Problem 1

1. Given an algorithm taking inputs

   (a) $G = (V, E)$ a connected, undirected graph

   (b) $w : E \to \mathbb{Z}^+$ a weight function

   (c) $T \subseteq E$, a MST of G

   (d) $e_1 = \{u, v\} \notin E$, an edge no in G

   (e) $w_1 \in \mathbb{Z}^+$, a weight for e_1

   and outputs a MST T_1 for $G_1 = (V, E \cup \{e_1\})$ with $w(e_1) = w_1$. For full marks, your algorithm must be more efficient than computing a MST for G_1 from scratch. Justify that this is the case by analysing your algorithms worst-case running time. Finally, write a detailed proof that your algorithm is correct.

   Solution. □

   Algorithm

   (a) Let $G_1 = (V, E \cup \{e_1\})$, let $e_1 = (s, t)$ for some $s, t \in V$

   (b) Find the unique simple cycle c in G_1. Starting from s (or t, the choice is arbitrary), run DFS on G_1 with slight modification. When looking at vertex $u \in V$, Check if $v.color$ is $GRAY$. Break from DFS if true, otherwise continue DFS.

   (c) Find the largest weight edge e_2 in the cycle. Iterate through $G_1.E$, find $e_2 = (u, v) \in G_1.E$ such that $u.color$ and $v.color$ are both $GRAY$ (in the cycle) and $w(u, v)$ is maximum of all edges with $GRAY$ vertices (max weighted edge).

   (d) Return MST $T_1 = T \cup \{e_1\} \setminus \{e_2\}$

   Analysis

   (a) Since we are adding a constant time check at every while iteration and DFS itself has a run time of $O(V + E)$, the part of algorithm for finding a cycle (modified DFS) has a worst case running time of $O(V + E)$

   (b) The loop over all vertices to locate the maximum weight edge runs for $|E|$ iterations, each time doing a constant time operation. hence has a worst case running time of $O(E)$

   (c) Altogether the algorithm runs for $O(V+E)$, which is more efficient by computing MST from ground up which takes $O(E \lg V)$

