$

$$\frac{\partial}{\partial z}(e^z \sin x - xz) = e^z \sin x - x.$$

Add them:

$$(2yz^3 - e^z \sin x) + (x - 2yz^3) + (e^z \sin x - x) = 0.$$

$$\boxed{\nabla \cdot (\nabla \times \mathbf{F}) = 0.}$$

[1 Mark]

(b) Calculate the work done by the force

$$\mathbf{F} = x\,\mathbf{i} - y\,\mathbf{j} + (x + y + z)\,\mathbf{k}$$

on a particle that moves along the parabola $y = 3x^2,\ z = 0$ from the origin to the point (2, 12, 0).

Solution: Step 1. Parameterize: $r(t) = (t, 3t^2, 0),\ 0 \le t \le 2$. Then $r'(t) = (1, 6t, 0)$.

Step 2. Evaluate F on curve:

$$F(r(t)) = (t,\ -3t^2,\ t + 3t^2).$$

Dot product:

$$F \cdot r' = t - 18t^3.$$

[1 Mark]

Step 3. Integrate:

$$W = \int_0^2 (t - 18t^3)\,dt = \left[\frac{t^2}{2} - \frac{18t^4}{4}\right]_0^2 = 2 - 72 = \boxed{-70}.$$

[1 Mark]

(c) $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + (xz + y)\,\mathbf{j} + (xy - z)\,\mathbf{k}$. Find an f such that $\nabla f = \mathbf{F}$.
Solution:

$$\nabla \times F = (0,\ -y,\ z - x),$$

which is not zero in general. Therefore F is not conservative and no scalar potential f exists:

[2 Mark]

6. Let $\mathbf{F}(x, y, z) = (xyz,\ y^2 + 1,\ z^3)$ and let S be the unit cube $S = [0, 1] \times [0, 1] \times [0, 1]$. Compute

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

using

(a) Stokes' theorem,

(b) the Divergence theorem.

Solution: (a) Using Stokes' theorem. For a closed surface S (boundary of a volume) the boundary of S is empty, so the circulation integral around ∂S is zero. Thus

$$\iint_S (\nabla \times F) \cdot n\, dS = \oint_{\partial S} F \cdot dr = 0.$$

[1 Mark]

Solution: (b) Using divergence theorem on $\nabla \times F$. Since $\nabla \cdot (\nabla \times F) \equiv 0$,

$$\iint_S (\nabla \times F) \cdot n\, dS = \iiint_{[0,1]^3} \nabla \cdot (\nabla \times F)\, dV = 0.$$

[1 Mark]

Both methods give:

$$\boxed{0.}$$

7. (a) Determine the values of k for which the improper integral

$$\int_1^\infty \left(\frac{kt}{1+t^2} - \frac{1}{2t} \right) dt$$

converges. [2 Marks]

Solution:

We have

$$\frac{kt}{1+t^2} - \frac{1}{2t} = \frac{(2k-1)t^2 - 1}{2t(1+t^2)}.$$

Case 1: $\boxed{k \neq \frac{1}{2}.}$

For large t the dominant term in the numerator is $(2k-1)t^2$. Compare the integrand with $1/t$. Compute the limit

$$\lim_{t\to\infty} \frac{\frac{(2k-1)t^2 - 1}{2t(1+t^2)}}{1/t} = \lim_{t\to\infty} \frac{(2k-1)t^2 - 1}{2(1+t^2)} = \frac{2k-1}{2} \neq 0 \quad \text{and finite as } k \neq \tfrac{1}{2}.$$

By the Limit Comparison Test with $g(t) = 1/t$ and using the fact that $\int_1^\infty \frac{1}{t}\, dt$ diverges, we conclude that

$$\int_1^\infty \left(\frac{kt}{1+t^2} - \frac{1}{2t} \right) dt$$

diverges for every $k \neq \frac{1}{2}$. [1 Mark]

Case 2: $\boxed{k = \frac{1}{2}}$

If $k = \frac{1}{2}$ the integrand becomes

$$\frac{\frac{1}{2}t}{1+t^2} - \frac{1}{2t} = -\frac{1}{2} \cdot \frac{1}{t(1+t^2)}.$$

