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1 Basics

1. **Functions**

   - **injective** $f : X \to Y$ injective or one-to-one if $\forall a, b \in X,\ f(a) = f(b) \Rightarrow a = b$
   - **surjective** $f : X \to Y$ is surjective if any $y \in Y \Rightarrow y = f(x)$ for some $x \in X$
   - composition of injective/surjective/bijective functions are injective/surjective/bijective
   - If f is injective with range Y, then its inverse function $f^{-1} : Y \to X$ is a bijective function

2. **Set Relations**

   - **De Morgan's Law**

   $$X \setminus \left(\bigcup_{\alpha \in I} A_\alpha \right) = \bigcap_{\alpha \in I} (X \setminus A_\alpha) \qquad X \setminus \left(\bigcap_{\alpha \in I} A_\alpha \right) = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$$

   - **how functions acts on sets** Let $A, B \subseteq X$ and $C, D \subseteq Y$
     - f well-behaved for union
     $$f(A \cup B) = f(A) \cup f(B)$$
     - f not well-behaved for intersection, difference
     $$\begin{aligned} f(A \cap B) &\subseteq f(A) \cap f(B) \\ f(A) \setminus f(B) &\subseteq f(A \setminus B) \end{aligned}$$
     - f^{-1} well-behaved with union, intersection, difference, and set complement
     $$\begin{aligned} f^{-1}(C \cup D) &= f^{-1}(C) \cup f^{-1}(D) \\ f^{-1}(C \cap D) &= f^{-1}(C) \cap f^{-1}(D) \\ f^{-1}(C \setminus D) &= f^{-1}(C) \setminus f^{-1}(D) \\ f^{-1}(Y \setminus C) &= X \setminus f^{-1}(C) \end{aligned}$$
     - f and f^{-1} mixed
     $$\begin{aligned} A &\subseteq f^{-1}(f(A)) && \text{(with equality if } f \text{ is injective)} \\ f(f^{-1}(C)) &\subseteq C && \text{(with equality if } f \text{ surjective)} \end{aligned}$$

Remark. Read

1. chapter 1, 1-8
2. chapter 2, 12-22
3. chapter 3, 23-29
4. chapter 4. 30-35

2 chapter 1 Topological Spaces and Continuous Functions

2.0.1 Relation

1. A relation on a set A is a subset C of cartesian product $A \times A$. xCy means $(x, y) \in C$
2. **equivalence relation** is a relation if it satisfies reflexivity, symmetry, transitivity
3. **equivalence class** a subset of A determined by some $x \in A$, i.e. $E = \{y \mid y \sim x\}$
4. **partition** of a set A is a collection of disjoint nonempty subsets of A whose union is all of A
5. **order relation** is a relation if it satisfies comparability (any $x \neq y \in A$ either xCy or yCx but not both) nonreflexivity (xCx does not hold for any $x \in A$) and transitivity
6. **dictionary order relation** Let A, B bet sets and $<_A$ and $<_B$ be order relations. The order relation on $A \times B$ is defined by $a_1 <_A a_2$ or if $a_1 = a_2$ and $b_1 <_B b_2$
7. **least upper bound property** An ordered set A has the property if every nonempty subset A_0 of A that is bounded above ($\exists b \in A$ s.t. $x \leq b$ for all $x \in A_0$) has a least upper bound (all bounds of A_0 has a smallest element)
   - $\mathbb{R}$ and $(-1, 1)$ has least upper bounde property
   - $B = (-1, 0) \cup (0, 1)$ does not heave least upper bound property, $\{-1/_{2n} \mid n \in \mathbb{Z}_+\}$ is bounded above by any $b \in (0, 1)$ but its least upper bound $0 \notin B$

2.0.2 Cartesian Product

1. **indexed family of sets** Let $\mathcal{A}$ be nonempty collection of sets, let $f : J \to \mathcal{A}$ be a surjective indexing function. $(\mathcal{A}, f)$ is called indexed family of sets, denoted by $\{\mathcal{A}_\alpha\}_{\alpha \in J} = \{\mathcal{A}_\alpha\}$ where $f(\alpha) = \mathcal{A}_\alpha$
2. **m-tuple** Let $m \in \mathbb{Z}_+$, Given a set X, define m-tuple of X to be a function $\mathbf{x} : \{1, \cdots, m\} \to X$ and denote $\mathbf{x} = (x_1, \cdots, x_m)$
3. **cartesian product** Let $\mathcal{A} = \{A_1, \cdots, A_m\}$ be indexed family of sets, let $X = \cup_{i=1}^m A_i$. Then cartesian product of $\mathcal{A}$ is
$$X^m = \prod_{i=1}^m A_i \qquad A_1 \times \cdots \times A_m$$
   to be the set of all m-tuples $\mathbf{x}$ of elements of X such that $x_i \in A_i$ for each i
4. **ω-tuple** Given a set X, define ω-tuple of elements of X be a function $\mathbf{x} : \mathbb{Z}_+ \to X$. $\mathbf{x}$ is an *infinite sequence*, of elements of X. Denote $x_i = \mathbf{x}(i)$ as i-th coordinate of $\mathbf{x}$. Denote $\mathbf{x}$ itself by $(x_1, x_2, \cdots)$ or $(x_n)_{n \in \mathbb{Z}_+}$
5. **cartesian product (infinite)** Let $\mathcal{A} = \{A_1, A_2, \cdots\}$ be indexed family of sets and X be union of sets in $\mathcal{A}$, the cartesian product of $\mathcal{A}$
$$X^\omega = \prod_{i \in \mathbb{Z}_+} A_i \qquad A_1 \times A_2 \times \cdots$$
   is defined to be the set of all ω-tuples $(x_1, x_2, \cdots)$ of elements of X such that $x_i \in A_i$ for each i

3 chapter 2

3.0.1 12 Topological Spaces

Definition. *(**Topology**) Topology on a set X is a collection $\mathcal{T}$ of subsets of X having properties*

1. $\emptyset, X \in \mathcal{T}$
2. *Arbitrary union of subcollection of $\mathcal{T}$ is in $\mathcal{T}$ (If $\forall \alpha \in I$, $U_\alpha \in \mathcal{T}$, then $\cap_{\alpha \in I} U_\alpha \in \mathcal{T}$)*
3. *Finite intersection of subcollection of $\mathcal{T}$ is in $\mathcal{T}$ (If $\forall 1 \leq i \leq n$, $U_i \in \mathcal{T}$, then $\cap_{i=1}^{n} U_i \in \mathcal{T}$)*

A topological space is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ are the open sets.

- *Standard topology on $\mathbb{R}^n$ is $\mathcal{T}_0 = \mathcal{T}_{std} = \{U \subset \mathbb{R}^n \mid \forall x \in U, \exists \epsilon > 0\, B_\epsilon(x) \subset U\}$*
- *Standard topology on $\mathbb{R}$ is generated by $\mathcal{B}_{std} = \{(a, b) = \{x \mid a < x < b\}$ the open intervals*
- *Lower limit topology on $\mathbb{R}$ is generated by $\mathcal{B}_{l.l.} = \{x \mid a \leq x < b\}$ the half-open intervals*
- *Discrete topology $\mathcal{T}_1 = \mathcal{T}_{disc} = \mathcal{P}(X)$ all subsets are open*
- *Trivial topology $\mathcal{T}_2 = \mathcal{T}_{triv} = \{\emptyset, X\}$ only empty set and X are open*
- *Finite complement topology $\mathcal{T}_{f.c.} = \{U \subseteq X \mid X - U$ is finite or all of $X\}$*
- *Countable complement topology $\mathcal{T}_c = \{U \subseteq X \mid X - U$ is countable or all of $X\}$*

Lemma. *(**Arbitrary intersection of topologies is a topology**) $\forall \alpha \in I$ $\mathcal{T}_\alpha$ is a topology, so is $\cap_{\alpha \in I} \mathcal{T}_\alpha$*

Definition. *(**Compare topology**) If $\mathcal{T}' \subset \mathcal{T}$, then $\mathcal{T}'$ is coarser / weaker / smaller than $\mathcal{T}$, $\mathcal{T}$ is finer / stronger / larger than $\mathcal{T}'$. $\mathcal{T}$ and $\mathcal{T}'$ are comparable if $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$*

$$\mathcal{T}_{triv} \subset \mathcal{T}_{f.c.} \subset \mathcal{T}_{std} \subset \mathcal{T}_{disc} \qquad \mathcal{T}_{std} \subset \mathcal{T}_{l.l.}$$

