

Due: Monday 12/8, 4:00 PM  
Grace period until Monday 12/8, 6:00PM  
Remember to show your work for all problems!

## Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.) If you used an LLM, place transcripts of your chats here.

## 1 Chebyshev's Inequality vs. Central Limit Theorem

Note 23  
Note 24

Let  $n$  be a positive integer. Let  $X_1, X_2, \dots, X_n$  be i.i.d. random variables with the following distribution:

$$\mathbb{P}[X_i = -1] = \frac{1}{12}; \quad \mathbb{P}[X_i = 1] = \frac{9}{12}; \quad \mathbb{P}[X_i = 2] = \frac{2}{12}.$$

- (a) Calculate the expectations and variances of  $X_1, \sum_{i=1}^n X_i, \sum_{i=1}^n (X_i - \mathbb{E}[X_i])$ , and

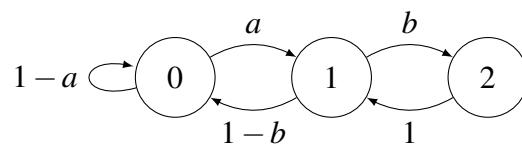
$$Z_n = \frac{\sum_{i=1}^n (X_i - \mathbb{E}[X_i])}{\sqrt{n/2}}.$$

- (b) Use Chebyshev's Inequality to find an upper bound  $b$  for  $\mathbb{P}[|Z_n| \geq 2]$ .  
(c) Use  $b$  from the previous part to bound  $\mathbb{P}[Z_n \geq 2]$  and  $\mathbb{P}[Z_n \leq -2]$ .  
(d) As  $n \rightarrow \infty$ , what is the distribution of  $Z_n$ ?  
(e) We know that if  $Z \sim \mathcal{N}(0, 1)$ , then  $\mathbb{P}[|Z| \leq 2] = \Phi(2) - \Phi(-2) \approx 0.9545$ . As  $n \rightarrow \infty$ , provide approximations for  $\mathbb{P}[Z_n \geq 2]$  and  $\mathbb{P}[Z_n \leq -2]$ .

## 2 Analyze a Markov Chain

Note 25

Consider a Markov chain with the state diagram shown below where  $a, b \in (0, 1)$ .



Here, we let  $X(n)$  denote the state at time  $n$ .

- (a) Is this Markov chain irreducible? Is this Markov chain aperiodic? Justify your answers.
- (b) Calculate  $\mathbb{P}[X(1) = 1, X(2) = 0, X(3) = 1, X(4) = 2 \mid X(0) = 0]$ .
- (c) Calculate the invariant distribution. Do all initial distributions converge to this invariant distribution? Justify your answer.

## 3 A Bit of Everything

Note 25  
Note 26

Suppose that  $X_0, X_1, \dots$  is a Markov chain with finite state space  $S = \{1, 2, \dots, n\}$ , where  $n > 2$ , and transition matrix  $P$ . Suppose further that

$$\begin{aligned} P(1, i) &= \frac{1}{n} \quad \text{for all states } i \text{ and} \\ P(j, j-1) &= 1 \quad \text{for all states } j \neq 1, \end{aligned}$$

with  $P(i, j) = 0$  everywhere else.

- (a) Prove that this Markov chain is irreducible and aperiodic.
- (b) Suppose you start at state 1. What is the distribution of  $T$ , where  $T$  is the number of transitions until you leave state 1 for the first time?
- (c) Again starting from state 1, what is the expected number of transitions until you reach state  $n$  for the first time?
- (d) Again starting from state 1, what is the probability you reach state 2 before you reach state  $n$ ?
- (e) Compute the stationary distribution of this Markov chain.

## 4 Playing Blackjack

Note 25

Suppose you start with \$1, and at each turn, you win \$1 with probability  $p$ , or lose \$1 with probability  $1 - p$ . You will continually play games of Blackjack until you either lose all your money, or you have a total of  $n$  dollars.

- (a) Formulate this problem as a Markov chain.
- (b) Let  $\alpha(i)$  denote the probability that you end the game with  $n$  dollars, given that you started with  $i$  dollars.

Notice that for  $0 < i < n$ , we can write  $\alpha(i+1) - \alpha(i) = k(\alpha(i) - \alpha(i-1))$ . Find  $k$ .

- (c) Using part (b), find  $\alpha(i)$ , where  $0 \leq i \leq n$ . (You will need to split into two cases:  $p = \frac{1}{2}$  or  $p \neq \frac{1}{2}$ .)

*Hint:* Try to apply part (b) iteratively, and look at a telescoping sum to write  $\alpha(i)$  in terms of  $\alpha(1)$ . The formula for the sum of a finite geometric series may be helpful when looking at

the case where  $p \neq \frac{1}{2}$ :

$$\sum_{k=0}^m a^k = \frac{1-a^{m+1}}{1-a}.$$

Lastly, it may help to use the value of  $\alpha(n)$  to find  $\alpha(1)$  for the last few steps of the calculation.

- (d) As  $n \rightarrow \infty$ , what happens to the probability of ending the game with  $n$  dollars, given that you start with  $i$  dollars, with the following values of  $p$ ?

- (i)  $p > \frac{1}{2}$
- (ii)  $p = \frac{1}{2}$
- (iii)  $p < \frac{1}{2}$