Discrete Probability Intro

Note 13 **Probability Space**: A probability space is a tuple $(\Omega, \mathbb{P})$, where Ω is the *sample space* and $\mathbb{P}$ is the *probability function* on the sample space.

Specifically, Ω is the set of all outcomes ω, and $\mathbb{P}$ is a function $\mathbb{P}\colon \Omega \to [0,1]$, assigning a probability to each outcome, satisfying the following conditions:

$$0 \le \mathbb{P}[\omega] \le 1 \quad \text{and} \quad \sum_{\omega \in \Omega} \mathbb{P}[\omega] = 1.$$

Event: an event A is a subset of Ω, i.e. a collection of some outcomes in the sample space. We define

$$\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega].$$

Uniform Probability Space: all outcomes are assigned the same probability, i.e. $\mathbb{P}[\omega] = \frac{1}{|\Omega|}$; this is just counting!

With an event A in a uniform probability space, $\mathbb{P}[A] = \frac{|A|}{|\Omega|}$, which is again more counting!

1 Flippin' Coins

Note 13 Suppose we have an unbiased coin, with outcomes H and T, with probability of heads $\mathbb{P}[H] = 1/2$ and probability of tails also $\mathbb{P}[T] = 1/2$. Suppose we perform an experiment in which we toss the coin 3 times. An outcome of this experiment is (X_1, X_2, X_3), where $X_i \in \{H, T\}$.

(a) What is the *sample space* for our experiment?

(b) Which of the following are examples of *events*? Select all that apply.

- $\{(H,H,T),(H,H),(T)\}$
- $\{(T,H,H),(H,T,H),(H,H,T),(H,H,H)\}$
- $\{(T,T,T)\}$
- $\{(T,T,T),(H,H,H)\}$
- $\{(T,H,T),(H,H,T)\}$

(c) What is the complement of the event $\{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,T,T)\}$?

(d) Let A be the event that our outcome has 0 heads. Let B be the event that our outcome has exactly 2 heads. What is $A \cup B$?

(e) What is the probability of the outcome (H,H,T)?

(f) What is the probability of the event that our outcome has exactly two heads?

(g) What is the probability of the event that our outcome has at least one head?

Solution:

(a) $\Omega = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$

(b) An event must be a subset of Ω, meaning that it must consist of possible outcomes.

- No
- Yes
- Yes
- Yes
- Yes

(c) $\{(T,H,H),(T,H,T),(T,T,H)\}$

(d) $\{(T,H,H),(H,H,T),(H,T,H),(T,T,T)\}$

(e) Since $|\Omega| = 2^3 = 8$ and every outcome has equal probability, $\mathbb{P}[(H,H,T)] = 1/8$.

(f) The event of interest is $E = \{(H,H,T),(H,T,H),(T,H,H)\}$, which has size 3. Whence $\mathbb{P}[E] = 3/8$.

(g) If we do not see at least one head, then we must see at exactly three tails. The event $\overline{E} = \{(T,T,T)\}$ of seeing exactly three tails is thus the complement of the event E that we see at least one head. $\overline{E}$ occurs with probability $(1/2)^3 = 1/8$, so its complement E must occur with probability $1 - 1/8 = 7/8$.

2 Sampling

Note 13

Suppose you have balls numbered $1,\ldots,n$, where n is a positive integer ≥ 2, inside a coffee mug. You pick a ball uniformly at random, look at the number on the ball, replace the ball back into the coffee mug, and pick another ball uniformly at random.

(a) What is the probability that the first ball is 1 and the second ball is 2?

(b) What is the probability that the second ball's number is strictly less than the first ball's number?

(c) What is the probability that the second ball's number is exactly one greater than the first ball's number?

(d) Now, assume that after you looked at the first ball, you did *not* replace the ball in the coffee mug (instead, you threw the ball away), and then you drew a second ball as before. Now, what are the answers to the previous parts?

Solution:

(a) Out of n^2 pairs of balls that you could have chosen, only one pair $(1,2)$ corresponds to the event we are interested in, so the probability is $1/n^2$.

(b) Again, there are n^2 total outcomes. Now, we want to count the number of outcomes where the second ball's number is strictly less than the first ball's number. Similarly to the last part, we can view any outcome as an ordered pair (n_1, n_2), where n_1 is the number on the first ball, and n_2 is the number on the second ball. There are $\binom{n}{2}$ outcomes where $n_1 > n_2$; select two distinct numbers from $[1,n]$, and assign the higher number to n_1. Thus, the probability is $\frac{\binom{n}{2}}{n^2} = \frac{n-1}{2n}$.

Alternate Solution: The probability that the two balls have the same number is $n/n^2 = 1/n$, so the probability that the balls have different numbers is $1 - 1/n = (n-1)/n$. By symmetry, it is equally likely for the first ball to have a greater number and for the second ball to have a greater number, so we take the probability $(n-1)/n$ and divide it by two to obtain $(n-1)/(2n)$.

(c) Again, there are n^2 pairs of balls that we could have drawn, but there are $n-1$ pairs of balls which correspond to the event we are interested in: $\{(1,2),(2,3),\dots,(n-1,n)\}$. So, the probability is $(n-1)/n^2$.

(d) There are a total of $n(n-1)$ pairs of balls that we could have drawn, and only the pair $(1,2)$ corresponds to the event that we are interested in, so the probability is $1/(n(n-1))$.

