1.1 Linear Systems and Their Solutions

Definition 1.1.9 A system of linear equations that has no solution is said to be *inconsistent*. A system that has at least one solution is called *consistent*.

Remark 1.1.10 Every system of linear equations has either *no solution*, *only one solution*, or *infinitely many solutions*.

1.2 Elementary Row Operations

Definition 1.2.6 Two augmented matrices are said to be *row equivalent* if one can be obtained from the other by a series of EROs.

Theorem 1.2.7 If augmented matrices of two linear systems are row equivalent, then the two systems have the same set of solutions.

Remark 1.2.9 EROs do not change the solution set.

1.3 Row-Echelon Forms

Definition 1.3.1 Row-Echelon Form (REF):

1. All zero rows are grouped together at the bottom.
2. For two successive non-zero rows, the leading entry in the lower row occurs farther to the right than that in the higher row.

Reduced Row-Echelon Form (RREF):

3. The leading entry of every nonzero row is 1.
4. In each pivot column, except the pivot point, all other entries are 0.

1.4 Gaussian Elimination

Algorithm 1.4.2 Gaussian Elimination

1. Locate leftmost non-zero column.
2. Interchange top row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.
3. Add a multiple of top row to rows below to make all other entries in that column 0.
4. Cover top row and repeat with the submatrix.

Algorithm 1.4.3 Gauss-Jordan Elimination

5. Multiply each row to make leading entries 1.
6. Start from the last nonzero row and add multiples of it to rows above to introduce zeros.

Remark 1.4.5 1. Each matrix has a unique RREF, but many REFs.

Remark 1.4.8

1. If the last column of the REF is a pivot column, the linear system is inconsistent i.e. has no solutions.
2. If except the last column, every column of the REF is a pivot column, the linear system is consistent and has only one solution.
3. If apart from the last column, the REF has at least one more non-pivot column, the linear system is consistent and has infinitely many solutions. In other words, the number of variables is greater than the number of nonzero rows.

1.5 Homogeneous Linear Systems

Definition 1.5.1 Homogeneous system of linear equations has the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \quad\vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

$x_1 = 0, x_2 = 0, \dots, x_n = 0$ is always a solution, called the *trivial solution*. Any other solution is a *non-trivial solution*.

Remark 1.5.4

1. A homogeneous system has either only the trivial solution or infinitely many solutions in addition to the trivial solution.
2. A homogeneous system with more unknowns than equations has infinitely many solutions.

2.1 Introduction to Matrices

Definition 2.1.3 A *column matrix* or *column vector* is a matrix with only one column. A *row matrix* or *row vector* is a matrix with only one row.

Definition 2.1.7

1. A matrix is called a *square matrix* if it has the same number of rows and columns. A $n \times n$ square matrix is called a square matrix of *order* n.
2. The *diagonal* of a matrix is the sequence of entries $a_{11}, a_{22}, \dots, a_{nn}$. They are called *diagonal entries*, and the other entries are *non-diagonal entries*. A square matrix is a *diagonal matrix* if all of its non-diagonal entries are 0.
3. If all diagonal entries are the same, it is a *scalar matrix*.
4. If all diagonal entries are 1, it is an *identity matrix*, represented as $\boldsymbol{I_n}$ or just $\boldsymbol{I}$.
5. A matrix with all entries equal to zero is called a *zero matrix*, represented by $\boldsymbol{0}_{m\times n}$ or just $\boldsymbol{0}$.
6. A square matrix is called symmetric if $a_{ij} = a_{ji}$ for all i, j.
7. A square matrix is upper triangular if $a_{ij} = 0$ whenever $i > j$; lower triangular if $a_{ij} = 0$ whenever $j > i$. Both are *triangular matrices*.

2.2 Matrix Operations

Definition 2.2.1 Two matrices are equal if they have the same size and their corresponding entries are equal.

Definition 2.2.3 Let $\boldsymbol{A} = (a_{ij})_{m\times n}, \boldsymbol{B} = (b_{ij})_{m\times n}$, c a real constant.