3.0.2 13 Basis for a Topology

A terser representation of $\mathcal{T}$

Definition. *(**Basis**) If X is a set, a basis for $(X, \mathcal{T})$ is a collection $\mathcal{B}$ (basis elements) of subsets of X s.t.*

1. *For each $x \in X$, exists at least one basis element $B \in \mathcal{B}$ containing it*
2. *If $x \in B_1 \cap B_2$, then exists $B_3 \in \mathcal{B}$ s.t. $x \in B_3 \in B_1 \cap B_2$*

*A topology $\mathcal{T}_\mathcal{B}$ **generated by** $\mathcal{B}$ is defined as*

$$\mathcal{T}_\mathcal{B} = \{U \subset X \mid \forall x \in U,\ \exists B \in \mathcal{B},\ x \in B \subset U\}$$

- *$\mathcal{T}_\mathcal{B}$ is the unique minimal topology containing $\mathcal{B}$*

$$\mathcal{T}_\mathcal{B} = \bigcap\nolimits_{\mathcal{T} \in \mathbb{T}} \mathcal{T}$$

  where $\mathbb{T} = \{\mathcal{T} \mid \mathcal{T} \supset \mathcal{B}$ and $\mathcal{T}$is a topology$\}$

- *For any X, all one point sets of X is a basis for $\mathcal{T}_{disc}$*

Lemma. *($\mathcal{B} \to \mathcal{T}_\mathcal{B}$) $\mathcal{T}_\mathcal{B}$ equals the collections of all unions of elements of $\mathcal{B}$*

$$\mathcal{T}_\mathcal{B} = \left\{\bigcup\nolimits_{\alpha \in I} B_\alpha \mid B_\alpha \in \mathcal{B} \quad \forall \alpha \in I\right\}$$

Lemma. *($\mathcal{T}_\mathcal{B} \to \mathcal{B}$) Let $(X, \mathcal{T})$ be topological space. Let $\mathcal{C}$ be a collection of open sets of X such that*

$$\forall U \subset X, \quad \forall x \in U, \quad \exists C \in \mathcal{C} \text{ s.t. } x \in C \subset U$$

Then $\mathcal{C}$ is a basis for $\mathcal{T}$ (handy in deciding $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for $Y \subset X$)

Lemma. *(**compare topology by basis**) Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then following equivalent*

1. *$\mathcal{T}' \supset \mathcal{T}$ ($\mathcal{T}'$ is finer than $\mathcal{T}$)*
2. *For each $x \in X$ and each $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset \mathcal{B}$*

3.0.3 14 Order Topology

Definition. *(**Order Topology**) Let X be a set with simple order relation. Let $\mathcal{B}$ be a collection of all sets of the following type*

$$\begin{aligned}\mathcal{B} = \{&(a,b) \mid a < b \quad a,b \in X\} \\ &\bigcup \{[a_0,b) \mid a_0 \textit{ is minimal element (if any) of } X \; b \in X \quad b \neq a_0\} \\ &\bigcup \{(a,b_0] \mid b_0 \textit{ is maximal element (if any) of } X \; a \in X \quad a \neq b_0\}\end{aligned}$$

Generated topology $\mathcal{T}_{\mathcal{B}}$ is called order topology

- *In $\mathbb{R}$, $\mathcal{T}_{ord} = \mathcal{T}_{std}$*
- *In $\mathbb{Z}_+$, $\mathcal{T}_{ord} = \mathcal{T}_{disc}$ (since any $\{n\} = (n-1, n+1) \in \mathcal{T}_{ord}$)*
- *In $\{1,2\} \times \mathbb{Z}_+$ in $\mathcal{T}_{dict}$ is not in $\mathcal{T}_{disc}$ (althogh most single point set are open, 2×1 is not open)*
- *In $\mathbb{R}^2$, both $\mathcal{B}$ and $\mathcal{B}'$ generates $\mathcal{T}_{dict}$*

$$\mathcal{B} = \{(a \times b, c \times d) \mid a < c \vee (a = c \wedge b < d)\} \quad \mathcal{B}' = \{(a \times b, a \times d) \mid b < d\}$$

3.0.4 15 Product Topology

Definition. *(**Product Topology**) Let X, Y be topological spaces, the product topology on $X \times Y$ is generated by the basis*

$$\mathcal{B} = \{U \times V \mid U \in \mathcal{T}_X \; V \in \mathcal{T}_Y\}$$

Alternatively define product topology with basis. If $\mathcal{B}$, $\mathcal{C}$ are basis for X and Y respectively. Then

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \; C \in \mathcal{C}\}$$

is a basis for topology of $X \times Y$

- $X \times \{y\} \cong X$
- $X \times Y \cong Y \times X$
- $(X \times Y) \times Z \cong X \times (Y \times Z)$
- *Product spaces does not work well with order topology.*
  - *Consider $X = \mathbb{R}^2$ and $Y = [0,1] \times [0,1]$, then $\{0.5\} \times [0,1]$ is not open in $\mathcal{T}_{ord}$ but is open in $\mathcal{T}_{subspace}$*

Definition. *(**Projection**) Let $\pi_1 : X \times Y \to X$ be defined by $\pi_1(x,y) = x$. π_1 is a projection of $X \times Y$ **onto** the first factor. (note projections are surjective)*

Definition. *(**Product Topology by Continuity of Functions**) Given X, Y, there exists unique topology on $X \times Y$ such that*

1. *projections π_X and π_Y are continuous*
2. *If $f : Z \to X$ and $g : Z \to Y$, hen $f \times g : Z \to X \times Y$ is continuous*

Proof. Define $\mathcal{B} = \{U \times V \mid U \subset X \text{ open } \; V \subset Y \text{ open }\}$. Show $\mathcal{T}_{\mathcal{B}}$ satisfies the above 2 conditions. Prove uniqueness by showing $id : (X, \mathcal{T}') \to (X, \mathcal{T}'')$ is a homeomorphism utilizing the above 2 conditions. □

3.0.5 16 Subspace Topology

Definition. *(**Subspace Topology**) Let $(X, \mathcal{T})$. Let $Y \subset X$, then*

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is the subspace topology on Y. Alternatively, define using basis. If $\mathcal{B}$ generates $\mathcal{T}$, then

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for $\mathcal{T}_Y$.

- ***(lemma)** $Y \subset X$. U open in Y and Y open in X, then U open in X*
- ***(lemma)** A subspace of a subspace is a subspace*
- ***(theorem)** A product of subspaces is a subspace of the product (subspace and product topology work well)*

- *X ordered and $Y \subset X$. order topology on Y may not be same as order topology of Y inherited as a subspace of X (subspace and order topology does not work well)*
  - *Let $Y = [0,1] \subset \mathbb{R}$. $\mathcal{B}_Y$ are of the form $(a,b) \cap [0,1]$. Note*
    1. *$[0,b)$ where $b \notin [0,1]$ is open in $[0,1]$ but not in $\mathbb{R}$*
    2. *$\mathcal{T}_{ord} \cong \mathcal{T}_{subspace}$ since the basis elements of the same form*
  - *Let $Y = [0,1) \cup \{2\} \subset \mathbb{R}$.*
    1. *$\{2\}$ open in $\mathcal{T}_{subspace}$ since $(1.5, 2.5) \cap Y = \{2\}$.*
    2. *$\{2\}$ not open in order topology since any basis of the form $\{x \in Y \mid a < x \leq 2 \; a \in Y\}$ contains some other point other than $\{2\}$*
    3. *$\mathcal{T}_{ord} \ncong \mathcal{T}_{subspace}$*
- ***(theorem)** subspace and order topology works well if the subspace is convex*

Definition. ***(convex)*** *Given ordered X and subset $Y \subset X$ is convex if $(a,b) \subset X$ lies in Y completely*

Theorem. ***(subspace and order topology works well if the subspace is convex)***
Let X be ordered and $Y \subset X$ be convex. Then order topology on Y same as topology Y inherits as a subspace of X.