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1  Function DFS-Visit (G, u)
2      u.color ← GRAY
3      for v ∈ G_1.Adj[u] do
4          if v.color is GRAY then
5              Exit-DFS
6          if v.color is WHITE then
7              DFS-Visit(G, v)
8      u.color ← BLACK
9  Function Find-MST-1 (G, w, T, e_1, w_1)
10     G_1 ← (V, E = E ∪ {e_1}) with w : E → ℝ and w(e_1) = w_1
11     (s, t) ← e_1
12     for v ∈ V do
13         v.color ← WHITE
14     DFS-Visit(G_1, s)
15     max_w ← 0
16     e_2 ← NIL
17     for (u, v) ∈ G_1.E do
18         if u.color = GRAY and v.color = GRAY and w(u, v) > max_w then
19             max_w ← w(u, v)
20             e_2 ← (u, v)
21     T_1 ← T ∪ {e_1} \ {e_2} return T_1
```

Lemma. *The graph $G_1 = (V, E = E \cup \{e_1\})$ has a unique simple cycle c, and e_1 is in c*

Proof. Let $T \subseteq E$, and $e_1 = (s, t)$. Since T is MST, s is reachable from t, i.e. exists a path p such that $t \overset{p}{\rightsquigarrow} s$. Consider $E_1 = E \cup \{e_1\}$, we have $c = s \to t \overset{p}{\rightsquigarrow} s = \langle v_0 = s, \cdots, v_k = s \rangle$. Therefore e_1 is in c. Now we prove c is unique. This is equivalent to proving that p is unique, since if adding e_1 to E produces more than 1 cycle, then there must exists p' such that $p' \neq p$. This is not possible since $s \overset{p}{\rightsquigarrow} t \overset{p'}{\rightsquigarrow} s$ forms a cycle, which is not possible in MST T. Also, c is simple because T is simple. □

Proof of correctness

Proposition. *The output T_1 is a MST for $G_1 = (V, E = E \cup \{e_1\})$*

Proof. By correctness of DFS, when we first explored (u, v), if $v.color$ is $GRAY$, then (u, v) is a back edge, indicating we have found a cycle c in the graph. By the previous lemma and the fact that we started from s where $e_1 = (s, t)$, we will always find such a cycle (i.e. discover t such that $s \in G_1.Adj[t]$ and $s.color$ is $GRAY$). When Exit-DFS is called, the vertices $v \in V$ such that $v.color$ is $GRAY$ represents vertices

constituting the cycle c. The claim holds since by the time EXIT-DFS is called, all ancestors of t has yet to finish (i.e. setting their color to $BLACK$). The subsequent loop over $G_1.E$ finds the maximum weighted edge e_2 in the cycle c. By the cycle property of MST, e_2 cannot be included in any MST of G_1. Now consider the return value $T_1 = T \cup \{e_1\} \setminus \{e_2\}$. Now we prove T_1 is a MST of G_1. Consider $A = T \setminus \{e_2\}$, note A breaks into 2 connected components as MST T is connected. Let $C = (P, Q)$ be the cut where $s \in P$ and $t \in Q$, $P \cup Q = A$ and $P \cap Q = \emptyset$. The fact that $e_2 \in T$ implies that e_2 is a light edge cross the cut C. Now we have $w(e_1) < w(e_2)$ since e_2 is the maximum weight edge in the cycle c, therefore e_1 is the light across the cut C. By corollary in CLRS, since P respects the cut C and e_2 is a light edge crossing the cut and $P \subseteq T$, therefore e_2 is safe for P. Therefore T_1 is some MST of G_0 □

2. Give an efficient algorithm that takes the following inputs:

   (a) $G = (V, E)$ a connected, undirected graph

   (b) $w : E \to \mathbb{Z}^+$ a weight function

   (c) $T \subseteq E$, a MST of G

   (d) $e_0 \in E$, an edge in G

   and that outputs a minimum spanning tree T_0 for the graph $G_0 = (V, E \setminus \{e_0\})$, if G_0 is still connected your algorithm should output the special value Nil if G_0 is disconnected. For full marks, your algorithm must be more efficient than computing a MST for G_0 from scratch. Justify that this is the case by analysing your algorithms worst-case running time. Finally, write a detailed proof that your algorithm is correct. (Note that this argument of correctness will be worth at least as much as the algorithm itself.)

   Solution. □

   Algorithm

   (a) Keep track of the cut $C = (P, Q)$ such that $P \cup Q = E \setminus \{e_0\}$

      i. Make set on every $v \in V$

      ii. Iterate over $T \subseteq E$, combine the sets if there is an edge connecting the two connected components

   (b) Find the set of edges $E' \subseteq E$ such that for all $e \in E'$, e crosses the cut C. This can be achieved by checking all $e = (u, v) \in E$ against cut C and see if the u and v are in different connected components

   (c) Find the light edge e_2 (lowest weight edge crossing the cut C) from the E'

   (d) Return MST $T_0 = T \setminus \{e_0\} \cup \{e_2\}$ if e_2 exists otherwise return Nil

