Problem Set 4

You are strongly encouraged to solve the following exercises before next week's tutorial:

On page 328, exercises 68, 70, 71 and 73.

Additional problems:

1. Let $X_1, \ldots, X_n$ be a sample from a truncated exponential distribution with

$$f_X(x|\theta) = \begin{cases} e^{-(x-\theta)} & x \geqslant \theta \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the MLE of θ, its cdf and pdf.

   (b) Is the MLE unbiased? If not, modify it to obtain an unbiased estimator of θ.

   (c) Find a sufficient statistic for θ.

   (d) Use the sufficient statistic of part (c) to Rao–Blackwellize the unbiased estimator of part (b). Comment on the result.

2. Which of the following distributions belong to the exponential family? Find their sufficient statistics.

   (a) The (continuous) Uniform distribution $U[0, \theta]$

   (b) Negative Binomial distribution $NB(r, p)$ (for known r)

   (c) Weibull distribution $f(x|\alpha, \beta) = \beta\alpha x^{\alpha-1} \exp\{-\beta x^{\alpha}\}$,

   $x > 0, \ \ \alpha > 0, \ \ \beta > 0$

      i. α is known

      ii. α is unknown

   (d) $f(x|a) = \dfrac{2(x+a)}{1+2a}, \ \ 0 < x < 1, \ \ a > 0$

Solutions:

1. (a) Writing

$$\mathcal{L}(\theta) = \prod_{i=1}^{n} f_X(x_i \big| \theta) = \begin{cases} \mathrm{e}^{-\sum x_i}\mathrm{e}^{n\theta} & x_1, \ldots, x_n \geqslant \theta \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \mathrm{e}^{-n\overline{x}}\mathrm{e}^{n\theta} & \theta \leqslant x_{\min} := \min(x_1, \ldots, x_n) \\ 0 & \text{otherwise} \end{cases}$$

we see that for a given sample (fixed $\overline{x}$) the likelihood is monotonically increasing in θ before being truncated at $x_{\min}$. Figure 1 makes it clear that the MLE is $\widehat{\theta} = X_{\min}$.

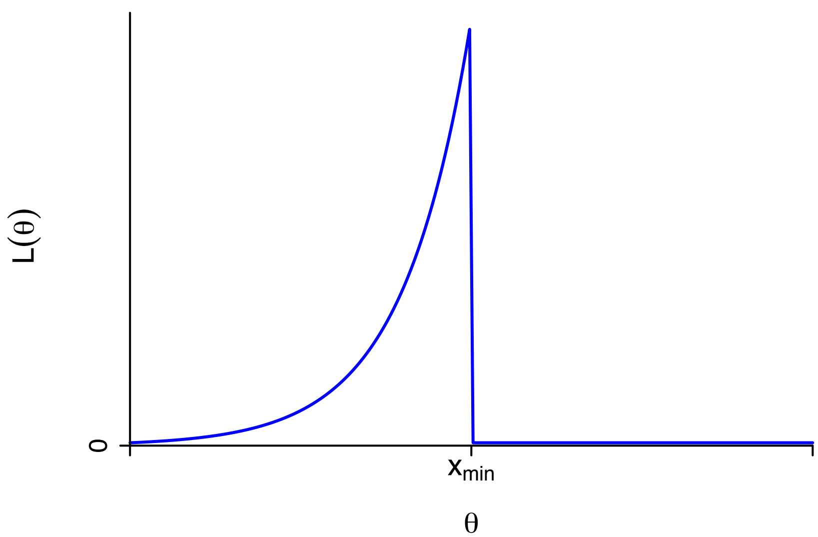

Figure 1: The likelihood function for the question.

Now, for $x \geqslant \theta$

$$F_X(x \big| \theta) = \mathbb{P}(X \leqslant x) = \int_{\theta}^{x} \mathrm{e}^{-(t-\theta)}\mathrm{d}t = 1 - \mathrm{e}^{-(x-\theta)} \ ,$$

hence

$$F_{X_{\min}}(x|\theta) = \mathbb{P}\left(\min\left(X_1, \ldots, X_n\right) \leqslant x\right) = 1 - \mathbb{P}\left(\min\left(X_1, \ldots, X_n\right) > x\right)$$

$$= 1 - \prod_{i=1}^{n} \mathbb{P}\left(X_i > x\right) = 1 - \left[1 - F_X(x|\theta)\right]^n$$

$$= \begin{cases} 1 - \mathrm{e}^{-n(x-\theta)} & x \geqslant \theta, \\ 0 & \text{otherwise,} \end{cases}$$

