1 Calculate These... or Else

Note 14

(a) A straight is defined as a 5 card hand such that the card values can be arranged in consecutive ascending order (i.e. $\{8,9,10,J,Q\}$ is a straight). Values do not loop around, so $\{Q,K,A,2,3\}$ is not a straight. However, an ace counts as both a low card and a high card, so both $\{A,2,3,4,5\}$ and $\{10,J,Q,K,A\}$ are considered straights.

When drawing a 5 card hand, what is the probability of drawing a straight from a 52-card deck?

(b) What is the probability of drawing a straight or a flush? (A flush is five cards of the same suit.)

(c) When drawing a 5 card hand, what is the probability of drawing at least one card from each suit?

(d) Two distinct squares are chosen at random on an 8×8 chessboard. What is the probability that they share a side?

(e) 8 rooks are placed randomly on an 8×8 chessboard. What is the probability none of them are attacking each other? (Two rooks attack each other if they are in the same row, or in the same column.)

Solution:

(a) The probability space is uniform over all possible 5-card hands, so we can use counting to solve this problem. There are $\binom{52}{5}$ possible hands, so that is our denominator. To count the number of possible straights, note that there are 4 choices of suit for each of the cards for a total of 4^5 suit choices. Also, observe that once we pick a starting card for the straight, the rest of the cards are determined (e.g. if we choose 3 as the first card, then our straight must be $\{3,4,5,6,7\}$). Therefore, we need to multiply by the number of possible starting cards.

$$\frac{10 \cdot 4^5}{\binom{52}{5}} \approx 0.00394.$$

(b) From part (a), we already know the probability of drawing a straight. We count the number of flushes in the following way: there are 4 choices for the suit, and $\binom{13}{5}$ choices for the cards within the suit, for a total of $4\binom{13}{5}$ flushes. Therefore, the probability of a flush is $4\binom{13}{5}/\binom{52}{5}$.

However, by the inclusion-exclusion principle, we must subtract the number of straight flushes. We now count the number of straight flushes: there are 4 choices for the suit and 10 choices for the starting card, for a total of 40 straight flushes.

The probability of drawing a straight or a flush is therefore:

$$\frac{10 \cdot 4^5 + 4 \cdot \binom{13}{5} - 40}{\binom{52}{5}} \approx 0.00591.$$

(c) A 5-card hand with at least one card from each suit will have 2 cards with the same suit and 3 cards with the other 3 suits. To count the number of such hands: there are 4 choices for which suit has 2 cards, $\binom{13}{2}$ choices for the 2 cards with that suit, and $13 \cdot 13 \cdot 13 = 13^3$ choices for the 3 cards with the 3 other suits.

Thus, the probability of drawing a 5-card hand with at least one card from each suit is:

$$\frac{4 \cdot \binom{13}{2} \cdot 13^3}{\binom{52}{5}} \approx 0.264$$

(d) In 64 squares, there are:

(1) 4 at-corner squares, each shares ONLY 2 sides with other squares.

(2) $6 \cdot 4 = 24$ side squares, each shares ONLY 3 sides with other squares.

(3) $6 \cdot 6 = 36$ inner squares, each shares 4 sides with other squares.

Notice that the three cases are mutually exclusive and we cannot choose the same square twice. So we just sum up the probabilities.

$$\frac{4}{64} \cdot \frac{2}{63} + \frac{24}{64} \cdot \frac{3}{63} + \frac{36}{64} \cdot \frac{4}{63} = \frac{1}{18} \approx 0.0556.$$

Alternatively, there are $\binom{64}{2}$ total pairs of squares, and for every pair of adjacent squares there is a unique edge associated with the pair (the edge that they share). Therefore, we can count the total number of edges in the chessboard, which is $8 \cdot 7 \cdot 2$ (if we only look at the horizontal edges, there are 8 edges per row, and 7 rows of edges, and then we multiply by 2 for the vertical edges too). Thus the probability is $8 \cdot 7 \cdot 2 / \binom{64}{2} = 1/18$ as before.

(e) To have none of the rooks attack each other, we need all the rooks to be in different rows and columns. Suppose we assign one rook per row.

There are 8 choices for the rook in the first row. Once the first-row rook is placed, there are 7 choices for the rook in the second row (namely any square not in the same column as the first-row rook). Then the rook in the third row has 6 choices, and so on. In total, there are $8 \cdot 7 \cdot 6 \cdots 1 = 8!$ ways to arrange the rooks without them attacking each other.

With $\binom{64}{8}$ ways to put 8 rooks on a chessboard, the probability that none of the rooks attack each other is

$$\frac{8!}{\binom{64}{8}} \approx 9.11 \cdot 10^{-6}$$

2 Box of Marbles

Note 14

You are given two boxes: one of them containing 900 red marbles and 100 blue marbles, the other one contains 500 red marbles and 500 blue marbles.

(a) If we pick one of the boxes randomly, and pick a marble what is the probability that it is blue?

