1 Counting Intro I

Note 10

Zeroth rule of counting: For sets A, B if there exists a bijection $f : A \to B$, then $|A| = |B|$.

First rule of counting: When counting the number of ways to count a sequences of k choices, if there are n_1 options for the first choice, n_2 options for the second choice regardless of the decision in the first choice, etc., then there are a total of $n_1 \cdot n_2 \cdots n_k$ ways to make all k choices.

Second rule of counting: Let B be the set of unordered objects for counting a sequence of choices, and let A be the ordered set of such objects. If there exists an m-to-1 mapping $f : A \to B$, then $|A| = m \cdot |B|$.

In this problem, we will derive the formulas for rerranging k items out of n total:

	with replacement	without replacement
order matters	n^k	$\frac{n!}{(n-k)!}$
order doesn't matter	$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$	$\frac{n!}{k!(n-k)!} = \binom{n}{k}$

- Order matters, without replacement: also known as *permutations*
- Order doesn't matter, without replacement: also known as *combinations*
- Order doesn't matter, with replacement: also known as *stars and bars*. Here, stars and bars is usually used to compute the number of ways to split k items into n categories (hence k stars, and $n-1$ bars to separate the n categories).

Most counting problems involve a mix of all 4 methods; don't box yourself into using just one of these!

In the below questions, do not blindly apply the above formulas—try to re-derive them yourself!

(a) How many 4-card hands are there in a standard 52-card deck? (Any rearrangement of the cards in your hand is counted as identical.)

(b) A US phone number consists of a country code (always "+1"), an area code (any 3 digits), a telephone prefix (any 3 digits), and a line number (any 4 digits). How many US phone numbers are there?

(c) How many anagrams of the word "COUNT" are there? What about "BERKELEY"? (An *anagram* of a word is a rearrangement of its letters; for example, "CTONU" and "TNUOC" are both anagrams of "COUNT")

(d) The concept of "stars and bars" seems at first glance quite different from the idea of "order doesn't matter, with replacement". We'll look at how these two concepts are actually the same.

Consider the problem of counting the number of ways of splitting ten \$1 bills among 4 people. How many ways of splitting the bills are there?

Compare this with the problem of counting the number of ways of arranging 10 numbers from the set $\{1,2,3,4\}$, where order doesn't matter and we have replacement. What is the relationship between the ways of counting these two scenarios? (*Hint*: Let each person be labeled by a digit from $\{1,2,3,4\}$. Can you represent a split of the ten bills with a sequence of digits?)

Solution:

(a) $\binom{52}{4}$. Order doesn't matter here, since different arrangements of the same hand are identical. We do not allow replacement, since we are drawing cards from a deck without putting them back. Hence, since we have 52 cards total, and we're picking 4 for our hand, there are $\binom{52}{4}$ possible hands.

(b) 10^{10}. The country code doesn't change, so we have a total of 10 digits in the phone number that can change. These can each be any of the digits 0–9, so we have 10^{10} possible phone numbers.

Here, order matters (the sequence "12" is different from the sequence "21" in the phone number), and we have replacement (we can reuse digits).

(c) There are 5! anagrams of the word "COUNT", since we can arrange the 5 distinct letters with order and without replacement.

There are $\frac{8!}{3!}$ anagrams of the word "BERKELEY". There are a total of 8! ways of arranging the letters, but this is an overcount: we have 3 E's, and re-ordering the E's by themselves does not change the resulting word. Since there are a total of 3! ways of arranging *just* the E's amongst themselves, we must divide by this quantity to fix our overcounting.

(d) We can think of each person as a "bin" for the 10 bills to fall into. We can then think of this problem as arranging the 10 bills (thought of as "stars") along with 3 dividers for the 4 bins (thought of as "bars"), in a line. For example, we could have the following split:

$$\star\star\star \mid \star\star \mid \star \mid \star\star\star\star$$

We have a total of 13 "objects" that we're arranging here. Notice that once we choose the location of the 10 stars, the location of the bars are fixed (similarly, once we choose the location of the 3 bars, the location of the 10 stars are fixed). There are a total of $\binom{13}{10}$ ways of choosing the location of the 10 stars (or $\binom{13}{3}$ ways of choosing the location of the 3 bars), so this is the number of ways of arranging these objects in a line.

This means that there are $\binom{13}{10} = \binom{13}{3}$ ways of splitting ten \$1 bills among 4 people.

Trying to relate this to the problem of arranging 10 numbers from the set $\{1,2,3,4\}$, suppose we label each person with a digit. A sequence of 10 of these digits then corresponds to a way of splitting the 10 bills, where each digit tells us which person the bill is given to. For example, the split from before could correspond to the following sequence:

$$(1,1,1,2,2,3,4,4,4,4)$$

Here, we don't care about order, since the sequence $(1,2)$ is the same as the sequence $(2,1)$. In either case, person 1 and person 2 both get one bill each—the order in which the person receives the bill doesn't matter. We also allow replacement, since each person can receive multiple bills (their digit can appear multiple times in the sequence).

