UNIT II

Discrete Fourier Transforms

INTRODUCTION :The DFT of a discrete-time signal $x(n)$ is a finite duration discrete frequency sequence. The DFT sequence is denoted by $X(k)$. The DFT is obtained by sampling one period of the Fourier transform $X(W)$ of the signal $x(n)$ at a finite number of frequency points. This sampling is conventionally performed at N equally spaced points in the period $0 \leq w \leq 2w$ or at $w_k = 2\pi k/N$; $0 \leq k \leq N-1$. We can say that DFT is used for transforming discrete-time sequence $x(n)$ of finite length into discrete frequency sequence $X(k)$ of finite length. The DFT is important for two reasons. First it allows us to determine the frequency content of a signal, that is to perform spectral analysis. The second application of the DFT is to perform filtering operation in the frequency domain. Let $x(n)$ be a discrete-time sequence with Fourier transform $X(W)$, then the DFT of $x(n)$ denoted by $X(k)$ is defined as

$$X(k) = X(\omega)\big|_{\omega=(2\pi k/N)}; \quad \text{for } k = 0, 1, 2, \ldots, N-1$$

The DFT of $x(n)$ is a sequence consisting of N samples of $X(k)$. The DFT sequence starts at $k = 0$, corresponding to $w = 0$, but does not include $k = N$ corresponding to $w = 2\pi$ (since the sample at $w = 0$ is same as the sample at $w = 2\pi$). Generally, the DFT is defined as

DFT The N-point DFT of a finite duration sequence $x(n)$ of length L, where $N \geq L$ is defined as:

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N} = \sum_{n=0}^{N-1} x(n)W_N^{nk}; \quad \text{for } k = 0, 1, 2, \ldots, N-1$$

IDFT The Inverse Discrete Fourier transform (IDFT) of the sequence $X(k)$ of length N is defined as:

$$\text{IDFT}\{X(k)\} = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi nk/N} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-nk}; \quad \text{for } n = 0, 1, 2, \ldots, N-1$$

where $W_N = e^{-j(2\pi/N)}$ is called the twiddle factor.

The N-point DFT pair $x(n)$ and $X(k)$ is denoted as:

$$x(n) \xleftrightarrow[N]{\text{DFT}} X(k)$$

EXAMPLE 2.1 (a) Find the 4-point DFT of $x(n) = \{1, -1, 2, -2\}$ directly.

(b) Find the IDFT of $X(k) = \{4, 2, 0, 4\}$ directly.

Solution:

(a) Given sequence is $x(n) = \{1, -1, 2, -2\}$. Here the DFT $X(k)$ to be found is $N = 4$-point and length of the sequence $L = 4$. So no padding of zeros is required. We know that the DFT $\{x(n)\}$ is given by

$$X(k)=\sum_{n=0}^{N-1} x(n)\,W_N^{nk}=\sum_{n=0}^{N-1} x(n)\,e^{-j(2\pi/N)nk}=\sum_{n=0}^{3} x(n)\,e^{-j(\pi/2)nk},\quad k=0,1,2,3$$

$$X(0)=\sum_{n=0}^{3} x(n)\,e^{0}=x(0)+x(1)+x(2)+x(3)=1-1+2-2=0$$

$$X(1)=\sum_{n=0}^{3} x(n)\,e^{-j(\pi/2)n}=x(0)+x(1)\,e^{-(j\pi/2)}+x(2)\,e^{-j\pi}+x(3)\,e^{-j(3\pi/2)}$$

$$=1+(-1)(0-j)+2(-1-j0)-2(0+j)$$

$$=-1-j$$

$$X(2)=\sum_{n=0}^{3} x(n)\,e^{-j\pi n}=x(0)+x(1)\,e^{-j\pi}+x(2)\,e^{-j2\pi}+x(3)\,e^{-j3\pi}$$

$$=1-1(-1-j0)+2(1-j0)-2(-1-j0)=6$$

$$X(3)=\sum_{n=0}^{3} x(n)\,e^{-j(3\pi/2)n}=x(0)+x(1)\,e^{-j(3\pi/2)}+x(2)\,e^{-j3\pi}+x(3)\,e^{-j(9\pi/2)}$$

$$=1-1(0+j)+2(-1-j0)-2(0-j)=-1+j$$

$X(k) = \{0, -1-j, 6, -1+j\}$

(b) Given DFT is $X(k) = \{4, 2, 0, 4\}$. The IDFT of $X(k)$, i.e. $x(n)$ is given by