(b) If we see that the marble is blue, what is the probability that it is chosen from box 1?

(c) Suppose we pick one marble from box 1 and without looking at its color we put it aside. Then we pick another marble from box 1. What is the probability that the second marble is blue?

Solution:

(a) Let B be the event that the picked marble is blue, R be the event that it is red, A_1 be the event that the marble is picked from box 1, and A_2 be the event that the marble is picked from box 2. Therefore we want to calculate $\mathbb{P}[B]$. By total probability,

$$\mathbb{P}[B] = \mathbb{P}[B \mid A_1]P[A_1] + \mathbb{P}[B \mid A_2]\,\mathbb{P}[A_2] = 0.5 \times 0.1 + 0.5 \times 0.5 = 0.3.$$

(b) In this part, we want to find $\mathbb{P}[A_1 \mid B]$. By Bayes rule,

$$\mathbb{P}[A_1 \mid B] = \frac{\mathbb{P}[B \mid A_1]\,\mathbb{P}[A_1]}{\mathbb{P}[B \mid A_1]\,\mathbb{P}[A_1] + \mathbb{P}[B \mid A_2]\,\mathbb{P}[A_2]} = \frac{0.1 \times 0.5}{0.5 \times 0.1 + 0.5 \times 0.5} = \frac{1}{6}.$$

(c) Let B_1 be the event that first marble is blue, R_1 be the event that the first marble is red, and B_2 be the event that second marble is blue without looking at the color of first marble. We want to find $\mathbb{P}[B_2]$. By total probability,

$$\mathbb{P}[B_2] = \mathbb{P}[B_2 \mid B_1]\,\mathbb{P}[B_1] + \mathbb{P}[B_2 \mid R_1]\,\mathbb{P}[R_1] = \frac{99}{999} \times 0.1 + \frac{100}{999} \times 0.9 = 0.1.$$

More generally, one can see that the probability that the n-th marble picked from box 1 is blue with probability 0.1. This is clear by symmetry: all the permutations of the 1000 marbles have the same probability, so the probability that the n-th marble is blue is the same as the probability that the first marble is blue.

3 Professional Crastination

Note 18
Note 19

You are a college student with a busy life, so your CS70 homework may be pushed aside. Each day, you have a 0.24 probability of working on your CS70 homework (once), independent of other days.

(a) On average, how many days does it take **before** you start working on homework *for the first time*?

(b) In the last 7 days, what is the probability that you do *not* work on your homework every day?

(c) You notice that your sleep schedule is suboptimal, and each *hour*, you have a 0.01 probability of working on your CS70 homework (once), independent of other hours.

In the last 7 days, how many times do you work on your homework, on average? (Recall that there are 24 hours in a day.)

(d) Every 7 days, you work on a new homework assignment. What is the probability that you do not work on *all four* of the last four homework assignments (at least once)?

Solution:

(a) Let X be a random variable denoting the number of days until you work on homework for the first time. $X \sim \text{Geometric}(0.24)$, so $\mathbb{E}[X] = \frac{1}{0.24}$, which is slightly more than 4 days.

X includes the day you worked on homework, so $Y = X - 1$ would denote the number of days before you work on homework for the first time. Using linearity of expectation,

$$\mathbb{E}[Y] = \mathbb{E}[X-1] = \mathbb{E}[X] - 1 = \frac{1}{0.24} - 1 \approx 3.16$$

(b) Let D be a random variable denoting the number of days you work on your homework in the last 7 days. $D \sim \text{Binomial}(7, 0.24)$. We want to find $\mathbb{P}[D \neq 7]$, the probability that we do not work on homework for 7 (out of 7) days.

$$\mathbb{P}[D \neq 7] = 1 - \mathbb{P}[D = 7] = 1 - (0.24)^7$$

(c) With H denoting the number of times you work on your homework, $H \sim \text{Binomial}(168, 0.01)$. (There are $7 \cdot 24 = 168$ hours in 7 days). We want to find $\mathbb{E}[H]$, which is $168 \cdot 0.01 = 1.68$.

(d) Let N denote the number of homeworks you do work on. $N \sim \text{Binomial}(4, p)$, where p is the probability that you work on an individual homework assignment. We want to find $\mathbb{P}[N \neq 4]$, but first we need to find p.

$p = \mathbb{P}[D \geq 1]$, as D is the number of days you work on homework in the last 7 days. Thus,

$$p = \mathbb{P}[D \geq 1] = 1 - \mathbb{P}[D = 0] = 1 - (1 - 0.24)^7$$

Then $\mathbb{P}[N \neq 4] = 1 - \mathbb{P}[N = 4] = 1 - p^4 = 1 - (1 - (1 - 0.24)^7)^4$, which is approximately 0.469. It's more likely for you to work on all four homeworks, so good job!

4 Balls in Bins

Note 19

You are throwing k balls into n bins. Let X_i be the number of balls thrown into bin i.

(a) What is $\mathbb{E}[X_i]$?

