Inferences in Regression and Correlation Analysis

2.1 Inference Concerning β_1

Definition. ***Inference concerning*** β_1 *Test concerning* β_1 *is of the form*

$$\begin{cases} \mathcal{H}_0 : \beta_1 = 0 \\ \mathcal{H}_\alpha : \beta_1 \neq 0 \end{cases}$$

$\beta_1 = 0$ *implies that there is no linear association between* Y *and* X*. On assumption of gaussian noise, there is also no relation of any type between* Y *and* X*, since probability of* Y *are identical for all levels of* X

Definition. ***Linear estimators*** *Least square estimator* $\hat{\beta}_1$ *is a linear estimator*

$$\hat{\beta}_1 = \sum_i k_i y_i \qquad k_i = \frac{x_i - \overline{x}}{\sum_j (x_j - \overline{x})^2}$$

with properties

$$\sum k_i = 0 \quad \sum k_i x_i = 1 \quad \sum k_i^2 = \frac{1}{S_{XX}}$$

Proof. Note

$$\sum (x_i - \overline{x})(y_i - \overline{y}) = \sum (x_i - \overline{x}) y_i - \sum (x_i - \overline{x})\overline{y} = \sum (x_i - \overline{x}) y_i \quad \text{by } \sum (x_i - \overline{x}) = 0$$

the result follows. Proof of properties are simple, i.e.

$$\sum k_i = \sum_i \left(\frac{x_i - \overline{x}}{\sum_j (x_j - \overline{x})^2} \right) = \frac{1}{\sum_j (x_j - \overline{x})^2} \sum_i (x_i - \overline{x}) = 0$$

For second property

$$\sum k_i x_i = \frac{1}{\sum_j (x_j - \overline{x})^2} \sum_i (x_i - \overline{x}) x_i = \frac{1}{S_{XX}} \sum_i x_i^2 - n\overline{x}^2 = \frac{1}{S_{XX}} S_{XX} = 1$$

□

Definition. ***Sampling distribution of*** $\hat{\beta}_1$ *The sampling distribution of* $\hat{\beta}_1$ *refers to different values of the estimator obtained with repeated sampling when the levels of the predictor variable* X *are held constant from sample to sample. Given* $\hat{\beta}_1$

$$\hat{\beta}_1 = \frac{\sum (x_i - \overline{x})(y_i - \overline{y})}{\sum (x_i - \overline{x})^2}$$

The sampling distribution of $\hat{\beta}_1$ *is* ***normal*** *with mean and variance,*

$$\mathbb{E}(\hat{\beta}_1 | X) = \beta_1 \quad Var(\hat{\beta}_1 | X) = \frac{\sigma^2}{\sum (x_i - \overline{x})^2} = \frac{\sigma^2}{S_{XX}}$$

$$\hat{\beta}_1 \sim \mathcal{N}(\beta_1, \frac{\sigma^2}{S_{XX}})$$

Proof.

1. **Mean and variance**

   $$\mathbb{E}(\hat{\beta}_1) = \mathbb{E}\{\sum k_i y_i\} \overset{ind}{=} \sum (k_i \mathbb{E}\{y_i\}) = \sum k_i(\beta_0 + \beta_1 x_i) = \beta_0 \sum k_i + \beta_1 \sum k_i x_i = \beta_1$$

   $$Var(\hat{\beta}_1) = Var\left(\sum k_i y_i\right) = \sum k_i^2 Var(y_i) = \sum k_i^2 \sigma^2 = \sigma^2 \sum k_i^2 = \frac{\sigma^2}{S_{XX}}$$

   with last step of both derivation given by properties of $\hat{\beta}_1$ as a linear estimator.

2. **Normality** of sampling distribution given by the fact that $\hat{\beta}_1$ is a linear combination of y_is. Since $y_i \overset{i.i.d}{\sim} \mathcal{N}(\beta_0 + \beta 1 x_i, \sigma^2)$, the estimator is normaly distributed because it is a linear combination of independent normal random variables

3. **Estimated variance** We can estimate the variance of $\hat{\beta}_1$ by substituting σ^2 (unknown) with its unbiased estimator MSE s^2

   $$s^2(\hat{\beta}_1) = \frac{MSE}{S_{XX}} \quad \text{where} \quad MSE = \frac{\sum e_i^2}{n-2}$$

   Note s^2 carries a denominator of S_{XX}, this is from the variance of $\hat{\beta}_1$, we are simply substituting the unknown σ^2 with MSE

