Integration

Integration in $\mathbb{R}$

Definition. ***Partition*** *A finite partition P of $[a,b]$ is an ordered collection of points $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$. The **order of** P is defined to be $|P| = n$ (i.e. the number of subintervals) and the **length of** P is*

$$l(P) = \max_{i=i,\cdots,|P|}[x_i - x_{i-1}]$$

that is, the length of P is the length of the longest interval whose endpoints are in P. In other words, if $\mathcal{P}_{[a,b]}$ is the set of all finite partitions of $[a,b]$ then $l : \mathcal{P}_{[a,b]} \to \mathbb{R}_+$ gives the worst case scenario for width of subintervals.

Definition. ***Refinement*** *If P and Q are two partitions of $[a,b]$ then Q is the refinement of P if $P \subseteq Q$*

Remark. Any two partition $P, Q \in \mathcal{P}_{[a,b]}$ admits a common partition R, where $R = P \cup Q$

Definition. ***Riemann sum*** *Given a function $f : [a,b] \to \mathbb{R}$, a Riemann sum of f with respect to the partition $P = \{x_0 < x_1 < \cdots < x_{n-1} < x_n\}$ is any sum of the form*

$$S(f,P) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}), \qquad t_i \in [x_{i-1}, x_i]$$

By how we pick t_i we have

1. ***Left- and Right-endpoint Riemann sums***

$$L(f,P) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}) \qquad R(f,P) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

2. ***Lower and Upper Riemann sums*** *Fix partition $P \in \mathcal{P}_{[a,b]}$ and $f : [a,b] \to \mathbb{R}$. Define*

$$m_i = \inf_{x\in[x_{i-1},x_i]} f(x) \qquad M_i = \sup_{x\in[x_{i-1},x_i]} f(x)$$

so m_i is the smallest value f takes on $[x_{i-1}, x_i]$ while M_i is the largest

$$u(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \qquad U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

Proposition. *If Q is a refinement of P then we have*

$$u(f,P) \leq u(f,Q) \qquad U(f,P) \geq U(f,Q)$$

Intuitively, u is increasing function over refinement of P while U is decreasing for refinement of P

Lemma. *Let A, B be sets such that $A \subseteq B$ then if infimum and supremum exists, we have*

$$\inf A \geq \inf B \qquad \sup A \leq \sup B$$

Definition. *The **Lower and Upper Integral** is defined to be*

$$u(f) = \sup_P [u(f, P)] \qquad U(f) = \inf_P [U(f, P)]$$

In other words, the lower integral is the lower Riemann sum for sufficiently fine P; while the upper integral is the upper Riemann sum for sufficiently fine P.

Definition. ***Riemann Integrable*** *We say that a function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ with integral I if for every $\epsilon > 0$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $P \in \mathcal{P}_{[a,b]}$ satisfies $l(P) < \delta$, then*

$$|S(f, P) - I| < \epsilon$$

where I is denoted as $I = \displaystyle\int_a^b f(x)dx$.

Remark. Roughly, a function is Riemann integrable with integral I if we can approximate I arbitrarily well by taking a sufficiently fine partition P.

The following definition are equivalent

1. *f is Riemann integrable*
2. *$\displaystyle\sup_{P \in \mathcal{P}_{[a,b]}} u(f, P) = \inf_{P \in \mathcal{P}_{[a,b]}} U(f, P)$. In other word, the lower and upper integral are equal*
3. *For every $\epsilon > 0$ there exists a partition $P \in \mathcal{P}_{[a,b]}$ such that $U(f, P) - u(f, P) < \epsilon$ (Cauchy Criterion)*
4. *For every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $P, Q \in \mathcal{P}_{[a,b]}$ satisfy $l(P) < \delta$ and $l(Q) < \delta$ then $|S(f, P) - S(f, Q)| < \epsilon$*

Remark. To prove that a function is Riemann integrable, we use the third definition. Specifically, we pick a partition P with $|P| = n$ and show that the difference between upper and lower Riemann sums converges. To prove that a function is *not* Riemann integrable, the characteristic function of rationals on $[0, 1]$ is not integrable

$$\chi_Q(x) = \begin{cases} 1 & x \in Q \cap [0, 1] \\ 0 & otherwise \end{cases}$$

Since Q is dense in $[0, 1]$, then $M_i = 1$ and $m_i = 0$ and so

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = x_1 - x_0 = 1 \neq 0 = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = u(f, P)$$

holds for any partition P, any $\epsilon < 1$ fails the definition of integrability

