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1. Functions

- **injective** $f : X \rightarrow Y$ injective or one-to-one if $\forall a, b \in X, f(a) = f(b) \Rightarrow a = b$
- **surjective** $f : X \rightarrow Y$ is surjective if any $y \in Y \Rightarrow y = f(x)$ for some $x \in X$
- composition of injective/surjective/bijective functions are injective/surjective/bijective
- If f is injective with range Y , then its inverse function $f^{-1} : Y \rightarrow X$ is a bijective function

2. Set Relations

- **De Morgan's Law**

$$X \setminus \left(\bigcup_{\alpha \in I} A_{\alpha} \right) = \bigcap_{\alpha \in I} (X \setminus A_{\alpha}) \quad X \setminus \left(\bigcap_{\alpha \in I} A_{\alpha} \right) = \bigcup_{\alpha \in I} (X \setminus A_{\alpha})$$

- **how functions acts on sets** Let $A, B \subseteq X$ and $C, D \subseteq Y$

- f well-behaved for union

$$f(A \cup B) = f(A) \cup f(B)$$

- f not well-behaved for intersection, difference

$$\begin{aligned} f(A \cap B) &\subseteq f(A) \cap f(B) \\ f(A) \setminus f(B) &\subseteq f(A \setminus B) \end{aligned}$$

- f^{-1} well-behaved with union, intersection, difference, and set complement

$$\begin{aligned} f^{-1}(C \cup D) &= f^{-1}(C) \cup f^{-1}(D) \\ f^{-1}(C \cap D) &= f^{-1}(C) \cap f^{-1}(D) \\ f^{-1}(C \setminus D) &= f^{-1}(C) \setminus f^{-1}(D) \\ f^{-1}(Y \setminus C) &= X \setminus f^{-1}(C) \end{aligned}$$

- f and f^{-1} mixed

$$\begin{aligned} A &\subseteq f^{-1}(f(A)) && \text{(with equality if } f \text{ is injective)} \\ f(f^{-1}(C)) &\subseteq C && \text{(with equality if } f \text{ surjective)} \end{aligned}$$

Remark. Read

1. chapter 1, 1-8
2. chapter 2, 12-22
3. chapter 3, 23-29
4. chapter 4. 30-35

1 General Topology

1.0.1 3 Relation

1. A relation on a set A is a subset C of cartesian product $A \times A$. xCy means $(x, y) \in C$
2. **equivalence relation** is a relation if it satisfies reflexivity, symmetry, transitivity
3. **equivalence class** a subset of A determined by some $x \in A$, i.e. $E = \{y \mid y \sim x\}$
4. **partition** of a set A is a collection of disjoint nonempty subsets of A whose union is all of A
5. **order relation** is a relation if it satisfies comparability (any $x \neq y \in A$ either xCy or yCx but not both) nonreflexivity (xCx does not hold for any $x \in A$) and transitivity
6. **dictionary order relation** Let A, B be sets and $<_A$ and $<_B$ be order relations. The order relation on $A \times B$ is defined by $a_1 <_A a_2$ or if $a_1 = a_2$ and $b_1 <_B b_2$
7. **least upper bound property** An ordered set A has the property if every nonempty subset A_0 of A that is bounded above ($\exists b \in A$ s.t. $x \leq b$ for all $x \in A_0$) has a least upper bound (all bounds of A_0 has a smallest element)
 - \mathbb{R} and $(-1, 1)$ has least upper bound property
 - $B = (-1, 0) \cup (0, 1)$ does not have least upper bound property, $\{-1/2^n \mid n \in \mathbb{Z}_+\}$ is bounded above by any $b \in (0, 1)$ but its least upper bound $0 \notin B$

1.0.2 5 Cartesian Product

1. **indexed family of sets** Let \mathcal{A} be nonempty collection of sets, let $f : J \rightarrow \mathcal{A}$ be a surjective indexing function. (\mathcal{A}, f) is called indexed family of sets, denoted by $\{\mathcal{A}_\alpha\}_{\alpha \in J} = \{\mathcal{A}_\alpha\}$ where $f(\alpha) = \mathcal{A}_\alpha$
2. **m-tuple** Let $m \in \mathbb{Z}_+$, Given a set X , define m-tuple of X to be a function $\mathbf{x} : \{1, \dots, m\} \rightarrow X$ and denote $\mathbf{x} = (x_1, \dots, x_m)$
3. **cartesian product** Let $\mathcal{A} = \{A_1, \dots, A_m\}$ be indexed family of sets, let $X = \cup_{i=1}^m A_i$. Then cartesian product of \mathcal{A} is

$$X^m = \prod_{i=1}^m A_i \quad A_1 \times \dots \times A_m$$

to be the set of all m-tuples \mathbf{x} of elements of X such that $x_i \in A_i$ for each i

4. **ω -tuple** Given a set X , define ω -tuple of elements of X be a function $\mathbf{x} : \mathbb{Z}_+ \rightarrow X$. \mathbf{x} is an *infinite sequence*, of elements of X . Denote $x_i = \mathbf{x}(i)$ as i -th coordinate of \mathbf{x} . Denote \mathbf{x} itself by (x_1, x_2, \dots) or $(x_n)_{n \in \mathbb{Z}_+}$
5. **cartesian product (infinite)** Let $\mathcal{A} = \{A_1, A_2, \dots\}$ be indexed family of sets and X be union of sets in \mathcal{A} , the cartesian product of \mathcal{A}

$$X^\omega = \prod_{i \in \mathbb{Z}_+} A_i \quad A_1 \times A_2 \times \dots$$

is defined to be the set of all ω -tuples (x_1, x_2, \dots) of elements of X such that $x_i \in A_i$ for each i

1.0.3 11 The Maximum Principle

Definition. (Strict Partial Ordering) Given a set A , and a relation \prec on A is a strict partial order if

1. (nonreflexivity) $a \prec a$ never holds
2. (transitivity) If $a \prec b$ and $b \prec c$, then $a \prec c$

Idea is not all $x, y \in A$ is comparable, i.e. either $x \prec y$ or $y \prec x$. Although a subset $S \subset A$ can be simply ordered.

Theorem. (The Maximum Principle) Let A be a set and \prec be a strict partial order on A . Then exists a maximal simply ordered subset B of A , i.e. does not exist $B' \subset A$ such that $B \subsetneq B'$ and B' simply ordered.

Definition. (upper bound and maximal element) Let A be a set and \prec be a strict partial order on A . If $B \subset A$, an **upper bound** on B is $c \in A$ such that $b = c$ or $b \prec c$ for all $b \in B$. A **maximal element** of A is an element $m \in A$ such that for no element $a \in A$ does $m \prec a$ hold.

Lemma. (Zorn's Lemma) Let A be a strictly partially ordered set. If every ordered subset of A has an upper bound in A , then A has a maximal element.

2 Topological Spaces and Continuous Functions

2.0.1 12 Topological Spaces

Definition. (Topology) Topology on a set X is a collection \mathcal{T} of subsets of X having properties

1. $\emptyset, X \in \mathcal{T}$
2. Arbitrary union of subcollection of \mathcal{T} is in \mathcal{T} (If $\forall \alpha \in I, U_\alpha \in \mathcal{T}$, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$)
3. Finite intersection of subcollection of \mathcal{T} is in \mathcal{T} (If $\forall 1 \leq i \leq n, U_i \in \mathcal{T}$, then $\bigcap_{i=1}^n U_i \in \mathcal{T}$)

A topological space is a pair (X, \mathcal{T}) , where \mathcal{T} are the open sets.

- Standard topology on \mathbb{R}^n is $\mathcal{T}_0 = \mathcal{T}_{std} = \{U \subset \mathbb{R}^n \mid \forall x \in U, \exists \epsilon > 0 B_\epsilon(x) \subset U\}$
- Standard topology on \mathbb{R} is generated by $\mathcal{B}_{std} = \{(a, b) = \{x \mid a < x < b\} \text{ the open intervals}$
- Lower limit topology on \mathbb{R} is generated by $\mathcal{B}_{l.l.} = \{x \mid a \leq x < b\} \text{ the half-open intervals}$
- Discrete topology $\mathcal{T}_1 = \mathcal{T}_{disc} = \mathcal{P}(X)$ all subsets are open
- Trivial topology $\mathcal{T}_2 = \mathcal{T}_{triv} = \{\emptyset, X\}$ only empty set and X are open
- Finite complement topology $\mathcal{T}_{f.c.} = \{U \subseteq X \mid X - U \text{ is finite or all of } X\}$
- Countable complement topology $\mathcal{T}_c = \{U \subseteq X \mid X - U \text{ is countable or all of } X\}$

Lemma. (Arbitrary intersection of topologies is a topology) $\forall \alpha \in I \mathcal{T}_\alpha$ is a topology, so is $\bigcap_{\alpha \in I} \mathcal{T}_\alpha$

Definition. (Compare topology) If $\mathcal{T}' \subset \mathcal{T}$, then \mathcal{T}' is coarser / weaker / smaller than \mathcal{T} , \mathcal{T} is finer / stronger / larger than \mathcal{T}' . \mathcal{T} and \mathcal{T}' are comparable if $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$

$$\mathcal{T}_{triv} \subset \mathcal{T}_{f.c.} \subset \mathcal{T}_{std} \subset \mathcal{T}_{disc} \quad \mathcal{T}_{std} \subset \mathcal{T}_{l.l.}$$

2.0.2 13 Basis for a Topology

A terser representation of \mathcal{T}

Definition. (Basis) If X is a set, a basis for (X, \mathcal{T}) is a collection \mathcal{B} (basis elements) of subsets of X s.t.