1. (Matrix Addition) $\boldsymbol{A} + \boldsymbol{B} = (a_{ij} + b_{ij})_{m\times n}$.
2. (Matrix Subtraction) $\boldsymbol{A} - \boldsymbol{B} = (a_{ij} - b_{ij})_{m\times n}$.
3. (Scalar Multiplication) $c\boldsymbol{A} = (ca_{ij})_{m\times n}$, c is a scalar.

Theorem 2.2.6 Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be matrices of the same size, c, d scalars.

1. (Commutative Law) $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$.
2. (Associative Law) $\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$.
3. $c(\boldsymbol{A} + \boldsymbol{B}) = c\boldsymbol{A} + c\boldsymbol{B}$.
4. $(c + d)\boldsymbol{A} = c\boldsymbol{A} + d\boldsymbol{A}$.
5. $c(d\boldsymbol{A}) = (cd)\boldsymbol{A} = d(c\boldsymbol{A})$.
6. $\boldsymbol{A} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{A} = \boldsymbol{A}$,
7. $\boldsymbol{A} - \boldsymbol{A} = \boldsymbol{0}$.
8. $0\boldsymbol{A} = \boldsymbol{0}$ (LHS 0 is the number, RHS $\boldsymbol{0}$ is a zero matrix).

Definition 2.2.8 Let $\boldsymbol{A} = (a_{ij})_{m\times p}$ and $\boldsymbol{B} = (b_{ij})_{p\times n}$. The product $\boldsymbol{AB}$ is defined to be an $m \times n$ matrix whose (i, j)-entry is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

Remark 2.2.10

1. We can only multiply two matrices when the number of columns of the first matrix is equals to the number of rows of the second.
2. Matrix multiplication is not commutative, i.e. in general $\boldsymbol{AB} \neq \boldsymbol{BA}$.
3. $\boldsymbol{AB}$ is the *pre-multiplication* of $\boldsymbol{A}$ to $\boldsymbol{B}$, and $\boldsymbol{BA}$ is the *post-multiplication*.
4. $\boldsymbol{AB} = \boldsymbol{0}$ does not imply $\boldsymbol{A} = \boldsymbol{0}$ or $\boldsymbol{B} = \boldsymbol{0}$.

Theorem 2.2.11

1. (Associative Law) $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$.
2. (Distributive Law) $\boldsymbol{A}(\boldsymbol{B_1} + \boldsymbol{B_2}) = \boldsymbol{AB_1} + \boldsymbol{AB_2}$ and $(\boldsymbol{C_1} + \boldsymbol{C_2})\boldsymbol{A} = \boldsymbol{C_1A} + \boldsymbol{C_2A}$.
3. $c(\boldsymbol{AB}) = (c\boldsymbol{A})\boldsymbol{B} = \boldsymbol{A}(c\boldsymbol{B})$, where c is a scalar.
4. $\boldsymbol{A0}_{n\times q} = \boldsymbol{0}_{m\times q}$, $\boldsymbol{0}_{p\times m}\boldsymbol{A} = \boldsymbol{0}_{p\times n}$ and $\boldsymbol{AI_n} = \boldsymbol{I_mA} = \boldsymbol{A}$.

Definition 2.2.12: Let $\boldsymbol{A}$ be a square matrix and n a nonnegative integer.

$$\boldsymbol{A}^n = \begin{cases} \boldsymbol{I}, & n = 0 \\ \boldsymbol{AA}\dots\boldsymbol{A}\ (n\ times), & n \geq 1 \end{cases}$$

Remark 2.2.14

1. $\boldsymbol{A^mA^n} = \boldsymbol{A^{m+n}}$, where m and n are nonnegative integers.
2. Since the matrix multiplication is not commutative, in general, for square matrices of the same order, $(\boldsymbol{AB})^n \neq \boldsymbol{A^nB^n}$.