Definition. ***Properties of Integral***

1. ***Additivity of Domain*** *If f is integrable on $[a,b]$ and $[b,c]$ then f is integrable on $[a,b]$ and*
$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$
2. ***Additivity of Integral*** *If f, g are integralon $[a,b]$ then $f+g$ also integrable on $[a,b]$ and*
$$\int_a^b f(x) + g(x)dx = \int_a^b g(x)dx + \int_a^b f(x)dx$$
3. ***Scalar Multiplication*** *If f integrable on $[a,b]$ and $c \in \mathbb{R}$ then cf is integrable on $[a,b]$*
$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$
4. ***Inherited Integrability*** *f is integrable on $[a,b]$ then f is integrable on any subinterval $[c,d] \subseteq [a,b]$*
5. ***Monotonicity of Integral*** *If f, g are integrable on $[a,b]$ and $f(x) \leq g(x)$ for all $x \in [a,b]$ then*
$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$
6. ***Subnormality*** *If f is integrable on $[a,b]$ then $|f|$ is integrable on $[a,b]$ and satisfies*
$$|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$$
7. *Let S_1, S_2 be two subsets. Let $S = S_1 \cup S_2$ and $T = S_1 \cap S_2$, Suppose f is integrable over S_1, S_2 then f is integrable over S and T, moreover*
$$\int_S f + \int_T f = \int_{S_1} f + \int_{S_2} f$$
8. *Let $S \subseteq \mathbb{R}^n$ Let $f, g : S \to \mathbb{R}$ Let $F(x) = max\{f(x), g(x)\}$ and $G(x) = min\{f(x), g(x)\}$ then*
   (a) *If f, g are continuous at x_0 then so are F and G*
   (b) *If f, g are integrable over S, so are F and G*

Definition. ***Fundamental Theorem of Calculus***

1. *If f is integrable on $[a, b]$ and $x \in [a, b]$ define $F(x) = \int_a^x f(t)dt$. The function F is continuous on $[a, b]$ and moreover, $F'(x)$ exists and equals $f(x)$ at every point x at which f is continuous*

2. *Let F be continuous function on $[a, b]$ that is differentiable except possibly at finitely many points in $[a, b]$ and take $f = F'$ at all such points. If f is integrable on $[a, b]$, then*
$$\int_a^b f(x)dx = F(b) - F(a)$$

Sufficient condition for integrability

Theorem. ***If f is bounded and monotone on $[a, b]$ then f is integrable***

Remark. It is easy to write down upper and lower Riemann sums for monotone functions. We select a partition P and prove that the upper and lower integral converges by bounding $l(P) \leq \delta$

Theorem. ***Every continous function on $[a, b]$ is integrable***

Proof. Cannot use the previous theorem because there are functions that is continuous on $[a, b]$ while not monotone, i.e. $\sin(\frac{1}{x})$ around origin. Use the fact that continuous function over a compact set is uniformly continuous. Given $\epsilon > 0$ we can find δ such that whenever $|x - y| < \delta$ we have $|f(x) - f(y)| < \frac{\epsilon}{b-a}$. So then we find a partition P such that $l(P) < \delta$ such that $M_i - m_i$ on every subinterval is bounded by $\frac{\epsilon}{b-a}$. We then have

$$\begin{aligned}
U(f, P) - u(f, P) &= \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^{n} \frac{\epsilon}{b-a}(x_i - x_{i-1}) \\
&= \frac{\epsilon}{b-a}\sum_{i=1}^{n}(x_i - x_{i-1}) \\
&= \frac{\epsilon}{b-a}(b-a) \\
&= \epsilon
\end{aligned}$$

□

Integral over a single point or any finite set of point is zero. Now we consider the integral infinitely many points with the idea of Jordan Measure.

