CS 70 Discrete Mathematics and Probability Theory
Fall 2025 Sabin, Hug

HW 11

Due: Saturday, 11/15, 4:00 PM
Grace period until Saturday, 11/15, 6:00 PM
Remember to show your work for all problems!

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.) If you used an LLM, place transcripts of your chats here.

1 Expectation and Variance Warm-Up

Note 19
Note 20

Let X be a random variable with a mean of 1. Show that $\mathbb{E}[5+9X+9X^2] \geq 23$.

2 More Socks

Note 19
Note 20

Gavin has n different pairs of socks (n left socks and n right socks, for $2n$ individual socks total) and is doing his laundry. He notices that the laundry machine spits out a uniformly random permutation of the $2n$ socks.

Let X be the number of matching pairs that are placed next to each other.

As an example, for the string 132231, $X = 1$ since only the 2nd pair of socks are placed together.

(a) What is the probability that the 1st pair of socks are placed together? We will denote this probability as p.

(b) What is $\mathbb{E}(X)$?

(c) What is the probability that both the 1st pair are placed together and the second pair are placed together? We will denote this probability as q.

(d) What is $\mathrm{Var}(X)$? Feel free to leave your answer in terms of p and q.

3 Coupon Collector Variance

Note 20

It's that time of the year again—Safeway is offering its Monopoly Card promotion. Each time you visit Safeway, you are given one of n different Monopoly Cards with equal probability. You need to collect them all to redeem the grand prize.

Let X be the number of visits you have to make before you can redeem the grand prize. Show that $\text{Var}(X) = n^2\left(\sum_{i=1}^{n} i^{-2}\right) - \mathbb{E}[X]$.

4 Unbiased Variance Estimation

Note 20

We have a random variable X and want to estimate its variance, σ^2 and mean, μ, by sampling from it. In this problem, we will derive an "unbiased estimator" for the variance.

(a) We define a random variable Y that corresponds to drawing n values from the distribution for X and averaging, or $Y = (X_1 + \ldots + X_n)/n$. What is $\mathbb{E}(Y)$? Note that if $\mathbb{E}(Y) = \mathbb{E}(X)$ then Y is an unbiased estimator of $\mu = \mathbb{E}(X)$.

Hint: There should not be much computation needed.

(b) Now let's assume the actual mean is 0 as variance doesn't change when one shifts the mean.

Before attempting to define an estimator for variance, show that $\mathbb{E}(Y^2) = \sigma^2/n$.

(c) In practice, we don't know the mean of X so following part (a), we estimate it as Y. With this in mind, we consider the random variable $Z = \sum_{i=1}^{n}(X_i - Y)^2$. What is $\mathbb{E}(Z)$?

(d) What is a good unbiased estimator for the $\text{Var}(X)$?

(e) How does this differ from what you might expect? Why? (Just tell us your intuition here, it is all good!)