Definition. ***(Subspace Topology by Continuity of Functions)*** *Let X be topological space, $Y \subset X$, there exists unique topology on Y such that*

1. *inclusion $i_Y : Y \hookrightarrow X$ is continuous*
2. *If $f : Z \to Y$ such that $i_Y \circ f : Z \to X$ is continuous, then f is continuous*

3.0.6 17 Closed Sets and Limit Points

Definition. ***(Closed)*** *A subset A of X is closed if $X - A$ is open.*

- *In $\mathcal{T}_{f.c.}$, all finite subsets and X are closed*
- *In $\mathcal{T}_{disc}$ every set is closed.*
- *In $Y = [0,1] \cup (2,3)$, $[0,1]$ and $(2,3)$ are open and closed in subspace topology of Y*
- *C closed in Y does not imply C is closed in X. However a closed set in a closed subspace is closed overall, i.e. in X (Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.)*

Theorem. ***(Topology by closed sets)***

1. *$\emptyset$ and X are closed*
2. *Arbitrary intersection of closed sets are closed*
3. *Finite unions of closed sets are closed*

Theorem. ***(Closedness in Subspace)*** *Let Y be a subspace of X. Then A is closed in Y if and only if it equals the intersection of a closed set X with Y ($Y \subset X$, then $A \subset Y$ closed in Y if exists $K \subset X$ s.t. $A = K \cap Y$)*

Definition. ***(Closure and Interior)*** *If $A \subset X$*

1. ***Interior** $Int_X A = \mathring{A}$ is*
   - *the union of all open sets in X contained in A, i.e. $Int_X A = \cup_{U \in \mathcal{T}_X : U \subset A} U$*
   - *maximal open subests of A in X*
2. ***Closure** $Cl_X A = \overline{A}$ is*
   - *the intersection of all closed sets containing A, i.e. $Cl_X A = \cap_{U \in \mathcal{T}_X : U \supset A} U$*
   - *minimal closed set containing A*

- *Set relationship*
$$IntA \subset A \subset \overline{A}$$
- *(theorem). Let $A \subset Y \subset X$, Let $\overline{A}$ denote closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$*
  - *In $\mathbb{R}$, $Y = (0,1]$, let $A = (0, 0.5) \subset Y$. $Cl_{\mathbb{R}} A = [0, 0.5]$, $Cl_Y A = Cl_{\mathbb{R}} A \cap Y = (0, 0.5]$*
- *If A open, then $Int(A) = A$; If A closed, then $\overline{A} = A$*
- *If $A \subset X$, then $(\mathring{A})^c = \overline{(A^c)}$ (Complement of interior is closure of the complement)*

- *In $\mathbb{R}$, let $A = \mathbb{Q}$ or $A = \mathbb{R} - \mathbb{Q}$, $int(A) = \emptyset$ and $\overline{A} = \mathbb{R}$ ($int(A) = \emptyset$ since no (a, b) contained fully in $\mathbb{Q}$ or $\mathbb{R} - \mathbb{Q}$)*

Definition. ***(neighborhood)*** *U is an open set containing x is equivalent to U is a neighborhood of x*

Definition. ***(intersects)*** *A intersects B if and only if $A \cap B \neq \emptyset$*

Theorem. ***(define closure using neighborhoods)*** *Let A be a subset of X,*

1. *then $x \in \overline{A}$ if and only if every neighborhood of x intersects A*
2. *Let $\mathcal{B}$ be basis of X, then $x \in \overline{A}$ if and only if every basic neighborhood B of x intersects A*

proof by contraposition. Following are examples which uses this theorem to test/determine the closure

- *If $A = (0, 1]$ then $\overline{A} = [0, 1]$ (since every neighborhood of $\{0\}$ intersects A)*
- *If $B = \{1/n \mid n \in \mathbb{Z}_+\}$, then $\overline{B} = \{0\} \cup B$*
- *If $C = \{0\} \cup (1, 2)$ then $\overline{C} = \{0\} \cup [1, 2]$*
- *In $\mathbb{R}$, $\overline{\mathbb{Q}} = \mathbb{R}$ since every neighborhood of $x \in \mathbb{R}$ contains some rational number, so intersects $\mathbb{Q}$*
- *In $\mathbb{Z}_+$, $\overline{\mathbb{Z}_+} = \mathbb{Z}_+$*

Definition. ***(Limit Point)*** *Let $A \subset X$ and $x \in X$, x is a limit point of A if every neighborhood of x intersects A in some point other than x itself. In other words,*

$$x \in \overline{A - \{x\}}$$

- *In $\mathbb{R}$, $A = (0, 1]$, then any $x \in [0, 1]$ is a limit point of A and no other point in $\mathbb{R}$ is a limit point*
- *In $\mathbb{R}$, $B = \{1/n \mid n \in \mathbb{Z}_+\}$, 0 is the only limit point of B (any other $x \in \mathbb{R}$ has neighborhood that does not intersect B or intersects at x itself)*
- *In $\mathbb{R}$, $C = \{0\} \cup (1, 2)$, all $x \in [1, 2]$ are limit points of C*
- *In $\mathbb{R}$, every $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$*
- *In $\mathbb{R}$, no point is a limit point of $\mathbb{Z}_+$*

Theorem. ***(define closure using limit point)*** *Let $A \subset X$, let A' be set of all limit points, then*

$$\overline{A} = A \cup A'$$

- *(corollary) $A \subset X$ is closed if and only if it contains all its limit points, i.e. $A' \subset A$*

Remark. ways to prove A is closed

1. show A^c is open
2. show $\overline{A} = A$ by proving that every $x \in A^c$ has open neighborhood that does not intersect A
   - $A = \{x_0\} \subset \mathbb{R}$ closed since every point different from x_0 has neighborhood not intersecting $\{x_0\}$

Definition. ***(Sequence Convergence)*** *A sequence (x_n) converges to a point $x \in X$, denoted $x_n \to x$ or $\lim_{n\to\infty} x_n = x$ if*

$$\forall \text{ neighborhood } U \text{ of } x\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N\ x_n \in U$$

For metric spaces

$$\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N\ |x_n - x| < \epsilon$$

Definition. ***(Separated)*** *Let $x, y \in X$, x and y can be separated if each lies in a neighborhood which does not contain the other point. (neighborhood not necessarily disjoint)*

Definition. ***(T_1 Space)*** *X is T_1 if any two distinct points in X are separated.*

- *(theorem) Let $A \subset X$ T_1. $x \in A'$ if and only if every neighborhood of x contains infinitely many points of A.*
- *(theorem) Every finite point set, specifically one-point set, in a T_1 space is closed*

Definition. ***(T_2 Hausdorff Space)*** *A topological space X is called Hausdorff space if for each pair x_1, x_2 of distinct points of X, there exists neighborhoods U_1 and U_2 of x_1 and x_2, respectively that are disjoint.*

$$\forall x \neq y \in X\ \exists \text{ neighborhoods } U, V \text{ of } x \text{ and } y \text{ respectively s.t. } U \cap V = \emptyset$$

- ***(motivation)*** *Generally, one point set not always closed; Sequences converges can converge to more than limit.*

- *(theorem) Every finite point set, specifically one-point set, in a Hausdorff space is closed*
- *(theorem) If X is T_2, then a sequence of points of X converges to at most 1 point of X*
- *(theorem) Every simply ordered set is T_2 in the order topology. Product of two T_2 space is T_2; Subspace of a T_2 space is T_2 (order/product/subspace topology well behaved with T_2)*
- *(examples)*
  - $\mathbb{R}^n_{std}$, X_{disc} *are* T_2
  - X_{triv} *not* T_2 *except when* $|X_{triv}| = 1$
  - $X_{f.c.}$ *not* T_2 *when* X *is infinite (since any* $x, y \in X_{f.c.}$ *are infinite and intersects)*

3.0.7 18 Continuous Functions

Definition. ***(Continuous)*** *A function $f : X \to Y$ is continuous if each open subset $V \subset Y$, the set $f^{-1}(V)$ is open. Alternatively formulated with basis, f is continuous if every basis element $B \in \mathcal{B}$, $f^{-1}(B)$ is open*

- *$id : \mathbb{R}_{std} \to \mathbb{R}_{l.l}$ is not continuous; $id : \mathbb{R}_{l.l.} \to \mathbb{R}_{std}$ is continuous*