The probability that the two balls are the same is now 0, but the symmetry described earlier still applies, so the probability that the second ball has a smaller number is $1/2$.

There are a total of $n(n-1)$ pairs of balls that we could have drawn, and we are interested in the $n-1$ pairs $(1,2),(2,3),\dots,(n-1,n)$ as before. Thus, the probability that the second ball is one greater than the first ball is $1/n$.

3 Monty Hall Variant

Recall the Monty Hall problem introduced in lecture. You are on a gameshow with 3 doors, behind which there are 2 goats and 1 new car. You choose a door. Then, the host reveals one door, which is a goat. Now, they ask if you if you would like to switch the door you have chosen to the other remaining one. Do you switch? Turns out, switching is the best choice from a probabilistic standpoint! Let's explore why this is and look at some modifications.

Note 14

(a) After many years, the standard version of the game with three doors becomes a little boring, so Monty decides to increase the number of doors to four (with one prize and three goats). What is now the probability that the contestant wins under the switching strategy?

(b) Monty Hall gets bored with the normal game with three doors, so he decides to increase the number of doors to five (one prize door and four with goats behind them). This time after the contestant selects a door, at least three of the remaining doors must have goats behind them. Monte picks at random two of the remaining doors with goats behind them and opens them. He then gives the contestant the option of switching to one of the two unopened doors.

What is the chance the contestant wins under the switching strategy? What is the chance the contestant wins under the sticking strategy? Should the contestant switch?

Solution:

(a) A contestant under the switching strategy wins only when (1) she picks the wrong door in the first step and (2) switches to the money door after Monty opened some goat door. There are many ways to compute this probability; we will use conditional probability here. Let event A be "she picks the wrong door in the first step" and event B be "she switches to the money door." We want to compute $Pr[B]$.

We note that $Pr[B] = Pr[B \cap A] + Pr[B \cap \neg A] = Pr[B|A] \cdot Pr[A] + Pr[B|\neg A] \cdot Pr[\neg A]$. Also, we know that the probability $Pr[B|\neg A]$ that she switches to the money door given that she picked the money door in the first place is 0. Therefore, we need only to compute the first term.

Note that $Pr[A] = 3/4$. Given that the contestant picks the wrong door, we can compute the probability $Pr[B|A]$ that she switches to the money door, which is $1/2$, because there are 3 door left, and one of them is the one she already picked. Therefore the probability that she wins is $Pr[B|A] \cdot Pr[A] = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$.

(b) First of all, we already know that if you stick, your probability of winning is $1/5$.

If you switch, then there is a $4/5$ chance that the prize is behind one of the other 4 doors. Monty opens 2 of them, so we must choose between the other 2 doors. Thus if the prize is behind one of the original 4 doors, then there is a $1/2$ chance you will get it. Thus your probability of winning is $4/5 \cdot 1/2 = 2/5$. So you should switch.

4 Intransitive Dice

Note 13 You're playing a game with your friend Bob, who has a set of three dice. You'll each choose a different die, roll it, and whoever had the higher result wins. The dice have sides as follows:

- Die A has sides 2, 2, 4, 4, 9, and 9.
- Die B has sides 1, 1, 6, 6, 8, and 8.
- Die C has sides 3, 3, 5, 5, 7, and 7.

(a) Suppose you have chosen die A and Bob has chosen die B. What is the probability that you win?
Hint: It may be easier to work with a sample space smaller than 6×6.

(b) Suppose you have chosen die B and Bob has chosen die C. What is the probability that you win?

(c) Suppose you have chosen die C and Bob has chosen die A. What is the probability that you win?

(d) Bob offers to let you choose your die first so that you can choose the best one. Is this an offer your should accept? Why or why not?

Solution: Each die has 6 sides, giving us $6 \times 6 = 36$ sample points. However, since each die has three sides repeated twice, we can treat the dice as 3 sided dice with each of their unique sides once. With this simplication, each die has 3 possible outcomes, so for each pair of dice, there are $3 \times 3 = 9$ possible outcomes. Each outcome occurs $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$ of the time.

(a) The event that Die A beats die B consists of the following sample points: (2,1), (4,1), (9,1), (9,6), (9,8). This gives us a total of 5 outcomes. The probability that you win is the sum of the probabilities of the outcomes, which is $5 \times \frac{1}{9} = \frac{5}{9}$.

(b) The event that Die B beats die C consists of the following sample points: (6,3), (6,5), (8,3), (8,5), (8,7). This gives us a total of 5 outcomes. The probability that you win is the sum of the probabilities of the outcomes, which is $5 \times \frac{1}{9} = \frac{5}{9}$.

(c) The event that Die C beats die A consists of the following sample points: (3,2), (5,2), (7,2), (5,4), (7,4). This gives us a total of 5 outcomes. The probability that you win is the sum of the probabilities of the outcomes, which is $5 \times \frac{1}{9} = \frac{5}{9}$.

(d) You should not accept! Bob is trying to trick you into choosing first. Notice that for each die, there is a different die that has an advantage against it. This means that if you choose first, Bob can always choose the die that beats your die. Therefore, you should not accept Bob's offer. This is due to the *intransitive* nature of the dice, just like rock-paper-scissors.