Compare with $g(t) = 1/t^3$. Compute the limit

$$\lim_{t\to\infty} \frac{-\frac{1}{2t(1+t^2)}}{1/t^3} = \lim_{t\to\infty} \left(-\frac{1}{2} \cdot \frac{t^3}{t(1+t^2)} \right) = \lim_{t\to\infty} \left(-\frac{1}{2} \cdot \frac{t^2}{1+t^2} \right) = -\frac{1}{2},$$

which is finite and nonzero. Since $\int_1^\infty \frac{1}{t^3}\,dt$ converges, by the Limit Comparison Test the given integral converges when $k = \frac{1}{2}$. [1 Mark]

$$\boxed{\text{The improper integral } \int_1^\infty \left(\frac{kt}{1+t^2} - \frac{1}{2t}\right) dt \text{ converges if and only if } k = \frac{1}{2}.}$$

(b) Consider the integral

$$\int_{-2}^{1} \frac{dx}{\sqrt[3]{3x-2}}$$

(i) Identify whether this integral is **proper** or **improper**. Explain the reason.

(ii) Check whether the integral is convergent or divergent. [2 Marks]

Solution: This is a type II improper integral as $y = \dfrac{1}{\sqrt[3]{3x-2}}$ is unbounded at $x = \frac{2}{3}$. [1 Mark]

We get

$$\begin{aligned}
\int_{-2}^{1} \frac{dx}{\sqrt[3]{3x-2}} &= \int_{-2}^{2/3} \frac{dx}{\sqrt[3]{3x-2}} + \int_{2/3}^{1} \frac{dx}{\sqrt[3]{3x-2}} \\
&= \lim_{b\to 2/3^-} \int_{-2}^{b} \frac{dx}{\sqrt[3]{3x-2}} + \lim_{a\to 2/3^+} \int_{a}^{1} \frac{dx}{\sqrt[3]{3x-2}} \\
&= \lim_{b\to 2/3^-} \left[\frac{(3x-2)^{2/3}}{2}\right]_{-2}^{b} + \lim_{a\to 2/3^+} \left[\frac{(3x-2)^{2/3}}{2}\right]_{a}^{1} \\
&= \lim_{b\to 2/3^-} \left(\frac{(3b-2)^{2/3}}{2} - 2\right) + \lim_{a\to 2/3^+} \left(1 - \frac{(3a-2)^{2/3}}{2}\right) \\
&= (0-2) + (1-0) \\
&= \boxed{-1}.
\end{aligned}$$

[1 Mark]

8. (a) Show that $\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right) = \pi \sec(\pi x)$.

Solution:

$$\begin{aligned}
\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right) &= \Gamma\left(\frac{1+2x}{2}\right)\Gamma\left(\frac{1-2x}{2}\right) \\
&= \Gamma\left(1-\frac{1-2x}{2}\right)\Gamma\left(\frac{1-2x}{2}\right) \qquad \text{[1 Mark]} \\
&= \frac{\pi}{\sin\left\{\pi\left(\frac{1-2x}{2}\right)\right\}} \\
&= \frac{\pi}{\sin\left(\frac{\pi}{2} - \pi x\right)} \\
&= \frac{\pi}{\cos(\pi x)} \\
&= \pi \sec(\pi x).
\end{aligned}$$

[1 Mark]

(b) Evaluate

$$\int_0^\infty \frac{x^3(1+x^4)}{(1+x)^{12}}\,dt$$

using gamma and beta functions. [2 Marks]

Solution:

$$\begin{aligned}
\int_0^\infty \frac{x^3(1+x^4)}{(1+x)^{12}}\,dx &= \int_0^\infty \frac{x^3}{(1+x)^{12}}\,dx + \int_0^\infty \frac{x^7}{(1+x)^{12}}\,dx \\
&= \int_0^\infty \frac{x^{4-1}}{(1+x)^{4+8}}\,dx + \int_0^\infty \frac{x^{8-1}}{(1+x)^{8+4}}\,dx \\
&= B(4,8) + B(8,4) \qquad \text{[1 Mark]} \\
&= 2B(4,8) \qquad [\because B(m,n) = B(n,m)] \\
&= 2\left[\frac{\Gamma(4)\,\Gamma(8)}{\Gamma(4+8)}\right] \qquad \left[\because B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right] \\
&= 2\left[\frac{3!\,7!}{11!}\right] \\
&= \frac{1}{660}.
\end{aligned}$$

[1 Mark]

9. (a) State whether the following statement is **True** or **False**, and explain why.