```
1  Function Find-MST-2 (G, w, T, e_0)
2      G_0 = (V, E \ {e_0})
3      for v ∈ V do
4          Make-Set(v)
5      for (u, v) ∈ T do
6          Union-Set(u, v)
7      E' ← ∅
8      for (u, v) ∈ G_0.E do
9          if Find-Set(u) ≠ Find-Set(v) then
10             E' ← E' ∪ {(u, v)}
11     if E' = ∅ then
12         return Nil
13     min_w ← ∞
14     e_2 ← Nil
15     for e ∈ E' do
16         if w(e) ≤ min_w then
17             min_w ← w(e)
18             e_2 ← e
19     T_0 ← T \ {e_0} ∪ {e_2}
20     return T_0
```

Proof of correctness

Proposition. *The algorithm returns T_0, a MST for $G_0 = (V, E \setminus \{e_0\})$ if exists,* NIL *otherwise*

Proof. Note $G = (V, E)$ has MST T implies G is connected. The removal of an edge e_0 disconnects E into 2 connected components. Let $C = (P, Q)$ be the cut such that $P \cup Q = E \setminus \{e_0\}$. $C = (P, Q)$ is kept track of in sets, where all $e_1 \in P$ belongs to one set and all $e_2 \in Q$ belongs to another. E' hence contains edges $e = (u, v) \in E$ such that $u \in P \wedge v \in Q$ or $v \in P \wedge u \in Q$. If there is no such edges other than e_0 that crosses the cut C, then E' is empty and G_0 is disconnected. In this case the algorithm returns NIL, which is correct. Otherwise we find a light edge e_2 and returns $T_0 = T \setminus \{e_0\} \cup \{e_2\}$. Here we claim T_0 is some MST of G_0. By corollary given in CLRS, since P respects the cut C and e_2 is a light edge crossing the cut and $P \subseteq T$, therefore e_2 is safe for P. Therefore T_0 is some MST of G_0 □

Problem 2

Consider the following MST with Fixed Leaves problem:

1. Input: A weighted graph $G = (V, E)$ with integer costs $c(e)$ for all edge $e \in E$, and a subset of vertices $L \subseteq V$

2. Output: A spanning tree T of G where every node of L is a leaf in T and T has the minimum total cost among all such spanning trees.

1. Does this problem always have a solution? In other words, are there inputs G, L for which there is no spanning tree T that satisfies the requirements? Either provide a counter-example (along with an explanation of why it is a counter-example), or give a detailed argument that there is always some solution.

   Solution. □

   The problem does not always have a solution. Consider $G = (V, E)$ where $V = \{1, 2, 3, 4\}$ and $E = \{(1, 2), (1, 3), (1, 4)\}$ with $L = \{1\}$ and arbitrary weights. Since there is only 3 edges which happen to connect to all 4 vertices, there is only one MST $T = E$ for G. A non-root node in a acyclic graph (tree) implies that that there is only one parent. Since vertex 1 connects to 3 other vertices, it cannot be non-root, implying it must be root. Therefore we have a counter example where the only MST possible $T = E$ do not have $v \in L$ as leaves.

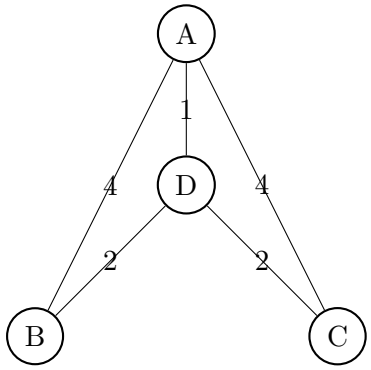
2. Let G, L be an input for the MST with Fixed Leaves problem for which there is a solution.

   (a) Is every MST of G an optimal solution to the MST with Fixed Leaves problem? Justify.

   (b) Is every optimal solution to the MST with Fixed Leaves problem necessarily a MST of G (if we remove the constraint that every node of L must be a leaf)? Justify.

   Solution. □

   Both claim are incorrect. Here we provide a counter example. Consider $G = (V, E)$ where $V = \{A, B, C, D\}$ and $E = \{(A, C), (A, B), (D, A), (D, B), (D, C)\}$ and weights labelled in the graph. Let $L = \{D\}$



There is only one MST $T = \{(D, A), (D, B), (D, C)\}$ for G with $w(T) = 4$. There is only one MST $T' = \{(A, C), (A, B), (D, A)\}$ given fixed leaves L for G with $w(T) = 9$. Note $T \neq T'$ and $w(T) < w(T')$.

(a) The existence of T, a MST of G but not the a solution to MST with fixed leaves problem, disproves this claim

(b) The existence of T', a MST of G with fixed leaves problem but not an MST for G, disproves this claim

3. Write a greedy algorithm to solve the MST with Fixed Leaves problem. Give a detailed pseudo-code implementation of your algorithm, as well as a high-level English description of the main steps in your algorithm. What is the worst-case running time of your algorithm? Justify briefly.

Algorithm

(a) Let $S = V \setminus L$

(b) Find MST for graph induced by S, let T_S be the ouput MST for S

(c) For each (u, v) where $u \in L$ and $v \in S$, adds the lowest weight edge for each u to T_S. In other words, find the light edge crossing the cut $(\{u\}, T_S)$ and adds it to T_S

(d) Return the augmented T, consists of T_S and edges added to it in the previous step

Analysis

(a) Assume sets S and T membership check can be done in $O(1)$ time, this is possible if keep an extra bit in adjacency list representation of the graph. Assume insertion to array T takes $O(1)$ time.

(b) MST-Kruskal has worst case running time of $O(E \lg V)$

(c) The for loop executes $O(E)$ times altogether, since sum of lengths of all adjacency list is $2|E|$ for undirected graphs and $|L| \leq |V|$. In each iteration, membership check and potential assignment operation takes $O(1)$.

(d) The entire algorithm has a worst time running time of $O(E \lg V)$

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1  Function Find-MST-Fixed-Leaves (G, w, L)
2      S ← V \ L
3      G' ← induced by S
4      T ← MST-Kruskal(G', w)
5      for u ∈ L do
6          e_light ← Nil
7          w_light ← ∞
8          for v ∈ G.Adj[u] do
9              if v ∈ S and w(u, v) < w_light then
10                 w_light ← w(u, v)
11                 e_light ← (u, v)
12         T ← T ∪ {e_light}
13     return T
```