And finally note that F and f are cdf and pdf of X_min

$$f_{X_{\min}}(x|\theta) = \frac{\mathrm{d}F_{X_{\min}}(x|\theta)}{\mathrm{d}x} = \begin{cases} n\mathrm{e}^{-n(x-\theta)} & x \geqslant \theta, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Calculating

$$\mathbb{E}[\hat{\theta}] = \mathbb{E}[X_{\min}] = \int_{\theta}^{\infty} nxe^{-n(x-\theta)}\mathrm{d}x = \int_{0}^{\infty} n(z+\theta)\mathrm{e}^{-nz}\mathrm{d}z$$

$$= \int_{0}^{\infty} nz\mathrm{e}^{-nz}\mathrm{d}z + \theta \int_{0}^{\infty} n\mathrm{e}^{-nz}\mathrm{d}z = \theta + \frac{1}{n},$$

or just differentiation by parts

therefore $\hat{\theta} = X_{\min}$ is biased, but $\hat{\theta}^* = X_{\min} - \dfrac{1}{n}$ is not.

(c) Writing the likelihood slightly different,

$$\mathcal{L}(\theta) = \underbrace{\mathrm{e}^{n\theta} I\left\{x_{\min} \leqslant \theta\right\}}_{g\left(x_{\min}, \theta\right)} \underbrace{\mathrm{e}^{-n\bar{x}}}_{h(\underline{x})},$$

where

$$I\{x_{\min} \leqslant \theta\} = \begin{cases} 1 & x_{\min} \leqslant \theta, \\ \\ 0 & \text{otherwise}, \end{cases}$$

we can apply the Neyman-Fisher Factorization Theorem to learn that $X_{\min}$ is a sufficient statistic for θ.

(d) Calculating

$$\widehat{\theta}_{\mathrm{RB}} = \mathbb{E}\left[\hat{\theta}^* \middle| X_{\min}\right] = \mathbb{E}\left[X_{\min} - \frac{1}{n} \middle| X_{\min}\right] = X_{\min} - \frac{1}{n} = \hat{\theta}^*,$$

since $\hat{\theta}^*$ is already a function of $X_{\min}$, hence in this case Rao–Blackwellization is of no use.

2. Recall that $f(x|\theta)$ belongs to an exponential family of distributions if we can write

$$f(x|\theta) = \begin{cases} \exp\{c(\theta)T(x) + d(\theta) + s(x)\} & x \in \mathcal{A}, \\ \\ 0 & x \notin \mathcal{A}, \end{cases}$$

where $\mathcal{A}$ (the *support* of $f(x|\theta)$) does not depend on θ, in which case $\sum_{i=1}^{n} T(x_i)$ is a sufficient statistic for θ.

(a) Here the support of $f(x|\theta)$ is depends on θ, thus, by definition $f(x|\theta)$ does not belong to an exponential family.

(b)

$$P(X = x|p) = \binom{x-1}{r-1} p^r (1-p)^{x-r} = \binom{x-1}{r-1} \left(\frac{p}{1-p}\right)^r (1-p)^x$$

$$= \exp\left\{ \underbrace{x}_{T(x)} \underbrace{\log(1-p)}_{c(p)} + \underbrace{r \log \frac{p}{1-p}}_{d(p)} + \underbrace{\log \binom{x-1}{r-1}}_{S(x)} \right\} ,$$

and the Negative Binomial distribution very much belongs to an exponential family, with sufficient statistic $\sum_{i=1}^n X_i$.

(c) First write

$$f(x|\alpha,\beta) = \beta\alpha x^{\alpha-1} \exp\{-\beta x^\alpha\} = \exp\Big\{ -\beta x^\alpha + \log(\alpha\beta) + (\alpha-1)\log x \Big\} \quad (1)$$

i. Here

$$f(x|\beta) = \exp\Big\{ \underbrace{-\beta}_{c(\beta)} \underbrace{x^\alpha}_{T(x)} + \underbrace{\log(\alpha\beta)}_{d(\beta)} + \underbrace{(\alpha-1)\log x}_{S(x)} \Big\}$$

thus if α is known the Weibull distribution forms an exponential family, with the sufficient statistic being $\sum_{i=1}^n X_i^\alpha$.

ii. Representation (1) of the likelihood shows that when α is unknown the Weibull distribution does not form an exponential family, since the term x^α cannot be factorized into $c(\alpha)T(x)$.

(d) Here

$$f(x|a) = \frac{2(x+a)}{1+2a} = \exp\{\log(x+a) - \log(1+2a) + \log 2\} \quad ,$$

and the inseparable term $\log(x+a)$ makes it clear this distribution does not form an exponential family.