MAT237 notes

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Contents

1 Structures in $\mathbb{R}^n$

Definition 1.1. A **Vector Space** is a collection of objects called vectors, which may be added together and multiplied ("scaled") by numbers, called scalars in this context. The set V and the operations of addition and multiplication must adhere to a number of requirements called axioms. let u, v and w be arbitrary vectors in V, and a and b scalars in F. First of all $u + v \in V$ and $au \in V$ and

$$u + (v + w) = (u + v) + w \tag{1}$$
$$u + v = v + u \tag{2}$$
$$\exists 0 \in V, v + 0 = v \forall v \in V \tag{3}$$
$$\forall v, \exists - v, v + (-v) = 0 \tag{4}$$
$$a(bv) = (ab)v \tag{5}$$
$$1v = v \tag{6}$$
$$a(u + v) = au + av \tag{7}$$
$$(a + b)v = av + bv \tag{8}$$

Definition 1.2. The **Euclidean inner product**, or dot product given two vectors $x = (x_1, \ldots, x_n)$, and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, which are two equal-length sequences of numbers, and returns a single number.
Algebraically, it is the sum of the products of the corresponding entries of the two sequences of numbers.

$$\langle x, y \rangle = x \cdot y := \sum_{i=1}^{n} x_i y_i = x_1 y_2 + x_2 y_2 + \cdots + x_n y_n$$

Geometrically, it is the product of the Euclidean magnitudes of the two vectors and the cosine of the angle between them.

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos(\theta)$$

where $\|x\|$ is the length of x, $\|y\|$ is the length of y, θ is the angle between x, y.

Note.
When $a \cdot b = 0$, a and b are **orthogonal**.
When $a \cdot b = \|a\|\|b\|$, a and b are **co-directional**.

Here is a list of properties of the **inner product space**. Given $a, b, c \in V$ and $r \in \mathbb{R}$

$$a \cdot b = b \cdot a \tag{Commutative}$$
$$a \cdot (b + c) = a \cdot b + a \cdot c \tag{Distributive over vector addition}$$
$$a \cdot (rb + c) = r(a \cdot b) + (a \cdot c) \tag{Bilinear}$$
$$(c_1 a) \cdot (c_2 b) = c_1 c_2 (a \cdot b) \tag{Scalar multiplication}$$
$$\text{two non-zero vectors are orthogonal} \iff a \cdot b = 0 \tag{Orthogonality}$$
$$a \cdot b = a \cdot c \text{ does not imply } b = c \tag{No cancellation}$$
$$a \cdot b \geq 0 \text{ and is euqal to zero if and only if } x = 0 \tag{Non-negative}$$

Remark. Mostly proved by using algebraic definition of inner dot product

Definition 1.3. The scalar projection is

$$comp_b a = \frac{a \cdot b}{||b||}$$

Proof. $b \cdot (a - b) = 0$ because they are orthogonal to each other. Arrange and with Biliear property for inner dot product we arrive at $comp_b a = \dfrac{a \cdot b}{||b||}$ □

Then the projection of a into b is the scalar projection multiply by the unit vector $\dfrac{b}{||b||}$

$$proj_b a = \frac{\langle a, b \rangle}{||b||^2} b$$

Definition 1.4. Norm is a way of measuring the length of a vector. Here we define $|| \cdot || : \mathbb{R}^n \to \mathbb{R}$ as the function

$$||x|| := \sqrt{\langle x, x \rangle} = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} = \sqrt{(x_1^2 + x_2^2 + \cdots + x_n^2)}$$

Proof. Use the algebraic definition of inner product $\mathbf{x} \cdot x = ||x|| \; ||x|| \cos(0) = ||x||^2$ □

The **normed space** has the following properties. Let $x, y \in \mathbb{R}^n$ and $c \in R$,

$$||x|| \geq 0 \text{ with equality if and only if } x = 0 \tag{Non-degeneracy}$$
$$||cx|| = |c|||x|| \tag{Normality}$$
$$||x + y|| \leq ||x|| + ||y|| \tag{Triangle Inequality}$$
$$|\langle x, y \rangle| \leq ||x||||y|| \tag{Couchy Schwarz Inequality}$$

Remark. proofs for Couchy Schwarz Inequality can be derived from geometric definition of inner dot product on the condition that $cos(x) \leq 1$. Proofs for Triangle Inequality requires Couchy Schwarz Inequality.

Definition 1.5. **Metric** is a method for determining the distance between the two vectors.

$$d(x,y) = \|x - y\| = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} = \sqrt{(x_1 - x_2)^2 + \cdots + (x_n - y_n)^2}$$

The metric space satisfies the following properties. Let $x, y, z \in \mathbb{R}^n$

$$d(x,y) = d(y,x) \tag{Symmetry}$$

$$d(x,y) \geq 0 \text{ with equality if and only if } x = y \tag{Non-degeneracy}$$

$$d(x,z) \leq d(x,y) + d(y,z) \tag{Triangle Inequality}$$

Definition 1.6. In $\mathbb{R}^3$ the cross product of two vectors is a way of determining a third vector which is orthogonal ot the original two. If $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ then

$$v \times w = (v_2 w_3 - w_2 v_3, w_1 v_3 - v_1 w_3, v_1 w_2 - w_1 v_2))$$

or we can use determinants to solve $v \times w$. Here i, j, k represent standard unit vectors in R^3

$$v \times w = det \begin{pmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = i \begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix} - j \begin{pmatrix} v_1 & v_3 \\ w_1 & w_3 \end{pmatrix} + k \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}$$

Note. Determinants of $2X2$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad - bc$

2 Open, closed and everything in between

Definition 2.1. Let $\mathbb{X}$ be a big (topological space) set. Let $S \subseteq \mathbb{X}$. Let fixed $x \in \mathbb{X}$ and fixed $r \in \mathbb{R}, r > 0$

We define the open ball of radius r at the point x as

$$B = B_r(x) = \{y \in \mathbb{X} : d(x, y) < r\}$$

Remark. Here $d(x, y)$ means distance, i.e. $d(x, y) = \|x - y\|$

Definition 2.2. The boundary of the open ball, ∂B, is the set

$$\partial B = \{y \in \mathbb{X} : d(y, x) = r\}$$

Definition 2.3. The closed ball, denoted $\overline{B}$, is defined as

$$\overline{B} = B \cup \partial B \qquad = \{y \in \mathbb{X} : d(y, x) \leq r\} \tag{9}$$

Definition 2.4. A set $S \subseteq \mathbb{X}$ is bounded if there exists a big enough ball $B \in \mathbb{X}$ that contains S

Definition 2.5. Let $x \in \mathbb{X}$, $S \in \mathbb{X}$

1. x is said to be an interior point of S if $\exists$ an open ball: $B = B_r(x)$ such that $B \in S$
2. x is said to be a boundary point of S if for every open ball

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