“ There exists a function $f:\mathbb{R}^2 \to \mathbb{R}$ such that $\dfrac{\partial f}{\partial x} = y$ and $\dfrac{\partial f}{\partial y} = x^2$.”

Solution: $\boxed{\text{FALSE}}$

If there were such a function, then its mixed second partial derivatives would be

$$\frac{\partial^2 f}{\partial y\,\partial x} = 1 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x\,\partial y} = 2x.$$

These functions are continuous and unequal. But we know that if a function has continuous mixed second partial derivatives then it must be equal.

[1 Mark]

(b) Let $u = x^2 - 2y^2, v = 2x^2 - y^2$ with $x = r\cos\theta, y = r\sin\theta$. Show that $\dfrac{\partial(u,v)}{\partial(r,\theta)} = 6r^3\sin 2\theta$. [1 Mark]

Solution:

$$\begin{aligned}
&x = r\cos\theta, && y = r\sin\theta, \\
&u = x^2 - 2y^2 = r^2(\cos^2\theta - 2\sin^2\theta), && v = 2x^2 - y^2 = r^2(2\cos^2\theta - \sin^2\theta), \\
&\frac{\partial u}{\partial r} = 2r(\cos^2\theta - 2\sin^2\theta), && \frac{\partial u}{\partial \theta} = -6r^2\sin\theta\cos\theta, \\
&\frac{\partial v}{\partial r} = 2r(2\cos^2\theta - \sin^2\theta), && \frac{\partial v}{\partial \theta} = -2r^2\sin\theta\cos\theta,
\end{aligned}$$

[$\frac{1}{2}$ Mark]

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix}$$
$$= \begin{vmatrix} 2r(\cos^2\theta - 2\sin^2\theta) & -6r^2\sin\theta\cos\theta \\ 2r(2\cos^2\theta - \sin^2\theta) & -2r^2\sin\theta\cos\theta \end{vmatrix}$$
$$= 6r^3\sin 2\theta.$$

[$\frac{1}{2}$ Mark]

(c) Consider

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

(i) Is f continuous at $(0,0)$?

(ii) Compute $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$. Are these continuous at $(0,0)$?

(iii) Is f differentiable at $(0,0)$? [3 Marks]

Solution: (a) Continuity at $(0,0)$.

For $(x,y) \neq (0,0)$,

$$|f(x,y)| = \left|\frac{x^2y}{x^2+y^2}\right| = \frac{x^2}{x^2+y^2}\,|y| \leq |y| \leq \sqrt{x^2+y^2}.$$

Given $\varepsilon > 0$, choose $\delta = \varepsilon$. If $\sqrt{x^2+y^2} < \delta$, then

$$|f(x,y) - 0| = |f(x,y)| \leq \sqrt{x^2+y^2} < \delta = \varepsilon.$$

Hence

$$\boxed{f \text{ is continuous at } (0,0).}$$

[1 Mark]

(b) Compute $\partial f/\partial x$ and $\partial f/\partial y$ and check continuity.

For $(x,y) \neq (0,0)$,

$$f_x(x,y) = \frac{2xy(x^2+y^2) - 2x x^2 y}{(x^2+y^2)^2} = \frac{2xy^3}{(x^2+y^2)^2}.$$

At $(0,0)$,

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-0}{h} = 0.$$

Along $y = x$,

$$f_x(x,x) = \frac{2x\cdot x^3}{(2x^2)^2} = \frac{1}{2} \neq 0.$$

Thus

$$\boxed{f_x \text{ is not continuous at } (0,0).}$$

[$\frac{1}{2}$ Mark]

