CS236 notes

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Definitions

Definition 1. Proof by **simple induction** is a method for proving statement

$$\forall n \in \mathbb{N}, P(n)$$

The method of induction consists of 2 steps
BASIS: Prove that $P(0)$ is true, ie. that predicate $P(n)$ holds for $n = 0$.
INDUCTION STEP: Prove that, for each $i \in \mathbb{N}$, if P(i) is true then $P(i + 1)$ is also true.

Remark. The assumption that $P(i)$ holds in the indcution step of the proof is called the *induction hypothesis.* Bases case can be non-zero.

Example 1.1. For any $m, n \in \mathbb{N}$ such that $n \neq 0$, there are unique $q, r \in \mathbb{N}$ such that $m = q\dot{n} + r$ and $r < n$

Remark. Think about 2 cases. either $r < n - 1$ or $r = n - 1$

Example 1.2. We can use an unlimited supply of 4-cent and 7-cent postage stamps to make exactly any amount of postage that is 18 cents or more. Or that $\exists a, b \in \mathbb{N}, i = 4a + 7b$

Remark. Intuitively, try to juggle around value of a, b so that there is an excess of 1-cent, which satisfies for $i + 1$. In this case prove by cases to make it happen. Otherwise use proof by complete induction which is easier.

Definition 2. a is **divisible** by b if the division of a by b has no remainder.

$$b \mid a : \exists k \in \mathbb{N} : a = bk$$

Remark. Read $b \mid a$ as b divides a

Definition 3. An integer n is **prime** if $n \geq 2$ and the only positive integers that divide n are 1 and itself.

$$\{n \in \mathbb{N} : m \mid n \Rightarrow m = 1 \vee m = 2\}$$

Remark. Prime factorization of a natural number n is a sequence of primes whose product is n

Definition 4. Proof by **complete induction** is a method for proving

$$\forall n \in \mathbb{N}, P(n)$$

BASIS: Prove that $P(n)$ holds for all $n \geq c$
INDUCTION STEP: Prove that, for each natural number $i > c$, if $P(j)$ holds for all natural numbers j such that $c \leq j < i$, then $P(i)$ holds as well.

Remark. It is important to ensure that both $j \geq c$ and $j < i$

Example 4.1. Any integer $n \geq 2$, has a prime factorization.

Proof. Define the predicate $P(n)$ as follows

$$P(n): \quad n \text{ has a prime factorization}$$

Use complete induction to prove that $P(n)$ holds for all integer $n \geq 2$. Let i be an arbitrary integer such taht $i \geq 2$. Assume that $P(j)$ holds for all integers j, such that $2 \leq j < i$. We muts prove that $P(i)$ holds as well. There are two cases
CASE 1: i is prime. Then $\langle i \rangle$ is a prime factorization of i. Thus $P(i)$ holds.
CASE 2: i is not prime. Thus there is a positive integer a that divides i such that $a \neq 1 \wedge a \neq i$. Let $b = {}^{i}/_{a}; i.e., i = a \cdot b$. Since $a \neq i \wedge a \lneq i$, it follows that a, b are both integers such that $2 \leq a, b \leq i$. Therefore, by the induction hypothesis, $P(a)$ and $P(b)$ both hold. That is, there is a prime factorization of a, say $\langle p_1, p_2, \ldots, p_k \rangle$, and there is a prime factorisation of b, say $\langle q_1, q_2, \ldots, q_l \rangle$. Since $i = a \cdot b$, it is obvious that concatenation of the prime factorisation of a and b, i.e. $\langle p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_l \rangle$, is a prime factorisation of i. Therefore, $P(i)$ holds in this case as well. Therefore $P(n)$ holds for all $n \geq 2$

Remark. However, if we know the factorisation of all numbers less than i, then we can easily find a prime factorisation of i: if i is prime, then it is its own prime factorisation, and we are done; if i is not prime, then we can get a prime factorisation of i by concatenating the prime factorisations of two factors (which are smaller than i and therefore whose prime factorisation we know by induction hypothesis).

□

Example 4.2. Prove that ppostage of exactly n cents can be made using only 5-cents and 8-cents stamps

Proof. Define the predicate $P(n)$ as follows

$$P(n): \exists a, b \in \mathbb{N}, n = 5a + 8b$$

Use proof by complete induction to prove $P(n)$ holds for $n \geq 28$. Let i be an arbitrary integer such that $i \geq 28$, and assume that $P(j)$ holds for all j such that $28 \leq j < i$. We

will prove that $P(i)$ holds as well.
CASE 1 or the BASIS: When $28 \leq i \leq 32$. We can make postage for all of them... Just have to calculate them...
CASE 2 or INDUCTION STEP: When $i \geq 32$. Let $j = i-5$ and therefore, by induction hypothesis, $P(j)$ holds. This means that $\exists a, b \in \mathbb{N}, j = 5a + 8b$.

$$\begin{aligned}
i &= j + 5 \\
&= 5a + 8b + 5 \\
&= 5(a+1) + 8b \qquad && a_1 = a + 1, b_1 = b \\
&= 5a_1 + 8b_1 && a_1, b_1 \in \mathbb{N}
\end{aligned}$$

Therefore, $P(i)$ holds as well. □

Remark. In this problem, a set of basis were discussed instead of one. This is to ensure that the choice of j satisfies $j \geq c$, which is required to use induction hypothesis.

Definition 5.

$$\lfloor x \rfloor = max\{m \in \mathbb{Z} : m \leq x\}$$
$$\lceil x \rceil = min\{m \in \mathbb{Z} : m \geq x\}$$