1. For each $x \in X$, exists at least one basis element $B \in \mathcal{B}$ containing it
2. If $x \in B_1 \cap B_2$, then exists $B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2$

A topology $\mathcal{T}_\mathcal{B}$ generated by \mathcal{B} is defined as

$$\mathcal{T}_\mathcal{B} = \{U \subset X \mid \forall x \in U, \exists B \in \mathcal{B}, x \in B \subset U\}$$

- $\mathcal{T}_\mathcal{B}$ is the unique minimal topology containing \mathcal{B}

$$\mathcal{T}_\mathcal{B} = \bigcap_{\mathcal{T} \in \mathbb{T}} \mathcal{T}$$

where $\mathbb{T} = \{\mathcal{T} \mid \mathcal{T} \supset \mathcal{B} \text{ and } \mathcal{T} \text{ is a topology}\}$

- For any X , all one point sets of X is a basis for \mathcal{T}_{disc}

Lemma. ($\mathcal{B} \rightarrow \mathcal{T}_\mathcal{B}$) $\mathcal{T}_\mathcal{B}$ equals the collections of all unions of elements of \mathcal{B}

$$\mathcal{T}_\mathcal{B} = \{\bigcup_{\alpha \in I} B_\alpha \mid B_\alpha \in \mathcal{B} \quad \forall \alpha \in I\}$$

Lemma. ($\mathcal{T}_\mathcal{B} \rightarrow \mathcal{B}$) Let (X, \mathcal{T}) be topological space. Let \mathcal{C} be a collection of open sets of X such that

$$\forall U \subset X, \quad \forall x \in U, \quad \exists C \in \mathcal{C} \text{ s.t. } x \in C \subset U$$

Then \mathcal{C} is a basis for \mathcal{T} (handy in deciding $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for $Y \subset X$)

Lemma. (compare topology by basis) Let \mathcal{B} and \mathcal{B}' be bases for \mathcal{T} and \mathcal{T}' , respectively, on X . Then following equivalent

1. $\mathcal{T}' \supset \mathcal{T}$ (\mathcal{T}' is finer than \mathcal{T})
2. For each $x \in X$ and each $B \in \mathcal{B}$ containing x , there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$

2.0.3 14 Order Topology

Definition. (Order Topology) Let X be a set with simple order relation. Let \mathcal{B} be a collection of all sets of the following type

$$\begin{aligned}\mathcal{B} = & \{(a, b) \mid a < b \quad a, b \in X\} \\ & \bigcup \{(a_0, b) \mid a_0 \text{ is minimal element (if any) of } X \quad b \in X \quad b \neq a_0\} \\ & \bigcup \{(a, b_0) \mid b_0 \text{ is maximal element (if any) of } X \quad a \in X \quad a \neq b_0\}\end{aligned}$$

Generated topology $\mathcal{T}_{\mathcal{B}}$ is called order topology

- In \mathbb{R} , $\mathcal{T}_{ord} = \mathcal{T}_{std}$
- In \mathbb{Z}_+ , $\mathcal{T}_{ord} = \mathcal{T}_{disc}$ (since any $\{n\} = (n-1, n+1) \in \mathcal{T}_{ord}$)
- In $\{1, 2\} \times \mathbb{Z}_+$ in \mathcal{T}_{dict} is not in \mathcal{T}_{disc} (althogh most single point set are open, 2×1 is not open)
- In \mathbb{R}^2 , both \mathcal{B} and \mathcal{B}' generates \mathcal{T}_{dict}

$$\mathcal{B} = \{(a \times b, c \times d) \mid a < c \vee (a = c \wedge b < d)\} \quad \mathcal{B}' = \{(a \times b, a \times d) \mid b < d\}$$

2.0.4 15 Product Topology

Definition. (Product Topology) Let X, Y be topological spaces, the product topology on $X \times Y$ is generated by the basis

$$\mathcal{B} = \{U \times V \mid U \in \mathcal{T}_X \quad V \in \mathcal{T}_Y\}$$

Alternatively define product topology with basis. If \mathcal{B}, \mathcal{C} are basis for X and Y respectively. Then

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \quad C \in \mathcal{C}\}$$

is a basis for topology of $X \times Y$

- $X \times \{y\} \cong X$
- $X \times Y \cong Y \times X$
- $(X \times Y) \times Z \cong X \times (Y \times Z)$
- Product spaces does not work well with order topology.
 - Consider $X = \mathbb{R}^2$ and $Y = [0, 1] \times [0, 1]$, then $\{0.5\} \times [0, 1]$ is not open in \mathcal{T}_{ord} but is open in $\mathcal{T}_{subspace}$

Definition. (Projection) Let $\pi_1 : X \times Y \rightarrow X$ be defined by $\pi_1(x, y) = x$. π_1 is a projection of $X \times Y$ **onto** the first factor. (note projections are surjective)

Definition. (Product Topology by Continuity of Functions) Given X, Y , there exists unique topology on $X \times Y$ such that

1. projections π_X and π_Y are continuous
2. If $f : Z \rightarrow X$ and $g : Z \rightarrow Y$, then $f \times g : Z \rightarrow X \times Y$ is continuous

Proof. Define $\mathcal{B} = \{U \times V \mid U \subset X \text{ open} \quad V \subset Y \text{ open}\}$. Show $\mathcal{T}_{\mathcal{B}}$ satisfies the above 2 conditions. Prove uniqueness by showing $id : (X, \mathcal{T}') \rightarrow (X, \mathcal{T}'')$ is a homeomorphism utilizing the above 2 conditions. \square

2.0.5 16 Subspace Topology

Definition. (Subspace Topology) Let (X, \mathcal{T}) . Let $Y \subset X$, then

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is the subspace topology on Y . Alternatively, define using basis. If \mathcal{B} generates \mathcal{T} , then

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for \mathcal{T}_Y .

- (**lemma**) $Y \subset X$. U open in Y and Y open in X , then U open in X
- (**lemma**) A subspace of a subspace is a subspace
- (**theorem**) A product of subspaces is a subspace of the product (subspace and product topology work well)

- X ordered and $Y \subset X$. order topology on Y may not be same as order topology of Y inherited as a subspace of X (subspace and order topology does not work well)
 - Let $Y = [0, 1] \subset \mathbb{R}$. \mathcal{B}_Y are of the form $(a, b) \cap [0, 1]$. Note
 1. $[0, b)$ where $b \notin [0, 1]$ is open in $[0, 1]$ but not in \mathbb{R}
 2. $\mathcal{T}_{ord} \cong \mathcal{T}_{subspace}$ since the basis elements of the same form
 - Let $Y = [0, 1) \cup \{2\} \subset \mathbb{R}$.
 1. $\{2\}$ open in $\mathcal{T}_{subspace}$ since $(1.5, 2.5) \cap Y = \{2\}$.
 2. $\{2\}$ not open in order topology since any basis of the form $\{x \in Y \mid a < x \leq 2, a \in Y\}$ contains some other point other than $\{2\}$
 3. $\mathcal{T}_{ord} \not\cong \mathcal{T}_{subspace}$
- **(theorem)** subspace and order topology works well if the subspace is convex

Definition. (convex) Given ordered X and subset $Y \subset X$ is convex if $(a, b) \subset X$ lies in Y completely

Theorem. (subspace and order topology works well if the subspace is convex)

Let X be ordered and $Y \subset X$ be convex. Then order topology on Y same as topology Y inherits as a subspace of X .

Definition. (Subspace Topology by Continuity of Functions) Let X be topological space, $Y \subset X$, there exists unique topology on Y such that

1. inclusion $i_Y : Y \hookrightarrow X$ is continuous
2. If $f : Z \rightarrow Y$ such that $i_Y \circ f : Z \rightarrow X$ is continuous, then f is continuous

2.0.6 17 Closed Sets and Limit Points

Definition. (Closed) A subset A of X is closed if $X - A$ is open.

- In $\mathcal{T}_{f.c.}$, all finite subsets and X are closed
- In \mathcal{T}_{disc} every set is closed.
- In $Y = [0, 1] \cup (2, 3)$, $[0, 1]$ and $(2, 3)$ are open and closed in subspace topology of Y
- C closed in Y does not imply C is closed in X . However a closed set in a closed subspace is closed overall, i.e. in X (Let Y be a subspace of X . If A is closed in Y and Y is closed in X , then A is closed in X .)

Theorem. (Topology by closed sets)

1. \emptyset and X are closed
2. Arbitrary intersection of closed sets are closed
3. Finite unions of closed sets are closed

Theorem. (Closedness in Subspace) Let Y be a subspace of X . Then A is closed in Y if and only if it equals the intersection of a closed set X with Y ($Y \subset X$, then $A \subset Y$ closed in Y if exists $K \subset X$ s.t. $A = K \cap Y$)

Definition. (Closure and Interior) If $A \subset X$

1. **Interior** $\text{Int}_X A = \overset{\circ}{A}$ is
 - the union of all open sets in X contained in A , i.e. $\text{Int}_X A = \bigcup_{U \in \mathcal{T}_X : U \subset A} U$
 - maximal open subsets of A in X
2. **Closure** $\text{Cl}_X A = \overline{A}$ is
 - the intersection of all closed sets containing A , i.e. $\text{Cl}_X A = \bigcap_{U \in \mathcal{T}_X : U \supset A} U$
 - minimal closed set containing A

- Set relationship

$$\text{Int} A \subset A \subset \overline{A}$$

- **(theorem).** Let $A \subset Y \subset X$, Let \overline{A} denote closure of A in X . Then the closure of A in Y is $\overline{A} \cap Y$
 - In \mathbb{R} , $Y = (0, 1]$, let $A = (0, 0.5) \subset Y$. $\text{Cl}_{\mathbb{R}} A = [0, 0.5]$, $\text{Cl}_Y A = \text{Cl}_{\mathbb{R}} A \cap Y = (0, 0.5]$
- If A open, then $\text{Int}(A) = A$; If A closed, then $\overline{A} = A$
- If $A \subset X$, then $(\overset{\circ}{A})^c = \overline{(A^c)}$ (Complement of interior is closure of the complement)

- In \mathbb{R} , let $A = \mathbb{Q}$ or $A = \mathbb{R} - \mathbb{Q}$, $\text{int}(A) = \emptyset$ and $\overline{A} = \mathbb{R}$ ($\text{int}(A) = \emptyset$ since no (a, b) contained fully in \mathbb{Q} or $\mathbb{R} - \mathbb{Q}$)

Definition. (*neighborhood*) U is an open set containing x is equivalent to U is a neighborhood of x

Definition. (*intersects*) A intersects B if and only if $A \cap B \neq \emptyset$

Theorem. (*define closure using neighborhoods*) Let A be a subset of X ,

1. then $x \in \overline{A}$ if and only if every neighborhood of x intersects A
2. Let \mathcal{B} be basis of X , then $x \in \overline{A}$ if and only if every basic neighborhood B of x intersects A

proof by contraposition. Following are examples which uses this theorem to test/determine the closure

- If $A = (0, 1]$ then $\overline{A} = [0, 1]$ (since every neighborhood of $\{0\}$ intersects A)
- If $B = \{1/n \mid n \in \mathbb{Z}_+\}$, then $\overline{B} = \{0\} \cup B$
- If $C = \{0\} \cup (1, 2)$ then $\overline{C} = \{0\} \cup [1, 2]$
- In \mathbb{R} , $\overline{\mathbb{Q}} = \mathbb{R}$ since every neighborhood of $x \in \mathbb{R}$ contains some rational number, so intersects \mathbb{Q}
- In \mathbb{Z}_+ , $\overline{\mathbb{Z}_+} = \mathbb{Z}_+$

Definition. (*Limit Point*) Let $A \subset X$ and $x \in X$, x is a limit point of A if every neighborhood of x intersects A in some point other than x itself. In other words,

$$x \in \overline{A - \{x\}}$$

- In \mathbb{R} , $A = (0, 1]$, then any $x \in [0, 1]$ is a limit point of A and no other point in \mathbb{R} is a limit point
- In \mathbb{R} , $B = \{1/n \mid n \in \mathbb{Z}_+\}$, 0 is the only limit point of B (any other $x \in \mathbb{R}$ has neighborhood that does not intersect B or intersects at x itself)
- In \mathbb{R} , $C = \{0\} \cup (1, 2)$, all $x \in [1, 2]$ are limit points of C
- In \mathbb{R} , every $x \in \mathbb{R}$ is a limit point of \mathbb{Q}
- In \mathbb{R} , no point is a limit point of \mathbb{Z}_+

Theorem. (*define closure using limit point*) Let $A \subset X$, let A' be set of all limit points, then

$$\overline{A} = A \cup A'$$

- (corollary) $A \subset X$ is closed if and only if it contains all its limit points, i.e. $A' \subset A$

Remark. ways to prove A is closed

1. show A^c is open
2. show $\overline{A} = A$ by proving that every $x \in A^c$ has open neighborhood that does not intersect A
 - $A = \{x_0\} \subset \mathbb{R}$ closed since every point different from x_0 has neighborhood not intersecting $\{x_0\}$

Definition. (*Sequence Convergence*) A sequence (x_n) converges to a point $x \in X$, denoted $x_n \rightarrow x$ or $\lim_{n \rightarrow \infty} x_n = x$ if

$$\forall \text{ neighborhood } U \text{ of } x \exists N \in \mathbb{N} \text{ such that } \forall n \geq N \ x_n \in U$$

For metric spaces

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \geq N \ |x_n - x| < \epsilon$$

Definition. (*Separated*) Let $x, y \in X$, x and y can be separated if each lies in a neighborhood which does not contain the other point. (neighborhood not necessarily disjoint)

Definition. (T_1 Space) X is T_1 if any two distinct points in X are separated.

