CS 70	Discrete Mathematics and Probability Theory	
Fall 2025	Sabin, Hug	HW 10

1 Combined Head Count

Note 18

Suppose you flip a fair coin twice.

(a) What is the sample space Ω generated from these flips?

(b) Define a random variable X to be the number of heads. What is the distribution of X?

(c) Define a random variable Y to be 1 if you get a heads followed by a tails and 0 otherwise. What is the distribution of Y?

(d) Compute the conditional probabilities $\mathbb{P}[Y = i \mid X = j]$ for all combinations of i and j.

(e) Define a third random variable $Z = X + Y$. Use the conditional probabilities you computed in part (d) to find the distribution of Z.

Solution:

(a) $\{(T,T),(H,T),(T,H),(H,H)\}$.

(b)

$$X = \begin{cases} 0 & \text{w.p.} \quad .25 \\ 1 & \text{w.p.} \quad .5 \\ 2 & \text{w.p.} \quad .25 \end{cases}$$

(c)

$$Y = \begin{cases} 0 & \text{w.p.} \quad .75 \\ 1 & \text{w.p.} \quad .25 \end{cases}$$

(d)
- $\mathbb{P}[Y = 0 \mid X = 0]$: Since $X = 0$, we have no heads; therefore, there is no chance that the first coin is heads, so Y must be 0. So $\mathbb{P}[Y = 0 \mid X = 0] = 1$.
- $\mathbb{P}[Y = 1 \mid X = 0] = 0$ as $\mathbb{P}[Y = 1 \mid X = 0] = 1 - \mathbb{P}[Y = 0 \mid X = 0] = 1 - 1 = 0$.
- $\mathbb{P}[Y = 0 \mid X = 1]$: If we have one head, then we have one of two outcomes, (H,T) or (T,H), and since this is a fair coin, both outcomes happen with equal probability. Only (T,H) makes $Y = 0$; thus $\mathbb{P}[Y = 0 \mid X = 1] = \frac{1}{2}$.
- $\mathbb{P}[Y = 1 \mid X = 1] = 0$ as $\mathbb{P}[Y = 1 \mid X = 1] = 1 - \mathbb{P}[Y = 0 \mid X = 1] = 1 - \frac{1}{2} = \frac{1}{2}$.
- $\mathbb{P}[Y = 0 \mid X = 2]$: Since $X = 0$, we have no tails; therefore, there is no chance that the second coin is tails, so Y must be 0. So $\mathbb{P}[Y = 0 \mid X = 2] = 1$.

- $\mathbb{P}[Y=1 \mid X=2] = 0$ as $\mathbb{P}[Y=1 \mid X=2] = 1 - \mathbb{P}[Y=0 \mid X=2] = 1-1 = 0$.

(e) Let's determine the values Z can take and the corresponding probabilities:

- $Z=0$: $\mathbb{P}(Z=0) = \mathbb{P}(X=0 \cap Y=0) = \mathbb{P}(X=0) \cdot \mathbb{P}(Y=0 \mid X=0) = .25 \cdot 1 = .25$
- $Z=1$:
$$\begin{aligned}\mathbb{P}(Z=1) &= \mathbb{P}(X=0 \cap Y=1) + \mathbb{P}(X=1 \cap Y=0)\\ &= \mathbb{P}(X=0) \cdot \mathbb{P}(Y=1 \mid X=0) + \mathbb{P}(X=1) \cdot \mathbb{P}(Y=0 \mid X=1)\\ &= .25 \cdot 0 + .5 \cdot .5 = .25\end{aligned}$$
- $Z=2$:
$$\begin{aligned}\mathbb{P}(Z=2) &= \mathbb{P}(X=1 \cap Y=1) + \mathbb{P}(X=2 \cap Y=0)\\ &= \mathbb{P}(X=1) \cdot \mathbb{P}(Y=1 \mid X=1) + \mathbb{P}(X=2) \cdot \mathbb{P}(Y=0 \mid X=2)\\ &= .5 \cdot .5 + .25 \cdot 1 = .5\end{aligned}$$
- $Z=3$: $\mathbb{P}(Z=3) = \mathbb{P}(X=2 \cap Y=1) = \mathbb{P}(X=2) \cdot \mathbb{P}(Y=1 \mid X=2) = .25 \cdot 0 = 0$

$$Z = \begin{cases} 0 & \text{w.p.} \quad .25 \\ 1 & \text{w.p.} \quad .25 \\ 2 & \text{w.p.} \quad .5 \end{cases}$$