Theorem. ***(TFAE for continuous function)*** *$f : X \to Y$*

1. *f is continuous*
2. *For every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$ (convergence: $x_n \to x \Rightarrow f(x_n) \to f(x)$ to $x \in \overline{A} \Rightarrow f(x) \in \overline{f(A)}$)*
3. *For every closed set B of Y, the set $f^{-1}(B)$ is closed*
4. *(generalized ϵ-δ) $\forall x \in X$ and $\forall$ neighborhood V of $f(x)$, $\exists$ neighborhood U of x such that $f(U) \subset V$*

- *In metric space, 4 can be reformulated with ϵ-δ definition. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if*

$$\begin{aligned} \forall x_0 \in \mathbb{R}^n, \quad \forall \epsilon > 0 \quad \exists \delta > 0 \ s.t.\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon \\ i.e.\ (x \in B_\delta(x_0) \Rightarrow f(x) \in B_\epsilon(f(x_0))) \end{aligned}$$

Definition. ***(Homeomorphism)*** *Let $f : X \to Y$ be a bijection If f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a homeomorphism. (continuous bijection)*

- *(theorem) If $X \cong Y$, then X and Y share topological property, i.e. property expressed in terms of topology only.*
- *(example) $(-1, 1) \cong \mathbb{R}$ since $f : (0, 1) \to \mathbb{R}$ by $f(x) = \tan(x)$ is a homeomorphism*
- *(example) A function can be continuous but not homeomorphic. Consider $S^1 = \{x \times y \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$ the unit circle. Let $f : [0, 1) \to S^1$ by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. f is bijective and continuous but f^{-1} not continuous*
- *If $\mathcal{T}_1 \subset \mathcal{T}_2$, then $Id : (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ is continuous*
- *If $\mathcal{T}_1 = \mathcal{T}_2$, then $Id : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is a homeomorphism*

Definition. ***(Imbedding)*** *An injective map $f : X \to Y$ is a topological imbedding of X in Y if $f' : X \to Z$ is a homeomorphism (note the image set $Z = f(X)$ carries subspace topology inherited from Y)*

- *Intuitively, an imbedding $f : X \to Y$ let us treat X as a subspace of Y.*
- *(theorem) Every map that is injective, continuous, and either open or closed is am imbedding.*
- *(example) $f : [0, 1) \to \mathbb{R}$ by $f(t) = (\cos(2\pi t), \sin(2\pi t))$ maps to S^1. f is a continuous injective map but not am imbedding.*

Theorem. ***(Constructing Continuous Functions)*** *Given X, Y, Z*

1. *(constant function) If $f : X \to Y$ by $f(X) = \{y_0\}$, then f is continuous*
2. *(inclusion) If $A \subset X$ with subspace topology, inclusion $i_A : A \hookrightarrow X$ is continuous*
3. *(composition) If $f : X \to Y$ and $g : Y \to Z$ are continous, then $g \circ f : X \to Z$ is continuous*
4. *(restricting domain) If $f : X \to Y$ is continous and $A \subset X$, then $f|_A : A \to Y$ is continuous ($f|_A = f \circ i_A$)*
5. *(restricting or expanding range) Let $f : X \to Y$ continuous. If $f(X) \subset A \subset Y \subset B$, then $g : X \to A$ obtained by restricting range of f is continuous. $h : X \to B$ obtained by expanding range of f also continuous ($h = i_Y \circ f$)*

6. *(local formulation of continuity)* $f : X \to Y$ is continuous if $X = \cup_{\alpha \in I} U_\alpha$ such that $f|_{U_\alpha}$ is continuous for each $\alpha \in I$

7. *(map to products)* Let $f : A \to X \times Y$ given by $f(t) = (f_1(t), f_2(t))$ and $f_1 : A \to X$, $f_2 : A \to Y$. Then f continuous if and only if f_1 and f_2 are continuous

8. *(Algebraic Operations)* If $f, g : X \to \mathbb{R}$ continous, then $f + g$, $f - g$, $f \cdot g$, f/g $(g(x) \neq 0)$ all continous

9. *(Uniform Limit Theorem)* If a sequence of continuous real-valued function of a real variable converges ***uniformly*** to a limit function, then the limit function is continuous

Theorem. ***(the pasting lemma)*** *Let $X = A \cup B$, where A and B are both closed or both open in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, set $h : X \to Y$*

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

then h is continuous

- $h(x) = x$ for $x \leq 0$ and $h(x) = x/2$ for $x \geq 0$ is continous

3.0.8 18 The Product Topology

Definition. ***(J-tuple)*** *Let J be index set. Define **J-tuple** of elements of X be a function $\mathbf{x} : J \to X$. Given $\alpha \in J$, denote **αth coordinate** as $\mathbf{x}$ at α by x_α instead of $\mathbf{x}(\alpha)$. Denote $\mathbf{x}$ as $(x_\alpha)_{\alpha \in J}$. Denote the **set of all J-tuples** of elements of X by X^J*

Definition. ***(Cartesian Product)*** *Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets; Let $X = \cup_{\alpha \in J} A_\alpha$. The cartesian product of this indexed family is given by*

$$\prod_{\alpha \in J} A_\alpha = \{(x_\alpha)_{\alpha \in J} \in X \mid x_\alpha \in A_\alpha\} = \{\mathbf{x} : J \to \bigcup_{\alpha \in J} X \mid \forall \alpha \in J \; \mathbf{x}(\alpha) \in A_\alpha\}$$

is the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$. Equivalently, the set of all functions $\mathbf{x} : J \to \cup_{\alpha \in J} X$ such that $\mathbf{x}(\alpha) = A_\alpha$ for each $\alpha \in J$

Definition. ***(Projection)*** *The projection mapping associated with index β is defined by*

$$\pi_\beta = \prod_{\alpha \in J} X_\alpha \to X_\beta \qquad \pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta$$

Definition. ***(Box Topology)*** *Let $\{X_\alpha\}_{\alpha \in J}$ be indexed family of topological spaces. Basis*

$$\mathcal{B}_{box} = \left\{ \prod_{\alpha \in J} U_\alpha \mid U_\alpha \text{ open in } X_\alpha \; \forall \alpha \in J \right\}$$

generates box topology

- *(theorem)* Given basis $\mathcal{B}_\alpha$ for each X_α, $\prod_{\alpha \in J} \mathcal{B}_\alpha$ where $B_\alpha \in \mathcal{B}_\alpha$ is a basis for $\prod_{\alpha \in J} X_\alpha$

Definition. ***(Product Topology)*** *Let $\{X_\alpha\}_{\alpha \in J}$ be indexed family of topological spaces. Basis*

$$\begin{aligned} \mathcal{B}_{prod} &= \left\{ \pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n}) \mid U_{\beta_i} \in \mathcal{T}_{X_{\beta_i}} \; \beta_1, \cdots, \beta_n \in J \right\} \\ &= \left\{ \prod_{\alpha \in J} U_\alpha \mid \forall \alpha \in J \; U_\alpha \in \mathcal{T}_{X_\alpha} \text{ for almost all } \alpha \; U_\alpha = X_\alpha \right\} \end{aligned}$$

generates product topology

- *(theorem)* Let $f : A \to \prod_{\alpha \in J} X_\alpha$ defined by $f(a) = (f_\alpha(a))_{\alpha \in J}$ where $f_\alpha : A \to X_\alpha$ for each α, Then f is continuous if and only if each function f_α is continuous

- *(theorem)* For finite products, $\mathcal{T}_{box} = \mathcal{T}_{prod}$

- *(example where product topology works while box topology does not)* In $\mathbb{R}^\omega = \prod_{n \in \mathbb{Z}_+} \mathbb{R}$, countably infinite product of $\mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}^\omega$ by $f(t) = (t, t, \cdots)$. Each $f_\alpha = \pi_\alpha \circ f$ continuous so f continuous in product topology. However f not continuous in box topology. Consider $B = \{(-1/n, 1/n) \mid n \in \mathbb{Z}_+\} \in \mathcal{B}_{box}$ open,

  $$f^{-1}(B) = \{t \mid (t, t, \cdots) \in B\} = \bigcap_{n \in \mathbb{Z}_+} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$$

  not open in $\mathbb{R}$

Definition. ***(product topology by continuity of functions)*** *There is a unique topology $\mathcal{T}_{prod}$ on $X = \prod_{\alpha \in J} X_\alpha$ with*

