CS 70	Discrete Mathematics and Probability Theory	
Fall 2025	Sabin, Hug	HW 02

1 Induction Starter

Note 3

Consider the inequality $2^n < n!$ (the right hand side is a factorial, not an exclamation mark).

(a) Make a conjecture as to which $n \in \mathbb{N}$ will have the inequality hold.

We will now prove your conjecture using induction.

(b) What is your base case?

(c) What is the inductive hypothesis for this proof?

(d) What do we want to show in the inductive step?

(e) Conclude the proof with the inductive step.

Solution:

(a) Intuitively, we can see that for larger values of n, as n increases, the RHS will be multiplied by n (which continues to grow larger), but the LHS will be multiplied by 2 (which is a constant value). Therefore, we expect that the inequality will hold for sufficiently large n. We can test out the first few values of n to see:

- For $n = 1$: $2^1 = 2$ and $1! = 1$, so $2^1 \not< 1!$.
- For $n = 2$: $2^2 = 4$ and $2! = 2$, so $2^2 \not< 2!$.
- For $n = 3$: $2^3 = 8$ and $3! = 6$, so $2^3 \not< 3!$.
- For $n = 4$: $2^4 = 16$ and $4! = 24$, so $2^4 < 4!$.

Thus, our conjecture is that the inequality $2^n < n!$ holds for all $n \geq 4$.

(b) The base case is $n = 4$. Copying from above, for $n = 4$: $2^4 = 16$ and $4! = 24$, so $2^4 < 4!$.

(c) The inductive hypothesis is that the inequality holds for some arbitrary integer $k \geq 4$, i.e., we assume that $2^k < k!$.

(d) In the inductive step, we want to show that if the inequality holds for k, then it also holds for $k+1$. Specifically, we want to prove that $2^{k+1} < (k+1)!$.

(e) We have:

$$2^{k+1} = 2 \cdot 2^k < 2 \cdot k! \quad \text{(by the inductive hypothesis)}$$

Now we need to show that $2 \cdot k! < (k+1)!$. We can rewrite $(k+1)!$ as:

$$(k+1)! = (k+1) \cdot k!$$

Thus, it suffices to show that $2 < k+1$, which is true for all $k \geq 4$.

Therefore, by induction, we have shown that $2^n < n!$ for all $n \geq 4$.

2 Airport

Note 3 Suppose that there are $2n+1$ airports, where n is a positive integer. The distances between any two airports are all different. For each airport, exactly one airplane departs from it and is destined for the closest airport. Prove by induction that there is an airport which has no airplanes destined for it.

Solution: We proceed by induction on n. For $n = 1$, let the 3 airports be A, B, C and without loss of generality suppose B, C is the closest pair of airports (which is well defined since all distances are different). Then the airplanes departing from B and C are flying towards each other. Since the airplane from A must fly to somewhere else, no airplanes are destined for airport A.

Now suppose the statement holds for $n = k$, i.e. when there are $2k+1$ airports. For $n = k+1$, i.e. when there are $2k+3$ airports, the airplanes departing from the closest two airports (say X and Y) must be destined for each other's starting airports. Removing these two airports reduce the problem to $2k+1$ airports.

From the inductive hypothesis, we know that among the $2k+1$ airports remaining, there is an airport with no incoming flights which we call airport Z. When we add back the two airports that we removed, there are two scenarios:

- Some of the flights get remapped to X or Y.
- None of the flights get remapped.

In either scenario, we conclude that the airport Z will continue to have no incoming flights when we add back the two airports, and so the statement holds for $n = k+1$. By induction, the claim holds for all $n \geq 1$.

3 Proving Inequality

Note 3 For all positive integers $n \geq 1$, prove with induction that

$$\frac{1}{3^1} + \frac{1}{3^2} + \ldots + \frac{1}{3^n} < \frac{1}{2}.$$

(Note: while you can use formula for an infinite geometric series to prove this, we require you to use induction. If direct induction seems difficult, consider strengthening the inductive hypothesis. Can you prove an equality statement instead of an inequality?)