2 Testing Model Planes

Note 18

Amin is testing model airplanes. He starts with n model planes which each independently have probability p of flying successfully each time they are flown, where $0 < p < 1$. Each day, he flies every single plane and keeps the ones that fly successfully (i.e. don't crash), throwing away all other models. He repeats this process for many days, where each "day" consists of Amin flying all remaining model planes and throwing away any that crash. Let X_i be the random variable representing how many model planes remain after i days. Note that $X_0 = n$. Justify your answers for each part.

(a) What is the distribution of X_1? That is, what is $\mathbb{P}[X_1 = k]$?

(b) What is the distribution of X_2? That is, what is $\mathbb{P}[X_2 = k]$? Recognize the distribution of X_2 as one of the famous ones and provide its name and parameters.

(c) Repeat the previous part for X_t for arbitrary $t \geq 1$.

(d) What is the probability that at least one model plane still remains (has not crashed yet) after t days? Do not have any summations in your answer.

(e) Considering only the first day of flights, is the event A_1 that the first and second model planes crash independent from the event B_1 that the second and third model planes crash? Recall that two events A and B are independent if $\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$. Prove your answer using this definition.

(f) Considering only the first day of flights, let A_2 be the event that the first model plane crashes *and* exactly two model planes crash in total. Let B_2 be the event that the second plane crashes on the first day. What must n be equal to in terms of p such that A_2 is independent from B_2? Prove your answer using the definition of independence stated in the previous part.

(g) Are the random variables X_i and X_j, where $i < j$, independent? Recall that two random variables X and Y are independent if $\mathbb{P}[X = k_1 \cap Y = k_2] = \mathbb{P}[X = k_1]\,\mathbb{P}[Y = k_2]$ for all k_1 and k_2. Prove your answer using this definition.

Solution:

(a) Since Amin is performing n trials (flying a plane), each with an independent probability of "success" (not crashing), we have $X_1 \sim \text{Binomial}(n, p)$, or $\mathbb{P}[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$, for $0 \le k \le n$.

(b) Each model plane independently has probability p^2 of surviving both days. Whether a model plane survives both days is still independent from whether any other model plane survives both days, so we can say $X_2 \sim \text{Binomial}(n, p^2)$, or $\mathbb{P}[X = k] = \binom{n}{k} p^{2k} (1-p^2)^{n-k}$, for $0 \le k \le n$.

(c) By extending the previous part, we see each model plane has probability p^t of surviving t days, so $X_t \sim \text{Binomial}(n, p^t)$, or $\mathbb{P}[X = k] = \binom{n}{k} (p^t)^k (1-p^t)^{n-k}$, for $0 \le k \le n$.

(d) We consider the complement, the probability that no model planes remain after t days. By the previous part we know this to be

$$\mathbb{P}[X_t = 0] = \binom{n}{0} (p^t)^0 (1-p^t)^{n-0} = (1-p^t)^n.$$

This means that the probability of at least model plane remaining after t days is $1-(1-p^t)^n$.

(e) No. $\mathbb{P}[A_1 \cap B_1]$ is the probability that the first three model planes crash, which is $(1-p)^3$. But $\mathbb{P}[A_1]\,\mathbb{P}[B_1] = (1-p)^2(1-p)^2 = (1-p)^4$. So $\mathbb{P}[A_1 \cap B_1] \neq \mathbb{P}[A_1]\,\mathbb{P}[B_1]$ and A_1 and B_1 are not independent.