4. Write a detailed proof that your algorithm always produces an optimal solution.

Lemma. *Given condition provided by the algorithm, provided that there is a valid solution for fixed leaves problem, then for any $u \in L$, there exists some $v \in S = V \setminus L$ such that $(u, v) \in E$.*

Proof. Prove by contradiciton. Assume there exists $u \in L$ such that for all $(u, v) \in E' = E$, we have $v \in L$. Pick any appropriate edge $(u, v) \in E'$ to be in some valid solution MST T. Now vertex u and v has one edge incident on it. We cannot include any other edge such that they incident on u or v since otherwise u or v would not be leaves (i.e. have degree of more than 1). Therefore $\{u, v\}$ is isolated from the rest of the vertices, hence no viable MST exists possibly as a solution. Since we are given there is some solution to the problem, the claim thus holds □

Proposition. *Given $G = (V, E)$ with cost function $c : E \to \mathbb{R}$ and a subset $L \subseteq V$, the algorithm returns MST T where every $u \in L$ is a leaf and T has the minimum cost amongst such spanning trees*

Proof. By correctness of Kruskal's algorithm, MST-Kruskal terminates and returns T_S, which is a MST for $S = V \setminus L$.

(a) Prove every $u \in L$ is a leaf in T. The algorithm finds and adds the least weight edge connecting u to some $v \in S$. Since T_S spans S and the fact that the previous lemma holds, such operation is always possible. Since the algorithm only adds one, specifically e_{light}, to T for each $u \in L$, u has degree of one, hence are leaves in MST T

(b) Prove by contradiction that the algorithm returns a MST amongst such spanning trees. Assume there is some other spanning tree T' such that $w(T) > w(T')$. We

can decompose $T' = T'_S \cup T'_O$ where T'_S is a subset of T' and spans S and T'_O be the rest of edges in T', specifically T'_O consists of edges that crosses the cut (L, S).

i. By correctness of MST-KRUSKAL we have

$$w(T_S) \leq w(T'_S)$$

ii. After finding T_S, the algorithm then construct the solution MST T by adding (u, v) to T_S where $v \in S$ for each $u \in L$ (Note this is always possible with previous lemma) such that $w(u, v)$ is minimized. Denote E' be such set of edges added to T_S Therefore

$$\sum_{e \in E'} w(e) \leq w(T'_O)$$

Althgether we have

$$w(T) = w(T_S) + \sum_{e \in E'} w(e) \leq w(T'_S) + w(T'_O) = w(T')$$

which contradicts the assumption that $w(T) > w(T')$. Therefore the claim holds

This completes the proof □

Problem 3

An edge in a flow network is called critical if decreasing the capacity of this edge reduces the maximum possible flow in the network. Give an efficient algorithm that finds a critical edge in a network. Give a rigorous argument that your algorithm is correct and analyse its running time.