Not 2.2.15 Let

$$\boldsymbol{A} = (a_{ij})_{m\times p} = \begin{pmatrix} \boldsymbol{a_1} \\ \boldsymbol{a_2} \\ \vdots \\ \boldsymbol{a_m} \end{pmatrix}, \boldsymbol{a_i} = (a_{i1}\ a_{i2} \cdots a_{ip})$$

and let

$$\boldsymbol{B} = (b_{ij})_{p\times n} = (\boldsymbol{b_1}\ \boldsymbol{b_2} \dots \boldsymbol{b_n}), \boldsymbol{b_j} = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{pmatrix}$$

Then

$$\boldsymbol{AB} = \begin{pmatrix} \boldsymbol{a_1b_1} & \cdots & \boldsymbol{a_1b_n} \\ \vdots & \ddots & \vdots \\ \boldsymbol{a_mb_1} & \cdots & \boldsymbol{a_mb_n} \end{pmatrix}$$

where $\boldsymbol{a_ib_j} = (a_{i1}\ a_{i2} \dots a_{ip})\begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{pmatrix} = a_{i1}b_{1j} + \cdots + a_{ip}b_{pj}$.

We can also write $\boldsymbol{AB} = (\boldsymbol{Ab_1}\ \boldsymbol{Ab_2} \cdots \boldsymbol{Ab_n})$, or $\boldsymbol{AB} = \begin{pmatrix} \boldsymbol{a_1B} \\ \boldsymbol{a_2B} \\ \vdots \\ \boldsymbol{a_mB} \end{pmatrix}$

Remark 2.2.17 We can rewrite a system of linear equations as $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A}$ is the *coefficient matrix*, $\boldsymbol{x}$ is the *variable matrix* and $\boldsymbol{b}$ is the *constant matrix*.

Definition 2.2.19 Let $\boldsymbol{A} = (a_{ij})$ be an $m \times n$ matrix. The *transpose* of $\boldsymbol{A}$, denoted by $\boldsymbol{A^T}$ or $\boldsymbol{A^t}$, is the $n \times m$ matrix whose (i, j)-entry is a_{ji}.

Remark 2.2.21 2. A square matrix $\boldsymbol{A}$ is symmetric iff $\boldsymbol{A} = \boldsymbol{A^T}$.

Theorem 2.2.22 Let $\boldsymbol{A}$ be an $m \times n$ matrix.

1. $(\boldsymbol{A^T})^T = \boldsymbol{A}$.
2. If $\boldsymbol{B}$ is a $m \times n$ matrix, then $(\boldsymbol{A} + \boldsymbol{B})^{\boldsymbol{T}} = \boldsymbol{A^T} + \boldsymbol{B^T}$.
3. $(c\boldsymbol{A})^T = c\boldsymbol{A}^T$.
4. If $\boldsymbol{B}$ is a $n \times p$ matrix, then $(\boldsymbol{AB})^T = \boldsymbol{B^TA^T}$.

2.3 Inverses of Square Matrices

Definition 2.3.2 A *square* matrix $\boldsymbol{A}$ is *invertible* if there exists another square matrix $\boldsymbol{B}$ such that $\boldsymbol{AB} = \boldsymbol{I}$ and $\boldsymbol{BA} = \boldsymbol{I}$. $\boldsymbol{B}$ is called an *inverse* of $\boldsymbol{A}$. A square matrix is *singular* if it has no inverse.

Remark 2.3.4

1. (Cancellation Law) If $\boldsymbol{A}$ is an invertible $m \times m$ matrix,

(a) $\boldsymbol{AB_1} = \boldsymbol{AB_2} \rightarrow \boldsymbol{B_1} = \boldsymbol{B_2}$

