

## Graph Theory II

Note 5

**Planar Graph:** A graph which can be drawn on a plane with no crossings. Planar graphs have faces, which are regions of the plane where any two points can be connected by a path without crossing the drawing of an edge. Note that it is fine if there is another drawing of a planar graph with crossings; as long as there exists a drawing of a graph without crossing, then the graph is planar.

A planar graph  $G$  with  $v$  vertices and  $e$  edges with a planar drawing with  $f$  faces satisfy the following:

- Euler's formula:  $v + f = e + 2$
- $\sum_{i=1}^f s_i = 2e$  where  $s_i$  is the number of edges (sides) bordering face  $i$ . (This is somewhat like the degree-sum formula in that each edge is a side to two faces.)
- If planar, then  $e \leq 3v - 6$
- If bipartite planar, then  $e \leq 2v - 4$
- Graphs are non-planar iff they contain  $K_5$  or  $K_{3,3}$  (the complete graph on 5 vertices or the complete bipartite graph on 3 vertices in each set) as a subgraph
- All planar graphs can be vertex colored in at most 4 colors

**Complete graph:** The complete graph on  $n$  vertices, denoted by  $K_n$ , contains an edge between every pair of vertices.

**Bipartite graph:** A graph  $G$  with two sets of vertices such that each edge is incident to one vertex from each set.

**Tree:** A graph is a tree iff it satisfies any of the following:

- connected and acyclic
- connected and has  $|V| - 1$  edges
- connected, and removing any edge disconnects the graph
- acyclic, and adding any edge creates a cycle

**Hypercube:** The hypercube of dimension  $n$  has  $2^n$  vertices, each labeled by a length  $n$  bitstring. Edges connect vertices that differ by exactly one bit. A hypercube of dimension  $n + 1$  can be recursively constructed by creating two copies of an  $n$ -dimensional hypercube and connecting corresponding vertices by an edge.

# 1 Always, Sometimes, or Never

Note 5

In each part below, you are given some information about a graph  $G$ . Using only the information in the current part, say whether  $G$  will always be planar, always be non-planar, or could be either. If you think it is always planar or always non-planar, prove it. If you think it could be either, give a planar example and a non-planar example.

- (a)  $G$  can be vertex-colored with 4 colors.
- (b)  $G$  requires 7 colors to be vertex-colored.
- (c)  $e \leq 3v - 6$ , where  $e$  is the number of edges of  $G$  and  $v$  is the number of vertices of  $G$ .
- (d)  $G$  is connected, and each vertex in  $G$  has degree at most 2.
- (e) Each vertex in  $G$  has degree at most 2.

## 2 Short Answers

Note 5

In each part below, provide the number/equation and brief justification.

(a) A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have?

(b) How many edges need to be removed from  $K_6$  to get a tree?

(c) The Euler's formula  $v - e + f = 2$  requires the planar graph to be connected. What is the analogous formula for planar graphs with  $k$  connected components?

## 3 Graph Coloring

Note 5

Prove that a graph with maximum degree at most  $k$  is  $(k+1)$ -colorable.

## 4 Hypercubes

Note 5

The vertex set of the  $n$ -dimensional hypercube  $G = (V, E)$  is given by  $V = \{0, 1\}^n$  (recall that  $\{0, 1\}^n$  denotes the set of all  $n$ -bit strings). There is an edge between two vertices  $x$  and  $y$  if and only if  $x$  and  $y$  differ in exactly one bit position.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Show that the edges of an  $n$ -dimensional hypercube can be colored using  $n$  colors so that no pair of edges sharing a common vertex have the same color.

(c) Show that for any  $n \geq 1$ , the  $n$ -dimensional hypercube is bipartite.