Definition. ***Jordan Meansure*** *If $I = [a, b]$ let the length of I be $l(I) = b - a$. If $\mathcal{P}(\mathbb{R})$ is the power-set of $\mathbb{R}$, we define* ***Jordan outer measure*** *as the function $m : \mathcal{P} \to \mathbb{R}_{\geq 0}$ given by*

$$m(S) = \inf \left\{ \sum_{k=1}^{n} l(I_k) : S \subseteq \cup_{k=1}^{n} I_k \text{ where } I \text{ is an interval} \right\}$$

If $m(S)$ exists and $m(\partial S) = 0$, we say that S is ***Jordan Measurable****. (If countable cover is used instead we say that the set is Lebesque measurable / zero)*
If $m(S) = 0$ we say that S has ***Jordan Measure Zero****.*

Remark. For proofs of Jordan Measure zero, it suffices to show that for all $\epsilon > 0$, $\sum_{k=1}^{n} l(I_k) < \epsilon$ by definition of infimum. Some examples below,

1. Jordan measure of any finite set is 0
2. Jordan measure of any interval $[a, b]$ is $m([a, b]) = b - a$
3. $m(\mathbb{R})$ does not exist because no finite cover for $\mathbb{R}$
4. If $S = \mathbb{Q} \cap [0, 1]$, then $\partial S = [0, 1]$ and $m(\partial S) = 1 \neq 0$, hence not Jordan measurable

In essense, Jordan measure is an extension of the notion of size (length, area, volume) to shape more complicated than say triangle, rectangles... Let M be a bounded set in the plane, i.e., M is contained entirely within a rectangle. The outer Jordan measure of M is the greatest lower bound of the areas of the coverings of M, consisting of finite unions of rectangles.

Proposition. ***Content (Jordan) zero implies Measure (Lebesque) Zero; Converse true only if the the set is compact and measure zero***

Theorem. ***Bounded and continuous function on a compact interval with a Jordan Measure Zero set of discontinuities is integrable*** *If $S \subseteq [a, b]$ is a Jordan measure zero set, and $f : [a, b] \to \mathbb{R}$ is bounded and continuous everywhere except possibly at S, then f is integrable.*

Proof. Given Jordan measure zero, we can find a finite cover I_k over the set of discontinuities $W = \cup_j I_j$ such that $\sum_j l(I_j) < \frac{\epsilon}{2(M - m)}$. We denote the set $V = [a, b] \setminus W$. By the fact that f is continuous over a compact set V, we can find a partition P such that $U(f|_V, P) - u(f|_V, P) < \frac{\epsilon}{2}$. Refine P such that subintervals contain endpoints of I_k. Then we can bound $U(f|_W, P) - u(f_W, P)$ so together $U(f, P) - u(f, P) < \epsilon$ $\square$

Corollary. *If f, g are integrable on $[a, b]$ and $f = g$ up to a set of Jordan measure zero, then $\int_a^b f(x)dx = \int_a^b g(x)dx$*

4.2 Integration in $\mathbb{R}^n$

Definition.
*A **Rectangle** $R \in \mathbb{R}^2$ is any set which can be written as $[a,b] \times [c,d]$. A **Partition** $P = P_x \times P_y$ is a partition of R where $P_x = \{a = x_0 < \cdots < x_n = b\}$ and $P_y = \{c = y_0 < \cdots < y_m = d\}$ are partitions of their respective intervals $[a,b]$ and $[c,d]$ with **subrectangles***

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \quad x = 1, \cdots, n \quad y = 1, \cdots, m$$

*The **Area of rectangle** R_{ij} is given by*

$$A(R_{ij}) = (x_i - x_{i-1})(y_j - y_{j-1})$$

*in which case the **Riemann Sum** for $f : \mathbb{R}^2 \to \mathbb{R}$ over partition P is given by*

$$S(f,P) = \sum_{\substack{i=1,\cdots,n \\ j=1,\cdots,m}} f(t_{ij})A(R_{ij}) \qquad t_{ij} \in R_{ij}$$

*and the **upper and lower Riemann sum** are defined as*

$$U(f,P) = \sum_{\substack{i=1,\cdots,n \\ j=1,\cdots,m}} \sup_{x \in R_{ij}} f(x)A(R_{ij}) \qquad u(f,P) = \sum_{\substack{i=1,\cdots,n \\ j=1,\cdots,m}} \inf_{x \in R_{ij}} f(x)A(R_{ij})$$