□

Definition. ***standardization of*** $\hat{\beta}_1$
Standardization of sampling distribution of $\hat{\beta}_1$ *gives*

$$Z = \frac{\hat{\beta}_1 - \beta_1}{\sigma(\hat{\beta}_1)} = \mathcal{N}(0,1) \quad \textit{wher} \quad \sigma(\hat{\beta}_1) = \frac{\sigma}{\sqrt{S_{XX}}}$$

Usually, have to estimate standard error $\sigma(\hat{\beta}_1)$ *with* $s(\hat{\beta}_1)$*. Standardization where the denominator is an estimated standard error is called* ***studentized statistic****, given by*

$$T = \frac{\hat{\beta}_1 - \beta_1}{s(\hat{\beta}_1)} \sim t_{n-2} \quad \textit{where} \quad s^2(\hat{\beta}_1) = \frac{MSE}{S_{XX}}$$

Proof. Assume proposition

$$\frac{\sum (\hat{e}_i)^2}{\sigma^2} \sim \chi^2_{n-2}$$

Then we have

$$\frac{\hat{\beta}_1 - \beta_1}{s(\hat{\beta}_1)} = \frac{\hat{\beta}_1 - \beta_1}{\sigma(\hat{\beta}_1)} \Big/ \frac{s(\hat{\beta}_1)}{\sigma(\hat{\beta}_1)} = \frac{\hat{\beta}_1 - \beta_1}{\sigma(\hat{\beta}_1)} \Big/ \sqrt{\frac{\sum e_i^2 S_{XX}}{(n-2) S_{XX} \sigma^2}} \sim \frac{Z}{\sqrt{\chi^2_{n-2} \Big/ (n-2)}} = t_{n-2}$$

□

Definition. ***Confidence Interval and test for*** β_1
We use the previously derived distribution as a pivot

$$\Pr\left(t_{\alpha/2,n-2} \leq \frac{\hat{\beta}_1 - \beta_1}{s(\hat{\beta}_1)} \leq t_{1-\alpha/2,n-2}\right) = 1-\alpha$$

$$\Pr\left(\hat{\beta}_1 - s(\hat{\beta}_1)t_{1-\alpha/2,n-2} \leq \beta_1 \leq \hat{\beta}_1 + s(\hat{\beta}_1)t_{1-\alpha/2,n-2}\right) = 1-\alpha$$

Hence the confidence interval is given by

$$\hat{\beta}_1 \pm s(\hat{\beta}_1)t_{1-\alpha/2,n-2} \quad \textit{where} \quad s^2(\hat{\beta}_1) = \frac{MSE}{S_{XX}}$$

For 2-sided tests

$$\begin{cases} \mathcal{H}_0 : \beta_1 = b \\ \mathcal{H}_\alpha : \beta_1 \neq b \end{cases}$$

We compute test statistics

$$t^* = \frac{\hat{\beta}_1 - b}{s(\hat{\beta}_1)} \quad \textit{and reject } \mathcal{H}_0 \textit{ if } |t^*| > t_{1-\alpha/2,n-2}$$

2.2 Inference Concerning of $\hat{\beta}_0$

Definition. ***Sampling Distribution of*** $\hat{\beta}_0$
Given point estimator

$$\hat{\beta}_0 = \overline{y} - \beta_1\overline{x}$$

$\hat{\beta}_0$ *refers to different values of* β_0 *that would be obtained with repeated sampling when levels of predictor variable* x *are held constant from sample to sample. The* ***sampling distribution of*** $\hat{\beta}_0$ *is* ***normal*** *with mean and variance*

$$\mathbb{E}(\hat{\beta}_0) = \beta_0 \qquad Var(\hat{\beta}_0) = \sigma^2\left[\frac{1}{n} + \frac{\overline{x}^2}{\sum(x_i - \overline{x})^2}\right] = \sigma^2\left[\frac{1}{n} + \frac{\overline{x}^2}{S_{XX}}\right]$$

Proof.

