CS 70 Discrete Mathematics and Probability Theory
Fall 2025 Sabin, Hug

DIS 12A

Concentration Inequalities Intro

Markov's Inequality: For any nonnegative random variable X and $t > 0$,

$$\mathbb{P}[X \geq t] \leq \frac{\mathbb{E}[X]}{t}.$$

Chebyshev's Inequality: For any random variable X and $c > 0$,

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq c] \leq \frac{\mathrm{Var}(X)}{c^2}.$$

Law of Large Numbers: Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with mean μ and variance σ^2. We have the following:

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \mu$$

$$\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{\sigma^2}{n}.$$

Applying Chebyshev's inequality on the sample mean $\frac{1}{n}\sum_{i=1}^{n} X_i$, we have that

$$\mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \varepsilon\right] \leq \frac{\sigma^2}{n\varepsilon^2}$$

which means that as $n \to \infty$, the probability of the sample mean deviating from the true mean by any $\varepsilon > 0$ approaches zero.

1 Probabilistic Bounds

Note 22

A random variable X has variance $\text{Var}(X) = 9$ and expectation $\mathbb{E}[X] = 2$. Furthermore, the value of X is never greater than 10.

Use Markov's or Chebyshev's inequalities to provide bounds on the probabilities. Remember that Markov's inequality requires a non-negative random variable, and Chebyshev's inequality provides a bound on the absolute deviation from the mean $|X - \mu|$.

Hint: If you want to use Markov's inequality, use information in the problem statement to define a new random variable Y that is non-negative.

(a) $\mathbb{P}[X \leq 1] \leq 8/9$.

(b) $\mathbb{P}[X \geq 6] \leq 9/16$.

2 Vegas

Note 22

On the planet Vegas, everyone carries a coin. Many people are honest and carry a fair coin (heads on one side and tails on the other), but a fraction p of them cheat and carry a trick coin with heads on both sides. You want to estimate p with the following experiment: you pick a random sample of n people and ask each one to flip their coin. Assume that each person is independently likely to carry a fair or a trick coin.

(a) Let X be the proportion of coin flips which are heads. Find $\mathbb{E}[X]$.

(b) Given the results of your experiment, how should you estimate p? (*Hint:* Construct an unbiased estimator for p using part (a). Recall that $\hat{p}$ is an unbiased estimator if $\mathbb{E}[\hat{p}] = p$.)

(c) How many people do you need to ask to be 95% sure that your answer is off by at most 0.05?

3 Working with the Law of Large Numbers

Note 22

(a) A fair coin is tossed multiple times and you win a prize if there are more than 60% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

(b) A fair coin is tossed multiple times and you win a prize if there are more than 40% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

(c) A fair coin is tossed multiple times and you win a prize if there are between 40% and 60% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

(d) A fair coin is tossed multiple times and you win a prize if there are exactly 50% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.