- (*theorem*) Let $A \subset X$ T_1 . $x \in A'$ if and only if every neighborhood of x contains infinitely many points of A .
- (*theorem*) Every finite point set, specifically one-point set, in a T_1 space is closed

Definition. (T_2 Hausdorff Space) A topological space X is called Hausdorff space if for each pair x_1, x_2 of distinct points of X , there exists neighborhoods U_1 and U_2 of x_1 and x_2 , respectively that are disjoint.

$$\forall x \neq y \in X \exists \text{ neighborhoods } U, V \text{ of } x \text{ and } y \text{ respectively s.t. } U \cap V = \emptyset$$

- (*motivation*) Generally, one point set not always closed; Sequences converges can converge to more than limit.

- **(theorem)** Every finite point set, specifically one-point set, in a Hausdorff space is closed
- **(theorem)** If X is T_2 , then a sequence of points of X converges to at most 1 point of X
- **(theorem)** Every simply ordered set is T_2 in the order topology. Product of two T_2 space is T_2 ; Subspace of a T_2 space is T_2 (order/product/subspace topology well behaved with T_2)
- **(examples)**
 - $\mathbb{R}_{std}^n, X_{disc}$ are T_2
 - X_{triv} not T_2 except when $|X_{triv}| = 1$
 - $X_{f.c.}$ not T_2 when X is infinite (since any $x, y \in X_{f.c.}$ are infinite and intersects)

2.0.7 18 Continuous Functions

Definition. (Continuous) A function $f : X \rightarrow Y$ is continuous if each open subset $V \subset Y$, the set $f^{-1}(V)$ is open. Alternatively formulated with basis, f is continuous if every basis element $B \in \mathcal{B}$, $f^{-1}(B)$ is open

- $id : \mathbb{R}_{std} \rightarrow \mathbb{R}_{l.l.}$ is not continuous; $id : \mathbb{R}_{l.l.} \rightarrow \mathbb{R}_{std}$ is continuous

Theorem. (TFAE for continuous function) $f : X \rightarrow Y$

1. f is continuous
 2. For every subset A of X , $f(\overline{A}) \subset \overline{f(A)}$ (convergence: $x_n \rightarrow x \Rightarrow f(x_n) \rightarrow f(x)$ to $x \in \overline{A} \Rightarrow f(x) \in \overline{f(A)}$)
 3. For every closed set B of Y , the set $f^{-1}(B)$ is closed
 4. (generalized ϵ - δ) $\forall x \in X$ and \forall neighborhood V of $f(x)$, \exists neighborhood U of x such that $f(U) \subset V$
- In metric space, 4 can be reformulated with ϵ - δ definition. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous if

$$\forall x_0 \in \mathbb{R}^n, \quad \forall \epsilon > 0 \quad \exists \delta > 0 \text{ s.t. } |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$

i.e. $(x \in B_\delta(x_0) \Rightarrow f(x) \in B_\epsilon(f(x_0)))$

Definition. (Homeomorphism) Let $f : X \rightarrow Y$ be a bijection. If f and the inverse function $f^{-1} : Y \rightarrow X$ are continuous, then f is called a homeomorphism. (continuous bijection)

- **(theorem)** If $X \cong Y$, then X and Y share topological property, i.e. property expressed in terms of topology only.
- **(example)** $(-1, 1) \cong \mathbb{R}$ since $f : (0, 1) \rightarrow \mathbb{R}$ by $f(x) = \tan(x)$ is a homeomorphism
- **(example)** A function can be continuous but not homeomorphic. Consider $S^1 = \{x \times y \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$ the unit circle. Let $f : [0, 1) \rightarrow S^1$ by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. f is bijective and continuous but f^{-1} not continuous
- If $\mathcal{T}_1 \subset \mathcal{T}_2$, then $Id : (X, \mathcal{T}_2) \rightarrow (X, \mathcal{T}_1)$ is continuous
- If $\mathcal{T}_1 = \mathcal{T}_2$, then $Id : (X, \mathcal{T}_1) \rightarrow (X, \mathcal{T}_2)$ is a homeomorphism

Definition. (Imbedding) An injective map $f : X \rightarrow Y$ is a topological imbedding of X in Y if $f' : X \rightarrow Z$ is a homeomorphism (note the image set $Z = f(X)$ carries subspace topology inherited from Y)

- Intuitively, an imbedding $f : X \rightarrow Y$ let us treat X as a subspace of Y .
- **(theorem)** Every map that is injective, continuous, and either open or closed is an imbedding.
- **(example)** $f : [0, 1) \rightarrow \mathbb{R}$ by $f(t) = (\cos(2\pi t), \sin(2\pi t))$ maps to S^1 . f is a continuous injective map but not an imbedding.

Theorem. (Constructing Continuous Functions) Given X, Y, Z

1. (constant function) If $f : X \rightarrow Y$ by $f(X) = \{y_0\}$, then f is continuous
2. (inclusion) If $A \subset X$ with subspace topology, inclusion $i_A : A \hookrightarrow X$ is continuous
3. (composition) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, then $g \circ f : X \rightarrow Z$ is continuous
4. (restricting domain) If $f : X \rightarrow Y$ is continuous and $A \subset X$, then $f|_A : A \rightarrow Y$ is continuous ($f|_A = f \circ i_A$)
5. (restricting or expanding range) Let $f : X \rightarrow Y$ continuous. If $f(X) \subset A \subset Y \subset B$, then $g : X \rightarrow A$ obtained by restricting range of f is continuous. $h : X \rightarrow B$ obtained by expanding range of f also continuous ($h = i_Y \circ f$)

6. (local formulation of continuity) $f : X \rightarrow Y$ is continuous if $X = \cup_{\alpha \in I} U_\alpha$ such that $f|_{U_\alpha}$ is continuous for each $\alpha \in I$
7. (map to products) Let $f : A \rightarrow X \times Y$ given by $f(t) = (f_1(t), f_2(t))$ and $f_1 : A \rightarrow X$, $f_2 : A \rightarrow Y$. Then f is continuous if and only if f_1 and f_2 are continuous
8. (Algebraic Operations) If $f, g : X \rightarrow \mathbb{R}$ are continuous, then $f + g$, $f - g$, $f \cdot g$, f/g ($g(x) \neq 0$) are all continuous
9. (Uniform Limit Theorem) If a sequence of continuous real-valued functions of a real variable converges **uniformly** to a limit function, then the limit function is continuous

Theorem. (the pasting lemma) Let $X = A \cup B$, where A and B are both closed or both open in X . Let $f : A \rightarrow Y$ and $g : B \rightarrow Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, set $h : X \rightarrow Y$

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

then h is continuous

- $h(x) = x$ for $x \leq 0$ and $h(x) = x/2$ for $x \geq 0$ is continuous

2.0.8 18 The Product Topology

Definition. (J-tuple) Let J be index set. Define **J-tuple** of elements of X be a function $\mathbf{x} : J \rightarrow X$. Given $\alpha \in J$, denote α th coordinate as \mathbf{x} at α by x_α instead of $\mathbf{x}(\alpha)$. Denote \mathbf{x} as $(x_\alpha)_{\alpha \in J}$. Denote the **set of all J-tuples** of elements of X by X^J

Definition. (Cartesian Product) Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets; Let $X = \cup_{\alpha \in J} A_\alpha$. The cartesian product of this indexed family is given by

$$\prod_{\alpha \in J} A_\alpha = \{(x_\alpha)_{\alpha \in J} \in X \mid x_\alpha \in A_\alpha\} = \{\mathbf{x} : J \rightarrow \bigcup_{\alpha \in J} X \mid \forall \alpha \in J \mathbf{x}(\alpha) \in A_\alpha\}$$

is the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$. Equivalently, the set of all functions $\mathbf{x} : J \rightarrow \cup_{\alpha \in J} X$ such that $\mathbf{x}(\alpha) \in A_\alpha$ for each $\alpha \in J$

Definition. (Projection) The projection mapping associated with index β is defined by

$$\pi_\beta : \prod_{\alpha \in J} X_\alpha \rightarrow X_\beta \quad \pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta$$

Definition. (Box Topology) Let $\{X_\alpha\}_{\alpha \in J}$ be indexed family of topological spaces. Basis

$$\mathcal{B}_{\text{box}} = \left\{ \prod_{\alpha \in J} U_\alpha \mid U_\alpha \text{ open in } X_\alpha \forall \alpha \in J \right\}$$

generates box topology

- (theorem) Given basis \mathcal{B}_α for each X_α , $\prod_{\alpha \in J} \mathcal{B}_\alpha$ where $B_\alpha \in \mathcal{B}_\alpha$ is a basis for $\prod_{\alpha \in J} X_\alpha$

Definition. (Product Topology) Let $\{X_\alpha\}_{\alpha \in J}$ be indexed family of topological spaces. Basis

$$\begin{aligned} \mathcal{B}_{\text{prod}} &= \left\{ \pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n}) \mid U_{\beta_i} \in \mathcal{T}_{X_{\beta_i}} \beta_1, \dots, \beta_n \in J \right\} \\ &= \left\{ \prod_{\alpha \in J} U_\alpha \mid \forall \alpha \in J U_\alpha \in \mathcal{T}_{X_\alpha} \text{ for almost all } \alpha U_\alpha = X_\alpha \right\} \end{aligned}$$

generates product topology

- (theorem) Let $f : A \rightarrow \prod_{\alpha \in J} X_\alpha$ defined by $f(a) = (f_\alpha(a))_{\alpha \in J}$ where $f_\alpha : A \rightarrow X_\alpha$ for each α , Then f is continuous if and only if each function f_α is continuous
- (theorem) For finite products, $\mathcal{T}_{\text{box}} = \mathcal{T}_{\text{prod}}$
- (example where product topology works while box topology does not) In $\mathbb{R}^\omega = \prod_{n \in \mathbb{Z}_+} \mathbb{R}$, countably infinite product of \mathbb{R} . Define $f : \mathbb{R} \rightarrow \mathbb{R}^\omega$ by $f(t) = (t, t, \dots)$. Each $f_\alpha = \pi_\alpha \circ f$ is continuous so f is continuous in product topology. However f is not continuous in box topology. Consider $B = \{(-1/n, 1/n) \mid n \in \mathbb{Z}_+\} \in \mathcal{B}_{\text{box}}$ open,

$$f^{-1}(B) = \{t \mid (t, t, \dots) \in B\} = \bigcap_{n \in \mathbb{Z}_+} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$$

not open in \mathbb{R}

Definition. (product topology by continuity of functions) There is a unique topology $\mathcal{T}_{\text{prod}}$ on $X = \prod_{\alpha \in J} X_\alpha$ with