$$\mathbb{E}(\hat{\beta}_0) = \mathbb{E}(\overline{y}) - \mathbb{E}(\hat{\beta}_1\overline{x}) = \frac{1}{n}\sum \mathbb{E}(y_i) - \beta_1\overline{x} = \beta_0 + \beta_1\frac{1}{n}\sum x_i - \beta_1\overline{x} = \beta_0$$

$$
\begin{aligned}
Var(\hat{\beta}_0) &= Var(\overline{y} - \hat{\beta}_1\overline{x}) \\
&= Var(\overline{y}) + \overline{x}^2 Var(\hat{\beta}_1) - 2\overline{x}Cov(\overline{y}, \hat{\beta}_1) \\
&= Var(\frac{1}{n}\sum y_i) + \frac{\sigma^2}{S_{XX}} - 2\overline{x}Cov(\frac{1}{n}\sum y_i, \sum_i k_i y_i) \\
&= \frac{\sigma^2}{n^2} + \frac{\sigma^2}{S_{XX}} - 2\overline{x}\frac{1}{n}\sum k_i Cov(y_i, y_i) \\
&= \frac{\sigma^2}{n^2} + \frac{\sigma^2}{S_{XX}} - 2\overline{x}\frac{\sigma^2}{n}\sum k_i \\
&= \sigma^2\left(\frac{1}{n} + \frac{\overline{x}^2}{S_{XX}}\right)
\end{aligned}
$$

The normality follows because $\hat{\beta}_0$ is also a linear estimator of y_is. We can estimate $Var(\hat{\beta}_0)$ by replacing σ^2 with MSE, as before

$$s^2(\hat{\beta}_0) = MSE\left[\frac{1}{n} + \frac{\overline{x}^2}{S_{XX}}\right]$$

□

Definition. ***Standardization of*** $\hat{\beta}_0$

$$Z = \frac{\hat{\beta}_0 - \beta_0}{\sigma(\hat{\beta}_0)} \sim t_{n-2} \quad \textit{where} \quad \sigma^2(\hat{\beta}_0) = \sigma^2\left[\frac{1}{n} + \frac{\overline{x}^2}{S_{XX}}\right]$$

$$T = \frac{\hat{\beta}_0 - \beta_0}{s(\hat{\beta}_0)} \sim t_{n-2} \quad \textit{where} \quad s^2(\hat{\beta}_0) = MSE\left[\frac{1}{n} + \frac{\overline{x}^2}{S_{XX}}\right]$$

Definition. ***Confidence Interval for*** $\hat{\beta}_0$

$$\hat{\beta}_0 \pm s(\hat{\beta}_0)t_{1-\alpha/2,n-2} \quad \textit{where} \quad s(\hat{\beta}_0) = \sqrt{\frac{\sum e_i^2}{n-2}}\sqrt{\frac{1}{n} + \frac{\overline{x}^2}{S_{XX}}}$$

Definition. ***Consideration about inference***

1. ***Spacing of*** *x* ***levels*** *Looking at variance of $\hat{\beta}_0$ and $\hat{\beta}_1$, the larger the spread in x levels, the larger S_{XX} and smaller the variance.*

2.4 Interval Estimation of $\mathbb{E}(Y|X = x^*)$, the population regression line

Definition. ***Mean estimation*** *Often times want to estimate mean for one or more probability distribution of Y. (i.e. mean Y for low and high X levels). Let x^* be level of X for which we want to estimate the mean response $\hat{y}^*$. The mean response is given by*

$$\mathbb{E}(Y|X = x^*) = E(y^*) = \beta_0 + \beta_1 x^*$$

The idea is that the expectation is a random variable because of the estimated correlation coefficients. We would want to do inference on the mean response. We have a point estimator of the the mean response

$$\hat{y}^* = \hat{\beta}_0 + \hat{\beta}_1 x^*$$

which is simply the estimated response from estimated correlation coefficients and regression function. Note $X = x^*$ is a known constant

Definition. ***Sampling Distribution of mean response estimator*** $\hat{y}^*$
The sampling distribution of $\hat{y}^*$ is ***normal*** *with mean and variance*

$$\mathbb{E}(\hat{y}^*) = \mathbb{E}(\hat{y}|X = x^*) = \beta_0 + \beta_1 x^* \quad (= \mathbb{E}(y^*) \text{ so unbiased})$$

$$Var(\hat{y}^*) = Var(\hat{y}|X = x^*) = \sigma^2 \left[\frac{1}{n} + \frac{(x^* - \overline{x})^2}{\sum(x_i - \overline{x})^2}\right] = \sigma^2 \left[\frac{1}{n} + \frac{(x^* - \overline{x})^2}{S_{XX}}\right]$$