(f) $\mathbb{P}[A_2 \cap B_2]$ is the probability that only the first model plane and second model plane crash, which is $(1-p)^2 p^{n-2}$. $\mathbb{P}[A_2]$ is the probability that the first model plane crashes, and exactly one of the remaining $n-1$ model planes crashes, so

$$\mathbb{P}[A_2] = (1-p) \cdot \binom{n-1}{1} (1-p) p^{n-1-1} = (n-1)(1-p)^2 p^{n-2}.$$

We also have $\mathbb{P}[B_2] = 1-p$, so we want to solve for n in

$$\begin{aligned}
\mathbb{P}[A_2 \cap B_2] &= \mathbb{P}[A_2]\,\mathbb{P}[B_2] \\
(1-p)^2 p^{n-2} &= \underbrace{(n-1)(1-p)^2 p^{n-2}}_{\mathbb{P}[A_2]}\underbrace{(1-p)}_{\mathbb{P}[B_2]} \\
(1-p)^2 p^{n-2} &= (n-1)(1-p)^3 p^{n-2} \\
1 &= (n-1)(1-p) \\
n &= 1 + \frac{1}{1-p}
\end{aligned}$$

(g) No. Let $k_1 = 0$ and $k_2 = 1$. Then, $\mathbb{P}[X_i = k_1 \cap X_j = k_2] = 0$ because you can't have 1 plane at the end of day 2 if there are no planes left at the end of day 1. However, $\mathbb{P}[X_i = k_1] > 0$ and $\mathbb{P}[X_j = k_2] > 0$, so $\mathbb{P}[X_i = k_1]\,\mathbb{P}[X_j = k_2] > 0$. Since $\mathbb{P}[X_i = k_1]\,\mathbb{P}[X_j = k_2] \neq \mathbb{P}[X_i = k_1 \cap X_j = k_2]$, they are not independent.

3 Fishy Computations

Note 18

Assume for each part that the random variable can be modelled by a Poisson distribution.

(a) Suppose that on average, a fisherman catches 20 salmon per week. What is the probability that he will catch exactly 7 salmon this week?

(b) Suppose that on average, you go to Fisherman's Wharf twice a year. What is the probability that you will go at most once in 2024?

(c) Suppose that in March, on average, there are 5.7 boats that sail in Laguna Beach per day. What is the probability there will be *at least* 3 boats sailing throughout the *next two days* in Laguna?

(d) Denote $X \sim \text{Pois}(\lambda)$. Prove that

$$\mathbb{E}[X f(X)] = \lambda\,\mathbb{E}[f(X+1)]$$

for any function f.

Solution:

(a) Let X be the number of salmon the fisherman catches per week. $X \sim \text{Poisson}(20 \text{ salmon/week})$, so

$$\mathbb{P}[X = 7 \text{ salmon/week}] = \frac{20^7}{7!} e^{-20} \approx 5.23 \cdot 10^{-4}.$$

(b) Similarly $X \sim \text{Poisson}(2)$, so

$$\mathbb{P}[X \leq 1] = \frac{2^0}{0!} e^{-2} + \frac{2^1}{1!} e^{-2} \approx 0.41.$$

(c) Let X_1 be the number of sailing boats on the next day, and X_2 be the number of sailing boats on the day after next. Now, we can model sailing boats on day i as a Poisson distribution $X_i \sim \text{Poisson}(\lambda = 5.7)$. Let Y be the number of boats that sail in the next two days. We are interested in $Y = X_1 + X_2$. We know that the sum of two independent Poisson random variables is Poisson. Thus, we have $Y \sim \text{Poisson}(\lambda = 5.7 + 5.7 = 11.4)$.

$$\begin{aligned}
\mathbb{P}[Y \geq 3] &= 1 - \mathbb{P}[Y < 3] \\
&= 1 - \mathbb{P}[Y = 0 \cup Y = 1 \cup Y = 2] \\
&= 1 - (\mathbb{P}[Y = 0] + \mathbb{P}[Y = 1] + \mathbb{P}[Y = 2]) \\
&= 1 - \left(\frac{11.4^0}{0!}e^{-11.4} + \frac{11.4^1}{1!}e^{-11.4} + \frac{11.4^2}{2!}e^{-11.4}\right) \\
&\approx 0.999.
\end{aligned}$$

(d) We apply the Law of the Unconscious Statistician,

$$\begin{aligned}
\mathbb{E}[Xf(X)] &= \sum_{x=0}^{\infty} x f(x)\,\mathbb{P}[X = x] \\
&= \sum_{x=0}^{\infty} x f(x) \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=1}^{\infty} x f(x) \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \lambda \sum_{x=1}^{\infty} f(x) \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} \\
&= \lambda \sum_{x=0}^{\infty} f(x+1) \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \lambda\,\mathbb{E}[f(X+1)]
\end{aligned}$$

as desired.