Solution: Try a few cases and come up with a stronger inductive hypothesis. For example:

- $\frac{1}{3} = \frac{1}{2} - \frac{1}{6}$

- $\frac{1}{3} + \frac{1}{9} = \frac{1}{2} - \frac{1}{18}$
- $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} = \frac{1}{2} - \frac{1}{54}$

One possible statement is

$$\frac{1}{3^1} + \frac{1}{3^2} + \ldots + \frac{1}{3^n} = \frac{1}{2} - \frac{1}{2 \cdot 3^n}$$

- *Base Case:* $n = 1$. $\frac{1}{3} = \frac{1}{2} - \frac{1}{6}$. True.
- *Inductive Hypothesis:* Assume the statement holds for $n \geq 1$.
- *Inductive Step:* Starting from the left hand side,

$$\begin{aligned} \frac{1}{3^1} + \frac{1}{3^2} + \ldots + \frac{1}{3^n} + \frac{1}{3^{n+1}} &= \frac{1}{2} - \frac{1}{2 \cdot 3^n} + \frac{1}{3^{n+1}} \\ &= \frac{1}{2} - \frac{3-2}{2 \cdot 3^{n+1}} \\ &= \frac{1}{2} - \frac{1}{2 \cdot 3^{n+1}}. \end{aligned}$$

Therefore, $\frac{1}{3^1} + \frac{1}{3^2} + \ldots + \frac{1}{3^n} = \frac{1}{2} - \frac{1}{2 \cdot 3^n} < \frac{1}{2}$.

Alternate Solution: Normal Induction without strengthening is viable for this problem.

Base Case: Suppose $n = 1$. We see that $\frac{1}{3^1} = \frac{1}{3} < \frac{1}{2}$.

Inductive Hypothesis: Suppose the statement is true for some arbitrary $n = k$.

Inductive Step: Utilizing the hypothesis we get

$$\begin{aligned} \frac{1}{3^1} + \frac{1}{3^2} + \ldots + \frac{1}{3^k} + \frac{1}{3^{k+1}} &= \frac{1}{3} + \frac{1}{3}\left(\frac{1}{3^1} + \ldots + \frac{1}{3^k}\right) \\ &< \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

which completes the induction.

4 Universal Preference

Note 4

Suppose that preferences in a stable matching instance are universal: all n jobs share the preferences $C_1 > C_2 > \cdots > C_n$ and all candidates share the preferences $J_1 > J_2 > \cdots > J_n$.

(a) What pairing do we get from running the algorithm with jobs proposing (Hint: Start with small examples and go through the algorithm. Do you see a pattern?)? Prove that this happens for all n.

(b) What pairing do we get from running the algorithm with candidates proposing? Explain.

(c) What does this tell us about the number of stable pairings? Justify your answer.

Solution:

(a) The pairing results in (C_i, J_i) for each $i \in \{1, 2, ..., n\}$. This result can be proved by induction:

Our base case is when $n = 1$, so the only pairing is (C_1, J_1), and thus the base case is trivially true.

Now assume this is true for some $n \in \mathbb{N}$. On the first day with $n+1$ jobs and $n+1$ candidates, all $n+1$ jobs will propose to C_1. C_1 prefers J_1 the most, and the rest of the jobs will be rejected. This leaves a set of n unpaired jobs and n unpaired candidates who all have the same preferences (after the pairing of (C_1, J_1)). By the process of induction, this means that every i^{th} preferred candidate will be paired with the i^{th} preferred job.

(b) The pairings will again result in (J_i, C_i) for each $i \in \{1, 2, ..., n\}$. This can be proved by induction in the same as above, but replacing "job" with "candidate" and vice-versa.

(c) We know that job-proposing produces a candidate-pessimal stable pairing. We also know that candidate-proposing produces a candidate-optimal stable pairing. We found that candidate-optimal and candidate-pessimal pairings are the same. This means that there is only one stable pairing, since both the best and worst pairings (for candidates) are the same pairings.

5 Pairing Up

Note 4

Prove that for every even $n \geq 2$, there exists an instance of the stable matching problem with n jobs and n candidates such that the instance has at least $2^{n/2}$ distinct stable matchings.

