5.6 Surface Integrals

Surface Area

Definition. ***Surface Area*** *Given a C^1 surface $S \in \mathbb{R}^3$, we approximate S with infinitesimal elements parallelograms and assign it to an* ***element of area***

$$dA = ||\, dx \times dy \,||$$

Let $G : R \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be parameterization of S, and fix some $(u_0, v_0) \in \mathbb{R}^2$ and apply infinitesimal translation du and dv to (u_0, v_0) to get corresponding vectors

$$G(u, v + dv) - G(u, v) = \frac{\partial G}{\partial v} dv \qquad G(u + du, v) - G(u, v) = \frac{\partial G}{\partial u} du$$

Note that magnitude of cross product of two vectors is the area of the parallelogram, so then

$$dA = \left|\left| \frac{\partial G}{\partial u} \times \frac{\partial G}{\partial v} \right|\right| dudv = ||\, \vec{n} \,||\, dudv$$

where $\vec{n}$ is the normal to the tangent plane. The surface area of surface S is given by just integrating over the area element

$$A(S) = \iint_S dA = \iint_R \left|\left| \frac{\partial G}{\partial u} \times \frac{\partial G}{\partial v} \right|\right| dudv$$

Let $G : (u, v) \mapsto (x, y, z)$ so that $\dfrac{\partial G}{\partial u} = (x_u, y_u, z_u)$ and $\dfrac{\partial G}{\partial v} = (x_v, y_v, z_v)$ so then

$$\left|\left| \frac{\partial G}{\partial u} \times \frac{\partial G}{\partial v} \right|\right| = ||y_u z_v - z_u y_v, z_u x_v - x_u z_v, x_u y_v - y_u x_v||$$

If we have instead given S as graph of C^1 function we can re-parameterize $z = f(x, y)$ with $G(u, v) = (u, v, f(u, v))$ in which case

$$\frac{\partial G}{\partial u} = (1, 0, f_u) \quad \frac{\partial G}{\partial v} = (0, 1, f_v) \quad \frac{\partial G}{\partial u} \times \frac{\partial G}{\partial v} = (-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1)$$

$$\left|\left| \frac{\partial G}{\partial u} \times \frac{\partial G}{\partial v} \right|\right| = \sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 + 1}$$

Surface Integrals over Vector Fields

Definition. ***Orientation of surface*** *An orientation of a surface S is a consistent choice of normal vector to the surface. A paramterization determines an orientation of the surface*

$$\frac{\partial G}{\partial u} \times \frac{\partial G}{\partial v} = \hat{n} dA$$

*where $\hat{n}$ is a unit normal vector and indicates positive orientation. Hence we can reverse orientation by exchanging roles of u and v. For S which bounds a 3-manifold, S has **Stoke's Orientation** if the normal vector of S points outwards with respect to the space it bounds. Although not all surface has orientation, such as the Mobius strip.*

Definition. ***Surface Integral*** *Given a vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ and a surface S, we want to compute the flux of the vector field through the surface, where flux represents the amount of force/fluid passing through S. The vector field travelilng in direction $\hat{n}$ is given by $F \cdot \hat{n}$ and so the surface integral is given by*

$$\iint_S F \cdot \hat{n} dA$$

where $F \cdot \hat{n}$ is the vector field projected onto the normal of the surface. Given parameterization $G : R \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ then

$$flux = \iint_S F \cdot \hat{n} dA = \iint_R F(G(u, v)) \cdot \left[\frac{\partial G}{\partial u} \times \frac{\partial G}{\partial v} \right] dudv$$

note that

$$\hat{n} dA = \frac{\vec{n}}{||\vec{n}||} ||\vec{n}|| dudv = \left[\frac{\partial G}{\partial u} \times \frac{\partial G}{\partial v} \right] dudv$$

5.7 Divergence Theorem

Theorem. ***Divergence Theorem*** *The outward flux of a vector field through a closed surface is equal to the volume integral of the divergence over the region inside the surface. Let $R \subseteq \mathbb{R}^3$ be a regular region with piecewise smooth boundary ∂R. If $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1 vector field and ∂R is positively oriented with respect to R then*

$$\iint_{\partial R} F \cdot \hat{n} dA = \iiint_R divF dV$$

Remark. A result that relates the flux of vector field through a surface to the behavior of the vector field inside the surface.

Theorem. ***Stokes' Theorem*** *Let S be a smooth surface with piecewise smooth (geometric, not topological) boundary ∂S, endowed with Stokes' Orientation. If $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1 vector field in a neighborhood of S, then*

$$\int_{\partial S} F \cdot dx = \iint_S (curlF) \cdot \hat{n} dA$$

Note ∂S has stoke's orientation if $n \times t$ points into S, where n is the orientation of S and t is the tangent vector of a parameterization of ∂S. Or that t points counterclockwise when the surface normal n points toward the viewer.

1. *If S is just a region (surface) in xy-plane, then $\hat{n} = (0,0,1)$ and so*

$$\int_{\partial S} F \cdot dx = \iint_S (curl F) \cdot \hat{n} dA = \iint_S \left[\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right] dA$$

   Hence, Stokes' Theorem in the xy-plane is just Green's theorem

2. *If $\partial S = \emptyset$ (S is closed surface) then*

$$\iint_S \nabla \times F \cdot dx = 0$$

3. *If S_1 and S_2 are 2 surfaces with a common boundary C, where C is oriented in such a way that S_1 and S_2 has Stoke's Orientatino then*

$$\iint_{S_1} \nabla \times F \cdot dx = \iint_{S_2} \nabla \times F \cdot dx$$