1. $\pi_\beta : X \rightarrow X_\beta$ continuous for every $\beta \in J$
2. Given $f : Z \rightarrow \prod_{\alpha \in J} X_\alpha$. If $f_\alpha = \pi_\alpha \circ f$ is continuous for every α , then f is continuous

Note $\mathcal{B}_{\text{prod}}$ satisfies the two condition

Proof. 1. by design. proof for 2 as follows

$$f^{-1} \left(\bigcap_{i=1}^n \pi_{\beta_i}^{-1}(U_{\beta_i}) \right) = \{z \in Z \mid \forall i \ f_{\beta_i} \in U_{\beta_i}\} = \bigcap_{i=1}^n f_{\beta_i}^{-1}(U_{\beta_i})$$

is open as finite intersection of open sets □

Theorem. (Both box and product topology works well with subspace/ T_2 /closure)

1. Let $A_\alpha \subset X_\alpha$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$
2. If X_α is T_2 , then $\prod X_\alpha$ is T_2
3. Let $A_\alpha \subset X_\alpha$, then $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$

Proof. proof of 2. Let $\mathbf{x} = (x_\alpha)$ and $\mathbf{y} = (y_\alpha)$ where $\mathbf{x} \neq \mathbf{y}$. So exists $\beta \in J$ such that $x_\beta \neq y_\beta$. Since X_β is T_2 , exists neighborhoods U_β, V_β for x_β and y_β s.t. $U_\beta \cap V_\beta = \emptyset$. Note $\pi^{-1}(U_\beta)$ and $\pi^{-1}(V_\beta)$ are disjoint neighborhoods of \mathbf{x}, \mathbf{y} . **proof of 3.** Mainly use the definition of closure using neighborhoods. (\Rightarrow) Let $\mathbf{x} = (x_\alpha) \in \prod \overline{A_\alpha}$. Need to show $\mathbf{x} \in \overline{\prod A_\alpha}$. Let $U = \prod U_\alpha$ be basic neighborhood of \mathbf{x} in \mathcal{T}_{box} or $\mathcal{T}_{\text{prod}}$. For each α , since $x_\alpha \in A_\alpha$, can find $y_\alpha \in U_\alpha \cap A_\alpha$. Note $\mathbf{y} = (y_\alpha) \in U \cap \prod A_\alpha$. Since U arbitrary, $\mathbf{x} \in \overline{\prod A_\alpha}$. (\Leftarrow) Let $\mathbf{x} = (x_\alpha) \in \overline{\prod A_\alpha}$. Want to show $x_\beta \in \overline{A_\beta}$ for all β . Let V_β be arbitrary neighborhood of x_β . Consider $\pi^{-1}(V_\beta)$, which is open in both \mathcal{T}_{box} and $\mathcal{T}_{\text{prod}}$. By definition of closure, exists $\mathbf{y} = (y_\alpha) \in \pi^{-1}(V_\beta) \cap \prod A_\alpha$. Hence $y_\beta \in V_\beta \cap A_\beta$. Hence, $x_\beta \in \overline{A_\beta}$ □

2.0.9 19 The Metric Topology

Definition. (Metric) A metric on X is a function

$$d : X \times X \rightarrow \mathbb{R}$$

with properties

1. (non-negativity) $d(x, y) \geq 0$ for all $x, y \in X$, with equality if $x = y$
2. (symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$
3. (triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$

Definition. (ϵ -ball centered at x)

$$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$$

Definition. (norm) Given $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, the norm of \mathbf{x} defined by $\|\mathbf{x}\| = (x_1^2 + \dots + x_n^2)^{1/2}$.

Definition. (Metric Topology) Given X and a metric d , basis

$$\mathcal{B}_d = \{B_d(x, \epsilon) \mid x \in X, \epsilon > 0\}$$

generates the metric topology \mathcal{T}_d induced by d . Therefore

$$\mathcal{T} = \{U \subset X \mid \forall x \in U \exists \epsilon > 0 \ B(x, \epsilon) \subset U\}$$

1. **discrete metric** d_{disc} , defined by $d_{\text{disc}}(x, y) = 1$ if $x \neq y$ and $d_{\text{disc}}(x, y) = 0$ if $x = y$ induces $\mathcal{T}_{\text{disc}}$
2. **standard metric** on \mathbb{R} defined by $d(x, y) = |x - y|$ induces \mathcal{T}_{ord}
3. **(diamond)** $d_1 = \sum_i |x_i - y_i|$
4. **euclidean metric (circle)** on \mathbb{R}^n , $d_2 = d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$
5. **square metric (square)** on \mathbb{R}^n , $d_\infty = \rho(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$ (furthest coordinate within ϵ)
6. (theorem) In \mathbb{R}^n , $\mathcal{T}_{d_1} = \mathcal{T}_{d_2} = \mathcal{T}_\infty$ induces same topology \mathcal{T}_{std} (proof by showing basis elements nests!)
7. (theorem) If X is metrizable, then X is T_2
8. (theorem) Subspaces of metric space behaves well $d|_{A \times A}$ induces subspace topology for $A \subset X$.

Definition. (Metrizable) If X is topological space, X is **metrizable** if there exists metric d that induces the topology of X . A **metric space** is a metrizable space X together with a specific metric d that gives the topology of X

- Metrizable is a topological property
- (not every topology comes with a metric) consider X_{triv} where $|X| \geq 2$, X not metrizable since it's not T_2
- \mathbb{R}^n is metrizable (d_1, d_2, d_∞ induces \mathbb{R}_{std}^n)
- \mathbb{R}^ω is metrizable under product topology but not box topology
- \mathbb{R}^J where J uncountable is not metrizable
- (theorem) countable products of metrizable spaces is metrizable
- (by sequence lemma) sequences are sufficient to describe metrizable spaces

Definition. (Bounded and Diameter) Given (X, d) , $A \subset X$ is **bounded** if there is some M such that

$$d(a_1, a_2) \leq M$$

for all $a_1, a_2 \in A$. The **diameter** of A is defined

$$\text{diam}(A, d) = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$$

- boundedness is not a topological property, since it depends on a specific d (consider d and \bar{d})

Definition. (Standard Bounded Metric) Given (X, d) , define $\bar{d} : X \times X \rightarrow \mathbb{R}$ by

$$\bar{d}(x, y) = \min\{d(x, y), 1\}$$

to be the standard bounded metric corresponding to d

- (theorem) d and \bar{d} induces the same topology, i.e. $\mathcal{T}_d = \mathcal{T}_{\bar{d}}$
- (trick) by above theorem, we can say $\text{diam}(A) \leq 1$ without loss of generality by replacing d with \bar{d}

Definition. (Uniform Metric and Uniform Topology) Generalize square metric to \mathbb{R}^J . Given $\mathbf{x} = (x_\alpha)_{\alpha \in J}$, $\mathbf{y} = (y_\alpha)_{\alpha \in J} \in \mathbb{R}^J$. Define metric $\bar{\rho}$ by

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$$

is called **uniform metric** on \mathbb{R}^J inducing **uniform topology**.

Theorem. (Relationship of Topologies on \mathbb{R}^J)

$$\mathcal{T}_{prod} \subset \mathcal{T}_{uniform} \subset \mathcal{T}_{box}$$

where the three topologies are all different if J is infinite.

Theorem. (Metric inducing product topology) If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^\omega$, define

$$D(x, y) = \sup \left\{ \frac{\bar{d}(x_i, y_i)}{i} \right\}$$

is a metric that induces product topology on \mathbb{R}^ω

Proof. Show D is a metric. Then show D gives product topology. Now show $\mathcal{T}_D = \mathcal{T}_{prod}$. To show \mathcal{T}_{prod} is finer, show there exists basis element in product topology that contains in a basis element of the metric topology. Let $B_D(\mathbf{x}, \epsilon)$ be basic neighborhood of \mathbf{x} . Let N be such that $1/N < \epsilon$. Consider $V \subset \mathcal{T}_{prod}$ defined as

$$V = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \mathbb{R} \cdots$$

Show $V \subset B_D(\mathbf{x}, \epsilon)$. Note for any $\mathbf{y} \in \mathbb{R}^\omega$, $\bar{d}(x_i, y_i)/i \leq 1/N$ for $i \geq N$. Therefore,

$$D(\mathbf{x}, \mathbf{y}) = \sup \left\{ \frac{\bar{d}(x_1, y_1)}{1}, \dots, \frac{\bar{d}(x_N, y_N)}{N}, \frac{1}{N} \right\}$$

If $\mathbf{y} \in V$, then $\bar{d}(x_i, y_i) < \epsilon$ for all $i < N$. So $D(\mathbf{x}, \mathbf{y}) < \epsilon$. Hence $V \subset B_D(\mathbf{x}, \epsilon)$. Conversely, want to show \mathcal{T}_D is finer. The key here is recognize that product topology $\prod U_i$ where each component is metrizable and induced by d . Let $U = \prod_{i \in I} U_i \times \prod_{i \notin I} \mathbb{R}$ be a basis element in \mathcal{T}_{prod} where I is finite. Let $\mathbf{x} \in U$, want to find a basic neighborhood $V \in \mathcal{T}_D$ such that $\mathbf{x} \in V \subset U$. For each $i \in I$, find an interval $(x_i - \epsilon_i, x_i + \epsilon_i)$ such that $i\epsilon < \epsilon_i$ and $\epsilon \leq 1$. We can achieve this by setting $\epsilon = \min\{\epsilon_i/i \mid i \in I\}$. Now we claim that $\mathbf{x} \in B_D(\mathbf{x}, \epsilon) \subset U$. Let $\mathbf{y} \in B_D(\mathbf{x}, \epsilon)$, then for all i ,

$$\frac{\bar{d}(x_i, y_i)}{i} \leq D(\mathbf{x}, \mathbf{y}) < \epsilon$$

We need to show that $\mathbf{y} \in U$. We only care about $i \in I$ since $y_i \in \mathbb{R}$ for all $i \notin I$. When $i \in I$, $\bar{d}(x_i, y_i) < i\epsilon < \epsilon_i \leq 1$. Therefore, $d(x_i, y_i) < \epsilon_i$ implying $\mathbf{y} \in U$ \square

2.0.10 21 The Metric Topology Continued

Definition. (Continuity in Metric Spaces) Let $f : X \rightarrow Y$ where (X, d_X) and (Y, d_Y) are metrizable. Then f is continuous if and only if $\forall x \in X$ and $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$$

Definition. (almost always) means all but finitely many

Definition. (Convergence) $(x_n) \rightarrow x$ if for all neighborhood U of x , almost always $x_n \in U$.