(b) $\boldsymbol{C_1A} = \boldsymbol{C_2A} \to \boldsymbol{C_1} = \boldsymbol{C_2}$
2. The above law will not hold if $\boldsymbol{A}$ is singular.
Theorem 2.3.5 If $\boldsymbol{B}$ and $\boldsymbol{C}$ are inverses of square matrix $\boldsymbol{A}$, then $\boldsymbol{B} = \boldsymbol{C}$.
Notation 2.3.6 The symbol $\boldsymbol{A}^{-1}$ denotes the unique inverse of $\boldsymbol{A}$.
Example 2.3.8 Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $ad - bc \neq 0$, then $\boldsymbol{A}$ is invertible and $\boldsymbol{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$.
Theorem 2.3.9 Let $\boldsymbol{A}, \boldsymbol{B}$ be two invertible matrices of the same size and c a nonzero scalar.
1. $c\boldsymbol{A}$ is invertible and $(c\boldsymbol{A})^{-1} = \frac{1}{c}\boldsymbol{A}^{-1}$.
2. $\boldsymbol{A}^T$ is invertible and $(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T$.
3. $\boldsymbol{A}^{-1}$ is invertible and $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$.
4. $\boldsymbol{AB}$ is invertible and $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.
Remark 2.3.10 If $\boldsymbol{A_1}, \boldsymbol{A_2}, \ldots, \boldsymbol{A_k}$ are invertible matrices of the same size, then $\boldsymbol{A_1A_2}\ldots\boldsymbol{A_k}$ is invertible and $(\boldsymbol{A_1A_2}\ldots\boldsymbol{A_k})^{-1} = \boldsymbol{A_k^{-1}}\ldots\boldsymbol{A_2^{-1}A_1^{-1}}$.
Definition 2.3.11
$$\boldsymbol{A}^{-n} = (\boldsymbol{A}^{-1})^n = \boldsymbol{A}^{-1}\boldsymbol{A}^{-1}\ldots\boldsymbol{A}^{-1}(n\ times)$$
Remark 2.3.13 Let $\boldsymbol{A}$ be an invertible matrix.
1. $\boldsymbol{A}^r\boldsymbol{A}^s = \boldsymbol{A}^{r+s}$
2. $\boldsymbol{A}^n$ is invertible and $(\boldsymbol{A}^n)^{-1} = \boldsymbol{A}^{-n}$ for any integer n.

2.4 Elementary Matrices

Discussion 2.4.2
1. Multiply a row by a constant k:
$$\boldsymbol{E_1} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & k & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$
where the k is at the ith row and ith column. The inverse is k being replaced by $\frac{1}{k}$.
2. Interchange two rows:
$$\boldsymbol{E_2} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} & \ddots & \ddots & \ddots & \mathbf{1} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 1 & \ddots & \ddots & \ddots & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & \ddots & \ddots & \ddots & 1 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \mathbf{1} & \ddots & \ddots & \ddots & \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$
where the highlighted 1s and 0s are at the intersection points between the ith row, jth row, ith column and jth column. This results in the ith and jth row of the matrix that $\boldsymbol{E_2}$ is pre-multiplied to have its ith and jth row interchange. The inverse of $\boldsymbol{E_2}$ is $\boldsymbol{E_2}$, since it will switch the rows back.
3. Add a multiple of one row to another:
$$\boldsymbol{E_3} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & k & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & 0 & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}, i < j$$
$$\boldsymbol{E_3} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & \ddots & k & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}, i > j$$
$\boldsymbol{E_3A}$ is the matrix obtained from $\boldsymbol{A}$ by adding k times of the ith row of $\boldsymbol{A}$ to the jth row. The inverse of $\boldsymbol{E_3}$ is the same except for k changing to $-k$.
Definition 2.4.3 A square matrix is called an *elementary matrix* if it can be obtained from an identity matrix by performing a single ERO.
Remark 2.4.4
1. Every elementary matrix is one of the three above types.
2. All elementary matrices are invertible, and their inverses are also elementary matrices.
Theorem 2.4.7 If $\boldsymbol{A}$ is a square matrix, the following statements are equivalent:
1. $\boldsymbol{A}$ is invertible
2. The linear system $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.
3. The RREF of $\boldsymbol{A}$ is an identity matrix.
4. $\boldsymbol{A}$ can be expressed as a product of elementary matrices.
Discussion 2.4.8 We can use EROs to transform $(\boldsymbol{A}|\boldsymbol{I})$ to $(\boldsymbol{I}|\boldsymbol{A}^{-1})$.
Remark 2.4.10 If the REF of a square matrix has no zero row, the matrix is invertible; otherwise, it is singular.
Theorem 2.4.12 Let $\boldsymbol{A}, \boldsymbol{B}$ be square matrices of the same size. If $\boldsymbol{AB} = \boldsymbol{I}$, then $\boldsymbol{A}, \boldsymbol{B}$ are both invertible, $\boldsymbol{A}^{-1} = \boldsymbol{B}$, $\boldsymbol{B}^{-1} = \boldsymbol{A}$ and $\boldsymbol{BA} = \boldsymbol{I}$.
Theorem 2.4.14 Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrices of the same order. If $\boldsymbol{A}$ is singular, then $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are singular.
Discussion 2.4.15 Post-multiplying elementary matrices result in *elementary column operations.*

