CS 70 Fall 2025 — Discrete Mathematics and Probability Theory — Sabin, Hug — HW 13

1 Short Answer

Note 23

(a) Let X be uniform on the interval $[0,2]$, and define $Y = 4X^2 + 1$. Find the PDF, CDF, expectation, and variance of Y.

(b) Let X and Y have joint distribution

$$f(x,y) = \begin{cases} cxy + \frac{1}{4} & x \in [1,2] \text{ and } y \in [0,2] \\ 0 & \text{otherwise.} \end{cases}$$

Find the constant c (Hint: remember that the PDF must integrate to 1). Are X and Y independent?

(c) Let $X \sim \text{Exp}(3)$.

(i) Find probability that $X \in [0, 1]$.

(ii) Let $Y = \lfloor X \rfloor$, where the floor operator is defined as: $(\forall x \in [k, k+1))(\lfloor x \rfloor = k)$. For each $k \in \mathbb{N}$, what is the probability that $Y = k$? Write the distribution of Y in terms of one of the famous distributions; provide that distribution's name and parameters.

(d) Let $X_i \sim \text{Exp}(\lambda_i)$ for $i = 1, \ldots, n$ be mutually independent. It is a (very nice) fact that $\min(X_1, \ldots, X_n) \sim \text{Exp}(\mu)$. Find μ.

Solution:

(a) Let's begin with the CDF. It will first be useful to recall that

$$F_X(t) = \mathbb{P}[X \le t] = \begin{cases} 0 & t \le 0 \\ \frac{t}{2} & t \in [0,2] \\ 1 & t \ge 2 \end{cases}.$$

Since Y is defined in terms of X, we can compute that

$$\begin{aligned}
F_Y(t) = \mathbb{P}[Y \leq t] &= \mathbb{P}[4X^2 + 1 \leq t] \\
&= \mathbb{P}\left[X^2 \leq \frac{t-1}{4}\right] \\
&= \mathbb{P}\left[X \leq \frac{1}{2}\sqrt{t-1}\right] \\
&= F_X\left(\frac{1}{2}\sqrt{t-1}\right) \\
&= \begin{cases} 0 & t \leq 1 \\ \frac{1}{4}\sqrt{t-1} & t \in [1, 17] \\ 1 & t \geq 17 \end{cases}
\end{aligned}$$

where in the third line we use that $X \in [0, 2]$, and in the final line we have used the PDF for X. We know that the PDF can be found by taking the derivative of the CDF, so

$$f_Y(t) = \frac{\mathrm{d}}{\mathrm{d}t} F_Y(t) = \begin{cases} \frac{1}{8\sqrt{t-1}} & t \in [1, 17] \\ 0 & \text{else} \end{cases}.$$

By linearity of expectation, we have $\mathbb{E}[Y] = \mathbb{E}[4X^2 + 1] = 4\,\mathbb{E}[X^2] + 1$. There are a couple ways to compute $\mathbb{E}[X^2]$.

One way is to use the fact that $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$, so $\mathbb{E}[X^2] = \mathrm{Var}(X) + \mathbb{E}[X]^2$. Since $X \sim \mathrm{Uniform}[0, 2]$, we know $\mathrm{Var}(X) = \frac{1}{3}$ and $\mathbb{E}[X] = 1$; this means

$$\mathbb{E}[X^2] = \mathrm{Var}(X) + \mathbb{E}[X]^2 = \frac{1}{3} + 1^2 = \frac{4}{3}.$$

Another way is to use LOTUS and integrate directly:

$$\mathbb{E}[X^2] = \int_0^2 t^2 f_X(t)\,\mathrm{d}t = \int_0^2 t^2 \cdot \frac{1}{2}\,\mathrm{d}t = \frac{1}{2}\left(\frac{1}{3}2^3\right) = \frac{4}{3}.$$

Plugging this in, we have $\mathbb{E}[Y] = 4\,\mathbb{E}[X^2] + 1 = 4 \cdot \frac{4}{3} + 1 = \frac{19}{3}$.

