Home Work 7

16.What is the probability density of the time between the arrival of the two packets of Example E in Section 3.4?

Ans:
Let $X=|T_1-T_2|$ Then

$$F_X(x)=P(X\leqslant x)=P(|T_1-T_2|\leqslant x)=P(-x\leqslant T_1-T_2\leqslant x)=\iint_A f_{T_1,T_2}(t_1,t_2)\,dt_1\,dt_2$$

with A is the area when $-x\leqslant T_1-T_2\leqslant x$, the shaded strip in Figure 3.12. From this example we have area of A is $T^2-(T-x)^2$.
Since T_1, T_2 are independent uniform random variables on [1,T] we have:

$$f_{T_1}(t_1)=\frac{1}{T},\forall t_1\in[0,T] \text{ and 0 otherwise,}$$

$$f_{T_2}(t_2)=\frac{1}{T},\forall t_2\in[0,T] \text{ and 0 otherwise,}$$

$$f_{T_1,T_2}(t_1,t_2)=f_{T_1}(t_1)f_{T_1}(t_1)=\frac{1}{T^2},\quad \forall t_1\in[0,T],t_2\in[0,T] \text{ and 0 otherwise.}$$

Hence: $$F_X(x)=\iint_A \frac{1}{T^2}dt_1\,dt_2=\frac{1}{T^2}\iint_A dt_1\,dt_2=\frac{1}{T^2}*Area\ of\ region\ \ A=\frac{T^2-(T-x)^2}{T^2}=1-(1-\frac{x}{T})^2$$

So probability density of X is $$f_X(x)=\frac{d}{dx}F_X(x)=\frac{d}{dx}(1-(1-\frac{x}{T})^2)=\frac{2}{T}(1-\frac{x}{T})$$

24. Let P have a uniform distribution on [0,1], and, conditional on P=p, let X have a Bernulli distribution with parameter p. Find the conditional distribution of P given X.

Ans:
By definition of conditional density we have conditional density of P given X is:

$$f_{P|X}(p|x)=\frac{f_{X,P}(x,p)}{f_X(x)}$$

So we need to find joint density X, P and marginal density of X.
From question we have:

- P have a uniform distribution on [0,1] so density (pdf) of P is:

$f_P(p)=1,\forall p\in[0,1]$ and 0 otherwise.

- Conditional on P=p, X has a Bernulli distribution with parameter p so conditional pdf of X given P is:

$f_{X|P}(x|p)=p^x(1-p)^{1-x}, if\ x=0\vee x=1$

Hence by multiplicative law we have joint pdf of X, P:

$$f_{X,P}(x,p)=f_{X|P}(x|p)f_P(p)=p^x(1-p)^{1-x},\quad if\ x=0\vee x=1,\ \ p\in[0,1] \text{ and 0 otherwise.}$$

By definition of marginal density we have:

$$f_X(x)=\int_0^1 f_{X,P}(x,p)\,dp=\int_0^1 p^x(1-p)^{1-x}, \quad \mathit{if}\ x=0 \vee x=1$$

Let x=1 we have $f_X(1)=\int_0^1 p(1-p)^0\,dp=\int_0^1 p\,dp=[\frac{p^2}{2}]_0^1=\frac{1}{2}$

Let x=0 we have $f_X(0)=\int_0^1 p^0(1-p)\,dp=\int_0^1 1-p\,dp=[p-\frac{p^2}{2}]_0^1=\frac{1}{2}$

so $f_X(x)=\frac{1}{2}, \mathit{if}\ x=0 \vee x=1$

Conditional density of P given X is:

$$f_{P|X}(p|x)=\frac{f_{X,P}(x,p)}{f_X(x)}=\frac{p^x(1-p)^{1-x}}{2}, \quad \mathit{if}\ x=0 \vee x=1, \quad p\in[0,1]$$

Noting that Bernulli is special case of binomial distribution, this solution resembles the method in Prof. Schroeder handout using in Bayesian Inference example (Text book page 94).

28. Show that C(u,v)=uv is a copula. Why is it called "the independence copula"?

Ans:

By $C(u,v)=uv$, the question means:

$C(u,v)=uv, \mathit{if}\ 0<u<1, 0<v<1$

$C(u,v)=u, \mathit{if}\ v\geqslant 1, 0<u<1$

$C(u,v)=v, \mathit{if}\ u\geqslant 1, 0<v<1$

$C(u,v)=1, \mathit{if}\ u\geqslant 1, v\geqslant 1$

$C(u,v)=0, \mathit{if}\ u\leqslant 0 \vee v\leqslant 0$

Hence $C(u,v)=uv$ is a joint cdf since $C(\infty,\infty)=C(1,1)=1$, $C(-\infty,-\infty)=C(0,0)=0$, and C is increasing wrt. u or wrt. v.