Definition. (Uniform Convergence) Let $f_n : X \rightarrow Y$ be a sequence of functions where Y is metrizable. Let d be metric for Y . The sequence (f_n) **converges uniformly** to the function $f : X \rightarrow Y$ if given $\epsilon > 0$, there exists $N > 0$ such that

$$d(f_n(x), f(x)) < \epsilon$$

for all $n > N$ and all $x \in X$.

- depends on \mathcal{T}_Y and metric d
- stronger than point-wise convergence
- (uniform limite theorem) Given $f_n : X \rightarrow Y$ where Y metrizable. If (f_n) converges uniformly to f , then f is continuous.

Definition. (Sequential Closure) Given $A \subset X$, sequential closure is given by

$$\text{seq-cl}(A) = \{x \in X \mid \exists (x_n) \rightarrow x, x_n \in A\}$$

- (fact) $A \subset \text{seq-cl}(A) \subset \bar{A}$
- (the sequence lemma) $\text{seq-cl}(A) \subset \text{cl}(A)$ and with equality if X is metrizable.

Lemma. (The sequence lemma) Let X be a topological space and let $A \subset X$. If there is a sequence of points of A converging to x , then $x \in \bar{A}$. The converse holds if X is metrizable.

Proof. (\Rightarrow) If $x_n \rightarrow x$ where $x_n \in A$. Let U be any neighborhood of x . By definition of convergent sequence, $\exists N$ such that $\forall n \geq N$, $x_n \in U$. Since U arbitrary, every neighborhood of x contains some $x_n \in A$, hence $x \in \bar{A}$. Conversely, let d be metric inducing topology of X . Consider neighborhood $B_d(x, 1/n)$ for all $n \in \mathbb{Z}_+$, pick n -th term for the sequence as $x_n \in B_d(x, 1/n) \cap A$. We claim that $(x_n)_{n \in \mathbb{Z}_+}$ is convergent. Indeed, take $B_d(x, \epsilon)$ be arbitrary basic neighborhood of x , take $N > 0$ such that $1/N < \epsilon$. Therefore, for all $i \geq N$, $x_i \in B_d(x, \epsilon)$ by construction. \square

Theorem. (Sequence Continuity) Let $f : X \rightarrow Y$. If f continuous, then every convergent sequence $x_n \rightarrow x$ in X , the sequence $f(x_n) \rightarrow f(x)$. The converse is true if X is metrizable.

Definition. (First countability axiom) A space X that has a countable basis at each point satisfies first countability axiom. A space X said to have **countable basis at the point** x if there is a countable collection $\{U_n\}_{n \in \mathbb{Z}_+}$ of neighborhoods of x such that any neighborhood U of x contains at least one of the sets U_n .

- used to prove the above lemma/theorem; metrizability is not necessary.

Definition. Spaces that are not metrizable

1. \mathbb{R}^ω in box topology is not metrizable
2. \mathbb{R}^J where J uncountable is not metrizable in product topology

Proof. Generally, to show that a space X is not metrizable, we can show that the space does not satisfy the sequence lemma. Specifically, if X is metrizable, then $\text{seq-cl}(A) = \text{cl}(A)$, to prove X not metrizable, find $A \subset X$ and $x \in X$ such that $x \in \text{cl}(A)$ and $x \notin \text{seq-cl}(A)$ (**Point 1**) Consider $A \subset \mathbb{R}^\omega$ be points with only positive coordinate values, i.e $A = \{(x_1, x_2, \dots) \mid x_i > 0 \text{ } i \in \mathbb{Z}_+\}$. Now we claim that $\mathbf{0} = (0, 0, \dots)$ is in $\text{cl}(A)$ but not $\text{seq-cl}(A)$. $\mathbf{0} \in \text{cl}(A)$: any neighborhood $B = (a_1, b_1) \times (a_2, b_2) \times \dots$ of $\mathbf{0}$ intersects A at point $(1/2b_1, 1/2b_2, \dots)$. Now we show there is no sequence of A converging to $\mathbf{0}$. Assume for contradiction that $(\mathbf{x}_n) \rightarrow \mathbf{0}$ where $\mathbf{x}_n = (x_{n,k})_{k=1}^\infty$. Let $U = \prod_n (-1, x_{n,n}) \subset \mathcal{T}_{\text{box}}$ be a neighborhood of $\mathbf{0}$. However not almost always $\mathbf{x}_n \in U$: in fact U contains no elements of the sequence (\mathbf{x}_n) since n -th coordinate $x_{n,n} \notin (-1, x_{n,n})$. Contradicts assumption that $(\mathbf{x}_n) \rightarrow \mathbf{0}$. Therefore $\mathbf{0} \notin \text{seq-cl}(A)$. (**Point 2**) Let $A = \{(x_\alpha) \mid x_\alpha = 1 \text{ except for finitely many } \alpha \in I\}$. Let $\mathbf{0} \in \mathbb{R}^J$ be origin. Now we show $\mathbf{0} \in \text{cl}(A)$. Let $U = \prod U_\alpha$ be neighborhood of $\mathbf{0}$ where $U_\alpha \neq \mathbb{R}$ for all $\alpha \in \{\alpha_1, \dots, \alpha_n\}$ in product topology. Now we show $U \cap A \neq \emptyset$. Consider $\mathbf{y} = (y_\alpha) \in \mathbb{R}^J$ where $y_\alpha = 0$ for all $\alpha \in \{\alpha_1, \dots, \alpha_n\}$ and $y_\alpha = 1$ otherwise. Note $\mathbf{y} \in A$ since all but finitely many y_α is 1; $\mathbf{y} \in U$ since $y_\alpha \in U_\alpha$ for $\alpha_1, \dots, \alpha_n$ and $y_\alpha \in \mathbb{R} = U_\alpha$ otherwise. Hence $\mathbf{y} \in U \cap A$ and therefore $\mathbf{0} \in \text{cl}(A)$. Now we show $\mathbf{0} \notin \text{seq-cl}(A)$. Let \mathbf{a}_n be a sequence of A . Let $J_n \subset J$ such that $\mathbf{a}_n(\alpha) \neq 1$ for all $\alpha \in J_n$. J finite by of A . Let $J' = \bigcup_{n \in \mathbb{Z}_+} J_n$. Note J' is a countable union of finite set and hence countable. Since J uncountable, exists $\beta \in J$ such that $\beta \notin J'$ and therefore every point in the sequence $\mathbf{a}_n(\beta) = 1$ for all $n \in \mathbb{Z}_+$. Let $U_\beta = (-1, 1) \in \mathbb{R}$. Consider a neighborhood $U = \pi^{-1}(U_\beta) \subset \mathbb{R}^J$ of $\mathbf{0}$ that contains no points in \mathbf{a}_n . Therefore \mathbf{a}_n does not converge to $\mathbf{0}$. Therefore $\mathbf{0} \notin \text{seq-cl}(A)$. \square

2.0.11 22 The Quotient Topology

Definition. (Quotient Map) Given X, Y and $p : X \rightarrow Y$ be a surjective map. p is a quotient map if a subset $U \subset Y$ is open if and only if $p^{-1}(U)$ is open in X .

- (theorem) p is a surjective continuous map that is either open or closed, then p is a quotient map
- (example) projection maps $\pi_1 : X \times Y \rightarrow X$ is surjective, continuous, open and therefore a quotient map. However π_1 is not a closed map (since $\pi_1(\{x \times y \mid xy = 1\}) = \mathbb{R} - \{0\}$ not closed)

Definition. (Quotient Topology and Quotient Space) Let X be a space and A be a set. If $p : X \rightarrow A$ is a surjective map, then there uniquely exists one topology \mathcal{T} on A relative to which p is a quotient map; \mathcal{T} is called the **quotient topology** induced by p .

$$\mathcal{T} = \{U \subset Y \mid p^{-1}(U) \in \mathcal{T}_X\}$$

As a special case. Given \sim be an equivalence relation on X and $Y = X/\sim = \{\{y : y \sim x_0\} \mid x_0 \in X\} \subset \mathcal{P}(X)$ are equivalence classes of X . Then $p : X \rightarrow Y$ exists and is a surjection. The space Y with the quotient topology is called a **quotient space** of X .

Definition. define quotient topology with continuity of functions Given topological space X and $\pi : X \rightarrow Y$ a surjection, there is a unique topology on Y satisfying

1. $\pi : X \rightarrow Y$ continuous
2. If Z is a topological space, $g : Y \rightarrow Z$ is a function. If $g \circ \pi$ is continuous, then g is continuous.

3 Connectedness and Compactness

theorems about continuous functions

- (intermediate value theorem) If $f : [a, b] \rightarrow \mathbb{R}$ continuous and $f(a) < r < f(b)$, then exists $c \in [a, b]$ s.t. $f(c) = r$
- (maximum value theorem) If $f : [a, b] \rightarrow \mathbb{R}$ continuous then exists $c \in [a, b]$ such that $f(x) \leq f(c)$ for all $x \in [a, b]$
- (uniform continuity theorem) If $f : [a, b] \rightarrow \mathbb{R}$ continuous, then given $\epsilon > 0$ exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \epsilon$ for every pair $x_1, x_2 \in [a, b]$ for which $|x_1 - x_2| < \delta$

3.0.1 23 Connected Spaces

Definition. (Separation and Connected) Given topological space X . A **separation** of X is a pair U, V where $U \cap V = \emptyset$ and $X = U \cup V$ and U, V both nonempty. The space X is **connected** if there does not exist a separation of X . Equivalently, X is connected if the only clopen sets are \emptyset and X .

- (fact) connectedness is a topological property
- (theorem) Image of a connected space under a continuous map is connected
- (theorem) Finite product $\prod X_\alpha$ connected if and only if X_α connected for all α

Definition. (Separation and Connected for subspaces) Given $Y \subset X$. A **separation** of Y is a pair of disjoint nonempty sets A and B whose union is Y , neither of which contains a limit point of the other (i.e. $cl_Y(A) \cap B = \overline{A} \cap B = \emptyset$). The space Y is connected if there exists no separation of Y .