1. *$\pi_\beta : X \to X_\beta$ continuous for every $\beta \in J$*
2. *Given $f : Z \to \prod_{\alpha \in J} X_\alpha$. If $f_\alpha = \pi_\alpha \circ f$ is continuous for every α, then f is continuous*

Note $\mathcal{B}_{prod}$ satisfies the two condition

Proof. 1. by design. proof for 2 as follows

$$f^{-1}\left(\bigcap_{i=1}^{n} \pi_{\beta_i}^{-1}(U_{\beta_i})\right) = \{z \in Z \mid \forall i \ f_{\beta_i} \in U_{\beta_i}\} = \bigcap_{i=1}^{n} f_{\beta_i}^{-1}(U_{\beta_i})$$

is open as finite intersection of open sets □

Theorem. ***(Both box and product topology works well with subspace/T_2/closure)***

1. *Let $A_\alpha \subset X_\alpha$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$*
2. *If X_α is T_2, then $\prod X_\alpha$ is T_2*
3. *Let $A_\alpha \subset X_\alpha$, then $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$*

Proof. **proof of 2.** Let $\mathbf{x} = (x_\alpha)$ and $\mathbf{y} = (y_\alpha)$ where $\mathbf{x} \neq \mathbf{y}$. So exists $\beta \in J$ such that $x_\beta \neq y_\beta$. Since X_β is T_2, exsits neighborhoods U_β, V_β for x_β and y_β s.t. $U_\beta \cap V_\beta = \emptyset$. Note $\pi^{-1}(U_\beta)$ and $\pi^{-1}(V_\beta)$ are disjoint neighborhoods of $\mathbf{x}, \mathbf{y}$. **proof of 3.** Mainly use the definition of closure using neighborhoods. ($\Rightarrow$) Let $\mathbf{x} = (x_\alpha) \in \prod \overline{A_\alpha}$. Need to show $\mathbf{x} \in \overline{\prod A_\alpha}$. Let $U = \prod U_\alpha$ be basic neighborhood of $\mathbf{x}$ in $\mathcal{T}_{box}$ or $\mathcal{T}_{prod}$. For each α, since $x_\alpha \in A_\alpha$, can find $y_\alpha \in U_\alpha \cap A_\alpha$. Note $\mathbf{y} = (y_\alpha) \in U \cap \prod A_\alpha$. Since U arbitrary,$\mathbf{x} \in \overline{\prod A_\alpha}$. ($\Leftarrow$) Let $\mathbf{x} = (x_\alpha) \in \overline{\prod A_\alpha}$. Want to show $x_\beta \in \overline{A_\beta}$ for all β. Let V_β be arbitrary neighborhood of x_β. Consider $\pi^{-1}(V_\beta)$, which is open in both $\mathcal{T}_{box}$ and $\mathcal{T}_{prod}$. By definition of closure, exsits $\mathbf{y} = (y_\alpha) \in \pi^{-1}(V_\beta) \cap \prod A_\alpha$. Hence $y_\beta \in V_\beta \cap A_\beta$. Hence, $x_\beta \in \overline{A_\beta}$ □

3.0.9 19 The Metric Topology

Definition. ***(Metric)*** *A metric on X is a function*

$$d : X \times X \to \mathbb{R}$$

with properties

1. *(non-negativity) $d(x, y) \geq 0$ for all $x, y \in X$, wth equality if $x = y$*
2. *(symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$*
3. *(triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$*

Definition. ***(ϵ-ball centered at x)***

$$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$$

Definition. ***(norm)*** *Given $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$, the norm of $\mathbf{x}$ defined by $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$.*

Definition. ***(Metric Topology)*** *Given X and a metric d, basis*

$$\mathcal{B}_d = \{B_d(x, \epsilon) \mid x \in X \ \epsilon > 0\}$$

generates the metric topology $\mathcal{T}_d$ induced by d. Therefore

$$\mathcal{T} = \{U \subset X \mid \forall x \in U \ \exists \epsilon > 0 \ B(x, \epsilon) \subset U\}$$

1. ***discrete metric*** *d_{disc}, defined by $d_{disc}(x, y) = 1$ if $x \neq y$ and $d_{disc}(x, y) = 0$ if $x = y$ induces $\mathcal{T}_{disc}$*
2. ***standard metric*** *on $\mathbb{R}$ defined by $d(x, y) = |x - y|$ induces $\mathcal{T}_{ord}$*
3. ***(diamond)*** *$d_1 = \sum_i |x_i - y_i|$*
4. ***euclidean metric (circle)*** *on $\mathbb{R}^n$, $d_2 = d(\mathbf{x}, \mathbf{y}) = \|x - y\| = \sqrt{(x_1 - y_2)^2 + \cdots + (x_n - y_n)^2}$*
5. ***square metric (square)*** *on $\mathbb{R}^n$, $d_\infty = \rho(\mathbf{x}, \mathbf{y}) = max\{|x_1 - y_1|, \cdots, |x_n - y_n|\}$ (furthest coordinate within ϵ)*
6. *(theorem) In $\mathbb{R}^n$, $\mathcal{T}_{d_1} = \mathcal{T}_{d_2} = \mathcal{T}_\infty$ induces same topology $\mathcal{T}_{std}$ (proof by showing basis elements nests!)*
7. *(theorem) If X is metrizable, then X is T_2*
8. *(theorem) Subspaces of metric space behaves well $d|_{A \times A}$ induces subspace topology for $A \subset X$.*

Definition. ***(Metrizable)*** *If X is topological space, X is **metrizable** if there exists metric d that induces the topology of X. A **metric space** is a metrizable space X together with a specific metric d that gives the topology of X*

- *Metrizability is a topological property*
- *(not every topology comes with a metric) consider X_{triv} where $|X| \geq 2$, X not metrizable since it's not T_2*
- *$\mathbb{R}^n$ is metrizable (d_1, d_2, d_∞ induces $\mathbb{R}^n_{std}$)*
- *$\mathbb{R}^\omega$ is metrizable under product topology but not box topology*
- *$\mathbb{R}^J$ where J uncountable is not metrizable*
- *(theorem) countable products of metrizable spaces is metrizable*
- *(by sequence lemma) sequences are sufficient to describe metrizable spaces*

Definition. ***(Bounded and Diameter)*** *Given (X, d), $A \subset X$ is **bounded** if there is some M such that*

$$d(a_1, a_2) \leq M$$

*for all $a_1, a_2 \in A$. The **diameter** of A is defined*

$$diam\,(A, d) = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$$

- *boundedness is not a topological property, since it depends on a specific d (consider d and $\overline{d}$)*

Definition. ***(Standard Bounded Metric)*** *Given (X, d), define $\overline{d} : X \times X \to \mathbb{R}$ by*

$$\overline{d}(x, y) = \min\{d(x, y), 1\}$$

to be the standard bounded metric corresponding to d

- *(theorem) d and $\overline{d}$ induces the same topology, i.e. $\mathcal{T}_d = \mathcal{T}_{\overline{d}}$*
- *(trick) by above theorem, we can say $\underline{diam(A) \leq 1 \text{ without loss of generality}}$ by replacing d with $\overline{d}$*

Definition. ***(Uniform Metric and Uniform Topology)*** *Generalize square metric to $\mathbb{R}^J$. Given $\mathbf{x} = (x_\alpha)_{\alpha \in J}$, $\mathbf{y} = (y_\alpha)_{\alpha \in J} \in \mathbb{R}^J$. Define metric $\overline{\rho}$ by*

$$\overline{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\overline{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$$

*is called **uniform metric** on $\mathbb{R}^J$ inducing **uniform topology**.*

Theorem. ***(Relationship of Topologies on $\mathbb{R}^J$)***

$$\mathcal{T}_{prod} \subset \mathcal{T}_{uniform} \subset \mathcal{T}_{box}$$

where the three topologies are all different if J is infinite.