Solution. □

Algorithm

1. Find a maximum flow f for G, let G_f be the residual flow network for G given f
2. Return the set of edges $E' \subseteq E$ for which $f(u, v) = c(u, v)$, $(u, v) \in E'$

Analysis

1. FORD-FULKERSON runs in $O(VE)$
2. The for loop iterates $|E|$ times, doing a constant $O(1)$ operation to update E'. Suppose E" is implemented with an array, the operation to insert an element at the end takes $O(1)$ time. Therefore the loop has a worst time running time of $O(E)$

3. Altogether the algorithm has a worst case running time of $O(VE)$

```
1 Function Find-Critical-Edge (G, s, t)
2     f ← Ford-Fulkerson(G, s, t)
3     E' ← ∅
4     for (u, v) ∈ G.E do
5         if f(u, v) = c(u, v) then
6             E' ← E' ∪ {(u, v)}
7     return E'
```

Proof of Correctness

Lemma. *Given a maximum flow f for flow network G and its residual flow network G_f. If any $(u, v) \in E$ such that $f(u, v) = c(u, v)$, then (u, v) is in the cut-set of some cut $C = (S, T)$, where $s \in S$, $t \in T$, and such that $|f| = c(S, T)$.*

Proof. Prove by contradiction. Assume there is no such cut C where (u, v) crosses $|f| = c(S, T)$. Then it must be that for all cut $C' = (S', T')$ that (u, v) crosses, $|f| < c(S, T)$ by the upper-bound property, therefore

$$|f| = f(S', T') = \sum_{u \in S'} \sum_{v \in T'} f(u, v) - \sum_{u \in S'} \sum_{v \in T'} f(v, u) < c(S, T) = \sum_{u \in S'} \sum_{v \in T'} c(u, v)$$

This implies that there is some other edge (x, y) in the cut-set of C' such that $f(x, y) < c(x, y)$. Since C' is an arbitrary and the above claim holds for all cuts, we can construct an augmenting path $p = \langle v_0, \cdots, v_k \rangle$ in G_f by picking appropriate cuts such that $f(v_{i-1}, v_i) > 0$ for all $i = 1, \cdots, k$. This contradicts with the assumption that the algorithm for finding max flow terminates, specifically when there is no augmenting paths left. Hence the claim holds. □

Proposition. *Given a maximum flow f for flow network G and its residual flow network G_f. the set of edges $E' \subseteq E$ for which $f(u, v) = c(u, v)$, $(u, v) \in E'$ is the set of critical edges*

Proof. Given assumption of the proposition and previous lemma, we have (u, v) in some cut $C = (S, T)$ such that $|f| = c(S, T)$. Since algorithm for finding the max flow f terminates, by the max-flow min-cut theorem, the cut C has minimal capacity, which bounds the value of the flow $|f|$. Hence, a reduction in $c(u, v)$ causes reduction in $C(S, T)$, and ultimately the maximum value a flow can achieve. Therefore (u, v) is a critical edge. Since (u, v) is an arbitrary edge in E', E' is the set of critical edges. □

Problem 4

1. Suppose we want to compute the shortest path from node s to node t in a directed graph $G = (V, E)$ with edge lengths $l_e > 0$ (assume integer valued lengths) for each

$e \in E$. Show that this is equivalent to finding a pseudo-flow f from s to t in G such that $|f| = 1$ and $\sum_{e \in E} l_e f_e$ is minimized. There are no capacity constraints. Part of this problem requires you to define precisely what we mean by pseudo-flow in a general, directed graph. This is a natural extension of the notion of flow in a network.