*$f : \mathbb{R}^2 \to \mathbb{R}$ is **Riemann Integrable** if for any $\epsilon > 0$ there exists a partition P (i.e. exists δ such that $A(P) < \delta$ where $A(P) = \max\{A(R_{ij})\}$ for all subrectangles R_{ij} of P) such that*

$$U(f,P) - u(f,P) < \epsilon$$

*The **Integral** is given by*

$$\iint_R f dA \quad or \quad \iint f(x,y)dxdy$$

Theorem. ***Properties of double integrals***

1. ***Linearity of Integral*** *If f_1, f_2 are integrable on R and $c_1, c_2 \in \mathbb{R}$ then $c_1 f_c + c_2 f_2$ is integrable on S and*

$$\iint_R [c_1 f_1 + c_2 f_2] dA = c_1 \iint_R f_1 dA + c_2 \iint_R f_2 dA$$

2. ***Additivity of Domain*** *If f is integrable on disjoint rectangles R_1 and R_2 th en f is integrable on $R_1 \cup R_2$ and*

$$\iint_{R_1 \cup R_2} f dA = \iint_{R_1} f dA + \iint_{R_2} f dA$$

3. ***Monotonicity*** *If $f_1 \leq f_2$ are integrable functions on R then*

$$\iint_R f_1 dA \leq \iint_R f_2 dA$$

4. ***Subnormality*** *If f is integrable on R and $|f|$ is integrable on R then*

$$|\iint f dA| \leq \iint |f| dA$$

5. ***If f is continuous, then f is integrable***

Definition. ***Generalized Jordan outer measure*** *of a set $S \in \mathbb{R}^2$ is defined to be*

$$m(S) = \inf \left\{ \sum_{i,j}^{n} A(R_{ij}) : S \subseteq \cup_{i,j}^{n} R_{ij} \textit{ where } R_{ij} \textit{ is an rectangle} \right\}$$

*If $m(S)$ exists and $m(\partial S) = 0$, we say that S is **Jordan Measurable**.*
*If $m(S) = 0$ we say that S has **Jordan Measure Zero**.*

Remark. One can think of Jordan measure as the area, and zero-measure set as one that does not have any area. In $\mathbb{R}^n$, sets of any sub-dimension S has no volumne, hence $m(S) = 0$.

1. $B^2 = \{(x, y) : x^2 + y^2 \leq 1\}$ the unit disk has $m(B^2) = \pi$
2. $S = [0, 1] \times \{0\} \subseteq \mathbb{R}^2$ has zero Jordan measure

Theorem. ***sub-manifolds is Jordan measure zero*** *If $f : \mathbb{R} \to \mathbb{R}^2$ is of C^1, then for every interval $I \subseteq \mathbb{R}$ we have that $f(I)$ has zero content. In other words, the image of a continuous C^1 function (curve) has Jordan measure zero, (i.e. covered by finitely many rectangles).*

Definition. ***Piecewise C^1 function*** *A function $f : [a, b] \to \mathbb{R}^2$ is piecewise C^1 if it is C^1 at all but a finite number of points.*

Corollary. *Any set $S \subseteq \mathbb{R}^2$ such that ∂S is defined by a piecewise C^1 curve is Jordan measurable.*

Proof. By the previous theorem, ∂S as the image of a C^1 curve has Jordan measure zero. i.e. $m(\partial S) = 0$ Hence S is Jordan measurable. □

Theorem. ***If R is a rectangle and f is continuous on R up to a set of Jordan measure zero, then f is integrable.*** *If $S \subseteq R$ is Jordan measure zero set, and $f : R \to \mathbb{R}$ is continuous every where except possibly at S, then f is integrable.*

Remark. A generalization of a previous theorem where function is continuous up to a set of Jordan measure zero interval is integrable. However, we still want to know integrability over non-rectangles, whose condition is given by the next theorem.

Theorem. ***If S is Jordan measurable and the set of discontinuities of $f : S \to \mathbb{R}^2$ has zero measure then f is Riemann integrable***

Proof. Fix a rectangle R such that $S \subseteq R$. Extend $f : S \to \mathbb{R}^2$ with a characteristic function

$$\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & otherwise \end{cases}$$

Thus the function $f\chi_S : \mathbb{R} \to \mathbb{R}^2$ is just $f(x)$ on S and 0 everywhere else inside R. Let D be the set of discontinuities of f and note that the set of discontinuities of χ_S is given by ∂S. Then we have $m(D) = 0$ as given and $m(\partial S) = 0$ by the fact that S is Jordan measurable. Then the set of discontinuities of f_{χ_S} on R is $D \cup \partial S$. Since the union of zero measure sets has zero measure, i.e. $m(D \cup \partial S) = 0$. Since $f\chi_S$ has zero-measure discontinuities (while continuous elsewhere) on rectangle R and hence Riemann Integrable by previous theorem. □

Corollary. ***If $S \subseteq R^2$ is Jordan measurable then $m(S) = \int_S \chi_S$***