For $(x,y) \neq (0,0)$,

$$f_y(x,y) = \frac{x^2(x^2+y^2) - 2yx^2y}{(x^2+y^2)^2} = \frac{x^2(x^2-y^2)}{(x^2+y^2)^2}.$$

At $(0,0)$,

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - 0}{h} = 0.$$

Along $y = x^2$,

$$f_y(x, x^2) = \frac{1 - x^2}{(1+x^2)^2} \to 1 \neq 0.$$

Thus

$$\boxed{f_y \text{ is not continuous at } (0,0).}$$

[$\frac{1}{2}$ Mark]

(c) Differentiability at $(0,0)$

f is differentiable at $(0,0)$ iff

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - 0 - 0 - 0}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{h^2k}{(h^2+k^2)^{3/2}} = 0.$$

Take the path $k = h$:

$$\frac{h^2h}{(h^2+h^2)^{3/2}} = \frac{h^3}{(2h^2)^{3/2}} = \frac{1}{2\sqrt{2}} \neq 0.$$

Therefore the required limit does not exist, so

$$\boxed{f \text{ is not differentiable at } (0,0).}$$

[1 Mark]

10. (a) Find the critical points of $f(x,y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$. [1 Mark]

Solution Given

$$f(x,y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}.$$

We have

$$\frac{\partial f}{\partial x} = 2x + y - \frac{1}{x^2}, \qquad \frac{\partial f}{\partial y} = x + 2y - \frac{1}{y^2}.$$

To find critical points:

$$2x + y - \frac{1}{x^2} = 0 \Rightarrow 2x^3 + x^2y - 1 = 0, \tag{1}$$

$$x + 2y - \frac{1}{y^2} = 0, \Rightarrow xy^2 + 2y^3 - 1 = 0. \tag{2}$$

[$\frac{1}{2}$ Mark]

$$\begin{aligned}
(1) - (2) \;&\Rightarrow\; (2x^3 + x^2y - 1) - (xy^2 + 2y^3 - 1) = 0, \\
&\Rightarrow\; 2(x^3 - y^3) + xy(x - y) = 0, \\
&\Rightarrow\; (x-y)\big(2(x^2 + xy + y^2) + xy\big) = 0, \\
&\Rightarrow\; (x-y)\big(2x^2 + 3xy + 2y^2\big) = 0.
\end{aligned}$$

Hence

$$x - y = 0 \quad \Rightarrow \quad x = y.$$

Substitute $x = y$ into (1) (or (2)):

$$2x + x - \frac{1}{x^2} = 0 \quad \Rightarrow \quad 3x - \frac{1}{x^2} = 0,$$

so

$$3x = \frac{1}{x^2} \quad \Longrightarrow \quad x^3 = \frac{1}{3}.$$

Therefore the critical point is $\big((1/3)^{1/3},\ (1/3)^{1/3}\big)$. [$\frac{1}{2}$ Mark]

(b) Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube. [2 Marks]

Solution: Let 2x, 2y, 2z be the length, breadth and height of the rectangular solid. Then its volume is V = 8xyz.

Let the sphere have a radius of r so that $x^2 + y^2 + z^2 = r^2$.

Taking $F(x, y, z) = V + \lambda(x^2 + y^2 + z^2 - r^2) = 8xyz + \lambda(x^2 + y^2 + z^2 - r^2)$, where λ is the multiplier.

$$\frac{\partial F}{\partial x} = 8yz + 2\lambda x = 0 \qquad (1)$$

$$\frac{\partial F}{\partial y} = 8xz + 2\lambda y = 0 \qquad (2)$$

$$\frac{\partial F}{\partial z} = 8xy + 2\lambda z = 0 \qquad (3)$$

[1 Mark]

Multiplying (1), (2), (3) by x, y, z respectively, we get:

$$8xyz + 2\lambda x^2 = 0 \qquad (4)$$

$$8xyz + 2\lambda y^2 = 0 \qquad (5)$$

$$8xyz + 2\lambda z^2 = 0 \qquad (6)$$

From (4), (5), (6) we have $x^2 = y^2 = z^2$. Since $x, y, z > 0$, we conclude $x = y = z$.

Thus the rectangular solid is a cube. [1 Mark]