4 Such High Expectations

Note 19

Suppose X and Y are independently drawn from a Geometric distribution with parameter p. For each of the below subparts, your answer must be simplified (i.e. NOT left in terms of a summation).

(a) Compute $\mathbb{E}[\min(X,Y)]$.

(b) Compute $\mathbb{E}[\max(X,Y)]$.

(c) Compute $\mathbb{P}[X + Y \geq t]$

Solution:

(a) By independence,

$$\mathbb{P}[\min(X,Y) \geq t] = \mathbb{P}[X \geq t]\,\mathbb{P}[Y \geq t] = (1-p)^{2(t-1)}.$$

By Tail Sum,

$$\mathbb{E}[\min(X,Y)] = \sum_{t=1}^{\infty} \mathbb{P}[\min(X,Y) \geq t] = \sum_{t=1}^{\infty} (1-p)^{2(t-1)} = \frac{1}{1-(1-p)^2}.$$

Alternate Solution: We can see that $\min(X,Y)$ is a geometric distribution by looking at the tail probability from earlier. In particular, we have that $\min(X,Y) \sim \text{Geom}(1-(1-p)^2)$. This means that

$$\mathbb{E}[\min(X,Y)] = \frac{1}{1-(1-p)^2},$$

from the expectation of a geometric distribution.

(b) We see that

$$\begin{aligned}
\mathbb{P}[\max(X,Y) \geq t] = 1 - \mathbb{P}[\max(X,Y) < t] &= 1 - \mathbb{P}[X < t]\,\mathbb{P}[Y < t] \\
&= 1 - (1 - \mathbb{P}[X \geq t])(1 - \mathbb{P}[Y \geq t]) \\
&= 1 - \left(1-(1-p)^{t-1}\right)\left(1-(1-p)^{t-1}\right) \\
&= 1 - \left(1 - 2(1-p)^{t-1} + (1-p)^{2(t-1)}\right) \\
&= 2(1-p)^{t-1} - (1-p)^{2(t-1)}.
\end{aligned}$$

Using the result from part (a),

$$\begin{aligned}
\mathbb{E}[\max(X,Y)] &= \sum_{t=1}^{\infty} \mathbb{P}[\max(X,Y) \geq t] \\
&= \sum_{t=1}^{\infty} 2(1-p)^{t-1} - (1-p)^{2(t-1)} \\
&= \sum_{t=1}^{\infty} 2(1-p)^{t-1} - \sum_{t=1}^{\infty} (1-p)^{2(t-1)} \\
&= \frac{2}{p} - \frac{1}{1-(1-p)^2}.
\end{aligned}$$

Alternate Solution: An extremely elegant one-liner with linearity:

$$\mathbb{E}[\max(X,Y)] = \mathbb{E}[X+Y-\min(X,Y)] = \mathbb{E}[X] + \mathbb{E}[Y] - \mathbb{E}[\min(X,Y)] = \frac{2}{p} - \frac{1}{1-(1-p)^2}.$$

(c) Note that if $X \geq t$, then regardless of the value of Y, $X+Y \geq t$ will be satisfied since $Y > 0$.