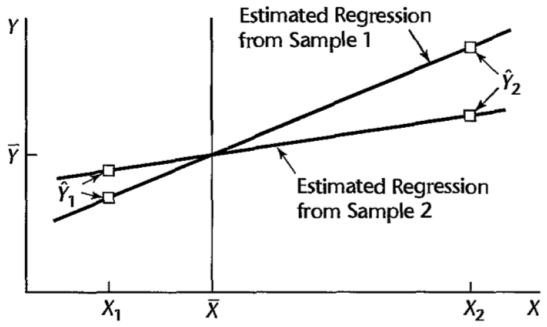
$$\hat{y}^* = (\hat{y}|X = x^*) \sim \mathcal{N}(\beta_0 + \beta_1 x^*, \sigma^2 \left[\frac{1}{n} + \frac{(x^* - \overline{x})^2}{S_{XX}}\right])$$

Note when $x^* = 0$, $Var(\hat{y}^*)$ reduces to variance of $\hat{\beta}_0$.

Proof. 3 parts

1. **Normality** of $\hat{y}^*$ follows from the fact that it is composed of $\hat{\beta}_0$ and $\hat{\beta}_1$, both of which are linear estimators of y_i
2. **Mean**

$$\mathbb{E}(\hat{y}^*) = \mathbb{E}\left(\hat{\beta}_0 + \hat{\beta}_1 x^*\right) = \mathbb{E}(\hat{\beta}_0) + x^* \mathbb{E}(\hat{\beta}_1) = \beta_0 + \beta_1 x^* = \mathbb{E}(y^*)$$



3. **Variance**

$$\begin{aligned} Var(\hat{y}|X = x^*) &= Var(\hat{\beta}_0 + \hat{\beta}_1 x|X = x^*) \\ &= Var(\hat{\beta}_0|X = x^*) + (x^*)^2 Var(\hat{\beta}_1|X = x^*) + 2x^* Cov(\hat{\beta}_0, \hat{\beta}_1|X = x^*) \end{aligned}$$

with

$$\begin{aligned} Cov(\hat{\beta}_0, \hat{\beta}_1 | X = x^*) &= Cov(\overline{y} - \hat{\beta}_1 \overline{x} | X = x^*) \\ &= Cov(\overline{y}, \hat{\beta}_1 | X = x^*) - \overline{x} Cov(\hat{\beta}_1, \hat{\beta}_1) \\ &= 0 - \overline{x} Var(\hat{\beta}_1) \\ &= \frac{-\overline{x}\sigma^2}{S_{XX}} \end{aligned}$$

So

$$Var(\hat{y}|X = x^*) = \sigma^2 \left(\frac{1}{2} + \frac{\overline{x}^2}{S_{XX}} \right) + (x^*)^2 \frac{\sigma^2}{S_{XX}} - \frac{2x^* \overline{x} \sigma^2}{S_{XX}} = \sigma^2 \left(\frac{1}{n} + \frac{(x^* - \overline{x})^2}{S_{XX}} \right)$$

Idea is variability of $\hat{y}^*$ is affected by how far x^* is from $\overline{x}$, via

$$(x^* - \overline{x})^2 = S_{XX}$$

The further x^* is from $\overline{x}$, the greater the variability. Note in plot, x_1 near $\overline{x}$, the fitted value $\hat{y}_1$ for two sample regression line (from 2 experiments) are close to each other; the fitted values $\hat{y}_2$ differ substantially due to the fact that x_2 is far from $\overline{x}$. In summary,

variation in $\hat{y}^*$ value from sample to sample will be greater when x^* is far from mean than when x^* is near mean

We can substitute MSE for σ^2 to obtain $s^2(\hat{y}^*)$. The estimated variance is given by

$$s^2(\hat{y}^*) = MSE \left[\frac{1}{n} + \frac{(x^* - \overline{x})^2}{S_{XX}} \right]$$

□

Definition. ***Standardization of mean response estimator*** $\hat{y}^*$

$$Z = \frac{\hat{y}^* - (\beta_0 + \beta_1 x^*)}{\sigma(\hat{y}^*)} \sim \mathcal{N}(0, 1)$$

note $\mathbb{E}(y^*) = \beta_0 + \beta_1 x^*$

$$T = \frac{\hat{y}^* - (\beta_0 + \beta_1 x^*)}{s(\hat{y}^*)} \sim t_{n-2}$$

Definition. ***Confidence Interval for*** $\hat{y}^*$
The $100(1-\alpha)\%$ *confidence interval for* $\mathbb{E}(Y|X = x^*) = \beta_0 + \beta_1 x^*$ *is given by*

$$\hat{y}^* \pm s(\hat{y}^*) t_{1-\alpha/2, n-2} = (\hat{\beta}_0 + \hat{\beta}_1 x^*) \pm t_{1-\alpha/2, n-2} \sqrt{\frac{\sum e_i^2}{n-2}} \sqrt{\frac{1}{n} + \frac{(x^* - \overline{x})^2}{S_{XX}}}$$