Theorem. ***(Metric inducing product topology)*** *If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^\omega$, define*

$$D(x, y) = \sup\left\{\frac{\overline{d}(x_i, y_i)}{i}\right\}$$

is a metric that induces product topology on $\mathbb{R}^\omega$

Proof. Show D is a metric. Then show D gives product topology. Now show $\mathcal{T}_D = \mathcal{T}_{prod}$. To show $\mathcal{T}_{prod}$ is finer, show there exists basis element in product topology that contains in a basis element of the metric topology. Let $B_D(\mathbf{x}, \epsilon)$ be basic neighborhood of $\mathbf{x}$. Let N be such that $1/N < \epsilon$. Consider $V \subset \mathcal{T}_{prod}$ defined as

$$V = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \mathbb{R} \cdots$$

Show $V \subset B_D(\mathbf{x}, \epsilon)$. Note for any $\mathbf{y} \in \mathbb{R}^\omega$, $\overline{d}(x_i, y_i)/i \leq 1/N$ for $i \geq N$. Therefore,

$$D(\mathbf{x}, \mathbf{y}) = \sup\left\{\frac{\overline{d}(x_1, y_1)}{1}, \cdots, \frac{\overline{d}(x_N, y_N)}{N}, \frac{1}{N}\right\}$$

If $\mathbf{y} \in V$, then $\overline{d}(x_i, y_i) < \epsilon$ for all $i < N$. So $D(\mathbf{x}, \mathbf{y}) < \epsilon$. Hence $V \subset B_D(\mathbf{x}, \epsilon)$. Conversely, want to show $\mathcal{T}_D$ is finer. The key here is recognize that product topology $\prod U_i$ where each component is metrizable and induced by d. Let $U = \prod_{i \in I} U_i \times \prod_{i \notin I} \mathbb{R}$ be a basis element in $\mathcal{T}_{prod}$ where I is finite. Let $\mathbf{x} \in U$, want to find a basic neighborhood $V \in \mathcal{T}_D$ such that $\mathbf{x} \in V \subset U$. For each $i \in I$, find an interval $(x_i - \epsilon_i, x_i + \epsilon_i)$ such that $i\epsilon < \epsilon_i$ and $\epsilon \leq 1$. We can achieve this by setting $\epsilon = \min\{\epsilon_i/i \mid i \in I\}$. Now we claim that $\mathbf{x} \in B_D(\mathbf{x}, \epsilon) \subset U$. Let $\mathbf{y} \in B_D(\mathbf{x}, \epsilon)$, then for all i,

$$\frac{\overline{d}(x_i, y_i)}{i} \leq D(\mathbf{x}, \mathbf{y}) < \epsilon$$

We need to show that $\mathbf{y} \in U$. We only care about $i \in I$ since $y_i \in \mathbb{R}$ for all $i \notin I$. When $i \in I$, $\overline{d}(x_i, y_i) < i\epsilon < \epsilon_i \leq 1$. Therefore, $d(x_i, y_i) < \epsilon_i$ implying $\mathbf{y} \in U$ $\square$

3.0.10 21 The Metric Topology Continued

Definition. ***(Continuity in Metric Spaces)*** *Let $f : X \to Y$ where (X, d_X) and (Y, d_Y) are metrizable. Then f continuous if and only if $\forall x \in X$ and $\forall \epsilon > 0$, $\exists \delta > 0$ such that*

$$d_X(x, y) < \delta \;\Rightarrow\; d_Y(f(x), f(y)) < \epsilon$$

Definition. ***(almost always)*** *means all but finitely many*

Definition. ***(Convergence)*** *$(x_n) \to x$ if for all neighborhood U of x, almost always $x_n \in U$.*

Definition. ***(Uniform Convergence)*** *Let $f_n : X \to Y$ be a sequence of functions where Y is metrizable. Let d be metric for Y. The sequence (f_n)* ***converges uniformly*** *to the function $f : X \to Y$ if given $\epsilon > 0$, there exists $N > 0$ such that*

$$d(f_n(x), f(x)) < \epsilon$$

for all $n > N$ and all $x \in X$.

- *depends on $\mathcal{T}_Y$ and metric d*
- *stronger than point-wise convergence*
- *(uniform limite theorem) Given $f_n : X \to Y$ where Y metrizable. If (f_n) converges uniformly to f, then f is continous.*

Definition. ***(Sequantial Closure)*** *Given $A \subset X$, sequential closure is given by*

$$seq\text{-}cl(A) = \{x \in X \mid \exists\, (x_n) \to x \; x_n \in A\}$$

- *(fact) $A \subset seq\text{-}cl(A) \subset \overline{A}$*
- *(the sequence lemma) $seq\text{-}cl(A) \subset cl(A)$ and with equality if X is metrizable.*

Lemma. ***(The sequence lemma)*** *Let X be a topological space and let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \overline{A}$. The converse holds if X is metrizable.*

Proof. $(\Rightarrow)$ If $x_n \to x$ where $x_n \in A$. Let U be any neighborhood of x. By definition of convergent sequence, $\exists N$ such that $\forall n \geq N$, $x \in U$. Since U arbitrary, every neighborhood of x contains some $x_n \in A$, hence $x \in \overline{A}$. Conversely, let d be metric inducing topology of X. Consider neighborhood $B_d(x, 1/n)$ for all $n \in \mathbb{Z}_+$, pick n-th term for the sequence as $x_n \in B_d(x, 1/n) \cap A$. We claim that $(x_n)_{n \in \mathbb{Z}_+}$ is convergent. Indeed, take $B_d(x, \epsilon)$ be arbitrary basic neighborhood of x, take $N > 0$ such that $1/N < \epsilon$. Therefore, for all $i \geq N$, $x_i \in B_d(x, \epsilon)$ by construction. □

Theorem. ***(Sequence Continuity)*** *Let $f : X \to Y$. If f continuous, then every convergent sequence $x_n \to x$ in X, the sequence $f(x_n) \to f(x)$. The converse is true if X is metrizable.*

Definition. ***(First countability axiom)*** *A space X that has a countable basis at each point satisfies first countability axiom. A space X said to have* ***countable basis at the point*** *x if there is a countable collection $\{U_n\}_{n \in \mathbb{Z}_+}$ of neighborhoods of x such that any neighborhood U of x contains at least one of the sets U_n.*

- *used to prove the above lemma/theorem; metrizability is not necessary.*

Definition. ***Spaces that are not metrizable***

1. *$\mathbb{R}^\omega$ in box topology is not metrizable*
2. *$\mathbb{R}^J$ where J uncountable is not metrizable in product topology*

Proof. Generally, to show that a space X is not metrizable, we can show that the space does not satisfy the sequence lemma. Specifically, if X is metrizable, then $seq\text{-}cl(A) = cl(A)$, to prove X not metrizable, find $A \subset X$ and $x \in X$ such that $x \in cl(A)$ and $x \notin seq\text{-}cl(A)$ **(Point 1)** Consider $A \subset \mathbb{R}^\omega$ be points with only positive coordinate values, i.e $A = \{(x_1, x_2, \cdots) \mid x_i > 0 \; i \in \mathbb{Z}_+\}$. Now we claim that $\mathbf{0} = (0, 0, \cdots)$ is in $cl(A)$ but not $seq\text{-}cl(A)$. $\mathbf{0} \in cl(A)$: any neighborhood $B = (a_1, b_1) \times (a_2, b_2) \times \cdots$ of $\mathbf{0}$ intersects A at point $({}^1/_2 b_1, {}^1/_2 b_2, \cdots)$. Now we show there is no sequence of A converging to $\mathbf{0}$. Assume for contradiction that $(\mathbf{x}_n) \to \mathbf{0}$ where $\mathbf{x}_n = (x_{n,k})_{k=1}^\infty$. Let $U = \prod_n (-1, x_{n,n}) \subset \mathcal{T}_{box}$ be a neighborhood of $\mathbf{0}$. However not almost always $\mathbf{x}_n \in U$: in fact U contains no elements of the sequence $(\mathbf{x}_n)$ since n-th coordinate $x_{n,n} \notin (-1, x_{n,n})$. Contradicts assumption that $(\mathbf{x}_n) \to \mathbf{0}$. Therefore $\mathbf{0} \notin seq\text{-}cl(A)$. **(Point 2)** Let $A = \{(x_\alpha) \mid x_\alpha = 1$ except for finitely many $\alpha \in I\}$. Let $\mathbf{0} \in \mathbb{R}^J$ be origin. Now we show $\mathbf{0} \in cl(A)$. Let $U = \prod U_\alpha$ be neighborhood of $\mathbf{0}$ where $U_\alpha \neq \mathbb{R}$ for all $\alpha \in \{\alpha_1, \cdots, \alpha_n\}$ in product topology. Now we show $U \cap A \neq \emptyset$. Consider $\mathbf{y} = (y_\alpha) \in \mathbb{R}^J$ where $y_\alpha = 0$ for all $\alpha \in \{\alpha_1, \cdots, \alpha_n\}$ and $y_\alpha = 1$ otherwise. Note $\mathbf{y} \in A$ since all but finitely many y_α is 1; $\mathbf{y} \in U$ since $y_\alpha \in U_\alpha$ for $\alpha_1, \cdots, \alpha_n$ and $y_\alpha \in \mathbb{R} = U_\alpha$ otherwise. Hence $\mathbf{y} \in U \cap A$ and therefore $\mathbf{0} \in cl(A)$. Now we show $\mathbf{0} \notin seq\text{-}cl(A)$. Let $\mathbf{a}_n$ be a sequence of A. Let $J_n \subset J$ such that $\mathbf{a}_n(\alpha) \neq 1$ for all $\alpha \in J_n$. J finite by of A. Let $J' = \cup_{n \in \mathbb{Z}_+} J_n$. Note J' is a countable union of finite set and hence countable. Since J uncountable, exists $\beta \in J$ such that $\beta \notin J'$ and therefore every point in the sequence $\mathbf{a}_n(\beta) = 1$ for all $n \in \mathbb{Z}_+$. Let $U_\beta = (-1, 1) \in \mathbb{R}$. Consider a neighborhood $U = \pi^{-1}(U_\beta) \subset R^J$ of $\mathbf{0}$ that contains no points in $\mathbf{a}_n$. Therefore $\mathbf{a}_n$ does not converge to $\mathbf{0}$. Therefore $\mathbf{0} \notin seq\text{-}cl(A)$. □