(*Hint*: It can help to start with some small examples; find an instance for $n = 2$, and think about how you can use these preference lists to construct an instance for $n = 4$. After this, you should be in a good position to generalize the construction for all even n. Additionally, $2^{n/2}$ is a very specific number; try to think about how your construction would build such a number as it is constructed with increasing n.)

Solution:

To prove that there exists such a stable matching instance for any even $n \geq 2$, it suffices to construct such an instance. But first, we look at the $n = 2$ case to generate some intuition. We can recognize that for the following preferences:

J_1	$C_1 > C_2$
J_2	$C_2 > C_1$

C_1	$J_2 > J_1$
C_2	$J_1 > J_2$

both $S = \{(J_1, C_1), (J_2, C_2)\}$ and $T = \{(C_1, J_2), (C_2, J_1)\}$ are stable pairings.

The $n/2$ in the exponent motivates us to consider pairing the n jobs into $n/2$ groups of 2 and likewise for the candidates. We pair up job $2k-1$ and $2k$ into a pair and candidate $2k-1$ and $2k$ into a pair, for $1 \le k \le n/2$.

From here, we recognize that for each pair (J_{2k-1}, J_{2k}) and (C_{2k-1}, C_{2k}), mirroring the preferences above would yield 2 stable matchings from the perspective of just these pairs. If we can extend this perspective to all $n/2$ pairs, this would be a total of $2^{n/2}$ stable matchings.

Our construction thus results in preference lists like follows:

J_1	$C_1 > C_2 > \dots$
J_2	$C_2 > C_1 > \dots$
$\vdots$	$\vdots$
J_{2k-1}	$C_{2k-1} > C_{2k} > \dots$
J_{2k}	$C_{2k} > C_{2k-1} > \dots$
$\vdots$	$\vdots$
J_{n-1}	$C_{n-1} > C_n > \dots$
J_n	$C_n > C_{n-1} > \dots$

C_1	$J_2 > J_1 > \dots$
C_2	$J_1 > J_2 > \dots$
$\vdots$	$\vdots$
C_{2k-1}	$J_{2k} > J_{2k-1} > \dots$
C_{2k}	$J_{2k-1} > J_{2k} > \dots$
$\vdots$	$\vdots$
C_{n-1}	$J_n > J_{n-1} > \dots$
C_n	$J_{n-1} > J_n > \dots$

Each match will have jobs in the kth pair paired to candidates in the kth pair for $1 \le k \le n/2$.

A job j in pair k will never form a rogue couple with any candidate c in pair $m \neq k$ since it always prefers the candidates in this pair over all candidates across other pairs. Since each job in pair k can be stably matched to either candidate in pair k, and there are $n/2$ total pairs, the number of stable matchings is $2^{n/2}$.

6 Optimal Candidates

Note 4

In the notes, we proved that the propose-and-reject algorithm always outputs the job-optimal pairing. However, we never explicitly showed why it is guaranteed that putting every job with its optimal candidate results in a pairing at all. Prove by contradiction that no two jobs can have the same optimal candidate. (Note: your proof should not rely on the fact that the propose-and-reject algorithm outputs the job-optimal pairing.)

Solution:

For the sake of contradiction, assume that we have some instance of the Stable Matching problem where both job J and job J' have candidate C as their optimal candidate. We further assume without loss of generality that C prefers J to J' (if this is not the case, we can just switch the names to make it so). This leads to preferences as follows:

J	$C > \dots$
J'	$C > \dots$

C	$J > J'$
C^*	$\dots$

Because C is J''s optimal partner, we know by definition that there must exist some stable pairing P in which J' is paired with C - i.e. $P = \{(J', C), (J, C^*), \ldots\}$.

Since C is J's optimal partner, we know by definition that J must prefer C to any candidate it is ever paired with in any stable pairing–including C^*. Moreover, we previously said that C prefers J to J'. Thus, J and C would form a rogue couple in P, which is a contradiction because P is stable. So our initial assumption must be false: there must never exist two jobs who have the same optimal candidate.