For the variance, we have $\mathrm{Var}(Y) = \mathrm{Var}(4X^2 + 1) = 16\,\mathrm{Var}(X^2) = 16(\mathbb{E}[X^4] - \mathbb{E}[X^2]^2)$. Here, we already know $\mathbb{E}[X^2] = \frac{4}{3}$, so we only need to compute $\mathbb{E}[X^4]$:

$$\mathbb{E}[X^4] = \int_0^2 t^4 f_X(t)\,\mathrm{d}t = \int_0^2 t^4 \cdot \frac{1}{2}\,\mathrm{d}t = \frac{1}{2}\left(\frac{1}{5}2^5\right) = \frac{16}{5}.$$

Putting this together, we have

$$\mathrm{Var}(Y) = 16(\mathbb{E}[X^4] - \mathbb{E}[X^2]^2) = 16\left(\frac{16}{5} - \frac{16}{9}\right) = \frac{1024}{45}.$$

(b) To find the correct constant, we use the fact that a PDF must integrate to one. In particular,

$$1 = \int_1^2 \int_0^2 (cxy + 1/4)\,\mathrm{d}y\,\mathrm{d}x = 3c + \frac{1}{2},$$

so $c = 1/6$. In order to check independence, we need to first find the marginal distributions of X and Y:

$$f_X(x) = \int_0^2 f(x,y)\,\mathrm{d}y = 1/2 + x/3$$
$$f_Y(y) = \int_1^2 f(x,y)\,\mathrm{d}x = 1/4 + y/4.$$

Since

$$f_X(x) f_Y(y) = \frac{1}{8} + \frac{y}{8} + \frac{x}{12} + \frac{xy}{12} \neq \frac{1}{4} + \frac{xy}{6} = f(x,y),$$

the random variables are not independent.

(c) (i) Since $X \sim \mathrm{Exp}(3)$, the CDF of X is $F(x) = 1 - e^{-3x}$. Thus we have

$$\mathbb{P}[X \in [0,1]] = \int_0^1 f(x)\,\mathrm{d}x = F(1) - F(0) = (1 - e^{-3}) - (1 - e^0) = 1 - e^{-3}.$$

(ii) Similarly, if $Y = \lfloor X \rfloor$, then $Y = k$ exactly when $X \in [k, k+1)$, so

$$\begin{aligned}
\mathbb{P}[Y = k] &= \mathbb{P}[X \in [k, k+1)] \\
&= \int_k^{k+1} f(x)\,\mathrm{d}x \\
&= F(k+1) - F(k) \\
&= (1 - e^{-3(k+1)}) - (1 - e^{-3k}) \\
&= e^{-3k} - e^{-3(k+1)} \\
&= e^{-3k}\left(1 - e^{-3}\right) = (e^{-3})^k \left(1 - e^{-3}\right).
\end{aligned}$$

In other words, $Y = W - 1$ for $W \sim \mathrm{Geometric}(1 - e^{-3})$.

(d) Since the X_i are independent,

$$\begin{aligned}
\mathbb{P}[\min(X_1, \ldots, X_n) \le t] &= 1 - \mathbb{P}[X_1 > t, X_2 > t, \cdots \cdot X_n > t] \\
&= 1 - \mathbb{P}[X_1 > t] \cdot \mathbb{P}[X_2 > t] \cdots \cdot \mathbb{P}[X_n > t] \qquad \text{(by independence)} \\
&= 1 - e^{-\lambda_1 t} e^{-\lambda_2 t} \cdots \cdot e^{-\lambda_n t} \\
&= 1 - e^{-(\lambda_1 + \lambda_2 + \cdots + \lambda_n) t}.
\end{aligned}$$

This is exactly the CDF of an $\mathrm{Exp}(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$ random variable, so $\mu = \lambda_1 + \cdots + \lambda_n$.

2 Darts with Friends

Note 23 Michelle and Alex are playing darts. Being the better player, Michelle's aim follows a uniform distribution over a disk of radius 1 around the center. Alex's aim follows a uniform distribution over a disk of radius 2 around the center.

(a) Let the distance of Michelle's throw from the center be denoted by the random variable X and let the distance of Alex's throw from the center be denoted by the random variable Y.

(i) What's the cumulative distribution function of X?

(ii) What's the cumulative distribution function of Y?