Solution. □

Define a pseudo-flow as a function $f : V \times V \to \mathbb{R}$ such that f_{uv} represents units of flow from vertex u to v. f still satisfies flow conservation but there is no capacity constraints on the amount of units of flow sent through the network. Given

$$|f| = \sum_{v \in V} f_{sv} = 1$$

we require exactly one unit of flow from the source, and since the graph is connected and by flow conservation, this one unit of flow is tranferred conceptually from s to t over vertices in some arbitrary path $p = \langle v_0 = s, v_1, \cdots, v_k = t \rangle$ such that $f_{v_{i-1}v_i} = 1$ for all $i = 1, \cdots, k$. We claim that

$$\text{there is not a flow } f_{uv} > 0 \text{ such that } (u, v) \text{ is not in } p.$$

Prove by contradiction, if there is such a flow $f_{uv} = 1$ for $u \to v$ where (u, v) is not in p, then to satisfy flow conservation, there must be some paths p_1 p_2 such that

$$p' = s \overset{p_1}{\rightsquigarrow} u \to v \overset{p_2}{\rightsquigarrow} t$$

Since (u, v) is not in p by assumption, then $p' \neq p$, we then have 2 paths originated from s, with $|f| = 2$, then all (u, v) with positive flow $f_{uv} > 0$ is in path p. Therefore,

$$\sum_{e \in E} l_e f_e = \sum_{e \text{ in } p} l_e \times 1 + \sum_{e \text{ not in } p} l_e \times 0 = \sum_{e \text{ in } p} l_e = w(p)$$

Minimizing $\sum_{e \in E} l_e f_e$ is equivalent to minimizing $w(p)$. Hence, the procedure of finding a pseudo-flow f from s to t such that $|f| = 1$ and $\sum_{e \in E} l_e f_e$ is minimized is equivalent to finding a path p with minimum weight $w(p)$, which is what shortest-path specifies

2. Write the shortest path problem as a linear or integer program where your objective function is minimized, based on your answer to the previous part. Give a detailed justification that your solution is correct.

Solution. □

$$\textbf{Minimize:} \quad \sum_{e \in E} l_e f_e$$

$$\textbf{Subject to:} \quad \sum_{v \in V} f_{uv} - \sum_{v \in V} f_{vu} = \begin{cases} 1 & \text{if } u = s \\ -1 & \text{if } u = t \\ 0 & \text{otherwise} \end{cases}$$

$$f_e \geq 0 \quad \text{for all } e \in E$$

The linear program is basically a formal specification of the assumptions made for finding pseudo-flow f from s to t in G. Since if $u = s$ we have

$$\sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs} = |f| = 1$$

as well as flow conservation given by

$$\sum_{v \in V} f_{uv} - \sum_{v \in V} f_{vu} = 0 \quad \iff \quad \sum_{v \in V} f_{uv} = \sum_{v \in V} f_{vu}$$

for all $u, v \in V \setminus \{s, t\}$. Note a feasible solution is simply values f_e for all $e \in E$, in other words, a function $f : V \times V \to \mathbb{R}$. Since flow conservation property is satisfied, this is saying that we are trying to find the pseudo-flow f such that $|f| = 1$ and $\sum_{e \in E} l_e f_e$ is minimized. This is equivalent to finding the solution to shortest path problem as justified in the previous question. Now we prove linear program yields correct solution. Let f be solution to the linear program. Let p be the path such that for all edges e in p, we have $f_e = 1$. Such path is always possible by the fact that we constrain source to admit 1 unit of flow and sink to accept 1 unit of flow. Now we prove that the path is the shortest. Prove by contradiction, assume there is some other path p' with

$$w(p') < w(p) = \sum_{e \text{ in } p} l_e = \sum_{e \text{ in } p} l_e \times 1 + \sum_{e \text{ not in } p} l_e \times 0 = \sum_{e \in E} l_e f_e$$

which contradicts the fact that $\sum_{e \in E} l_e f_e$ is minimal. Hence the solution to linear programming is correct. Note the transformation in the equation is possible as proved in the previous question

Problem 5

1. In the Traveling Salesman Problem (TSP), we are given a directed graph $G = (V, E)$ with an integer weight $w(e)$ for each edge $e \in E$, and we are asked to find a simple

cycle (more strict than MST, since MST cannot have cycle) over all the vertices (a circuit) with minimum total weight. (Note that the weights $w(e)$ can be positive or negative.) Show how to represent an arbitrary instance of the TSP as an integer program. Justify that your representation is correct, and describe how to obtain a solution to the instance of the TSP from solutions to your integer program.

