1. **279 #15** Let $\underset{n\times m}{A}$ be such that

$$x'(t) = Ax(t) \qquad x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad x'(t) = \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}$$

Suppose A is diagonalizable with distinct eigenvalues of A, $\lambda_1, \cdots, \lambda_k$. Prove $x : \mathbb{R} \to \mathbb{R}^n$ is a solution to the system if and oly if x is of the form

$$x(t) = e^{\lambda_1 t} z_i + \cdots + e^{\lambda_k t} z_k \qquad \forall i \quad z_i \in E_{\lambda_i}$$

Use the result to prove that the set of solutions to the system is an n-dimensional real vector space.

Proof. Since A is diagonalizable, let v_i for $i = 1, \cdots, n$ be the corresponding eigenvector to $\lambda_1, \cdots, \lambda_k$. Let Q whose ith column is v_i, we have

$$Q^{-1}AQ = D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix}$$

Then $A = QDQ^{-1}$, so

$$x' = QDQ^{-1}x \qquad \to \qquad Q^{-1}x = DQ^{-1}x$$

Let $y : \mathbb{R} \to \mathbb{R}^n$ be defined by $y(t) = Q^{-1}y(t)$, so then $y' = Qx'$. Substitute to $Q^{-1}x = DQ^{-1}x$, we have

$$y' = Dy = \begin{pmatrix} \lambda_1 y_1(t) \\ \vdots \\ \lambda_n y_n(t) \end{pmatrix} \qquad \to \qquad y_i(t) = c_i e^{\lambda_i t} \quad \text{for some } c_i \in \mathbb{R}$$

$$x(t) = Qy(t) = c_i e^{\lambda_i}$$

$\square$