(iii) What's the probability density function of X?

(iv) What's the probability density function of Y?

(b) What's the probability that Michelle's throw is closer to the center than Alex's throw? What's the probability that Alex's throw is closer to the center?

(c) What's the cumulative distribution function of $U = \max(X,Y)$?

Solution:

(a) (i) To get the cumulative distribution function of X, we'll consider the ratio of the area where the distance to the center is less than x, compared to the entire available area. This gives us the following expression:

$$\mathbb{P}[X \le x] = \frac{\pi x^2}{\pi} = x^2, \qquad x \in [0,1].$$

(ii) Using the same approach as the previous part:

$$\mathbb{P}[Y \le y] = \frac{\pi y^2}{\pi \cdot 4} = \frac{y^2}{4}, \qquad y \in [0,2].$$

(iii) We'll take the derivative of the CDF to get the following:

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}[X \le x] = 2x, \qquad x \in [0,1].$$

(iv) Using the same approach as the previous part:

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}\mathbb{P}[Y \le y] = \frac{y}{2}, \qquad y \in [0,2].$$

(b) We'll condition on Alex's outcome and then integrate over all the possibilities to get the marginal $\mathbb{P}[X \le Y]$ as following:

$$\begin{aligned}\mathbb{P}[X \le Y] &= \int_0^2 \mathbb{P}[X \le Y \mid Y = y] f_Y(y)\,\mathrm{d}y = \int_0^1 y^2 \times \frac{y}{2}\,\mathrm{d}y + \int_1^2 1 \times \frac{y}{2}\,\mathrm{d}y \\ &= \frac{1}{8} + \frac{3}{4} = \frac{7}{8}.\end{aligned}$$

Note the range within which $\mathbb{P}[X \leq Y] = 1$. This allowed us to separate the integral to simplify our solution. Using this, we can get $\mathbb{P}[Y \leq X]$ by the following:

$$\mathbb{P}[Y \leq X] = 1 - \mathbb{P}[X \leq Y] = \frac{1}{8}$$

A similar approach to the integral above could be used to verify this result:

$$\mathbb{P}[Y \leq X] = \int_0^1 \mathbb{P}[Y \leq X \mid X = x] f_X(x)\,\mathrm{d}x = \int_0^1 \frac{x^2}{4} 2x\,\mathrm{d}x = \frac{1}{2}\int_0^1 x^3\,\mathrm{d}x = \frac{1}{8}.$$

(c) Getting the CDF of U relies on the insight that for the maximum of two random variables to be smaller than a value, they both need to be smaller than that value. Using this we can get the following result for $u \in [0,1]$:

$$\mathbb{P}[U \leq u] = \mathbb{P}[X \leq u]\,\mathbb{P}[Y \leq u] = \left(u^2\right)\left(\frac{u^2}{4}\right) = \frac{u^4}{4}.$$

For $u \in [1,2]$ we have $\mathbb{P}[X \leq u] = 1$; this makes

$$\mathbb{P}[U \leq u] = \mathbb{P}[Y \leq u] = \frac{u^2}{4}.$$

For $u > 2$ we have $\mathbb{P}[U \leq u] = 1$ since CDFs of both X and Y are 1 in this range.

3 Predictable Gaussians

Note 24

Let Y be the result of a fair coin flip, and X be a normally distributed random variable with parameters dependent on Y. That is, if $Y = 1$, then $X \sim N(\mu_1, \sigma_1^2)$, and if $Y = 0$, then $X \sim N(\mu_0, \sigma_0^2)$.