3.0.11 22 The Quotient Topology

Definition. ***(Quotient Map)*** *Given X, Y and $p : X \to Y$ be a surjective map. p is a quotient map if a subset $U \subset Y$ is open if and only if $p^{-1}(U)$ is open in X.*

- *(theorem) p is a surjective continuous map that is either open or closed, then p is a quotient map*
- *(example) projection maps $\pi_1 : X \times Y \to X$ is surjective, continuous, open and therefore a quotient map. However π_1 is not a closed map (since $\pi_1(\{x \times y \mid xy = 1\}) = \mathbb{R} - \{0\}$ not closed)*

Definition. ***(Quotient Topology and Quotient Space)*** *Let X be a space and A be a set. If $p : X \to A$ is a surjective map, then there uniquely exists one topology $\mathcal{T}$ on A relative to which p is a quotient map; $\mathcal{T}$ is called the **quotient topology** induced by p.*

$$\mathcal{T} = \{U \subset Y \mid p^{-1}(U) \in \mathcal{T}_X\}$$

*As a special case. Givecn $\sim$ be an equivalence relation on X and $Y = {}^{X}/_{\sim} = \{\{y : y \sim x_0\} \mid x_0 \in X\} \subset \mathcal{P}(X)$ are equivalence classes of X. Then $p : X \to Y$ exists and is a surjection. The space Y with the quotient topology is called a **quotient space** of X.*

Definition. ***define quotient topology with continuity of functions*** *Given topological space X and $\pi : X \to Y$ a surjection, there is a unique topology on Y satisfying*

1. *$\pi : X \to Y$ continuous*
2. *If Z is a topological space, $g : Y \to Z$ is a function. If $g \circ \pi$ is continuous, then g is continuous.*

4 Connectedness and Compactness

theorems about continuous functions

- (intermediate value theorem) If $f : [a, b] \to \mathbb{R}$ continuous and $f(a) < r < f(b)$, then exists $c \in [a, b]$ s.t. $f(c) = r$
- (maximum value theorem) If $f : [a, b] \to \mathbb{R}$ continuous then exists $c \in [a, b]$ such that $f(x) \leq f(c)$ for all $x \in [a, b]$
- (uniform continuity theorem) If $f : [a, b] \to \mathbb{R}$ continuous, then given $\epsilon > 0$ exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \epsilon$ for every pair $x_1, x_2 \in [a, b]$ for which $|x_1 - x_2| < \delta$

4.0.1 23 Connected Spaces

Definition. ***(Separation and Connected)*** *Given topological space X. A* ***separation*** *of X is a pair U, V where $U \cap V = \emptyset$ and $X = U \cup V$ and U, V both nonempty. The space X is* ***connected*** *if there does not exists a separation of X. Equivalently, X is connected if the only clopen sets are $\emptyset$ and X.*

- *(fact) connectedness is a topological property*

Definition. ***(Separation and Connected for subspaces)*** *Given $Y \subset X$. A* ***separation*** *of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other (i.e. $cl_Y(A) \cap B = \overline{A} \cap B = \emptyset$). The space Y is connected if there exits no separation of Y*

Proof. Suppose A, B forms a seperation of Y. Then $cl_Y(A) = \overline{A} \cap Y$. Since A closed in Y, $A = cl_Y(A) = \overline{A} \cap Y$. Since A, B disjoint, $\overline{A} \cap B = \emptyset$. Since $\overline{A}$ contains all its limit point, B has no limit points of A. Conversely, given assumption we have $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$. Hence $\overline{A} \cap Y = A$ and $\overline{B} \cap Y = B$. Then A, B are closed in Y. Since A, B partitions Y, A, B are open in Y as well. □

- *(examples)*
  - *$\{a, b\}$ with $\mathcal{T}_{triv}$ is connected*
  - *$Y = [-1, 0) \cup (0, 1] \subset \mathbb{R}$ is connected since $[-1, 0)$ and $(0, 1]$ is a seperation. (0 is a limit point to both, but does not matter since 0 is not contained in $[-1, 0)$ or $(0, 1]$)*
  - *$X = [-1, 1] \subset \mathbb{R}$. $[-1, 0]$ and $(0, 1]$ is not a seperation (0 is a limit point of $(0, 1]$ but contained in $[-1, 0]$)*
  - *$\mathbb{Q}$ not connected. Only one point subsets of $\mathbb{Q}$ are connected.*
  - *$X = \{(x, y) \mid y = 0\} \cup \{(x, y) \mid y = 1/x\} \subset \mathbb{R}^2$ not connected (neither contain a limit point of each other)*
- *(lemma) If C, D forms a separation of X and Y is a connected subspace of X, then Y lies entirely in C or D*
- *(theorem) Union of connected subspaces of X with a commont point is connected ($\cap A_\alpha \neq \emptyset$ where A_α connected for all α then $\cup A_\alpha$ connected)*
- *(theorem) If $A \subset X$ be a connected subspace. If $A \subset B \subset \overline{A}$, then B also connected*
- *(theorem) Image of a connected space under a continuous map is connected*
- *(theorem) Finite product $\prod X_\alpha$ connected if and only if X_α connected for all α*

Definition. ***(Connectedness for Infinite Products)***

1. *$\mathbb{R}^\omega$ is not connected in box topology*
2. *$\mathbb{R}^\omega$ is connected in product topology*

Proof. **(Point 1)** Enough to find a separation of $\mathbb{R}^\omega$. Interprete $\mathbb{R}^\omega$ as the collection of all real numbered sequences and is partitioned by the set of all bounded sequences of real numbers A and the set of all unbounded sequences of real numbers B. Now we show A, B both open. Consider $\mathbf{a} \in \mathbb{R}^\omega$, we can find a neighborhood of $\mathbf{a}$ in the box topology by $U = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \cdots$. If $\mathbf{a}$ is bounded, U consists of only bounded sequencs so $\mathbf{a} \in U \subset A$. If $\mathbf{a}$ is unbounded, then U consists of only unbounded sequences and $\mathbf{a} \in U \subset B$. Therefore $\mathbb{R}^\omega$ not connected in box topology. **(Point 2)** To show $\mathbb{R}^\omega$ in product topology is connected, we find some connected $C \subset \mathbb{R}^\omega$ where $C \subset \mathbb{R}^\omega \subset \overline{C}$ and use the lemma to show that $\mathbb{R}^\omega$ is connected. Consider $\tilde{\mathbb{R}}^n \subset \mathbb{R}^\omega$ defined to be the set of all sequences fixed to 0 beyond n: $\mathbf{x} \in \mathbb{R}^\omega$ such that $x_i > 0$ for all $i > n$. Since $\tilde{\mathbb{R}}^n \cong \mathbb{R}^n$ and $\mathbb{R}^n$ is connected, $\tilde{\mathbb{R}}^n$ is connected. Since each $\tilde{\mathbb{R}}^n$ is connected and $\cap_n \tilde{\mathbb{R}}^n = \{0, 0, \cdots\} = \mathbf{0}$, we have $\mathbb{R}^\infty = \cup_n \tilde{\mathbb{R}}^n$ connected. To complete the proof, we show $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$. Consider $\mathbf{a} = (a_1, a_2, \cdots) \in \mathbb{R}^\omega$ and $U = \prod U_\alpha$ be neighborhood of $\mathbf{a}$ in box topology. We show $U \cap \mathbb{R}^\infty \neq \emptyset$. Let N be such that $U_i = \mathbb{R}$ for all $i > N$. Consider $\mathbf{x} = \{a_1, a_2, \cdots, a_N, 0, 0, \cdots\} \in \mathbb{R}^\infty$. $\mathbf{x} \in U$ since $x_i \in U_i$ for all $i \leq N$ and $x_i \in \mathbb{R} = U_i$ for all $i > N$. Therefore $\mathbf{x} \in U \cap \mathbb{R}^\infty$ hence $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$ □

