University of Toronto
Faculty of Arts and Science
Final Examinations, April-May 2015
MAT247HS – Algebra II
Instructor: Stephen S. Kudla
Duration – 3 hours
No aids allowed or needed

Please write clearly and show all of your work.
The point value of each problem is indicated.

1. (*40 points*) Let V, $<\,,\,>$ be an inner product space over $\mathbb{C}$. State and prove the Cauchy-Schwarz inequality.

2. (*40 points*) (i) Suppose that S and T are linear transformations of a vector space V such that $ST = TS$. Prove that T preserves the eigenspaces E_λ and the generalized eigenspaces K_λ of S.
(ii) Now suppose that V is a real vector space of dimension 4 and that the minimal polynomial of S is $(t-1)^2(t-2)^2$. What is the JCF of S?
(iii) Prove that the characteristic polynomial $P_T(t)$ of T splits completely and has at most two distinct roots.

3. (*40 points*) Consider the linear transformation $T = L_A$ of $\mathbb{C}^2$ defined by the matrix

$$A = \begin{pmatrix} 4 & 5i \\ -5i & 4 \end{pmatrix}.$$

Use the standard inner product on $\mathbb{C}^2$.
(i) Show that T is normal.
(ii) Find the eigenvalues λ_1 and λ_2 of T.
(iii) Find the eigenprojections T_i, $i = 1, 2$ onto the two eigenspaces of T.
(iv) Write T^{-1} as a linear combination of these eigenprojections.

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4. (*40 points*) (i) Find the Jordan Canonical Form (JCF) of the matrix

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix}.$$

(ii) Find a matrix Q such that $Q^{-1}AQ$ is in JCF.

5. (*60 points*) Suppose that $T : V \longrightarrow V$ is a linear transformation of a finite dimensional vector space V and that $W \subset V$ is a T-invariant subspace.
(i) Suppose that some vector $w \in W$ can be written as

$$w = v_1 + \cdots + v_k,$$

where $T(v_i) = \lambda_i v_i$ with $\lambda_i \neq \lambda_j$ for $i \neq j$. Show, by induction on k, that $v_i \in W$ for all i.
(ii) Suppose that T is diagonalizable. Prove that the restriction T_W of T to W is diagonalizable.

6. (*50 points*) Suppose that V, $\langle\, , \rangle$ is an inner product space and that $T : V \longrightarrow V$ is a linear transformation.
(i) Define what it means for T to be a projection.
(ii) Prove that if $T = T^2$, then T is a projection.
(iii) Define what it means for T to be an orthogonal projection.
(iv) Assume that V is finite dimensional. Prove that if $T^2 = T = T^*$, then T is an orthogonal projection.

7. (*30 points*) Suppose that a linear transformation $T : \mathbb{R}^6 \longrightarrow \mathbb{R}^6$ has characteristic polynomial $P_T(t) = (t^2 + 5)^2(t - 1)^2$.
(i) What are the possibilities for the minimal polynomial $M_T(t)$ of T?
(ii) What are the possibilities for the rational canonical form for T?

UNIVERSITY OF TORONTO
The Faculty of Arts and Science

APRIL 2015 EXAMINATIONS

MAT257Y1Y
Analysis II

Duration – 3 hours

NO AIDS ALLOWED

Last Name: ______________________

First Name: ______________________

Student Number: ______________________

Section: ______________________

Note: *there are 11 pages, excluding this cover page*

Note 2: *There are nine questions, all of equal value; do any seven.*
Total Marks Possible =70

Question	Mark
1	
2	
3	
4	
5	
6	
7	
8	
9	
TOTAL	

There are nine questions, all of equal value; do any seven. ***Show all your work.***

1. In each of the following, say whether the statement is true or false; if false, give a counterexample; if true, give a *brief* explanation for your answer.

   a) If $f : \mathbb{R}^k \to \mathbb{R}^n$ is continuous and $C \subset \mathbb{R}^k$ is compact, then $f(C)$ is compact.