Definition. ***Obervations***

1. *variance of $\hat{y}^*$ is smallest when $x^* = \overline{x}$. So in an experiment to estimate mean response at a particular level x^* of predictor variable, the* ***precision of the estimate is greatest*** *if (everything else remain equal) the observations on X are spaced so that $\overline{x} = x^*$*

2. *confidence interval for $\hat{y}^*$ not sensitive to moderate departures from assumption of error being normally distributed. The robustness in estimating mean response is related to robustness of Confidence interval for β_0 and β_1*

2.5 Prediction of New Observations

Definition. ***Prediction of New Observations***

1. ***Motivation*** *Idea is that we have a model set up given a set of data, and we would want to extrapolate to new data points. The new observation Y is viewed as the result of a new trial, independent of trials on which the regression analysis is based. Let level of X be x^* and the new observation y^*_{new} (which is unknown, and which we want to characterize), assuming that the underlying regression model is still appliacble for basic sample data*

2. ***Estimate of mean response*** $\mathbb{E}(y^*)$ ***vs. Prediction of new response*** y^*_{new}
   In the former case, we estimate ***mean of distribution of*** Y*. In latter case, we predict an* ***individual outcome*** *drown from the distribution of Y. Idea is we have to take into account of the fact that the majority of individual outcomes deviate from the mean response*

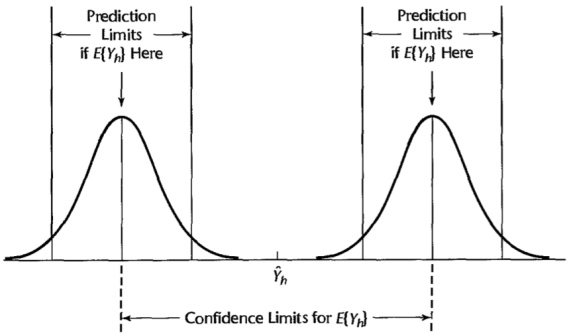
Definition. ***Prediction Interval for*** y^*_{new} ***when parameter is known***
Assume all parameters are known, we have y^* follow a normal distribution

$$y^*_{new} \sim \mathcal{N}(\beta_0 + \beta_1 x^*, \sigma^2)$$

So we have confidence interval,

$$(\beta_0 + \beta_1 x^*) \pm \sigma z_{1-\alpha/2}$$

Definition. ***Prediction Interval for*** y^*_{new} ***when parameter is unknown***
When parameter is unknown, we must estimate regression parameters. We might want to estimate mean distribution of Y with $\hat{y^*}$ and variance of distribution of Y with MSE.



However, we cannot substitute these estimate into the previous distribution because $\mathbb{E}(y^*)$ *is a random variable. Since we do not know the mean* $\mathbb{E}(y^*)$*, and only estiamte it by a confidence interval (shown previously), we cannot be certain of the distribution of* Y*. It could be anywhere along within its confidence intervals (for* $\mathbb{E}\{Y_h\}$*). Hence* ***prediction limit*** *for* y^*_{new} *must take into account two elements*

1. *variation in possible location of distribution of* Y *(i.e. sampling distributino of* $\hat{y}^*$*)*
2. *variation within the probability distribution of* Y *(namely* σ^2*, same as that of error terms')*

$$y^*_{new} - \hat{y}^* = \beta_0 + \beta_1 x^* + e^* - \hat{y}^* = \mathbb{E}(y_{new}|X = x^*) - \hat{y}^* + e^*$$

We can prove that

$$\mathbb{E}(y^*_{new} - \hat{y}^*) = \mathbb{E}(y_{new} - \hat{y}|X = x^*) = \beta_0 + \beta_1 x^* + \mathbb{E}(e^*) - \mathbb{E}(\hat{y}^*) = 0$$

$$\begin{aligned} Var(y^*_{new} - \hat{y}^*) &= Var(\hat{\beta}_0 + \hat{\beta}_1 x^*) + Var(\hat{y}^*) - Cov(y^*_{new}, \hat{y}^*) \\ &= \sigma^2 + \sigma^2 \left[\frac{1}{n} + \frac{(x^* - \overline{x})^2}{S_{XX}}\right] = \sigma^2 \left[1 + \frac{1}{n} + \frac{(x^* - \overline{x})^2}{S_{XX}}\right] \end{aligned}$$