2.5 Determinants

Definition 2.5.2 The determinant of $\boldsymbol{A}$ is defined as
$$det(\boldsymbol{A}) = \begin{cases} a_{11}, & if\ n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}, & if\ n > 1 \end{cases}$$
where $A_{ij} = (-1)^{i+j}\det(\boldsymbol{M_{ij}})$, and $\boldsymbol{M_{ij}}$ is a $(n-1)\times(n-1)$ matrix obtained from $\boldsymbol{A}$ by deleting the ith row and jth column. The number A_{ij} is called the (i,j)-cofactor of $\boldsymbol{A}$.
Not 2.5.3 $\det(\boldsymbol{A})$ is usually written as
$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$
Remark 2.5.5 Let $\boldsymbol{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. Then
$$\det(\boldsymbol{A}) = aei + bfg + cdh - ceg - afh - bdi$$
Theorem 2.5.6 Cofactor Expansions $\det(\boldsymbol{A})$ can be expressed as a cofactor expansion along any row or column:
$$\det(A) = a_{i1}A_{i1} + \cdots + a_{in}A_{in} = a_{ij}A_{ij} + \cdots + a_{nj}A_{nj}$$
for any row $i = 1,2,\ldots n$ and column $j = 1,2,\ldots n$.
Theorem 2.5.8 The determinant of a triangular matrix is the product of the diagonal entries.
Theorem 2.5.10 For square matrix $\boldsymbol{A}$, $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^T)$.
Theorem 2.5.12
1. The det of a square matrix with two identical rows is zero.
2. The det of a square matrix with two identical columns is zero.
Theorem 2.5.15 Let $\boldsymbol{A}$ be a square matrix.
1. If $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by multiplying one row of $\boldsymbol{A}$ by a constant, then $\det(\boldsymbol{B}) = k\det(\boldsymbol{A})$.
2. If $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by interchanging two rows of $\boldsymbol{A}$, then $\det(\boldsymbol{B}) = -\det(\boldsymbol{A})$.
3. If $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by adding a multiple of one row of $\boldsymbol{A}$ to another row, then $\det(\boldsymbol{B}) = \det(\boldsymbol{A})$.
4. Let $\boldsymbol{E}$ be an elementary matrix of the same size as $\boldsymbol{A}$. Then $\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$.
Remark 2.5.16 By Theorem 2.5.15, We can use EROs to transform a square matrix to a triangular matrix then calculate the determinant by Theorem 2.5.8.
Remark 2.5.18 The 4 rules above still hold if 'rows' are changed to 'columns'.
Theorem 2.5.19 A square matrix A is invertible iff $\det(\boldsymbol{A}) \neq 0$.
Theorem 2.5.22 Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two square matrices of order n and c a scalar. Then
1. $\det(c\boldsymbol{A}) = c^n\det(\boldsymbol{A})$;
2. $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$; and
3. If $\boldsymbol{A}$ is invertible, $\det(\boldsymbol{A}^{-1}) = \frac{1}{\det(\boldsymbol{A})}$.
Definition 2.5.24 Adjoint Let $\boldsymbol{A}$ be a square matrix of order n. Then the (*classical*) *adjoint* of $\boldsymbol{A}$ is the $n \times n$ matrix
$$\mathbf{adj}(\boldsymbol{A}) = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}^T = \begin{pmatrix} A_{11} & \cdots & A_{n1} \\ \vdots & \ddots & \vdots \\ A_{1n} & \cdots & A_{nn} \end{pmatrix}$$
where A_{ij} is the (i,j)-cofactor of $\boldsymbol{A}$.
Theorem 2.5.25 Let $\boldsymbol{A}$ be a square matrix. If $\boldsymbol{A}$ is invertible, then
$$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\mathbf{adj}(\boldsymbol{A})$$
Theorem 2.5.27 Cramer's Rule Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ is a linear system where $\boldsymbol{A}$ is an $n \times n$ matrix. Let $\boldsymbol{A_i}$ be the matrix obtained from $\boldsymbol{A}$ by replacing the ith column of $\boldsymbol{A}$ by $\boldsymbol{b}$. If $\boldsymbol{A}$ is invertible, then the system only has one solution
$$\boldsymbol{x} = \frac{1}{\det(\boldsymbol{A})}\begin{pmatrix} \det(\boldsymbol{A_1}) \\ \det(\boldsymbol{A_2}) \\ \vdots \\ \det(\boldsymbol{A_n}) \end{pmatrix}$$

