CS 70	Discrete Mathematics and Probability Theory	
Fall 2025	Sabin, Hug	HW 11

1 Expectation and Variance Warm-Up

Note 19
Note 20

Let X be a random variable with a mean of 1. Show that $\mathbb{E}[5+9X+9X^2] \geq 23$.

Solution:

$$\begin{aligned}
\mathbb{E}[5+9X+9X^2] &\geq 23 \\
\mathbb{E}[5]+9\,\mathbb{E}[X]+9\,\mathbb{E}[X^2] &\geq 23 \\
5+9\,\mathbb{E}[X]+9\,\mathbb{E}[X^2] &\geq 23 \\
9(\mathbb{E}[X]+\mathbb{E}[X^2]) &\geq 18 \\
\mathbb{E}[X]+\mathbb{E}[X^2] &\geq 2
\end{aligned}$$

Notice that $0 \leq \mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

$$\mathbb{E}[X^2] \geq \mathbb{E}[X]^2 = 1.$$

Thus we have:

$$\begin{aligned}
\mathbb{E}[X]+\mathbb{E}[X^2] &\geq 2 \\
1+1 &\geq 2.
\end{aligned}$$

2 More Socks

Note 19
Note 20

Gavin has n different pairs of socks (n left socks and n right socks, for $2n$ individual socks total) and is doing his laundry. He notices that the laundry machine spits out a uniformly random permutation of the $2n$ socks.

Let X be the number of matching pairs that are placed next to each other.

As an example, for the string 132231, $X = 1$ since only the 2nd pair of socks are placed together.

(a) What is the probability that the 1st pair of socks are placed together? We will denote this probability as p.

(b) What is $\mathbb{E}(X)$?

(c) What is the probability that both the 1st pair are placed together and the second pair are placed together? We will denote this probability as q.

(d) What is $\text{Var}(X)$? Feel free to leave your answer in terms of p and q.

Solution:

(a) Consider the ith matching pair as a single, condensed unit. As an example, in for $n = 3$, an original permutation could look like 132213. Let us condense both the 2's together, and label it as B. Then, a resulting string would look like $13B13$. Then, there are $2n-1$ 'units' left that we can order, and thus $(2n-1)!$ ways to order them. Also, when we condensed them, either the left sock or the right sock could've came first, so there are 2 ways to condense this pair. Thus, the probability is $\frac{2(2n-1)!}{(2n)!}$.

(b) Let X_i be an indicator for the ith pair of matching socks, $X_i = 1$ if the socks are placed together, and 0 otherwise. Then, $X = \sum_{i=1}^{n}(X_i)$, since there are n pairs. Thus, $\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}(X_i) = n \cdot \mathbb{P}[X_i = 1] = \frac{2n(2n-1)!}{(2n)!} = 1$.

(c) Again, we consider condensing both pairs. There are 2^2 ways to condense both pairs. Once condensed, there are $2n-2$ units left, and thus $(2n-2)!$ ways to order them, so the probability becomes $2^2\frac{(2n-2)!}{(2n)!}$

(d) We have

$$\mathbb{E}[X^2] = \mathbb{E}[(X_1+\cdots+X_n)^2] = \sum_{i=1}^{n} \mathbf{E}[X_i^2] + \sum_{i \neq j} \mathbb{E}[X_i X_j] = n\mathbb{E}[X_1^2] + n\cdot(n-1)\mathbb{E}[X_1 X_2] = np + (n-1)q$$

where p is the answer to part a, and q is the answer to part c. So, $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = np + n(n-1)q - n^2p^2$.

3 Coupon Collector Variance

Note 20

It's that time of the year again—Safeway is offering its Monopoly Card promotion. Each time you visit Safeway, you are given one of n different Monopoly Cards with equal probability. You need to collect them all to redeem the grand prize.

Let X be the number of visits you have to make before you can redeem the grand prize. Show that $\text{Var}(X) = n^2\left(\sum_{i=1}^{n} i^{-2}\right) - \mathbb{E}[X]$.