$$x(n)=\frac{1}{N}\sum_{k=0}^{N-1} X(k)\,W_N^{-nk}=\frac{1}{N}\sum_{k=0}^{N-1} X(k)\,e^{j(2\pi/N)nk}$$

i.e.
$$x(n)=\frac{1}{4}\sum_{k=0}^{3} X(k)e^{j(\pi/2)nk}$$

$\therefore$
$$x(0)=\frac{1}{4}\sum_{k=0}^{3} X(k)e^{0}=\frac{1}{4}[X(0)+X(1)+X(2)+X(3)]$$

$$=\frac{1}{4}[4+2+0+4]=2.5$$

$$x(1)=\frac{1}{4}\sum_{k=0}^{3} X(k)\,e^{j(\pi/2)k}=\frac{1}{4}\left[X(0)+X(1)\,e^{j(\pi/2)}+X(2)\,e^{j\pi}+X(3)e^{j(3\pi/2)}\right]$$

$$=\frac{1}{4}[4+2(0+j)+0+4(0-j)]=1-j0.5$$

$$x(2)=\frac{1}{4}\sum_{k=0}^{3} X(k)\,e^{j\pi k}=\frac{1}{4}\left[X(0)+X(1)\,e^{j\pi}+X(2)\,e^{j2\pi}+X(3)e^{j3\pi}\right]$$

$$=\frac{1}{4}[4+2(-1+j0)+0+4(-1+j0)]=-0.5$$

$$x(3)=\frac{1}{4}\sum_{k=0}^{3} X(k)\,e^{j(3\pi/2)k}=\frac{1}{4}\left[X(0)+X(1)\,e^{j(3\pi/2)}+X(2)\,e^{j3\pi}+X(3)e^{j(9\pi/2)}\right]$$

$$=\frac{1}{4}[4+2(0-j)+0+4(0+j)]=1+j0.5$$

$x_3(n) = \{2.5,\ 1-j0.5,\ -0.5,\ 1+j0.5\}$

EXAMPLE 2.2 (a) Find the 4-point DFT of $x(n) = \{1, -2, 3, 2\}$.
(b) Find the IDFT of $X(k) = \{1, 0, 1, 0\}$.

Solution:

(a) Given $x(n) = \{1, -2, 3, 2\}$.

Here $N = 4$, $L = 4$. The DFT of $x(n)$ is $X(k)$.

$$\therefore \quad X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n=0}^{3} x(n) e^{-j(2\pi/4)nk} = \sum_{n=0}^{3} x(n) e^{-j(\pi/2)nk}, \quad k = 0, 1, 2, 3$$

$$X(0) = \sum_{n=0}^{3} x(n) e^{0} = x(0) + x(1) + x(2) + x(3) = 1 - 2 + 3 + 2 = 4$$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j(\pi/2)n} = x(0) + x(1)e^{-j(\pi/2)} + x(2)e^{-j\pi} + x(3)e^{-j(3\pi/2)}$$

$$= 1 - 2(0 - j) + 3(-1 - j0) + 2(0 + j) = -2 + j4$$

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j\pi n} = x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$

$$= 1 - 2(-1 - j0) + 3(1 - j0) + 2(-1 - j0) = 4$$

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j(3\pi/2)n} = x(0) + x(1)e^{-j(3\pi/2)} + x(2)e^{-j3\pi} + x(3)e^{-j(9\pi/2)}$$

$$= 1 - 2(0 + j) + 3(-1 - j0) + 2(0 - j) = -2 - j4$$

$$\therefore \quad X(k) = \{4, -2 + j4, 4, -2 - j4\}$$

(b) Given $X(k) = \{1, 0, 1, 0\}$

Let the IDFT of $X(k)$ be $x(n)$.

$$\therefore \quad x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk} = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j(2\pi/N)nk}$$

$$x(0) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{0} = \frac{1}{4}\left[X(0) + X(1) + X(2) + X(3)\right] = \frac{1}{4}\left[1 + 0 + 1 + 0\right] = 0.5$$

$$x(1) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{j(\pi/2)k} = \frac{1}{4}\left[X(0) + X(1)e^{j(\pi/2)} + X(2)e^{j\pi} + X(3)e^{j(3\pi/2)}\right]$$

$$= \frac{1}{4}[1 + 0 + e^{j\pi} + 0] = \frac{1}{4}[1 + 0 - 1 + 0] = 0$$

$$x(2) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{j\pi k} = \frac{1}{4}\left[X(0) + X(1)e^{j\pi} + X(2)e^{j2\pi} + X(3)e^{j3\pi}\right]$$