   b) If $f : \mathbb{R}^k \to \mathbb{R}^n$ is continuous and $O \subset \mathbb{R}^k$ is open, then $f(O)$ is open.

   c) If $f : \mathbb{R}^k \to \mathbb{R}^n$ is continuous and $D \subset \mathbb{R}^n$ is compact, then $f^{-1}(D)$ is compact.

   d) If $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and one-to-one, then $Df(x)$ is invertible for all x.

   e) If $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and $Df(x)$ is invertible for all x, then $f(\mathbb{R}^n)$ is open.

2. Define $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x) = x$ and

$$g(x,y) = \begin{cases} y - x^2 & y \geq x^2 \\ \frac{y(y-x^2)}{x^2} & 0 \leq y < x^2 \\ -\frac{y(y+x^2)}{x^2} & -x^2 \leq y < 0 \\ y + x^2 & y < -x^2 \end{cases}$$

a) Show that F is differentiable at the origin.

b) Show that $DF(0,0)$ is nonsingular, but F is not $1:1$ in any neighbourhood of the origin.

c) Why does this not contradict the Inverse Function Theorem?

3. a) Show that the set of solutions to the equations

$$\begin{cases} \cos(x+y)e^z & = e \\ \sin(x-2y) + 2z - z^2 & = 1 \end{cases}$$

can be parametrized near $(x, y, z) = (0, 0, 1)$ by smooth functions $y = \phi(x)$, $z = \psi(x)$. Compute the derivatives $\phi'(0)$ and $\psi'(0)$.

b) Can you also solve for x, z in terms of y? For x, y in terms of z? Explain your reasons.

4. Evaluate $\int\int\int_R (y-x)\,dx\,dy\,dz$ where R is the tetrahedron with vertices $(0,0,0), (0,0,2), (1,1,1), (1,3,1)$. (Suggestion: find a linear transformation mapping R to the tetrahedron with vertices $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$.)

5. a) Show that $M = \{(x, y, z, w) : x^2 + y^2 + z^2 + w^2 = 1; x + y = 1\}$ is a compact 2–manifold in $\mathbb{R}^4$.
b) Find the extreme values of $f(x, y, z, w) = -x + z$ on M. (You do not need to find where they occur.)

6. a) Evaluate the line integral

$$\int_C (y^2 \sin(xy^2)dx + 2xy \sin(xy^2)dy)$$

where C is the unit circle, oriented counter-clockwise.

b) Find a function $f(x, y)$ such that $df = y^2 \sin(xy^2)dx + 2xy \sin(xy^2)dy$.

7. a) State the classical Stokes' theorem and divergence theorem.

b) Compute

$$\int_C (2xy^2zdz + 2x^2yzdy + (x^2y^2 - 2z)dz)$$

where C is the curve $x = \cos t, y = \sin t, z = \sin t$, oriented in the direction of increasing $t, 0 \leq t \leq 2\pi$.

c) Compute $\int_S < F, n > dA$ where $F(x,y,z) = (e^y \cos z, e^x \sin z, e^x \cos y)$ and S is the unit sphere with outward pointing normal n.

8. a) Define *closed* form and *exact* form on an open set $U \subset \mathbb{R}^n$
b) Suppose that α and β are closed forms on the open unit ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$. Prove that $\alpha \wedge \beta$ is exact on B.
c) Suppose that $U \subset \mathbb{R}^n$ is an arbitrary open set, and that α is a closed form and β is an exact form on U. Show that $\alpha \wedge \beta$ is exact on U.

9. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function such that $M := \{x \in \mathbb{R}^3 : f(x) \geq 0\}$ is a nonempty compact 3-dimensional manifold with boundary. Let $g := f^2$. Show that

$$\int_M \Delta g \, dV = 0$$

where $\Delta g(x) := \sum_{i=1}^{3} D_i(D_i g)$.

Extra Page.

Extra Page.