Note $Cov(y^*_{new}, \hat{y}^*) = 0$ *by independence*

$$y^*_{new} - \hat{y}^* \sim \mathcal{N}(0, \sigma^2 \left[1 + \frac{1}{n} + \frac{(x^* - \overline{x})^2}{S_{XX}}\right])$$

An unbiased estimtor of the variance is given by

$$s^2(y^*_{new} - \hat{y}^*) = MSE \left[1 + \frac{1}{n} + \frac{(x^* - \overline{x})^2}{S_{XX}}\right]$$

Standardization of y^*_{new} *gives*

$$T = \frac{y^*_{new} - \hat{y}^*}{s^2(y^*_{new} - \hat{y}^*)} \sim t_{n-2}$$

The $(100-\alpha)\%$ ***Prediction limit*** *for* y^*_{new} *at* $X = x^*$ *is thus given by,*

$$(\hat{\beta}_0 + \hat{\beta}_1 x^*) \pm t_{1-\alpha/2,n-2} MSE \sqrt{1 + \frac{1}{n} + \frac{(x^* - \overline{x})^2}{S_{XX}}}$$

Definition. ***Comments***

1. *prediction limit is subject to departure from normality of error term distributions*
2. *A confidence interval represents an inference on a paramter and is an interval that is intended to cover the value of* ***parameter****. A prediction interval, is a statement about the value to be taken by a* ***random variable****, the new observation* y^*_{new}

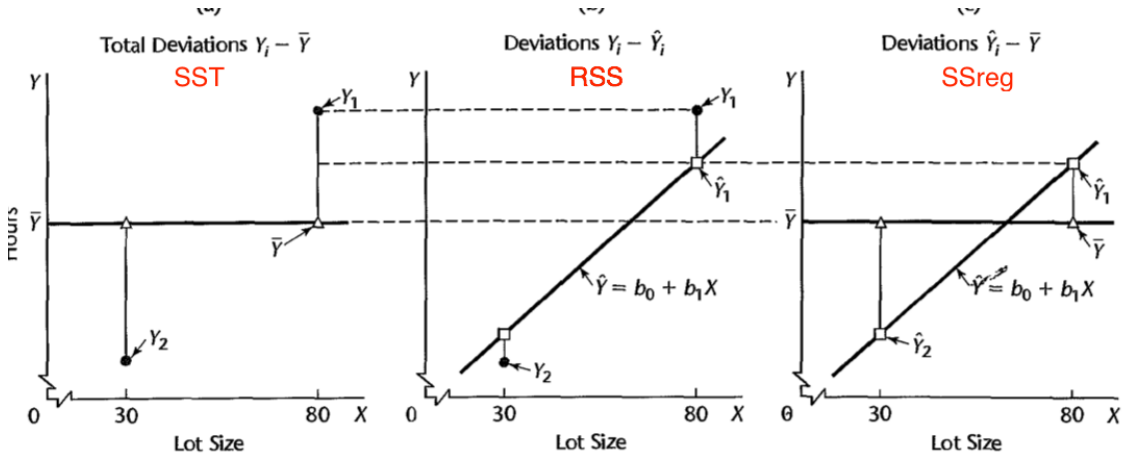
Confidence-band for Regression line

Definition. ***Confidence-band*** *represents uncertainty in the estimate of regression line (i.e.* $\mathbb{E}(Y) = \beta_0 + \beta_1 X$*). Used to determine appropriateness of a fitted regression function.* ***The Working-Hotelling*** $1-\alpha$ *confidence band for regression line has boundary values at any level* x^*

$$\hat{y}^* \pm W s\{\hat{y}^*\} \quad \text{where} \quad W^2 = 2F_{1-\alpha,n-2}$$

Analysis of Variance Approach to Regression Analysis

Definition. ***Partitioning Total sum of Squares*** *Idea is to partititon sums of squares and degree of freedom associated with response variable* Y



1. ***Total sum of squares*** *(SST)*

$$SST = \sum (y_i - \overline{y})^2$$

is a measure of uncertainty of Y*, without taking* X *into account*

2. ***Residual sum of squares*** *(RSS, SSE)*

$$RSS = \sum (y_i - \hat{y}_i)^2$$

is a measure of variation of Y, with X taken into account. The greater variation of Y around fitted regression line, the larger the RSS.