$$= \frac{1}{4}[1 + 0 + e^{j2\pi} + 0] = \frac{1}{4}[1 + 0 + 1 + 0] = 0.5$$

$$x(3) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{j(3\pi/2)k} = \frac{1}{4}\left[X(0) + X(1)e^{j(3\pi/2)} + X(2)e^{j3\pi} + X(3)e^{j(9\pi/2)}\right]$$

$$= \frac{1}{4}[1 + 0 + e^{j3\pi} + 0] = \frac{1}{4}[1 + 0 - 1 + 0] = 0$$

The IDFT of $X(k) = \{1, 0, 1, 0\}$ is $x(n) = \{0.5, 0, 0.5, 0\}$.

EXAMPLE 2.3 Compute the DFT of the 3-point sequence $x(n) = \{2, 1, 2\}$. Using the same sequence, compute the 6-point DFT and compare the two DFTs.

Solution: The given 3-point sequence is $x(n) = \{2, 1, 2\}$, $N = 3$.

$$\text{DFT}\, x(n) = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n=0}^{2} x(n) e^{-j(2\pi/3)nk}, \quad k = 0, 1, 2$$

$$= x(0) + x(1) e^{-j(2\pi/3)k} + x(2) e^{-j(4\pi/3)k}$$

$$= 2 + \left(\cos\frac{2\pi}{3}k - j\sin\frac{2\pi}{3}k\right) + 2\left(\cos\frac{4\pi}{3}k - j\sin\frac{4\pi}{3}k\right)$$

When $k = 0$, $\quad X(k) = X(0) = 2 + 1 + 2 = 5$

When $k = 1$, $\quad X(k) = X(1) = 2 + \left(\cos\frac{2\pi}{3} - j\sin\frac{2\pi}{3}\right) + 2\left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}\right)$

$$= 2 + (-0.5 - j0.866) + 2(-0.5 + j0.866)$$

$$= 0.5 + j0.866$$

When $k = 2$, $\quad X(k) = X(2) = 2 + \left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}\right) + 2\left(\cos\frac{8\pi}{3} - j\sin\frac{8\pi}{3}\right)$

$$= 2 + (-0.5 + j0.866) + 2(-0.5 - j0.866)$$

$$= 0.5 - j0.866$$

$\therefore$ 3-point DFT of $x(n) = X(k) = \{5, 0.5 + j0.866, 0.5 - j0.866\}$

To compute the 6-point DFT, convert the 3-point sequence $x(n)$ into 6-point sequence by padding with zeros.

$$x(n) = \{2, 1, 2, 0, 0, 0\}, \quad N = 6$$

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n=0}^{5} x(n) e^{-j(2\pi/N)nk}, \quad k = 0, 1, 2, 3, 4, 5$$

$$= x(0) + x(1) e^{-j(2\pi/6)k} + x(2) e^{-j(4\pi/6)k} + x(3) e^{-j(6\pi/6)k} + x(4) e^{-j(8\pi/6)k}$$
$$+ x(5) e^{-j(10\pi/6)k}$$

$$= 2 + e^{-j(\pi/3)k} + 2e^{-j(2\pi/3)k}$$

When $k = 0$, $\quad X(0) = 2 + 1 + 2 = 5$

When $k = 1$, $$X(1) = 2 + e^{-j(\pi/3)} + 2e^{-j(2\pi/3)}$$
$$= 2 + (0.5 - j0.866) + 2(-0.5 - j0.866) = 1.5 - j2.598$$

When $k = 2$, $$X(2) = 2 + e^{-j(2\pi/3)} + 2e^{-j(4\pi/3)}$$
$$= 2 + (-0.5 - j0.866) + 2(-0.5 + j0.866) = 0.5 + j0.866$$

When $k = 3$, $$X(3) = x(0) + x(1)e^{-j(3\pi/3)} + x(2)e^{-j(6\pi/3)}$$
$$= 2 + (\cos\pi - j\sin\pi) + 2(\cos 2\pi - j\sin 2\pi)$$
$$= 2 - 1 + 2 = 3$$

When $k = 4$, $$X(4) = x(0) + x(1)e^{-j(4\pi/3)} + x(2)e^{-j(8\pi/3)}$$
$$= 2 + \left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}\right) + 2\left(\cos\frac{8\pi}{3} - j\sin\frac{8\pi}{3}\right)$$
$$= 2 + (-0.5 + j0.866) + 2(-0.5 - j0.866)$$
$$= 0.5 - j0.866$$