Proof. Suppose A, B forms a separation of Y . Then $cl_Y(A) = \overline{A} \cap Y$. Since A closed in Y , $A = cl_Y(A) = \overline{A} \cap Y$. Since A, B disjoint, $\overline{A} \cap B = \emptyset$. Since \overline{A} contains all its limit point, B has no limit points of A . Conversely, given assumption we have $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$. Hence $\overline{A} \cap Y = A$ and $\overline{B} \cap Y = B$. Then A, B are closed in Y . Since A, B partitions Y , A, B are open in Y as well. \square

- (examples)
 - $\{a, b\}$ with \mathcal{T}_{triv} is connected
 - $Y = [-1, 0) \cup (0, 1] \subset \mathbb{R}$ is connected since $[-1, 0)$ and $(0, 1]$ is a separation. (0 is a limit point to both, but does not matter since 0 is not contained in $[-1, 0)$ or $(0, 1]$)
 - $X = [-1, 1] \subset \mathbb{R}$. $[-1, 0]$ and $(0, 1]$ is not a separation (0 is a limit point of $(0, 1]$ but contained in $[-1, 0]$)
 - \mathbb{Q} not connected. Only one point subsets of \mathbb{Q} are connected.
 - $X = \{(x, y) \mid y = 0\} \cup \{(x, y) \mid y = 1/x\} \subset \mathbb{R}^2$ not connected (neither contain a limit point of each other)
- (lemma) If C, D forms a separation of X and Y is a connected subspace of X , then Y lies entirely in C or D
- (theorem) Union of connected subspaces of X with a common point is connected
($\cap A_\alpha \neq \emptyset$ where A_α connected for all α then $\cup A_\alpha$ connected)
- (theorem) If $A \subset X$ be a connected subspace. If $A \subset B \subset \overline{A}$, then B also connected

Definition. (Connectedness for Infinite Products)

1. \mathbb{R}^ω is not connected in box topology
2. \mathbb{R}^ω is connected in product topology

Proof. (Point 1) Enough to find a separation of \mathbb{R}^ω . Interpret \mathbb{R}^ω as the collection of all real numbered sequences and is partitioned by the set of all bounded sequences of real numbers A and the set of all unbounded sequences of real numbers B . Now we show A, B both open. Consider $\mathbf{a} \in \mathbb{R}^\omega$, we can find a neighborhood of \mathbf{a} in the box topology by $U = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \dots$. If \mathbf{a} is bounded, U consists of only bounded sequences so $\mathbf{a} \in U \subset A$. If \mathbf{a} is unbounded, then U consists of only unbounded sequences and $\mathbf{a} \in U \subset B$. Therefore \mathbb{R}^ω not connected in box topology. **(Point 2)** To show \mathbb{R}^ω in product topology is connected, we find some connected $C \subset \mathbb{R}^\omega$ where $C \subset \mathbb{R}^\omega \subset \overline{C}$ and use the lemma to show that \mathbb{R}^ω is connected. Consider $\tilde{\mathbb{R}}^n \subset \mathbb{R}^\omega$ defined to be the set of all sequences fixed to 0 beyond n : $\mathbf{x} \in \tilde{\mathbb{R}}^n$ such that $x_i = 0$ for all $i > n$. Since $\tilde{\mathbb{R}}^n \cong \mathbb{R}^n$ and \mathbb{R}^n is connected, $\tilde{\mathbb{R}}^n$ is connected. Since each $\tilde{\mathbb{R}}^n$ is connected and $\cap_n \tilde{\mathbb{R}}^n = \{0, 0, \dots\} = \mathbf{0}$, we have $\mathbb{R}^\omega = \cup_n \tilde{\mathbb{R}}^n$ connected. To complete the proof, we show $\overline{\mathbb{R}^\omega} = \mathbb{R}^\omega$. Consider $\mathbf{a} = (a_1, a_2, \dots) \in \mathbb{R}^\omega$ and $U = \prod U_\alpha$ be neighborhood of \mathbf{a} in box topology. We show $U \cap \mathbb{R}^\omega \neq \emptyset$. Let N be such that $U_i = \mathbb{R}$ for all $i > N$. Consider $\mathbf{x} = \{a_1, a_2, \dots, a_N, 0, 0, \dots\} \in \mathbb{R}^\omega$. $\mathbf{x} \in U$ since $x_i \in U_i$ for all $i \leq N$ and $x_i = 0 \in \mathbb{R} = U_i$ for all $i > N$. Therefore $\mathbf{x} \in U \cap \mathbb{R}^\omega$ hence $\overline{\mathbb{R}^\omega} = \mathbb{R}^\omega$ \square

3.0.2 24 Connected Subspaces of the Real Line

Definition. (convex) $Y \subset X$ is convex if every $a < b \in Y$, $[a, b] \in Y$

Definition. (Linear Continuum) A simply ordered set L having more than 1 element is called a linear continuum if

1. L has least upper bound property
 2. If $x < y$, there exists z such that $x < z < y$
- (fact) condition for connectedness to hold on \mathbb{R}
 - (theorem) If L is linear continuum in order topology, then L is connected, and so are intervals and rays in L
 - (theorem) If $Y \subset \mathbb{R}$, then Y is connected if and only if Y is convex and nonempty
 - (corollary) \mathbb{R} , intervals and rays in \mathbb{R} are all connected

Theorem. (Unit interval in $I = [0, 1] \subset \mathbb{R}$ is connected)

Proof. Idea is to show any separation (A, A^c) gives $A = I$. Let $A \subset I$ be clopen. without loss of generality let $0 \in A$. Define $G = \{x \in I \mid [0, x] \in A\} \subset A$ and $g = \sup G$. Goal is to show $1 = g \in G$ such that $A = G$ and therefore $I - A = \emptyset$ which contradicts assumption of separation. **First** we show $g > 0$, note $0 \in A$ where A is open, hence $[0, \epsilon) \in A$ and therefore $\epsilon/2 \in G$. So $g = \sup G \geq \epsilon/2 > 0$. **Second** we show $g \not< 1$. We first show $g \in A$. Since $G \subset A$, we have $\overline{G} \subset \overline{A} = A$. Then $g = \sup G = \overline{G} \in A$. Hence $g \in A$. Since A open, can find $(g - \epsilon, g + \epsilon) \in A$. Easily, $[0, g + \epsilon/2] \in A$ and hence $g + \epsilon/2 \in G$ which contradicts $g = \sup G$. **Third** we show $g \in G$. Same as before we show $g \in A$. Can find open neighborhood $(1 - \epsilon, 1] \in A$. Easily $[0, 1] \in A$ hence $A = I$. To conclude (A, A^c) is not a separation. \square

Theorem. (Intermediate Value Theorem) Let $f : X \rightarrow Y$ be continuous map, where X is connected and Y is in order topology. If $a, b \in X$ and $r \in Y$ such that $f(a) < r < f(b)$, then there exists $c \in X$ such that $f(c) = r$.

Proof. Let $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, \infty)$. Note $A \cap B = \emptyset$ and neither are empty since $f(a) \in A$ and $f(b) \in B$. Note $A, B \subset f(X)$ are open in the subspace topology by definition. If no $c \in X$ such that $f(c) = r$, then $f(X) = A \cup B$ then (A, B) constitutes a separation of $f(X)$, contradicting the fact that image of a connected space under a continuous map $f(X)$ is connected. \square

Definition. (Path Connectedness) Let $x, y \in X$, a **path** in X from x to y is a continuous map $f : [a, b] \rightarrow X$ such that $f(a) = x$ and $f(b) = y$. A space X is **path-connected** if every pair of points in X can be joined by a path.

$$\forall x, y \in X \exists \text{ continuous } f : [0, 1] \rightarrow X \text{ } f(0) = x \text{ } f(1) = y$$

- (theorem) continuous image of path-connected space is path-connected
- (theorem) path-connected space is connected. (converse not always true: see topologist's sine curve)
- (proposition) connectedness and path-connected subsets of \mathbb{R} are the same
- (theorem) If X_α path-connected, then $\prod X_\alpha$ is also path-connected (in product topology)
- (example) unit ball $B^n = \{\mathbf{x} \mid \|\mathbf{x}\| \leq 1\} \subset \mathbb{R}^n$ is path-connected ($f : [0, 1] \rightarrow \mathbb{R}^n \text{ } t \rightarrow (1-t)\mathbf{x} + t\mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$)
- (example) punctured euclidean space $\mathbb{R} - \{0\}$ is path-connected
- (example) unit sphere $S^{n-1} = \{\mathbf{x} \mid \|\mathbf{x}\| = 1\} \subset \mathbb{R}^n$ is path-connected ($g : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1} \text{ } \mathbf{x} \rightarrow \mathbf{x} / \|\mathbf{x}\|$ is continuous)
- (example) Let $S = \{x \times \sin(1/x) \mid 0 < x \leq 1\} \subset \mathbb{R}^2$. The **topologist's sine curve** \overline{S} is connected but not path-connected

$$\overline{S} = (\{0\} \times [-1, 1]) \cup \{x \times \sin(1/x) \mid 0 < x \leq 1\} \subset \mathbb{R}^2$$

Theorem. (Topologist's sine curve is connected but not path-connected.)

Proof. Let $S' = (\{0\} \times [-1, 1])$ and let $S = \{x \times \sin(1/x) \mid 0 < x \leq 1\} \subset \mathbb{R}^2$ and hence $\overline{S} = S' \cup S$. Since S is connected and \overline{S} is image of a connected set $(0, 1]$ under continuous map, \overline{S} is also connected. Now we show \overline{S} is not path-connected. Let $f : [a, c] \rightarrow \overline{S}$, $a < 0$ be a path connecting $(0, 0)$ and $(1, 0)$. Since S' is closed, $f^{-1}(S')$ is closed and has a largest element b . Therefore $f' : [b, c] \rightarrow \overline{S}$ where f' maps b to S' and rest of points to S . Replace $[b, c]$ with $[0, 1]$ for convenience. Let $f(t) = (x(t), y(t))$ which has to be continuous. Then $x(0) = 0$ and $x(t) > 0$ and $y(t) = \sin(1/x(t))$ for $t > 0$. There exists t_n a sequence such that $y(t_n) = (-1)^n$ does not converge, contradicting continuity of f . We construct t_n as follows. For each n , pick u in range $0 < u < x(1/n)$ such that $\sin(1/u) = (-1)^n$. Use intermediate value theorem to find t_n such that $x(t_n) = u$. \square

3.0.3 26 Compact Spaces

Definition 3.1. (Cover and Compact) A collection $\mathcal{A} \subset \mathcal{P}(X)$ **cover** X , or be a **covering** of X , if $\cup_{A \in \mathcal{A}} A = X$. \mathcal{A} is **open cover**, if it is a cover and all $A \in \mathcal{A}$ are open. A space X is said to be **compact** if every open cover \mathcal{A} contains a finite subcollection that also covers X , i.e. if $\{A_\alpha\}$ is an open cover, exists $I = \{\alpha_1, \dots, \alpha_n\}$ such that $\cup_{\alpha \in I} A_\alpha = X$