Now I prove that $C(u,v)=uv$ has 2 marginal density that are uniform. First find the joint pdf of U,V

$$c_{U,V}(u,v)=\frac{d^2}{dx\,dy}C(u,v)=1, \quad \mathit{if}\ 0\leqslant u\leqslant 1, 0\leqslant v\leqslant 1 \quad \text{and 0 otherwise.}$$

So marginal pdf of U is: $f_U(u)=\int_0^1 1\,dv=1$ if $0\leqslant u\leqslant 1$ and 0 otherwise, marginal pdf of V is:

$f_V(v)=\int_0^1 1\,du=1$ if $0\leqslant v\leqslant 1$ and 0 otherwise. Hence U, V have uniform density.

So C(u,v) is a copula by definition on page 78.

It is called "the independence copula" because U and V are indepent Rvs:

$c_{U,V}(u,v)=1=f_U(u)f_V(v)$

29. Use the Farlie-Morgenstern copula to construct a bivariate density whose marginal densities are exponential. Find an expression for the joint density.

Ans:

Farlie- Morgenstern Family (Example C, page 77)

$H(x,y)=F(x)G(y)\{1+\alpha[1-F(x)][1-G(x)]\}$ are a joint cdf which has marginal cdf F and G.

Let $F(x)=1-e^{-\alpha x}$ $G(y)=1-e^{-\beta y}$ then

$H(x,y)=(1-e^{-\lambda x})(1-e^{-\mu y})\{1+\alpha[1-(1-e^{-\lambda x})][1-(1-e^{-\mu y})]\}=(1-e^{-\lambda x})(1-e^{-\mu y})(1+\alpha e^{-\lambda x}e^{-\mu y})$

has two marginal densities which are exponential. Joint pdf of X, Y is:

$$
\begin{aligned}
h(x,y)&=\frac{d^2}{dx\,dy}(1-e^{-\lambda x})(1-e^{-\mu y})(1+\alpha e^{-\lambda x}e^{-\mu y})=\frac{d}{dx}\left(\frac{d}{dy}(1-e^{-\lambda x})(1-e^{-\mu y})(1+\alpha e^{-\lambda x}e^{-\mu y})\right)\\
&=\frac{d}{dx}\left((1-e^{-\lambda x})\frac{d}{dy}(1-e^{-\mu y})(1+\alpha e^{-\lambda x}e^{-\mu y})\right)\\
&=\frac{d}{dx}\left((1-e^{-\lambda x})[\mu e^{-\mu y}(1+\alpha e^{-\lambda x}e^{-\mu y})+(1-e^{-\mu y})(-\alpha\mu e^{-\lambda x}e^{-\mu y})]\right)\\
&=\frac{d}{dx}\left((1-e^{-\lambda x})[\mu e^{-\mu y}+\mu\alpha e^{-\lambda x}e^{-2\mu y}-\alpha\mu e^{-\lambda x}e^{-\mu y}+\alpha\mu e^{-\lambda x}e^{-2\mu y}]\right)\\
&=\frac{d}{dx}\left((1-e^{-\lambda x})[\mu e^{-\mu y}(1-\alpha e^{-\lambda x}+2\alpha e^{-\lambda x}e^{-\mu y})]\right)\\
&=\mu e^{-\mu y}\frac{d}{dx}(1-e^{-\lambda x})(1-\alpha e^{-\lambda x}+2\alpha e^{-\lambda x}e^{-\mu y})\\
&=\mu e^{-\mu y}[\lambda e^{-\lambda x}(1-\alpha e^{-\lambda x}+2\alpha e^{-\lambda x}e^{-\mu y})+(1-e^{-\lambda x})(\alpha\lambda e^{-\lambda x}-2\alpha\lambda e^{-\lambda x}e^{-\mu y})]\\
&=\mu e^{-\mu y}\lambda e^{-\lambda x}[(1-\alpha e^{-\lambda x}+2\alpha e^{-\lambda x}e^{-\mu y})+\alpha(1-e^{-\lambda x})(1-2e^{-\mu y})]\\
&=\mu e^{-\mu y}\lambda e^{-\lambda x}[1-\alpha e^{-\lambda x}(1-2e^{-\mu y})+\alpha(1-e^{-\lambda x})(1-2e^{-\mu y})]\\
&=\mu e^{-\mu y}\lambda e^{-\lambda x}[1+\alpha(1-2e^{-\lambda x})(1-2e^{-\mu y})]
\end{aligned}
$$