4.0.2 24 Connected Subspaces of the Real Line

Definition. *(convex)* $Y \subset X$ *is convex if every* $a < b \in Y$, $[a, b] \in Y$

Definition. *(Linear Continuum) A simply ordered set* L *having more than 1 eleent is called a linear continuum if*

1. *L has least upper bound property*
2. *If* $x < y$*, there exists* z *such that* $x < z < y$

- *(fact) condition for connectedness to hold on* $\mathbb{R}$
- *(theorem) If* L *is linear continuum in order topology, then* L *is connected, and so are intervals and rays in* L
- *(theorem) If* $Y \subset \mathbb{R}$*, then* Y *is connected if and only if* Y *is convex and nonempty*
- *(corollary)* $\mathbb{R}$*, intervals and rays in* $\mathbb{R}$ *are all connected*

Theorem. *(Unit interval in* $I = [0, 1] \subset \mathbb{R}$ *is connected)*

Proof. Idea is to show any separation (A, A^c) gives $A = I$. Let $A \subset I$ be clopen. without loss of generality let $0 \in A$. Define $G = \{x \in I \mid [0, x] \in A\} \subset A$ and $g = \sup G$. Goal is to show $1 = g \in G$ such that $A = G$ and therefore $I - A = \emptyset$ which contradicts assumption of separation. **First** we show $g > 0$, note $0 \in A$ where A is open, hence $[0, \epsilon) \in A$ and therefore $\epsilon/2 \in G$. So $g = \sup G \geq \epsilon/2 > 0$. **Second** we show $g \not< 1$. We first show $g \in A$. Since $G \subset A$, we have $\overline{G} \subset \overline{A} = A$. Then $g = \sup G = \overline{G} \in A$. Hence $g \in A$. Since A open, can find $(g - \epsilon, g + \epsilon) \in A$. Easily, $[0, g + \epsilon/2] \in A$ and hence $g + \epsilon/2 \in G$ which contradicts $g = \sup G$. **Third** we show $g \in G$. Same as before we show $g \in A$. Can find open neighborhood $(1 - \epsilon, 1] \in A$. Easily $[0, 1] \in A$ hence $A = I$. To conclude (A, A^c) is not a separation. □

Theorem. *(Intermediate Value Theorem) Let* $f : X \to Y$ *be continuous map, where* X *is connected and* Y *is in order topology. If* $a, b \in X$ *and* $r \in Y$ *such that* $f(a) < r < f(b)$*, then there exists* $c \in X$ *such that* $f(c) = r$*.*

Proof. Let $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, \infty)$. Note $A \cap B = \emptyset$ and neither are empty since $f(a) \in A$ and $f(b) \in B$. Note $A, B \subset f(X)$ are open in the subspace topology by definition. If no $c \in X$ such that $f(c) = r$, then $f(X) = A \cup B$ then (A, B) constitutes a separation of $f(X)$, contradicting the fact that image of a connected space under a continuous map $f(X)$ is connected. □

Definition. *(Path Connectedness) Let* $x, y \in X$*, a* ***path*** *in* X *from* x *to* y *is a continuous map* $f : [a, b] \to X$ *such that* $f(a) = x$ *and* $f(b) = y$*. A space* X *is* ***path-connected*** *if every pair of points in* X *can be joined by a path.*

$$\forall x, y \in X \ \exists \text{ continuous } f : [0, 1] \to X \ f(0) = x \ f(1) = y$$

- *(theorem) continuous image of path-connected space is path-connected*
- *(theorem) path-connected space is connected. (converse not always true: see topologist's sine curve)*
- *(proposition) connectedness and path-connected subsets of* $\mathbb{R}$ *are the same*
- *(theorem) If* X_α *path-connected, then* $\prod X_\alpha$ *is also path-connected (in product topology)*
- *(example) unit ball* $B^n = \{\mathbf{x} \mid \|\mathbf{x}\| \leq 1\} \subset \mathbb{R}^n$ *is path-connected (*$f : [0, 1] \to \mathbb{R}^n \ t \to (1-t)\mathbf{x} + t\mathbf{y}$ *for any* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$*)*
- *(example) punctured euclidean space* $\mathbb{R} - \{0\}$ *is path-connected*
- *(example) unit sphere* $S^{n-1}\{\mathbf{x} \mid \|\mathbf{x}\| = 1\} \subset \mathbb{R}^n$ *is path-connected (*$g : \mathbb{R}^n \setminus \{0\} \to S^{n-1} \ \mathbf{x} \to \mathbf{x}/\|\mathbf{x}\|$ *is continuous)*
- *(example) Let* $S = \{x \times \sin(1/x) \mid 0 < x \leq 1\} \subset \mathbb{R}^2$*. The* ***topologist's sine curve*** $\overline{S}$ *is connected but not path-connected*
$$\overline{S} = (\{0\} \times [-1, 1]) \bigcup \{x \times \sin(1/x) \mid 0 < x \leq 1\} \subset \mathbb{R}^2$$

Theorem. *(Topologist's sine curve is connected but not path-connected.)*

Proof. Let $S' = (\{0\} \times [-1, 1])$ and let $S = \{x \times \sin(1/x) \mid 0 < x \leq 1\} \subset \mathbb{R}^2$ and hence $\overline{S} = S' \cup S$. Since S is connected and $\overline{S}$ is image of a connected set $(0, 1]$ under continuous map, $\overline{S}$ is also connected. Now we show $\overline{S}$ is not path-connected. Let $f : [a, c] \to \overline{S}$, $a < 0$ be a path connecting $(0, 0)$ and $(1, 0)$. Since S' is closed, $f^{-1}(S')$ is closed and has a largest element b. Therefore $f' : [b, c] \to \overline{S}$ where f' maps b to S' and rest of points to S. Replace $[b, c]$ with $[0, 1]$ for convenience. Let $f(t) = (x(t), y(t))$ which has to be continuous. Then $x(0) = 0$ and $x(t) > 0$ and $y(t) = \sin(1/x(t))$ for $t > 0$. There exists t_n a sequence such that $y(t_n) = (-1)^n$ does not converge, contradicting continuity of f. We construct t_n as follows. For each n, pick u in range $0 < u < x(1/n)$ such that $\sin(1/u) = (-1)^n$. Use intermediate value theorem to find t_n suc that $x(t_n) = u$. □

4.0.3 26 Compact Spaces

Definition 4.1. ***(Cover and Compact)*** *A collection $\mathcal{A} \subset \mathcal{P}(X)$ **cover** X, or be a **covering** of X, if $\cup_{A \in \mathcal{A}} A = X$. $\mathcal{A}$ is open cover, if it is a cover and all $A \in \mathcal{A}$ are open. A space X is said to be **compact** if every open cover $\mathcal{A}$ contains a finite subcollection that also covers X, i.e. if $\{A_\alpha\}$ is an open cover, exists $I = \{\alpha_1, \cdots, \alpha_n\}$ such that $\cup_{\alpha \in I} A_\alpha = X$*

- *(example)* $\mathbb{R}$ *not compact, since* $\mathcal{A} = \{(n, n+2) \mid n \in \mathbb{Z}_+\}$ *does not have a finite subcover*
- *(example)* $X = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\} \subset \mathbb{R}$ *is compact.*