Solution:

Note that this is the coupon collector's problem, but now we have to find the variance. Let X_i be the number of visits we need to make before we have collected the ith unique Monopoly card actually obtained, given that we have already collected $i-1$ unique Monopoly cards. Then $X = \sum_{i=1}^{n} X_i$ and each X_i is geometrically distributed with $p = (n-i+1)/n$. Moreover, the indicators themselves are independent, since each time you collect a new card, you are starting from a clean slate.

$$\begin{aligned}
\mathrm{Var}(X) &= \sum_{i=1}^{n} \mathrm{Var}(X_i) && \text{(as the } X_i \text{ are independent)} \\
&= \sum_{i=1}^{n} \frac{1-(n-i+1)/n}{[(n-i+1)/n]^2} && \text{(variance of a geometric r.v. is } (1-p)/p^2\text{)} \\
&= \sum_{j=1}^{n} \frac{1-j/n}{(j/n)^2} && \text{(by noticing that } n-i+1 \text{ takes on all values from 1 to } n\text{)} \\
&= \sum_{j=1}^{n} \frac{n(n-j)}{j^2} \\
&= \sum_{j=1}^{n} \frac{n^2}{j^2} - \sum_{j=1}^{n} \frac{n}{j} \\
&= n^2\left(\sum_{j=1}^{n} \frac{1}{j^2}\right) - \mathbb{E}[X] && \text{(using the coupon collector problem expected value).}
\end{aligned}$$

4 Unbiased Variance Estimation

Note 20

We have a random variable X and want to estimate its variance, σ^2 and mean, μ, by sampling from it. In this problem, we will derive an "unbiased estimator" for the variance.

(a) We define a random variable Y that corresponds to drawing n values from the distribution for X and averaging, or $Y = (X_1 + \ldots + X_n)/n$. What is $\mathbb{E}(Y)$? Note that if $\mathbb{E}(Y) = \mathbb{E}(X)$ then Y is an unbiased estimator of $\mu = \mathbb{E}(X)$.

Hint: There should not be much computation needed.

(b) Now let's assume the actual mean is 0 as variance doesn't change when one shifts the mean.

Before attempting to define an estimator for variance, show that $\mathbb{E}(Y^2) = \sigma^2/n$.

(c) In practice, we don't know the mean of X so following part (a), we estimate it as Y. With this in mind, we consider the random variable $Z = \sum_{i=1}^{n}(X_i - Y)^2$. What is $\mathbb{E}(Z)$?

(d) What is a good unbiased estimator for the $\mathrm{Var}(X)$?

(e) How does this differ from what you might expect? Why? (Just tell us your intuition here, it is all good!)

Solution:

(a) By linearity of expecation, the value is $(\sum_{i=1}^{n} \mathbb{E}(X_i))/n = \mathbb{E}(X)$.

(b) The variables X_i are independent, so

$$\mathbb{E}(Y^2) = \mathbb{E}((Y - \mathbb{E}(Y))^2) = \mathrm{Var}\left(\frac{1}{n}\left(\sum_{i=1}^{n} X_i\right)\right) = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}(X_i) = \frac{\sigma^2}{n}.$$

The first equality follows the fact that $\mathbb{E}(Y) = \mathbb{E}(X) = 0$, the second from the definition of variance, the third from linearity of variance for independent variables, and the others by substitution.

(c)

$$\begin{aligned}
\mathbb{E}(Z) &= \sum_{i=1}^{n} (\mathbb{E}(X_i^2) - \mathbb{E}(2YX_i) + \mathbb{E}(Y^2)) \\
&= (n+1)\sigma^2 - 2\sum_{i=1}^{n} \mathbb{E}(X_i Y) \\
&= (n+1)\sigma^2 - \frac{2}{n}\left(\sum_{i=1}^{n} \mathbb{E}(X_i^2) + \sum_{j \neq i} \mathbb{E}(X_i X_j)\right) \\
&= (n-1)\sigma^2 - \frac{2}{n}\left(\sum_{j \neq i} \mathbb{E}(X_i X_j)\right) \\
&= (n-1)\sigma^2
\end{aligned}$$

The first equality is plugging in definition of Z and uses linearity of expectation. The second line uses $\mathbb{E}(X_i^2) = \sigma^2$ and $\mathbb{E}(Y^2) = \sigma^2/n$. The third line plugs in the definition of Y and uses linearity of expectation. The fourth again uses $\mathbb{E}(X_i^2) = \sigma^2$. The final line follows from $\mathbb{E}(X_i X_j) = \mathbb{E}(X_i)\,\mathbb{E}(X_j) = 0$ since the X_i are chosen independently and have expectation 0.

(d) $Z/(n-1)$, since $\mathbb{E}(Z/(n-1)) = \sigma^2$.

(e) Maybe one could guess Z/n since there are n terms in Z. But in fact, each term is a bit smaller than expected as Y contains a bit of X_i/n in it. So a term, $(X_i - Y)^2$ is actually

$$\left(\frac{n-1}{n} X_i - \frac{1}{n}\sum_{i \neq j} X_j\right)^2,$$

so it is a bit smaller than the variance of X_i.