Linear Combinations And Span

Linear Combinations :

- A **linear combination** of vectors refers to a new vector obtained by multiplying each vector by a scalar (numerical value) and then adding them together.

Checking if a vector (B) is a linear combination of two other vectors (V and M):

1. **Setting up the system of equations:** You're on the right track with representing the scenario as a linear system of equations:
    - B = aV + bM
2. **Solving the system:**
    - This step is crucial. You can solve the system using various methods like Gaussian elimination or matrix inversion.
    - There are indeed **3 possible solutions**:
        - **No solution:**
            - This occurs when the system is inconsistent, meaning there are no values for `a` and `b` that satisfy the equation.
            - Geometrically, this implies **V and M are parallel** (lie on the same line) OR V and M at the same span and **B lies outside the span** created by V and M.
        - **Infinite solutions:**
            - This happens when the system has infinitely many solutions.
            - Geometrically, it represents **V and M being dependent vectors OR B at the same span V and M**. This means one vector can be expressed as a scalar multiple of the other. In this case, any combination of `a` and `b` (except when `a = b = 0`) will satisfy the equation.
        - **One unique solution:**

- This is the case when the system has a single solution for `a` and `b`.
- Geometrically, it signifies that **V and M span a plane**, and **B lies within this plane**.
- There exists only one specific way to express B as a linear combination of V and M.

Example :

The previous activity also shows that questions about linear combinations lead naturally to linear systems. Let's ask how we can describe the vector $\mathbf{b} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ as a linear combination of $\mathbf{v}$ and $\mathbf{w}$. We need to find weights a and b such that

$$a\begin{bmatrix} 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2a \\ a \end{bmatrix} + \begin{bmatrix} b \\ 2b \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2a+b \\ a+2b \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Equating the components of the vectors on each side of the equation, we arrive at the linear system

$$\begin{aligned} 2a + b &= -1 \\ a + 2b &= 4 \end{aligned}$$

This means that $\mathbf{b}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$ if this linear system is consistent.

To solve this linear system, we construct its corresponding augmented matrix and find its reduced row echelon form.

$$\left[\begin{array}{rr|r} 2 & 1 & -1 \\ 1 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 0 & -2 \\ 0 & 1 & 3 \end{array}\right],$$

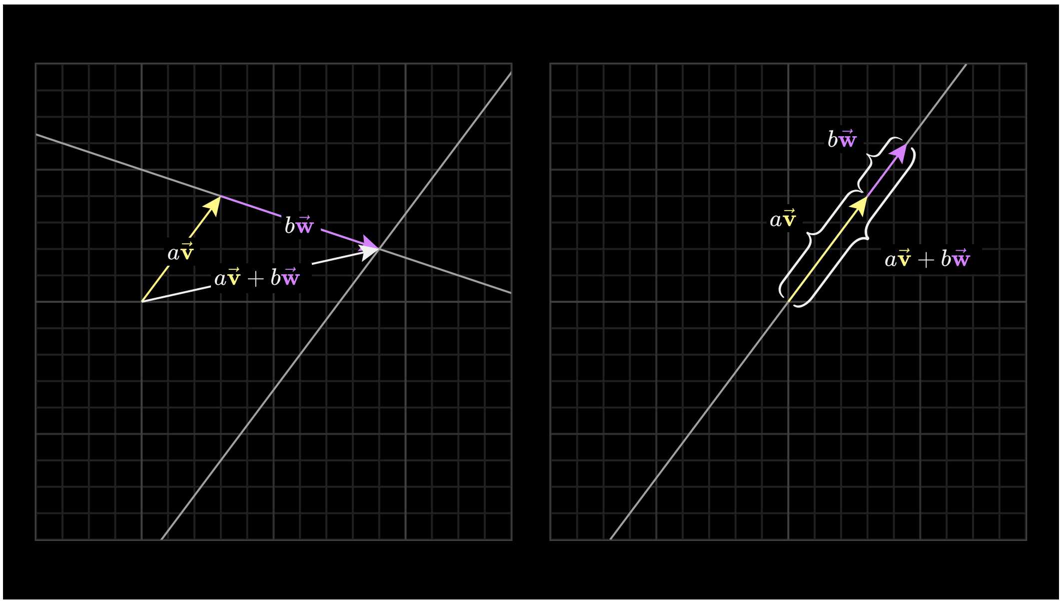
which tells us the weights $a = -2$ and $b = 3$; that is,

$$-2\mathbf{v} + 3\mathbf{w} = \mathbf{b}.$$

In fact, we know even more because the reduced row echelon matrix tells us that these are the only possible weights. Therefore, $\mathbf{b}$ may be expressed as a linear combination of $\mathbf{v}$ and $\mathbf{w}$ in exactly one way.

Span

- **Span (denoted by span(S)):** This refers to the collection of all possible linear combinations of vectors in a set S.



Examples:

- In ℝ² (2-dimensional space), consider vectors $v_1 = [1, 2]$ and $v_2 = [3, 1]$. Their span includes all vectors of the form:
  - $c_1v_1 + c_2v_2 = [c_1 + 3c_2, 2c_1 + c_2]$
  - This represents all points that lie on the line connecting v_1 and v_2 (and extending infinitely in both directions).
- In ℝ³, consider vectors $v_1 = [1, 0, 0]$, $v_2 = [0, 1, 0]$, and $v_3 = [0, 0, 1]$. Their span encompasses all possible vectors in ℝ³, as any 3D vector can be expressed as a linear combination of these three basis vectors.