Tutorials and Exercises

Tutorial 2.3 (a) A diagonal matrix raised to a positive power is the same matrix with the diagonal entries raised to that power.
Tutorial 2.5 (a) $\boldsymbol{A}(\boldsymbol{B_1}\ \boldsymbol{B_2}) = (\boldsymbol{AB_1}\ \boldsymbol{AB_2})$ (c) $\begin{pmatrix} \boldsymbol{D_1} \\ \boldsymbol{D_2} \end{pmatrix}\boldsymbol{A} = \begin{pmatrix} \boldsymbol{D_1A} \\ \boldsymbol{D_2A} \end{pmatrix}$
(d) $\begin{pmatrix} \boldsymbol{A_1} & \boldsymbol{A_2} \\ \boldsymbol{A_3} & \boldsymbol{A_4} \end{pmatrix} + \begin{pmatrix} \boldsymbol{B_1} & \boldsymbol{B_2} \\ \boldsymbol{B_3} & \boldsymbol{B_4} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A_1} + \boldsymbol{B_1} & \boldsymbol{A_2} + \boldsymbol{B_2} \\ \boldsymbol{A_3} + \boldsymbol{B_3} & \boldsymbol{A_4} + \boldsymbol{B_4} \end{pmatrix}$
Tutorial 2.6 (b) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n \times n$ symmetric matrices and if $\boldsymbol{AB} = \boldsymbol{BA}$, then $\boldsymbol{AB}$ is symmetric. (d) If $\boldsymbol{A}$ is a matrix such that $\boldsymbol{AA}^T = \boldsymbol{0}$, then $\boldsymbol{A} = \boldsymbol{0}$.
Tutorial 3.2 If $\boldsymbol{A}^n = \boldsymbol{0}$ for $n > 1$, then $\boldsymbol{I} - \boldsymbol{A}$ is invertible and $(\boldsymbol{I} - \boldsymbol{A})^{-1} = \boldsymbol{I} + \boldsymbol{A} + \cdots + \boldsymbol{A}^{n-1}$.
Tutorial 3.5 $\boldsymbol{B}$ and $\boldsymbol{C}$ are row equivalent iff $\exists \boldsymbol{A}, \boldsymbol{AB} = \boldsymbol{C}$.
Tutorial 4.6 Let $\boldsymbol{A}$ be an invertible matrix. (a) $\mathbf{adj}(\boldsymbol{A})$ is invertible. (b) $\mathbf{adj}(\boldsymbol{A})^{-1} = \frac{1}{\det(\boldsymbol{A})}\boldsymbol{A}$. $\det(\mathbf{adj}(\boldsymbol{A})) = \det(\boldsymbol{A})^{n-1}$(c) If $\det(\boldsymbol{A}) = 1$, $\mathbf{adj}(\mathbf{adj}(\boldsymbol{A})) = \boldsymbol{A}$.
Exercise 2.11 The *trace* of square matrix $\boldsymbol{A}$ is the sum of its diagonal entries.
Exercise 2.12 A square matrix $\boldsymbol{A}$ is *orthogonal* if $\boldsymbol{AA^T} = \boldsymbol{I}$ and $\boldsymbol{A^TA} = \boldsymbol{I}$.
Exercise 2.13 A square matrix $\boldsymbol{A}$ is *nilpotent* if $\boldsymbol{A^k} = \boldsymbol{0}$ for some positive integer $\boldsymbol{k}$.
Exercise 2.24 (d) $(\boldsymbol{A} + \boldsymbol{B})^2 = \boldsymbol{A}^2 + \boldsymbol{AB} + \boldsymbol{BA} + \boldsymbol{B}^2$