Put another way, these constraints essentially tell us that since order does not matter, we really only care about *count* of each digit. For stars and bars, each bin corresponds to a possible digit, and the number of stars in each bin corresponds to the number of times that digit appears in the sequence.

2 Counting Produce

Note 10

(a) How many solutions does $y_0 + y_1 + \cdots + y_k = n$ have, if each y must be a non-negative integer?

(b) How many solutions does $y_0 + y_1 + \cdots + y_k = n$ have, if each y must be a positive integer?

Armed with the above knowledge, you visit Berkeley Bowl. Suppose you want to buy k fruits. Count how many ways you can do this, assuming:

(c) There are peaches and apples at the market.

(d) There are peaches, apples, oranges, and pears at the market.

(e) There are n kinds of fruits at the market, and you want to end up with at least two different types of fruit.

Solution:

(a) $\binom{n+k}{k}$. We can imagine this as a sequence of n ones and k plus signs: y_0 is the number of ones before the first plus, y_1 is the number of ones between the first and second plus, etc. We can now count the number of sequences using the "balls and bins" method (also known as "stars and bars").

(b) $\binom{(n-(k+1))+k}{k} = \binom{n-1}{k}$. By subtracting 1 from all $k+1$ variables, and $k+1$ from the total required, we reduce it to problem with the same form as the previous problem. Once we have a solution to that we reverse the process, and adding 1 to all the non-negative variables gives us positive variables.

Alternatively, we can derive a method similar to stars and bars/balls and bins; here, the restriction to positive integers means that we cannot have any empty groups. In particular, instead of arranging all of the objects (i.e. all the stars and all the bars), we can instead choose where to place the bars.

Looking at the "gaps" between the stars (i.e. the 1's), we have a total of $n-1$ places to put the bars in between the n stars. Selecting k of these positions (we can't have two bars occupy the same gap, otherwise we'd have an empty group), we have a total of $\binom{n-1}{k}$ ways to group the 1's.

(c) $k+1$. We can have 0 peaches and k apples, or 1 peach and $k-1$ apples, etc. all the way to k peaches and 0 apples. We can equivalently think about this as k balls and 2 bins, or k stars and 1 bar, giving us $\binom{k+1}{1} = \binom{k+1}{k}$.

(d) $\binom{k+3}{3}$. We have k balls and 4 bins, or k stars and 3 bars.

(e) There are $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$ ways to choose k fruits of n types with no additional restrictions (i.e. k balls and n bins, or k stars and $n-1$ bars). n of these combinations, however, contain only one variety of fruit, so we subtract them for a total of $\binom{n+k-1}{n-1} - n = \binom{n+k-1}{k} - n$.

3 The Count

Note 10

(a) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have?

(b) Now instead of non-increasing, they must be strictly decreasing. So 9983220 is no longer valid, while 9753210 is valid. How many choices for a new phone number does he have now?

(c) The Count now wants to make a password to secure his phone. His password must be exactly 10 digits long and can only contain the digits 0 and 1. On top of that, he also wants it to contain at least five consecutive 0's. How many possible passwords can he make?

Solution:

(a) This is actually a stars and bars problem in disguise! We have seven positions for digits, and nine dividers to partition these positions into places for nines, places for eights, etc. This is because we know that the digits are non-increasing, so all the nines (if any) must come first, then all the eights (if any), and so on. That means there are a total of 16 objects and dividers, and we are looking for where to put the nine dividers, so our answer is $\binom{16}{9}$.

(b) This can be found from just combinations. For any choice of 7 digits, there is exactly one arrangement of them that is strictly decreasing. Thus, the total number of strictly decreasing strings is exactly $\binom{10}{7}$.

(c) This problem is a bit trickier to approach, since there is a strong possibility of overcounting - it is not sufficient to just choose 5 consecutive positions to be 00000, and let the rest of the positions be arbitrary values.

One counting strategy is strategic casework - we will split up the problem into exhaustive cases based on where the run of 0's begins (we look at the leftmost zero of a run of at least 5 zeros). It can begin somewhere between the first digit and the sixth digit, inclusively.

If the run begins with the first digit, the first five digits are 0, and there are $2^5 = 32$ choices for the other 5 digits.

If the run begins after the i^{th} digit, then the $i-1^{th}$ digit must be a 1, and the other $(10-5-1 = 4)$ digits can be chosen arbitrarily. The other four digits can be freely chosen with $2^4 = 16$ possibilities. Thus the total number of valid passwords is $2^5 + 5 \cdot 2^4 = 112$. Note that, since there are only 10 digits, there can only be one occurence of the "100000" pattern.