Solution. □

We formulate similarly to problem 4. Pick arbitrary $s \in V$, add another vertex t and its incident edges to G such that t has the exact same incident edges (including their weights) of s. Let $G' = (E', V')$ be the resulting graph. Consider pseudo-flow $f : V \times V \to \mathbb{R}$ be units of flow from s to t. the value of f_e determines if an edge is used in the circuit.

$$\textbf{Miminize:} \quad \sum_{e \in E'} w(e) f_e$$

$$\textbf{Subject to:} \quad \sum_{v \in V'} f_{uv} - \sum_{v \in V'} f_{vu} = \begin{cases} 1 & \text{if } u = s \\ -1 & \text{if } u = t \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{v \in V'} f_{uv} = 2 \quad \text{for all } u \in V' \setminus \{s, t\}$$

$$f_e \geq 0 \quad \text{for all } e \in E$$

The first constraint given states that the model follows flow conservation and that the demand is 1 unit of flow. The second constraint states that any vertex has 2 incident edges for which the flow go through, therefore any edge in $E' \setminus \{s, t\}$ has degree 2. Altogether the feasible region consists of a flow f such that for all $(u, v) \in E'$ for which $f_{uv} \neq 0$ forms a simple path from s to t. The path is simple and spanning precisely because every vertex except from start and end has degree 2. Since we are minimizing $\sum_{e \in E'} w(e) f_e$, the path will be least weight. Therefore, if LP has a solution, the resulting f_e specifies the least weight flow from s to t. Let $E_p \subseteq E'$ be a set of edges where $f_e = 1$ for all $e \in E_p$. Now to get the simple cycle, we first replace the instance of vertex t with s in edges in E', then start from s, and add u to the sequence of vertices if $(s, u) \in E_p$.

2. Your company has recently discovered that several of its problems can be solved using linear programming. Your company doesnt want to write their own solver program, since many efficient programs are available for sale. But your boss has been reading his spam again and went out and purchased the MELPSE system (Most Efficient Linear Program Solver Ever!) in a fit of misdirected leadership. As expected, the

claim is slightly overstated, and the package comes with some serious limita- tions. From the advertisement:

> MELPSE is the fastest and most streamlined LP solver ever! Using the latest technology, it will find the best non-negative values for all your variables, get the biggest value for your objective function, and it will even make sure that $Ax \leq b$!

In fact, this is all that MELPSE can do: it only supports non-negative variables (it implicitly enforces a constraint $x_1 \geq 0$ on all variables x_i), only allows maximizations of the linear objective function, and insists that all constraints are in the form $\sum_{i=1}^{n} a_{j,i}x_i \leq bj$ (where $a_{j,i}$ and b_j are real numbers, perhaps negative). `Exit-DFS` The linear programs you need to solve are usually minimization problems, where variables occasionally take negative values, and some constraints are expressed with equality or greater- than-or-equal. The boss has already spent your entire budget, so to save your team you will need to figure out how to use the MELPSE system to solve your problems.

(a) One of your problems fits the restrictions of MELPSE except that you need to minimize your objective function. Describe precisely how to convert your program into one MELPSE can solve; that is, describe how to create an equivalent LP where the objective is being maximized. Briefly explain how to use a solution to your new program to find a solution to your original problem.

Solution. □

Given a minimization linear program L, we can convert L to an equivalent maximization linear program L' by negating the coefficients in the objective function. L and L' are equivalent since they share the same feasible solution (since constraints unchanged) and for each feasible solution, the objective value in L is the negative of the objective value in L', therefore by definition of equivalence, L and L' are equivalent. The solution to the converted L' is the solution to L without any modification

(b) Another of your problems contains a constaint of the form $a_1x_1 + a_2x_2 + \cdots + a_xx_n \geq b$ Describe precisely how to convert this to a constraint of the form $a'_1x_1 + a'_2x_2 + \cdots + a'_xx_n \leq b'$ as required by MELPSE.

