QR decomposition

QR decomposition is a fundamental technique in linear algebra used to decompose a matrix (A) into two specific matrices:

1. **Q (orthogonal matrix):** This matrix has square dimensions (m x m) and possesses special properties. Its columns are orthonormal vectors, meaning they have a unit length (norm) and are perpendicular (orthogonal) to each other. Imagine an m-dimensional space where Q's columns define a set of mutually perpendicular axes
2. Q is the matrix E in Gram-Schmidt
3. **R (upper triangular matrix):** This matrix has the same dimensions as the original matrix (m x n). However, all entries below the diagonal are zero. It represents the coefficients used to project the original data points (columns of A) onto the basis vectors defined by Q.

The Decomposition:

The core idea is to express the original matrix (A) as a product of these two matrices:

A = Q * R

Here, Q acts as a transformation that rotates and stretches the space, and R captures the scaling factors needed to project the data points onto the transformed axes.

Benefits of QR Decomposition:

- **Solving Linear Systems:** QR decomposition provides a numerically stable and efficient way to solve systems of linear equations (Ax = b).
- **Least Squares Problems:** It helps find solutions to least squares problems, where you aim to minimize the squared difference between a set of data points and a fitted model.

- **Eigenvalue Computations:** QR decomposition can be used as a preliminary step for computing eigenvalues and eigenvectors of a matrix.
- **QR Algorithm:** It forms the basis for the QR algorithm, a popular iterative method for finding eigenvalues.

Example Solving a Linear System

Consider a simple linear system:

```
 1x + 2y = 5
2x + 4y = 10
```

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This can be represented as a matrix equation:

```
[[1, 2], [2, 4]] * [x, y] = [5, 10]
```

1. **Decompose A:** Perform QR decomposition on the coefficient matrix:

```
[[1, 2], [2, 4]] = Q * R =
[[1/√5, -1/√5]], [[√5, 0], [√5, √5]]
```

2. **Solve for Qy:** Multiply both sides by Q^T (transpose of Q):

```
Q^T * A * [x, y] = Q^T * [5, 10]
[√5, √5] * [[1, 2], [2, 4]] * [x, y] = [√5, √5] * [5, 10]
[√5x + √5y, √5x + 5√5y] = [5√5, 10√5]
```

3. **Solve the Upper Triangular System:** This system has become upper triangular, making it easier to solve for y and then x:

```
√5x + √5y = 5√5
√5x + 5√5y = 10√5

Solving for y: 4√5y = 5√5 --> y = 1
Substitute y = 1 back into the first equation: √5x + √5 = 5√5 --> x = 2
```

Therefore, the solution to the system is x = 2 and y = 1.