Here, if you apply directly the **formula in Prof. Schroeder's handout**, you get it faster:

$$
\begin{aligned}
h(x,y)&=f(x)g(y)[1+\alpha(1-2\mathrm{F}(x))(1-2\mathrm{G}(y))]\\
&=\mu e^{-\mu y}\lambda e^{-\lambda x}[1+\alpha(1-2(1-e^{-\lambda x}))(1-2(1-e^{-\mu y}))]\\
&=\mu e^{-\mu y}\lambda e^{-\lambda x}[1+\alpha(-1+2e^{-\lambda x})(-1+2e^{-\mu y})]\\
&=\mu e^{-\mu y}\lambda e^{-\lambda x}[1+\alpha(1-2e^{-\lambda x})(1-2e^{-\mu y})]
\end{aligned}
$$

32. Continuing Ex E. Section 3.5.2. Find θ that maximizes the posterior density. Does the result make intuitive sense?

Ans:

From Example E. we have the posterior:

$$f\,\theta|X(\theta|x)=\frac{f\,\theta,X(\theta,x)}{f_X(x)}=\frac{\Gamma(n+2)}{\Gamma(x+1)\Gamma(n-x+1)}\theta^x(1-\theta)^{n-x}$$

In order to maximize the posterior, I find its derivative. Actully I don't need to pay attention on constant part so I tak the derivative of $\theta^x(1-\theta)^{n-x}$ wrt. θ and get:

$$\frac{d}{d}\theta\,\theta^{x}(1-\theta)^{n-x}=x\theta^{x-1}(1-\theta)^{n-x}-\theta^{x}(n-x)(1-\theta)^{n-x-1}$$
$$=\theta^{x-1}(1-\theta)^{n-x-1}(x(1-\theta)-\theta(n-x))$$
$$=\theta^{x-1}(1-\theta)^{n-x-1}(x-\theta n)$$

So solutions for the derivative is $\theta=0,\theta=1,\theta=\frac{x}{n}$ .Moreover, the sign of derivative function changes from positive to negative at $\theta=\frac{x}{n}$. So $\theta=\frac{x}{n}$ maximizes the posterior. The result make sense since the intuitive way to guess probability of success is taking the ratio between number of succeses over number of trials.

35. Ans: When I increase the number of spins, Ratio $\frac{x}{n}$ approximate 0.5 and my graph becomes more symmetric.

43. Let U_1 , U_2 be independent and uniform on [0,1]. Find and sketch the density function of $S=U_1+U_2$.

Ans:

U_1 , U_2 be uniform on [0,1] hence pdfs of U_1 and U_2 are
$f_{U_1}(u_1)=1,\forall u_1\in[0,1]$ and 0 otherwise,
$f_{U_2}(u_2)=1,\forall u_2\in[0,1]$ and 0 otherwise.
Since they are independent, joint density of U_1 and U_2 is
$f_{U_1,U_2}(u_1,u_2)=f_{U_1}(u_1)f_{U_2}(u_2)=1,\quad \forall u_1\in[0,1]\ \forall u_2\in[0,1]$
There are 2 methods to to this:

- Direct Method (find cdf then take derivative to find pdf)

$$F_S(s)=P(S\leqslant s)=P(U_1+U_2\leqslant s)=\iint_A f_{U_1,U_2}(u_1,u_2)\,du_1\,du_2=\iint_A 1\,du_1\,du_2=Area\ \ of\ A$$