- (examples)
 - \mathbb{R} is not compact ($\mathcal{A} = \{(n, n+2) \mid n \in \mathbb{Z}_+\}$ does not have a finite subcover)
 - $(0, 1]$ (and similarly $(0, 1)$) is not compact ($\mathcal{A} = \{(1/n, 1] \mid n \in \mathbb{Z}_+\}$ emits no finite cover)
 - $X = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\} \subset \mathbb{R}$ is compact. (idea: $U \in \mathcal{A}$ covering 0 covers all but finitely many points of $1/n$. Take U and $U_i \in \mathcal{A}$ for all $1/i$ not in U forms a finite cover for X)
 - X with finitely many points is compact (all covers are finite)
 - $[0, 1]$ is compact
- (theorem) Image of a compact space under a continuous map is compact
- (theorem) Let $f : X \rightarrow Y$ be bijective continuous map. If X is compact and Y is T_2 , then f is a homeomorphism (idea: $K \subset X$ closed. Since X compact, K also compact, $f(K)$ is then compact, which is closed in T_2 space Y)
- (theorem) Product of finitely many compact spaces is compact (tube lemma)
- (theorem) Product of infinitely many compact spaces in product topology is compact (Tychonoff theorem)
- (theorem) A continuous function $f : X \rightarrow \mathbb{R}$ where X is compact is bounded. (idea: $X = \cup_{i=1}^{\infty} f^{-1}((-\epsilon_i, \epsilon_i))$, by compactness $X = \cup_{i=1}^n f^{-1}((-\epsilon_i, \epsilon_i)) = f^{-1}((-M, M))$ where $M = \max_{i=1}^n \epsilon_i$)

Definition. (Compactness for subspace) Let $Y \subset X$. Y is compact (i.e. every covering of Y by open sets in Y has a finite subcover) if and only if every covering of Y by sets open in X contains a finite subcollection covering Y

- (theorem) Closed subspace of a compact space is compact, i.e. $Y \subset X$ where X is compact and Y is closed, then Y is compact (idea: Given covering \mathcal{A} of Y by open sets in X , $\mathcal{A} \cup \{X - Y\}$ emits finite subcover for X , and therefore for Y by definition of compactness for subspace)
- (example) Given $[a, b] \in \mathbb{R}$ compact, any closed subset of $[a, b]$ is also compact.
- (theorem) Compact subspace of a T_2 space is closed, i.e. $Y \subset X$ where X is T_2 and Y is compact, then Y is closed (idea: fix $x_0 \in X \setminus Y$, find U s.t. $x_0 \in U \subset X \setminus Y$. By T_2 , exists disjoint open pair U_y, V_y separating x_0, y for all $y \in Y$. By compactness of Y , $\{V_y \mid y \in Y\}$ emits finite cover $V = V_{y_1} \cup \dots \cup V_{y_n}$ which are disjoint from $U = U_{y_1} \cap \dots \cap U_{y_n}$. $X \setminus Y$ open hence Y closed)
- (lemma) If X is T_2 and $Y \subset X$ is compact and $x_0 \notin Y$, exists disjoint $U, V \in \mathcal{T}_X$ that covers x_0 and Y , respectively
- (example) $(a, b] \in \mathbb{R}$ not compact since its not closed. (In T_2 , closedness is a necessary condition for compactness)

Definition. (T_3 Space) A space X is T_3 if

1. it is T_1 (singletons are closed)
 2. if $A \subset X$ is closed and $y \in A^c$, then there exists disjoint open $U, V \in \mathcal{T}_X$ such that $A \subset U$ and $y \in V$
- (theorem) X is compact and T_2 , then X is T_3 (straight from previous lemma)

Lemma. The Tube Lemma Consider $X \times Y$, where Y is compact. If $N \subset X \times Y$ is open and contains the slice $\{x_0\} \times Y$, then N contains some **tube** $W \times Y$ about $\{x_0\} \times Y$, where W is a neighborhood of x_0 in X

Proof. Idea is try to cover the slice $\{x_0\} \times Y$ with basis elements $U \times V \in N$, which happens to cover some tube about the slice. Since $\{x_0\} \times Y \cong Y$ and Y compact, then the slice is compact. Hence can cover $\{x_0\} \times Y$ with finitely basis elements $\mathcal{A} = \{U_1 \times V_1, \dots, U_n \times V_n\}$ where $U_i \times V_i \in N$. Note $W = U_1 \cap \dots \cap U_n$ is open and contains x_0 . We claim that \mathcal{A} actually covers not only the slice but also the tube $W \times Y$. Let $x \times y \in W \times Y$. Some $V_i \supset y$ and since $x \in \cap_j U_j$, $x \in U_i$. Hence $x \times y \in U_i \times V_i$. \square

- (example) $N = \{x \times y \mid |x| < 1/(y^2 + 1)\}$ is an open set containing $\{0\} \times \mathbb{R}$ but it contains no tube about the slice. (since \mathbb{R} is not compact, tube lemma does not hold)

Theorem. (Product of finitely many compact spaces is compact)

Proof. Show $X \times Y$ compact given X, Y compact and do induction from here. Let \mathcal{A} be any open cover for $X \times Y$. For any $x_0 \in X$, $\{x_0\} \times Y \subset X \times Y$ is compact and therefore covered by finitely many open sets of $X \times Y$, specifically $\{A_1, \dots, A_m\} \in \mathcal{A}$. Let $N = \{A_1, \dots, A_m\}$. Since $N \supset \{x_0\} \times Y$ and N is open, there exists tube $W \times Y$ where W is a neighborhood of x_0 . For each $x \in X$, we can find such neighborhood W_x of x such that the tube $W_x \times Y$ is covered by finitely many elements of \mathcal{A} . Since X is compact, there is a finite covering $\{W_1, \dots, W_k\}$ of X . Union of the tubes covers $X \times Y$, i.e. $\bigcup_{j=1}^k W_j \times Y = X \times Y$. Since each tube covered by finitely many elements of \mathcal{A} and there is finitely many tubes, we can cover $X \times Y$ with finitely many elements of \mathcal{A} \square

Definition. (Finite intersection property - FIP) A collection $\mathcal{C} \subset \mathcal{P}(X)$ is said to have **finite intersection property** if intersection of every finite subcollection is nonempty

$$\{C_1, \dots, C_n\} \subset \mathcal{C} \Rightarrow \bigcap_{i=1}^n C_i \neq \emptyset$$

Theorem. (define compactness using finite intersection property) X is compact if and only if every collection of closed sets having the finite intersection property has nonempty intersection

$$X \text{ compact} \iff \forall \mathcal{C} \text{ of closed sets in } X \text{ with FIP} \Rightarrow \bigcap_{C \in \mathcal{C}} C \neq \emptyset$$

Proof. Given $\mathcal{A} \subset \mathcal{P}(X)$, let $\mathcal{C} = \{X - A \mid A \in \mathcal{A}\}$. Then the following holds

1. \mathcal{A} is a collection of open sets if and only if \mathcal{C} is a collection of closed sets
2. \mathcal{A} covers X if and only if intersection $\bigcap_{C \in \mathcal{C}} C$ of all elements of \mathcal{C} is empty (by DeMorgan's Law)
3. $\{A_1, \dots, A_n\} \subset \mathcal{A}$ covers X if and only if intersection of corresponding elements $C_i = X - A_i$ of \mathcal{C} is empty

Idea is to take contrapositive of normal definition of compactness: Given any \mathcal{A} of open sets, if no finite subcollection of \mathcal{A} covers X , then \mathcal{A} does not cover X . Let \mathcal{C} be defined as above, apply above 3 points. We get: Given any \mathcal{C} of closed sets (1), if every finite collection of elements of \mathcal{C} is nonempty (3), then the intersection of all the elements of \mathcal{C} is nonempty (2) \square

- (example) nested sequence of closed sets $C_1 \supset C_2 \supset \dots$, where each $C_n \neq \emptyset$, then $\mathcal{C} = \{C_n\}_{n \in \mathbb{Z}_+}$ has finite intersection property and the intersection $\bigcap_{n \in \mathbb{Z}_+} C_n \neq \emptyset$

3.0.4 27 Compact Subspaces of the Real Line

Theorem. (Compactness for Ordered Set with Least Upper Bound Property) Let X be a simply ordered set having least upper bound property. In order topology, each closed interval in X is compact.

Theorem. (Characterize Compact Subspace of \mathbb{R}^n)

1. (\mathbb{R}) every closed interval in \mathbb{R} is compact (follows from previous theorem)
 - $[0, 1] \in \mathbb{R}$ is compact
2. (\mathbb{R}^n) $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded in euclidean d_2 or square d_∞ metric
 - (note) compact sets in metric space is **not** equivalent to the set of closed and bounded sets. boundedness depends on metric d whereas compactness is purely a topological property
 - Unit sphere and closed unit ball $S^{n-1}, B^n \subset \mathbb{R}^n$ are compact (closed and bounded)
 - $A = \{x \times 1/x \mid 0 < x \leq 1\} \subset \mathbb{R}^2$ is not compact (closed but not bounded)
 - $S = \{x \times \sin(1/x)\}$ is not compact (bounded but not closed since does not contain limit points $S' = \{0\} \times [-1, 1]$)

Proof. (Part 1) proof similar to how we proved $I = [0, 1]$ is connected: 3 cases, by contradiction. **(Part 2)** Note $d_\infty(\mathbf{x}, \mathbf{y}) \leq d_2(\mathbf{x}, \mathbf{y}) \leq \sqrt{n} d_\infty(\mathbf{x}, \mathbf{y})$. So A is bounded under d_2 if and only if it is bounded under d_∞ . So we need to consider square metric d_∞ only. (\Rightarrow) Assume A compact. Since $A \subset \mathbb{R}^n$ is compact and \mathbb{R}^n is T_2 , A is closed. To show it is bounded, consider an open cover $\mathcal{A} = \{B_{d_\infty}(\mathbf{0}, m) \mid m \in \mathbb{Z}_+\}$ for \mathbb{R}^n . Since $A \subset \mathbb{R}^n$ is compact, some finite collections covers A . So $A \subset B_{d_\infty}(\mathbf{0}, M)$ for some $M < \infty$. So any $\mathbf{x}, \mathbf{y} \in A$ has $d_\infty(\mathbf{x}, \mathbf{y}) \leq 2M$. Hence A is bounded. (\Leftarrow) Idea is to find a compact superset of A , when combined with the fact that A is closed, gives A is compact. Assume for any $\mathbf{x}, \mathbf{y} \in A$, $d_\infty(\mathbf{x}, \mathbf{y}) \leq N$. Let $\mathbf{x}_0 \in A$ and let $b = d_\infty(\mathbf{x}_0, \mathbf{0})$. Then for any $\mathbf{x} \in A$, we have $d_\infty(\mathbf{0}, \mathbf{x}) \leq d_\infty(\mathbf{0}, \mathbf{x}_0) + d_\infty(\mathbf{x}_0, \mathbf{x}) \leq N + b = P$. Therefore $A \subset \prod_{i=1}^n [-P, P]$ which is compact since $[-P, P] \subset \mathbb{R}$ is compact and finite product of compact set is compact. Since A is closed and $A \subset [-P, P]^n$, A is also compact. \square

Theorem. (Extreme Value Theorem) Let $f : X \rightarrow Y$ be continuous, Y is in order topology. If X is compact, then f attains its maximum and minimum, i.e. $\exists c, d \in X$ such that $f(c) = \inf_{x \in X} f(x)$ and $f(d) = \sup_{x \in X} f(x)$

Proof. Now show f attains its maximum, proof for minimum is similar. (\Rightarrow) Since f continuous and X compact, then $A = f(X)$ is compact. Enough to show that A contains its largest element M , and since $M \in A$, $M = f(d)$ for some $d \in X$. By contradiction assume A has no largest element, the $\mathcal{A} = \{(-\infty, a) \mid a \in A\}$ forms an open covering of A . Since A compact, $\{(-\infty, a_1), \dots, (-\infty, a_n)\}$ covers A . Let $a = \max\{a_1, \dots, a_n\}$. Note $a \notin (-\infty, a_i)$ for all $1 \leq i \leq n$. Since $a \in A$, a is not covered by \mathcal{A} which is a contradiction. (Alternatively, simply use the fact that $f(X)$ is a compact subspace of T_2 space Y so closed, and therefore $\sup_{x \in X} f(x) \in f(X)$) \square

Definition. (point-set distance) Let (X, d) be metric, let $A \subset X$ be nonempty. For each $x \in X$, define **distance from x to A** by $d(x, A) = \inf\{d(x, a) \mid a \in A\}$.