(a) Sketch the two distributions of X overlaid on the same graph for the following cases:

(i) $\sigma_0^2 = \sigma_1^2, \mu_0 \neq \mu_1$

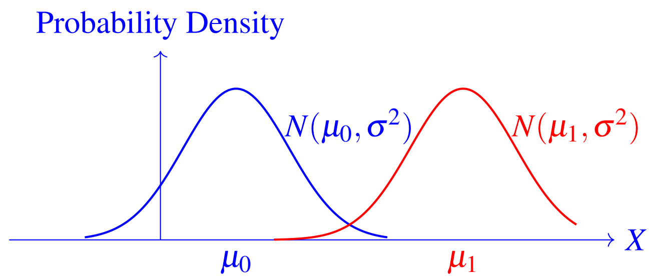
(ii) $\sigma_0^2 \neq \sigma_1^2, \mu_0 = \mu_1$

(b) Bayes' rule for mixed distributions can be formulated as $\mathbb{P}[Y = 1 \mid X = x] = \frac{\mathbb{P}[Y=1] f_{X|Y=1}(x)}{f_X(x)}$ where Y is a discrete distribution and X is a continuous distribution. Compute $\mathbb{P}[Y = 1 \mid X = x]$, and show that this can be expressed in the form of $\frac{1}{1+e^{\gamma}}$ for some expression γ. (Hint: any value z can be equivalently expressed as $e^{\ln(z)}$)

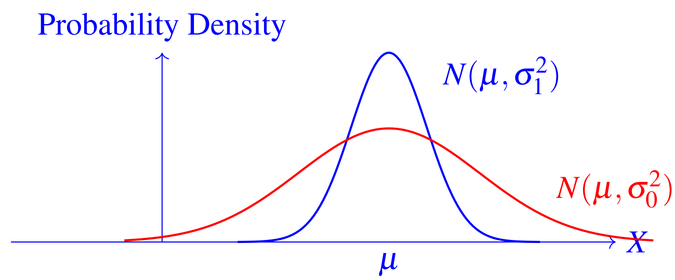
(c) In the special case where $\sigma_0^2 = \sigma_1^2$ find a simple expression for the value of x where $\mathbb{P}[Y = 1 \mid X = x] = \mathbb{P}[Y = 0 \mid X = x] = 1/2$, and interpret what the expression represents. (Hint: the identity $(a+b)(a-b) = a^2 - b^2$ may be useful)

Solution:

(a) (i) In this case, there are two bell curves with the same spread/width due to the variances being equal, but being centered at different means.



(ii) In this case, there will be two bell curves centered at the same mean, but the one with lower variance will be skinnier and taller, due to more of the probability density being centered closer to the mean.



(b)

$$\mathbb{P}[Y = 1 \mid X = x]$$

$$\begin{aligned}
&= \frac{\mathbb{P}[Y=1] f_{X|Y=1}(x)}{\mathbb{P}[Y=1] f_{X|Y=1}(x) + \mathbb{P}[Y=0] f_{X|Y=0}(x)} \\
&= \frac{\frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right)}{\frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right) + \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{(x-\mu_0)^2}{2\sigma_0^2}\right)} \\
&= \frac{1}{1 + \frac{\sigma_1}{\sigma_0} \exp\left(\frac{(x-\mu_1)^2}{2\sigma_1^2} - \frac{(x-\mu_0)^2}{2\sigma_0^2}\right)} \\
&= \frac{1}{1 + \exp\left(\ln\left(\frac{\sigma_1}{\sigma_0}\right) + \frac{(x-\mu_1)^2}{2\sigma_1^2} - \frac{(x-\mu_0)^2}{2\sigma_0^2}\right)}.
\end{aligned}$$

Which is of the desired form, with $\gamma = \ln\left(\frac{\sigma_1}{\sigma_0}\right) + \left(\frac{(x-\mu_1)^2}{2\sigma_1^2} - \frac{(x-\mu_0)^2}{2\sigma_0^2}\right)$

(c) Note that $\mathbb{P}[Y = 1 \mid X = x] = \frac{1}{2}$ implies that $\exp(\gamma) = 1$, which means that $\gamma = 0$. Thus, $\ln\left(\frac{\sigma_1}{\sigma_0}\right) + \left(\frac{(x-\mu_1)^2}{2\sigma_1^2} - \frac{(x-\mu_0)^2}{2\sigma_0^2}\right) = 0$. Using the conditions from the problem statement, we can simplify this expression.