When $A=\{(u_1,u_2)\in[0,1]\times[0,1],u_1+u_2\leqslant s\}$.
If $s<0$ $Area\ of\ A=0$
If $0\leqslant s\leqslant 1$ $Area\ of\ A=\frac{s^2}{2}$ Hence cdf of S: $F_S(s)=Area\ \ of\ A=\frac{s^2}{2}$ so density of S:

$$f_S(s)=\frac{d}{ds}F_S(s)=\frac{d}{ds}\frac{s^2}{2}=s$$

If $1\leqslant s\leqslant 2$ $Area\ of\ A=1-\frac{(2-s)^2}{2}$ Hence cdf of S: $F_S(s)=Area\ \ of\ A=1-\frac{(2-s)^2}{2}$ so density of S:

$$f_S(s)=\frac{d}{ds}F_S(s)=\frac{d}{ds}(1-\frac{(2-s)^2}{2})=2-s$$

If $s>2$ $Area\ \ of\ A=1$ Hence cdf of S: $F_S(s)=Area\ \ of\ A=1$ so density of S:

$f_S(s) = \frac{d}{ds} F_S(s) = 0$

- Convolution method: use convolution formula (page 97)

$f_S(s) = \int_{\infty}^{\infty} f_{U_1}(u_1) f_{U_2}(s-u_1)\, du_1 = \int_0^1 f_{U_1}(u_1) f_{U_2}(s-u_1)\, du_1 = \int_0^1 1 * f_{U_2}(s-u_1)\, du_1$

If $s<0$ then $s-u_1<0 \; \forall u_1 \in [0,1]$ and $f_{U_2}(s-u_1)=0, \; \forall u_1 \in [0,1]$. So $f_S(s)=0$

If $0 \leqslant s \leqslant 1$, then $0 \leqslant s-u_1 \leqslant 1 \; \forall u_1 \in [0,s]$ and $f_{U_2}(s-u_1)=1, \; \forall u_1 \in [0,s]$ and 0 otherwise.

So $f_S(s) = \int_0^1 f_{U_2}(s-u_1)\, du_1 = \int_0^s 1\, du_1 = s$

If $1 \leqslant s \leqslant 2$, then $0 \leqslant s-u_1 \leqslant 1 \; \forall u_1 \in [s-1,1]$ and $f_{U_2}(s-u_1)=1, \; \forall u_1 \in [s-1,1]$ and 0 otherwise. So $f_S(s) = \int_0^1 f_{U_2}(s-u_1)\, du_1 = \int_{s-1}^1 1\, du_1 = 2-s$

If $s>2$ then $s-u_1>1 \; \forall u_1 \in [0,1]$ and $f_{U_2}(s-u_1)=0, \; \forall u_1 \in [0,1]$. So $f_S(s)=0$

44. Let N_1 , N_2 be independent random variables following Poisson distributions with parameters λ_1 and λ_2 . Show that the distribution of $N = N_1 + N_2$ is Poisson with parameter $\lambda_1 + \lambda_2$.

Ans:

N_1 , N_2 be independent random variables following Poisson distributions with parameters λ_1 and λ_2 :

$p_{N_1}(n_1) = \frac{\lambda_1^{n_1} e^{-\lambda_1}}{n_1!}, \quad n_1 = 0,1,2,.. \; , \quad p_{N_2}(n_2) = \frac{\lambda_2^{n_2} e^{-\lambda_2}}{n_2!}, \quad n_2 = 0,1,2,.. \; ,$

Using convolution, I have:

$$p_N(n) = \sum_{n_1=-\infty}^{\infty} p_{N_1}(n_1) p_{N_2}(n-n_1) = \sum_{n_1=0}^{n} p_{N_1}(n_1) p_{N_2}(n-n_1) = \sum_{n_1=0}^{n} \frac{\lambda_1^{n_1} e^{-\lambda_1}}{n_1!} \frac{\lambda_2^{n-n_1} e^{-\lambda_2}}{(n-n_1)!} = \frac{e^{-\lambda_1-\lambda_2}}{n!} \sum_{n_1=0}^{n} \frac{n!\lambda_1^{n_1} \lambda_2^{n-n_1}}{n_1!(n-n_1)!}$$

So $p_N(n) = \frac{e^{-\lambda_1-\lambda_2}}{n!} \sum_{n_1=0}^{n} \frac{n!\lambda_1^{n_1} \lambda_2^{n-n_1}}{n_1!(n-n_1)!} = \frac{e^{-\lambda_1-\lambda_2}}{n!} \sum_{n_1=0}^{n} \binom{n}{n_1} \lambda_1^{n_1} \lambda_2^{n-n_1} = \frac{e^{-\lambda_1-\lambda_2}}{n!} (\lambda_1+\lambda_2)^n, \quad n = 0,1,2,...$

So N is Poisson with parameter $\lambda_1 + \lambda_2$.