- (lemma) Fixing A , $d : X \rightarrow \mathbb{R} \ x \mapsto d(x, A)$ is a continuous function

Definition. (diameter) For metric space (X, d) and $A \subset X$, $\text{diam}(A, d) = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$

Lemma. (Lebesgue Number Lemma) Let \mathcal{A} be open covering of metric space (X, d) . If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ , there exists an element of \mathcal{A} containing it. The number δ is a **Lebesgue number** for the covering \mathcal{A} . In other words,

$$X \text{ compact} \quad \Rightarrow \quad \exists \delta > 0 \quad \forall B \subset \mathcal{P}(X) \quad (\text{diam}(B, d) < \delta \Rightarrow \exists A \in \mathcal{A} \ B \subset A)$$

Proof. Let \mathcal{A} be a cover for X . If $X \in \mathcal{A}$, then $\delta > 0$ could be any since X contains any subsets in X . Now assume $X \notin \mathcal{A}$. Choose finite $\{A_1, \dots, A_n\} \subset \mathcal{A}$ covering X . For each i , let $C_i = X \setminus A_i$. Define $f : X \rightarrow \mathbb{R}$ by

$$f(x) = \frac{1}{n} \sum_{i=1}^n d(x, C_i)$$

Key idea is $f(x) \geq 0$. Consider any $x \in X$, pick ϵ such that $B_\epsilon(x) \subset A_i$, then $d(x, C_i) \geq \epsilon$, so then $f(x) \geq \epsilon/n$. Note f is continuous and by extreme value theorem attains its minimum δ . Now we show δ is the Lebesgue number for \mathcal{A} . Let $B \subset X$ having diameter less than δ , let $x_0 \in B$ be arbitrary. Note $B \subset B_\delta(x_0)$. By definition of f as the average distance to all C_i s, since $f(x_0) \geq \delta$, we have $d(x_0, C_m) \geq \delta$ for some C_m . Therefore $B \subset X \setminus C_m = A_m \in \mathcal{A}$. \square

Definition. (Uniform Continuity) Given (X, d_X) and (Y, d_Y) metric, then $f : X \rightarrow Y$ **uniformly continuous** if

$$\forall \epsilon > 0 \exists \delta > 0 \forall x_0, x_1 \in X \quad (d_X(x_0, x_1) < \delta \Rightarrow d_Y(f(x_0), f(x_1)) < \epsilon)$$

- (note) Idea is one $\delta > 0$ works for all $x, y \in X$ nearby such that $f(x), f(y) \in Y$ are also nearby
- (theorem) Given continuous $f : X \rightarrow Y$, where X, Y are metric and X is compact, then f is uniformly continuous

Definition. (continuous function on compact space is uniformly continuous)

Proof. Let $\epsilon > 0$. take open covering of Y by balls $B(y, \epsilon/2)$. Let $\mathcal{A} = \{f^{-1}(B(y, \epsilon/2)) \mid y \in Y\}$ be open covering for X . Let δ be Lebesgue Number for \mathcal{A} on compact X . Let $x_0, x_1 \in X$ such that $d_X(x_0, x_1) \leq \delta$. Then $\text{diam}(\{x_0, x_1\}) \leq \delta$, and by Lebesgue Number Lemma, $\exists A \in \mathcal{A}$ such that $\{x_0, x_1\} \in A = f^{-1}(B(y, \epsilon/2))$ for some $y \in Y$. Therefore, $d_Y(f(x_0), f(x_1)) < \epsilon$ \square

3.0.5 28 Limit Point Compactness

4 Countability and Separation Axioms

5 The Tychonoff Theorem

5.0.1 37 The Tychonoff Theorem

Lemma. (Get Maximal $\mathcal{D} \subset \mathcal{P}(X)$ with FIP using Zorn's Lemma) Let X be a set; $\mathcal{A} \subset \mathcal{P}(X)$ having finite intersection property. Then there exists a collection $\mathcal{D} \subset \mathcal{P}(X)$ such that

1. $\mathcal{D} \supset \mathcal{A}$
2. \mathcal{D} has finite intersection property
3. no collection $\mathcal{D}' \subset \mathcal{P}(X)$ where $\mathcal{D}' \supsetneq \mathcal{D}$ has finite intersection property

Call \mathcal{D} as maximal with respect to FIP

Proof. Let \mathbb{C} be a superset whose elements are collections of subsets of X , i.e. $\mathcal{A}, \mathcal{D} \in \mathbb{C}$. Let

$$\mathbb{A} = \{ \mathcal{B} \subset \mathcal{P}(X) \mid \mathcal{B} \supset \mathcal{A} \text{ and } \mathcal{B} \text{ has FIP} \}$$

Use \subsetneq as strict ordering on \mathbb{A} . In order to apply Zorn's lemma on (\mathbb{A}, \subsetneq) , we show if $\mathbb{B} \subset \mathbb{A}$ that is simply ordered by \subsetneq , then \mathbb{B} has an upper bound in \mathbb{A} . Let $\mathcal{C} = \cup_{\mathcal{B} \in \mathbb{B}} \mathcal{B}$ and we show that \mathcal{C} is an element of \mathbb{A} , and then it is the required upper bound for \mathbb{B} . To show $\mathcal{C} \in \mathbb{A}$. We show

1. $(\mathcal{A} \subset \mathcal{C})$ obvious since any $\mathcal{B} \in \mathbb{B}$ has $\mathcal{A} \subset \mathcal{B}$
2. (\mathcal{C} has FIP) Let $C_1, \dots, C_n \subset \mathcal{C}$ a finite subset, we want to show that their intersection is nonempty. Since $\mathcal{C} = \cup_{\mathcal{B} \in \mathbb{B}} \mathcal{B}$, $C_i \in \mathcal{B}_i$ for each i . Hence, $\{C_1, \dots, C_n\} \subset \mathbb{B}$. Since \mathbb{B} is assumed to be simply ordered, there exists \mathcal{B}_k such that $\mathcal{B}_i \subset \mathcal{B}_k$ for all $i \neq k$. Therefore $\{C_1, \dots, C_n\} \subset \mathcal{B}_k$. Since \mathcal{B}_k has FIP, $\cap_{i=1}^n C_i \neq \emptyset$ as desired.

Hence the upper bound \mathcal{C} is inside \mathbb{A} □

Lemma. (Intersection on Maximal \mathcal{D} w.r.t FIP) Let X be a set and \mathcal{D} be a collection of subsets of X that is maximal with respect to the finite intersection property. Then,

1. Any finite intersection of elements of \mathcal{D} is an element of \mathcal{D}
2. If $A \subset X$ that intersects every element of \mathcal{D} , then A is an element of \mathcal{D}

Proof. (Part 1) Let B be intersection of finitely many elements of \mathcal{D} . Let $\mathcal{E} = \mathcal{D} \cup \{B\}$. To prove $B \in \mathcal{D}$, we show \mathcal{E} has FIP so by maximality of \mathcal{D} , $\mathcal{E} = \mathcal{D}$ and $B \in \mathcal{D}$. Now take finitely many elements of \mathcal{E} , if none of them is B , then their intersection is nonempty since they are a subset of \mathcal{D} which has FIP. Now if one of finitely many elements of \mathcal{E} is B , then the intersection $D_1 \cap \dots \cap D_m \cap B$ is nonempty since B is the intersection of finitely many elements of \mathcal{D} . **(Part 2)** Let $X \subset X$ s.t. $A \cap D \neq \emptyset$ for all $D \in \mathcal{D}$. Similar to idea of part 1, define $\mathcal{E} = \mathcal{D} \cup \{A\}$. We show that \mathcal{E} has FIP. Now take finitely many elements of \mathcal{E} . If none of them is A , then they are nonempty by FIP of \mathcal{D} . Otherwise, $D_1 \cap \dots \cap D_m \cap A$ is nonempty since $A \cap D_j$ for all $1 \leq j \leq m$ by assumption. □

Theorem. (Tychonoff Theorem) Arbitrary product of compact spaces is compact in product topology

Proof. Let $X = \prod_{\alpha \in J} X_\alpha$ where each space X_α is compact. Let $\mathcal{A} \subset \mathcal{P}(X)$ be arbitrary and have FIP, now we show that $\cap_{A \in \mathcal{A}} A$ is nonempty, thereby proving that X is compact. We apply lemma 1 to find a maximal collection \mathcal{D} such that $\mathcal{A} \in \mathcal{D}$ and \mathcal{D} has FIP. Now we show that $\cap_{D \in \mathcal{D}} \overline{D}$ is nonempty. Let $\pi_\alpha : X \rightarrow X_\alpha$ be projection map. For each α , consider $\{\pi_\alpha(D) \mid D \in \mathcal{D}\} \subset X_\alpha$ which has FIP since \mathcal{D} does. By compactness, we can choose $x_\alpha \in \cap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$, which is possible by FIP. Let $\mathbf{x} = (x_\alpha)_{\alpha \in J} \in X$. To complete the proof, we show that $\mathbf{x} \in \cap_{D \in \mathcal{D}} \overline{D}$. Consider any coordinate index β and any $D \in \mathcal{D}$, let $U_\beta \subset X_\beta$ be a neighborhood of x_β . Since $x_\beta \in \overline{\pi_\beta(D)}$, U_β intersects $\pi_\beta(D)$. Let $\pi_\beta(\mathbf{y}) \in U_\beta \cap \pi_\beta(D)$ for some $\mathbf{y} \in D$. Hence $\mathbf{y} \in \pi_\beta^{-1}(U_\beta) \cap D$. Since D is arbitrary, by lemma 2.2, $\pi_\beta^{-1}(U_\beta) \in \mathcal{D}$. Since β arbitrary, this holds for all $\beta \in J$. Then any basis element containing \mathbf{x} as finite intersection of subbasis $\pi_\beta^{-1}(U_\beta) \in \mathcal{D}$, then by lemma 2.1, is also in \mathcal{D} . Now use \mathcal{D} FIP again, $B \cap D \neq \emptyset$ where $D \in \mathcal{D}$ and $B \in \mathcal{D}$ is any basic element containing \mathbf{x} . $\mathbf{x} \in \cap_{D \in \mathcal{D}} \overline{D}$ □