When $k = 5$, $$X(5) = x(0) + x(1)e^{-j(5\pi/3)} + x(2)e^{-j(10\pi/3)}$$
$$= 2 + \left(\cos\frac{5\pi}{3} - j\sin\frac{5\pi}{3}\right) + 2\left(\cos\frac{10\pi}{3} - j\sin\frac{10\pi}{3}\right)$$
$$= 2 + (0.5 - j0.866) + 2(-0.5 + j0.866) = 1.5 + j0.866$$

Tabulating the above 3-point and 6-point DFTs, we have

DFT	$X(0)$	$X(1)$	$X(2)$	$X(3)$	$X(4)$	$X(5)$
3-point	5	$0.5 + j0.866$	$0.5 - j0.866$	–	–	–
6-point	5	$1.5 - j2.598$	$0.5 + j0.866$	3	$0.5 - j0.866$	$1.5 + j0.866$

MATRIX FORMULATION OF THE DFT AND IDFT

If we let $W_N = e^{-j(2\pi/N)}$, the defining relations for the DFT and IDFT may be written as:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \qquad k = 0, 1, \ldots, N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-nk}, \qquad n = 0, 1, 2, \ldots, N-1$$

The first set of N DFT equations in N unknowns may be expressed in matrix form as:

$$\mathbf{X} = \mathbf{W_N}\mathbf{x}$$

Here $\mathbf{X}$ and $\mathbf{x}$ are $N \times 1$ matrices, and $\mathbf{W_N}$ is an $N \times N$ square matrix called the DFT matrix. The full matrix form is described by

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ W_N^0 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

THE IDFT FROM THE MATRIX FORM

The matrix **x** may be expressed in terms of the inverse of **W**N as:

$$\mathbf{x} = \mathbf{W_N^{-1}}\ \mathbf{x}$$

WN is called the IDFT matrix. We may also obtain **x** directly from the IDFT relation in matrix form, where the change of index from *n* to *k* and the change in the sign of the exponent in $e^{j(2\pi/N)nk}$ lead to the conjugate transpose of **W**N. We then have

$$\mathbf{x} = \frac{1}{\mathbf{N}}\left[\mathbf{W_N^*}\right]^T \mathbf{x}$$

EXAMPLE 2.4 Find the DFT of the sequence $x(n) = \{1, 2, 1, 0\}$
Solution: The DFT $X(k)$ of the given sequence $x(n) = \{1, 2, 1, 0\}$ may be obtained by solving the matrix product as follows. Here $N = 4$.

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & W_N^3 \\ W_N^0 & W_N^2 & W_N^4 & W_N^6 \\ W_N^0 & W_N^3 & W_N^6 & W_N^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -j2 \\ 0 \\ j2 \end{bmatrix}$$

The result is DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$.

EXAMPLE 2.5 Find the DFT of $x(n) = \{1, -1, 2, -2\}$.
Solution: The DFT, $X(k)$ of the given sequence $x(n) = \{1, -1, 2, -2\}$ can be determined using matrix as shown below.

$$X(k) = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1-j \\ 6 \\ -1+j \end{bmatrix}$$

$\therefore \quad \text{DFT}\ \{x(n)\} = X(k) = \{0, -1-j, 6, -1+j\}$

EXAMPLE 2.6. Find the 4-point DFT of $x(n) = \{1, -2, 3, 2\}$.
Solution: Given $x(n) = \{1, -2, 3, 2\}$, the 4-point DFT$\{x(n)\} = X(k)$ is determined using matrix as shown below.

$$\text{DFT}\ \{x(n)\} = X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2+j4 \\ 4 \\ -2-j4 \end{bmatrix}$$

$\text{DFT}\ \{x(n)\} = X(k) = \{4, -2+j4, 4, -2-j4\}$

EXAMPLE 2.6 Find the IDFT of $X(k)=\{4, -j2, 0, j2\}$ using DFT.
Solution: Given $X(k) = \{4, -j2, 0, j2\}$ _ $X^*(k) = \{4, j2, 0, -j2\}$
The IDFT of $X(k)$ is determined using matrix as shown below.
To find IDFT of $X(k)$ first find $X^*(k)$, then find DFT of $X^*(k)$, then take conjugate of DFT $\{X^*(k)\}$ and divide by N.

$$\text{DFT}\{X^*(k)\} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ j2 \\ 0 \\ -j2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 4 \\ 0 \end{bmatrix}$$

$$\therefore \quad \text{IDFT}[X(k)] = x(n) = \frac{1}{4}[4, 8, 4, 0]^* = \frac{1}{4}[4, 8, 4, 0] = [1, 2, 1, 0]$$