Solution. □

Let $a'_i = -a_i$ for all $i = 1, \cdots, n$ and let $b' = -b$

(c) This problem also has a constraint of the form

$$a_1x_1 + a_2x_2 + \cdots + a_xx_n = b$$

Describe precisely how to create an equivalent LP that can be solved by MELPSE (or in the form of part (b)).

Solution. □

Replace the given equality constraint with

$$a_1x_1 + a_2x_2 + \cdots + a_xx_n \leq b \quad \text{and} \quad a_1x_1 + a_2x_2 + \cdots + a_xx_n \geq b$$

The resulting LP are equivalent because the objective function is unchanged and the constraints are equivalent because of the fact that

$$a = b \iff a \geq b \wedge a \leq b$$

(d) From the previous parts we know how to solve a minimization problem with $\geq$ or $=$ constraints. We can assume that all constraints are in the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b$. But we have a problem where one of the variables, x_1, could be negative. Describe precisely how to construct an equivalent LP where we replace x_1 with two new variables which can only take non-negative values. Briefly explain how to use a solution to your new program to find a solution to your original problem.

Solution. □

Replace each occurrence of x_1 in original LP L with $x_1' - x_1''$ in both the objective function and the constraints, add the constraints $x_1', x_1'' \geq 0$. let L' be the resulting LP. A feasible solution $\hat{x}$ to L' corresponds to a feasible solution $\bar{x}$ to L where $\bar{x}_1 = \hat{x}_1' - \hat{x}_1''$. Conversely, a feasible solution $\bar{x}$ to L corresponds to a feasible solution $\hat{x}$ given by

i. $\hat{x}_1' = \bar{x}_1$ and $\hat{x}_1'' = 0$ if $\bar{x}_1 \geq 0$.
ii. $\hat{x}_1' = 0$ and $\hat{x}_1'' = \bar{x}_1$ if $\bar{x}_1 < 0$.

Therefore, L and L' are equivalent. Note the above 2 points are always the case because x_1' and x_1'' are always of different signs. And if both are in the objective function, the positively signed variable will be optimized in the objective function, to the point such that the other variable is 0.

(e) Write an efficient high-level algorithm that will convert a linear program that might be a maximization problem, might contain greater-than-or-equal or equality constraints, and may contain variables lacking a non-negativity constraint, into an equivalent linear program that can be solved by MELPSE. Give a good bound on the size (the size being the number of variables and number of constraints) of your new linear program and briefly justify it.

Solution. □

Algorithm Let $L' = (A', b', c')$ be the result given $L = (A, b, c)$. Let there be n variables in the solution and m constraints

i. If the algorithm is as minimization linear program, let $c' = -c$

ii. If there exists a constraint $\sum_{j=1}^{n} a_{ij}x_j = b_i$ for some $i = 1, \cdots, m$, replace the constriant with $\sum_{j=1}^{n} a_{ij}x_j \geq b_i$ and $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$

iii. If there exists a constraint $\sum_{j=1}^{n} a_{ij}x_j \geq b_i$ for some $i = 1, \cdots, m$, let $a'_{ij} = -a_{ij}$ for all $j = 1, \cdots, n$, and revert the sign from $\geq$ to $\leq$

iv. if exists a negative variable $x_j < 0$ for some $j = 1, \cdots, n$, then replace x_j with $x'_j - x''_j$ in LP and add constraints $x'_j, x''_j \geq 0$

Now we discuss worst case size of the resulting L'. The worst case happens when we have an unconstrained x_j for all $j = 1, \cdots, n$ variables and that every constaint is an equality constraint.

i. For each x_j lacking a nonnegative constraint, the conversion doubles the size of variables by removing one variable x_j and adding 2 variables x'_j, x''_j into L'. Hence size of variable at most doubles. The conversion also introduces $2n$ nonnegative constraints for x'_j and x''_j

ii. For each equality constraint, the conversion replaces the constraint with 2 inequality constraints, therefore the size of constraint at most doubles

Altogether, the size of L' is bounded by $2n + 2n + 2m = 4n + 2m$