$$
\begin{aligned}
&\ln\left(\frac{\sigma_1}{\sigma_0}\right) + \left(\frac{(x-\mu_1)^2}{2\sigma^2} - \frac{(x-\mu_0)^2}{2\sigma^2}\right) = 0 \\
&0 + \left(\frac{(x-\mu_1)^2}{2\sigma^2} - \frac{(x-\mu_0)^2}{2\sigma^2}\right) = 0 \\
&(x-\mu_1)^2 = (x-\mu_0)^2 \\
&x^2 - 2\mu_1 x + \mu_1^2 = x^2 - 2\mu_0 x + \mu_0^2 \\
&2(\mu_0 - \mu_1)x = \mu_0^2 - \mu_1^2 \\
&x = \frac{\mu_0^2 - \mu_1^2}{2(\mu_0 - \mu_1)} = \frac{\mu_0 + \mu_1}{2}
\end{aligned}
$$

Notice that x becomes the average, or center, of the two means.

4 Moments of the Gaussian

Note 24

For a random variable X, the quantity $\mathbb{E}[X^k]$ for $k \in \mathbb{N}$ is called the *kth moment* of the distribution. In this problem, we will calculate the moments of a standard normal distribution.

(a) Prove the identity

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{tx^2}{2}\right) \mathrm{d}x = t^{-1/2}$$

for $t > 0$.

Hint: Consider a normal distribution with variance $\frac{1}{t}$ and mean 0.

(b) For the rest of the problem, X is a standard normal distribution (with mean 0 and variance 1). Use part (a) to compute $\mathbb{E}[X^{2k}]$ for $k \in \mathbb{N}$.

Hint: Try differentiating both sides with respect to t, k times. You may use the fact that we can differentiate under the integral without proof.

(c) Compute $\mathbb{E}[X^{2k+1}]$ for $k \in \mathbb{N}$.

Solution:

(a) Note that a normal distribution with mean 0 and variance t^{-1} has the density function

$$f(x) = \frac{\sqrt{t}}{\sqrt{2\pi}} \exp\left(-\frac{tx^2}{2}\right),$$

and since the density must integrate to 1, we see that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{tx^2}{2}\right) \mathrm{d}x = t^{-1/2}.$$

(b) Differentiating the identity from (a) k times with respect to t, we obtain a LHS of

$$\begin{aligned}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{tx^2}{2}\right)\,\mathrm{d}x\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[\exp\left(-\frac{tx^2}{2}\right)\right]\mathrm{d}x\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(-1)^k\frac{x^{2k}}{2^k}\exp\left(-\frac{tx^2}{2}\right)\mathrm{d}x\\ &= \frac{1}{\sqrt{2\pi}}\frac{(-1)^k}{2^k}\int_{-\infty}^{\infty}x^{2k}\exp\left(-\frac{tx^2}{2}\right)\mathrm{d}x\end{aligned}$$

Here, we use the fact that everything involving x is a constant with respect to t.

Looking at the RHS, we have

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[t^{-1/2}\right] = (-1)^k\frac{1\cdot 3\cdots(2k-3)\cdot(2k-1)}{2^k}t^{-(2k+1)/2}$$

Together, this means that

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\frac{\cancel{(-1)^k}}{\cancel{2^k}}\int_{-\infty}^{\infty}x^{2k}\exp\left(-\frac{tx^2}{2}\right)\mathrm{d}x &= \cancel{(-1)^k}\frac{1\cdot 3\cdots(2k-3)\cdot(2k-1)}{\cancel{2^k}}t^{-(2k+1)/2}\\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}x^{2k}\exp\left(-\frac{tx^2}{2}\right)\mathrm{d}x &= (1\cdot 3\cdots(2k-3)\cdot(2k-1))t^{-(2k+1)/2}\end{aligned}$$

If we set $t=1$, we get

$$\mathbb{E}[X^{2k}] = \int_{-\infty}^{\infty}x^{2k}\cdot\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)\mathrm{d}x = \prod_{i=1}^{k}(2i-1).$$

This is sometimes denoted $(2k-1)!!$. Note that we can also write the result as

$$\mathbb{E}[X^{2k}] = (2k-1)!! = \frac{(2k)!}{2\cdot 4\cdots(2k-2)\cdot(2k)} = \frac{(2k)!}{2^k k!}.$$

(c) $\mathbb{E}[X^{2k+1}] = 0$, since the density function is symmetric around 0.