3. ***Regression sum of squares*** *(SSR, SSreg)*

$$SSreg = \sum (\hat{y}_i - \overline{y})^2$$

Alternatively we can expand and get to an equivalent expression

$$SSreg = \sum ((\hat{\beta}_0 - \hat{\beta}_1 x_i) - (\hat{\beta}_0 + \hat{\beta}_1 \overline{x}))^2 = \hat{\beta}_1^2 \sum (x_i - \overline{x})^2$$

SST can be broken down into 2 components, i.e. deviation of fitted value $\hat{Y}_i$ around mean $\overline{Y}$ and deviation of observed Y_i around fitted regression line

$$\underbrace{Y_i - \bar{Y}}_{\text{Total deviation}} = \underbrace{\hat{Y}_i - \bar{Y}}_{\substack{\text{Deviation of fitted} \\ \text{regression value} \\ \text{around mean}}} + \underbrace{Y_i - \hat{Y}_i}_{\substack{\text{Deviation} \\ \text{around} \\ \text{fitted} \\ \text{regression} \\ \text{line}}}$$

The sum of these squared deviations have the same relationship!

$$\sum (Y_i - \hat{Y})^2 = \sum (\hat{Y}_i - \overline{Y})^2 + \sum (Y_i - \hat{Y}_i)^2$$

$$SST = RSS + SSreg$$

$$\textit{total variability} = \textit{unexplained variability} + \textit{variability explained by model}$$

Proof.

$$\begin{aligned} \sum (y_i - \overline{y})^2 &= \sum [(\hat{y}_i - \overline{y}) + (y_i - \hat{y}_i)]^2 \\ &= \sum (\hat{y}_i - \overline{y})^2 + \sum (y_i - \hat{y}_i)^2 + 2 \sum (\hat{y}_i - \overline{y})(y_i - \hat{y}_i) \\ &= SSreg + RSS \end{aligned}$$

where

$$\begin{aligned} 2 \sum (\hat{y}_i - \overline{y})(y_i - \hat{y}_i) &= 2 \sum \hat{y}_i (y_i - \hat{y}_i) - 2\overline{y} \sum (y_i - \hat{y}_i) \\ &= 2 \sum \hat{y}_i e_i - 2\overline{y} \sum e_i \\ &= 0 \end{aligned}$$

□

Definition. ***Break down of degress of freedom***

1. *SST has* $n-1$ *degrees of freedom (1 df is lost because sample mean* $\overline{y}$ *is used to estimate population mean)*
2. *RSS has* $n-2$ *degrees of freedom (2 df lost because* β_0 *and* β_1 *are estimated)*
3. *SSreg has* 1 *degree of freedom.*

$$n-1 = 1 + (n-2)$$

Definition. ***Mean Squares*** *A sum of squares divided by its associated degree of freedom is mean square*

$$MSR = \frac{SSreg}{1} = SSreg \qquad MSE = \frac{RSS}{n-2}$$

MSE of an estimator measures the average of the squares of the errors or deviations, i.e. the difference between the estimator and what is estimated. MSE is a measure of quality of estimator

Definition. ***Analysis of Variance Table*** *displays breakdowns of total sum of squares and associated degrees of freedom*

Source of Variation	SS	df	MS	E{MS}
Regression	$SSR = \sum(\hat{Y}_i - \bar{Y})^2$	1	$MSR = \frac{SSR}{1}$	$\sigma^2 + \beta_1^2 \sum(X_i - \bar{X})^2$
Error	$SSE = \sum(Y_i - \hat{Y}_i)^2$	$n-2$	$MSE = \frac{SSE}{n-2}$	σ^2
Total	$SSTO = \sum(Y_i - \bar{Y})^2$	$n-1$		

Definition. ***Expected Mean square*** *Expected value of a mean square is the mean of its sampling distribution and tells us what is being estimated by the mean square.*

$$\mathbb{E}(MSE) = \sigma^2$$
$$\mathbb{E}(MSR) = \sigma^2 + \beta_1^2 \sum (x_i - \overline{x})^2$$

