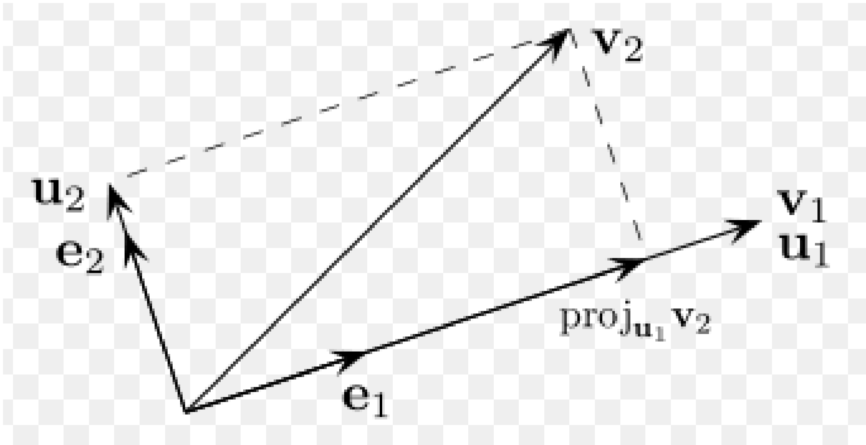
Orthonormal



Orthogonality:

The vectors are perpendicular to each other. Imagine two straight lines intersecting at a 90-degree angle. This perpendicularity is mathematically captured by the concept of the dot product. Two vectors are orthogonal if their dot product is zero.

Unit norm:

Each vector in the set has a length of 1. This means the magnitude of the vector, which is often calculated using the square root of the sum of squares of its components, is equal to

Why are orthonormal sets important?

- **Simplicity in calculations:** Because the vectors are perpendicular, their dot product with each other is zero. This simplifies calculations involving projections and inner products.

- **Basis for vector spaces:** An orthonormal set can be used as a basis for a vector space. A basis is a set of vectors that can be linearly combined to represent any other vector in that space. Orthonormal bases offer advantages in tasks like coordinate transformations and projections.
- **Gram-Schmidt process:** This process allows you to take a set of linearly independent vectors and transform them into an orthonormal set. This is useful for many applications in linear algebra.

Gram-Schmidt Process

1. Start with a set of linearly independent vectors: These vectors can span a particular subspace, but they don't necessarily need to be orthogonal or of unit length.

2. Iterative process: The process works its way through the vectors one by one. For each vector (let's call it v_i):

- **Projection:** Project v_i onto the subspace spanned by all the previously processed vectors (v_1, v_2, ..., v_(i-1)). This projection removes any component of v_i that lies in the direction of the already processed vectors.
- **Normalization:** Take the resulting vector from the projection step and normalize it by dividing it by its magnitude (length). This ensures the final vector has a unit norm (length of 1).

THEOREM 5.12 Gram-Schmidt Orthonormalization Process

1. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for an inner product space V.

2. Let $B' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$, where

$$\mathbf{w}_1 = \mathbf{v}_1$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2$$

$$\vdots$$

$$\mathbf{w}_n = \mathbf{v}_n - \frac{\langle \mathbf{v}_n, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_n, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2 - \cdots - \frac{\langle \mathbf{v}_n, \mathbf{w}_{n-1} \rangle}{\langle \mathbf{w}_{n-1}, \mathbf{w}_{n-1} \rangle}\mathbf{w}_{n-1}.$$

Then B' is an *orthogonal* basis for V.

3. Let $\mathbf{u}_i = \dfrac{\mathbf{w}_i}{\|\mathbf{w}_i\|}$. Then $B'' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an *orthonormal* basis for V.

Also, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ for $k = 1, 2, \ldots, n$.

EXAMPLE

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \in \mathbb{R}^3$$

Use the Gram-Schmidt algorithm to find an orthonormal set of vectors $\{\vec{w}_1, \vec{w}_2\}$ having the same span.

Solution

We already remarked that the set of vectors in $\{\vec{u}_1, \vec{u}_2\}$ is linearly independent, so we can proceed with the Gram-Schmidt algorithm:

$$\begin{aligned} \vec{v}_1 &= \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ \vec{v}_2 &= \vec{u}_2 - \left(\frac{\vec{u}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2} \right) \vec{v}_1 \\ &= \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} \end{aligned}$$

Now to normalize simply let

$$\begin{aligned} \vec{w}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \\ \vec{w}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \end{aligned}$$

You can verify that $\{\vec{w}_1, \vec{w}_2\}$ is an orthonormal set of vectors having the same span as $\{\vec{u}_1, \vec{u}_2\}$, namely the XY-plane.