EXAMPLE 2.7 Find the IDFT of $X(k) = \{4, 2, 0, 4\}$ using DFT.
Solution: Given $X(k) = \{4, 2, 0, 4\}$
$X^*(k) = \{4, 2, 0, 4\}$
The IDFT of $X(k)$ is determined using matrix as shown below.
To find IDFT of $X(k)$, first find $X^*(k)$, then find DFT of $X^*(k)$, then take conjugate of DFT $\{X^*(k)\}$ and divide by N

$$\text{DFT}[X^*(k)] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 4+j2 \\ -2 \\ 4-j2 \end{bmatrix}$$

$$\therefore \quad \text{IDFT}\{X(k)\} = x(n) = \frac{1}{4}[10, 4+j2, -2, 4-j2]^* = \{2.5, 1-j0.5, -0.5, 1+j0.5\}$$

EXAMPLE2.8 Find the IDFT of $X(k) = \{1, 0, 1, 0\}$.
Solution: Given $X(k) = \{1, 0, 1, 0\}$, the IDFT of $X(k)$, i.e. $x(n)$ is determined using matrix as shown below.

$$X^*(k) = \{1, 0, 1, 0\}^* = \{1, 0, 1, 0\}$$

$$\text{DFT}\{X^*(k)\} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\therefore \quad \text{IDFT}\{X(k)\} = x(n) = \frac{1}{4}[\text{DFT}\{X^*(k)\}]^* = \frac{1}{4}\{2, 0, 2, 0\} = \{0.5, 0, 0.5, 0\}$$

PROPERTIES OF DFT

Like the Fourier and Z-transforms, the DFT has several important properties that are used to process the finite duration sequences. Some of those properties are discussed as follows

Periodicity:

If a sequence $x(n)$ is periodic with periodicity of N samples, then N-point DFT of the sequence, $X(k)$ is also periodic with periodicity of N samples.
Hence, if $x(n)$ and $X(k)$ are an N-point DFT pair, then

$$x(n+N) = x(n) \qquad \text{for all } n$$

$$X(k+N) = X(k) \qquad \text{for all } k$$

Proof: By definition of DFT, the $(k + N)$th coefficient of $X(k)$ is given by

$$X(k+N) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi n(k+N)/N} = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}e^{-j2\pi nN/N}$$

But $e^{-j2\pi n} = 1$ for all n (Here n is an integer)

$$\therefore \qquad X(k+N) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N} = X(k)$$

Linearity

If $x_1(n)$ and $x_2(n)$ are two finite duration sequences and if

$$\text{DFT}\ \{x_1(n)\} = X_1(k)$$

and

$$\text{DFT}\ \{x_2(n)\} = X_2(k)$$

Then for any real valued or complex valued constants a and b,

$$\text{DFT}\{ax_1(n) + bx_2(n)\} = aX_1(k) + bX_2(k)$$

Proof:

$$\text{DFT}\{ax_1(n) + bx_2(n)\} = \sum_{n=0}^{N-1}[ax_1(n) + bx_2(n)]\, e^{-j2\pi nk/N}$$

$$= a\sum_{n=0}^{N-1} x_1(n)e^{-j2\pi nk/N} + b\sum_{n=0}^{N-1} x_2(n)e^{-j2\pi nk/N}$$

$$= aX_1(k) + bX_2(k)$$

DFT of Even and Odd Sequences

The DFT of an even sequence is purely real, and the DFT of an odd sequence is purely imaginary. Therefore, DFT can be evaluated using cosine and sine transforms for even and odd sequences respectively.

For even sequence, $X(k) = \sum_{n=0}^{N-1} x(n)\cos\left(\frac{2\pi nk}{N}\right)$

For odd sequence, $X(k) = \sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi nk}{N}\right)$

Time Reversal of the Sequence

The time reversal of an N-point sequence $x(n)$ is obtained by wrapping the sequence $x(n)$ around the circle in the clockwise direction. It is denoted as $x[(-n), \text{mod } N]$ and

$$x[(-n), \text{mod } N] = x(N-n), \qquad 0 \le n \le N-1$$

If DFT $\{x(n)\} = X(k)$, then

$$\text{DFT}\ \{x(-n), \text{mod } N\} = \text{DFT}\ \{x(N-n)\}$$

$$= X[(-k), \text{mod } N] = X(N-k)$$

Proof: $\text{DFT}\ \{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)e^{-j2\pi nk/N}$