(b) What is the expected number of empty bins?

(c) Define a collision to occur when a ball lands in a nonempty bin. What is the expected number of collisions? *Hint: If there are n balls in a bin, $n-1$ collisions have occurred in that bin.*

Solution:

(a) We will use linearity of expectation. Note that the expectation of an indicator variable is just the probability the indicator variable = 1. (Verify for yourself that is true).

$$\begin{aligned}\mathbb{E}[X_i] &= \mathbb{P}[\text{ball 1 falls into bin } i] + \mathbb{P}[\text{ball 2 falls into bin } i] + \cdots + \mathbb{P}[\text{ball } k \text{ falls into bin } i] \\ &= \frac{1}{n} + \cdots + \frac{1}{n} = \frac{k}{n}.\end{aligned}$$

(b) Let I_i be the indicator variable denoting whether bin i ends up empty. This can happen if and only if all the balls end in the remaining $n-1$ bins, and this happens with a probability of $\left(\frac{n-1}{n}\right)^k$. Hence the expected number of empty bins is

$$\mathbb{E}[I_1 + \ldots + I_n] = \mathbb{E}[I_1] + \ldots + \mathbb{E}[I_n] = n\left(\frac{n-1}{n}\right)^k$$

(c) The number of collisions is the number of balls minus the number of occupied bins, since the first ball of every occupied bin is not a collision.

$$\begin{aligned}\mathbb{E}[\text{collisions}] &= k - \mathbb{E}[\text{occupied bins}] = k - n + \mathbb{E}[\text{empty locations}] \\ &= k - n + n\left(1 - \frac{1}{n}\right)^k\end{aligned}$$

5 Diversify Your Hand

Note 19
Note 20

You are dealt 5 cards from a standard 52 card deck. Let X be the number of distinct values in your hand. For instance, the hand (A, A, A, 2, 3) has 3 distinct values.

(a) Calculate $\mathbb{E}[X]$. (Hint: Consider indicator variables X_i representing whether i appears in the hand.)

(b) Calculate $\text{Var}(X)$. The answer expression will be quite involved; you do not need to simplify anything.

Solution:

(a) Let X_i be the indicator of the ith value appearing in your hand. Then, $X = X_1 + X_2 + \ldots + X_{13}$. (Here we let 13 correspond to K, 12 correspond to Q, and 11 correspond to J.) By linearity of expectation, $\mathbb{E}[X] = \sum_{i=1}^{13} \mathbb{E}[X_i]$.

We can calculate $\mathbb{P}[X_i = 1]$ by taking the complement, $1 - \mathbb{P}[X_i = 0]$, or 1 minus the probability that the card does not appear in your hand. This is $1 - \frac{\binom{48}{5}}{\binom{52}{5}}$.

Then, $\mathbb{E}[X] = 13\,\mathbb{P}[X_1 = 1] = 13\left(1 - \frac{\binom{48}{5}}{\binom{52}{5}}\right)$.

(b) To calculate variance, since the indicators are not independent, we have to use the formula $\mathbb{E}[X^2] = \sum_{i=j} \mathbb{E}[X_i^2] + \sum_{i \neq j} \mathbb{E}[X_i X_j]$.

First, we have

$$\sum_{i=j} \mathbb{E}[X_i^2] = \sum_{i=j} \mathbb{E}[X_i] = 13\left(1 - \frac{\binom{48}{5}}{\binom{52}{5}}\right).$$

Next, we tackle $\sum_{i \neq j} \mathbb{E}[X_i X_j]$. Note that $\mathbb{E}[X_i X_j] = \mathbb{P}[X_i X_j = 1]$, as $X_i X_j$ is either 0 or 1.

To calculate $\mathbb{P}[X_i X_j = 1]$ (the probability we have both cards in our hand), we note that $\mathbb{P}[X_i X_j = 1] = 1 - \mathbb{P}[X_i = 0] - \mathbb{P}[X_j = 0] + \mathbb{P}[X_i = 0, X_j = 0]$. Then

$$\begin{aligned}
\sum_{i \neq j} \mathbb{E}[X_i X_j] &= 13 \cdot 12\,\mathbb{P}[X_i X_j = 1] \\
&= 13 \cdot 12(1 - \mathbb{P}[X_i = 0] - \mathbb{P}[X_j = 0] + \mathbb{P}[X_i = 0, X_j = 0]) \\
&= 156\left(1 - 2\frac{\binom{48}{5}}{\binom{52}{5}} + \frac{\binom{44}{5}}{\binom{52}{5}}\right)
\end{aligned}$$

Putting it all together, we have

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= 13\left(1 - \frac{\binom{48}{5}}{\binom{52}{5}}\right) + 156\left(1 - 2\frac{\binom{48}{5}}{\binom{52}{5}} + \frac{\binom{44}{5}}{\binom{52}{5}}\right) - \left(13\left(1 - \frac{\binom{48}{5}}{\binom{52}{5}}\right)\right)^2.
\end{aligned}$$