Hence,

$$\begin{aligned}
\mathbb{P}[X+Y \geq t] &= \sum_{x=1}^{\infty} \mathbb{P}[X=x]\,\mathbb{P}[Y \geq t-x] \\
&= \sum_{x=1}^{t-1} \mathbb{P}[X=x]\,\mathbb{P}[Y \geq t-x] + \sum_{x=t}^{\infty} \mathbb{P}[X=x]\,\mathbb{P}[Y \geq t-x] \\
&= \sum_{x=1}^{t-1} (1-p)^{x-1} p (1-p)^{t-x-1} + \sum_{x=t}^{\infty} \mathbb{P}[X=x] \\
&= \sum_{x=1}^{t-1} (1-p)^{t-2} p + \mathbb{P}[X \geq t] \\
&= (t-1)(1-p)^{t-2} p + (1-p)^{t-1}
\end{aligned}$$

5 Swaps and Cycles

Note 19

A permutation of n objects is a bijection from $(1,\ldots,n)$ to itself. For example, the permutation $\pi = (2,1,4,3)$ of 4 objects is the mapping $\pi(1) = 2$, $\pi(2) = 1$, $\pi(3) = 4$, and $\pi(4) = 3$. We'll say that a permutation $\pi = (\pi(1),\ldots,\pi(n))$ contains a *swap* if there exist $i, j \in \{1,\ldots,n\}$ so that $\pi(i) = j$ and $\pi(j) = i$, where $i \neq j$. The example above contains two swaps: $(1,2)$ and $(3,4)$.

(a) In terms of n, what is the expected number of swaps in a random permutation?

(b) In the same spirit as above, we'll say that π contains a *k-cycle* if there exist $i_1,\ldots,i_k \in \{1,\ldots,n\}$ with $\pi(i_1) = i_2, \pi(i_2) = i_3, \ldots, \pi(i_k) = i_1$. Compute the expectation of the number of k-cycles.

Solution:

(a) As a warm-up, let's compute the probability that 1 and 2 are swapped. There are $n!$ possible permutations, and $(n-2)!$ of them have $\pi(1) = 2$ and $\pi(2) = 1$. This means

$$\mathbb{P}[(1,2) \text{ are a swap}] = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}.$$

There was nothing special about 1 and 2 in this calculation, so for any $\{i, j\} \subset \{1,...,n\}$, the probability that i and j are swapped is the same as above. Let's write $I_{i,j}$ for the indicator that i and j are swapped, and N for the total number of swaps, so that

$$\mathbb{E}[N] = \mathbb{E}\left[\sum_{\{i,j\} \subset \{1,\ldots,n\}} I_{i,j}\right] = \sum_{\{i,j\} \subset \{1,\ldots n\}} \mathbb{P}[(i,j) \text{ are swapped}] = \frac{1}{n(n-1)}\binom{n}{2} = \frac{1}{2}.$$

(b) The idea here is quite similar to the above, so we'll be a little less verbose in the exposition. However, as a first aside we need the notion of a *cyclic ordering* of k elements from a set

$\{1, \ldots n\}$. We mean by this a labelling of the k beads of a necklace with elements of the set, where we say that labellings of the beads are the same if we can move them along the string to turn one into the other. For example, $(1,2,3,4)$ and $(1,2,4,3)$ are different cyclic orderings, but $(1,2,3,4)$ and $(2,3,4,1)$ are the same. There are

$$\binom{n}{k}\frac{k!}{k} = \frac{n!}{(n-k)!}\frac{1}{k}$$

possible cyclic orderings of length k from a set with n elements, since if we first count all subsets of size k, and then all permutations of each of those subsets, we have overcounted by a factor of k.

Now, let N be a random variable counting the number of k-cycles, and for each cyclic ordering $(i_1, \ldots, i_k)$ of k elements of $\{1, \ldots, n\}$, let $I_{(i_1, \ldots, i_k)}$ be the indicator that $\pi(i_1) = i_2, \pi(i_2) = i_3, \ldots, \pi(i_k) = i_1$. There are $(n-k)!$ permutations in which $(i_1, \ldots, i_k)$ form an k-cycle (since we are free to do whatever we want to the remaining $(n-k)$ elements of $\{1, \ldots, n\}$), so the probability that $(i_1, \ldots, i_k)$ are such a cycle is $\frac{(n-k)!}{n!}$, and

$$\mathbb{E}[N] = \mathbb{E}\left[\sum_{(i_1, \ldots, i_k) \text{ cyclic ordering}} I_{(i_1, \ldots, i_k)}\right] = \frac{n!}{(n-k)!}\frac{1}{k}\frac{(n-k)!}{n!} = \frac{1}{k}.$$