Proof. Given

$$\frac{RSS}{\sigma^2} \sim \chi^2_{n-2}$$

We have

$$\mathbb{E}(\frac{RSS}{\sigma^2}) = n-2 \quad \rightarrow \quad \mathbb{E}(\frac{RSS}{n-2}) = \mathbb{E}(MSE) = \sigma^2$$

To prove $\mathbb{E}(MSR)$, we first note form of $SSreg$,

$$SSreg = \hat{\beta}_1^2 \sum (x_i - \overline{x})^2$$

to find its expected value, we first find

$$\mathbb{E}(\hat{\beta}_1^2) = Var\hat{\beta}_1 + (\mathbb{E}(\hat{\beta}_1))^2 = \frac{\sigma^2}{S_{XX}} + \beta_1^2$$

Then we have

$$\mathbb{E}(SSreg) = \mathbb{E}(\hat{\beta}_1^2) \sum (x_i - \overline{x})^2 = (\frac{\sigma^2}{S_{XX}} + \beta_1^2) S_{XX} = \sigma^2 + \beta_1^2 S_{XX}$$

Then we find mean squared regression

$$\mathbb{E}(MSR) = \mathbb{E}(\frac{SSreg}{1}) = \sigma^2 + \beta_1^2 S_{XX}$$

□

Definition. *F **test of** $\beta_1 = 0$ **versus** $\beta_1 \neq 0$*
To test for if there is a linear relationship between X and Y, We can test for this with t test

$$T = \frac{\hat{\beta}_1 - 0}{\sigma / \sqrt{S_{XX}}} \sim t_{n-2}$$

Alternatively, to generalize the test to multiple regression, we can use F test where test statistics is given by

$$F^* = \frac{MSR}{MSE} = \frac{SSreg}{RSS/(n-2)}$$

Intuitively, larger values of F^* (larger MSR) supports H_α and values of F^* near 1 supports H_0. It can be shown that sampling distribution for F^* is given by

$$F^* \sim F_{1,n-2}$$

The test is then given by

$$\textbf{\textit{Reject if}}\ F^* > F_{1-\alpha;1,n-2}$$

Proof.

$$F^* = \frac{MSR}{MSE} = \frac{MSR\,/\,\sigma^2}{MSE\,/\,\sigma^2} = \frac{\chi_1^2}{\chi_{n-2}^2} \sim F_{1,n-2}$$

□

Definition. ***Equivalence of** F **test and** t **test***
Given α confidence level, F test of $\beta_1 = 0$ versus $\beta_1 \neq 0$ is equivalent to two-sided t test

Proof.

$$F^* = \frac{SSreg/1}{RSS/(n-2)} = \frac{\hat{\beta}_1 \sum (x_i - \overline{x})^2}{MSE} = \frac{\hat{\beta}_1 S_{XX}}{MSE}$$

Note

$$s^2(\hat{\beta}_1) = \frac{MSE}{S_{XX}}$$

we have

$$F^* = \left(\frac{\hat{\beta}_1^2 - 0}{s(\hat{\beta}_1)} \right)^2 = (t^*)^2$$

where t is the t statistics □

2.9 Descriptive Measures of Linear Association between X and Y

Definition. ***Coefficient of Determination***

1. *SST measures variation in Y without taken into account of X*
2. *RSS is a measure of variation in Y when taken into account of X*
3. *A natural measure of effect of X in reducing variation in Y is in the difference between SST and RSS as a proportion of total variation*

The ***coefficient of determination*** *R^2 is given by*

$$R^2 = \frac{SST - RSS}{SST} = \frac{SSreg}{SST}$$

with

$$0 \leq R^2 \leq 1$$

R^2 can be interpreted as the proportionate reduction of total variation associated with use of X predictor variable

Definition. ***Coefficient of Correlation***
A measure of linear association between Y and X when both Y and X are random is coefficient of correlation.

$$r = \pm\sqrt{R^2} = \frac{S_{XY}}{\sqrt{S_{XX} S_{YY}}}$$

Proof.

$$\sqrt{R^2} = \sqrt{\frac{SSReg}{SST}} = \sqrt{\frac{\hat{\beta}_1^2 S_{XX}}{S_{YY}}} = \sqrt{(\frac{S_{XY}}{S_{XX}})^2 \frac{S_{XX}}{S_{YY}}} = \frac{S_{XY}}{\sqrt{S_{XX} S_{YY}}}$$

□

Dummy variable regression

Definition. *1.* ***Dummy variable*** *is one that takes the value 0 or 1 to indicate the absence or presence of some categorical effect that may expect to shift the outcome*