# HW 6 Solutions

**7.** Find the joint and marginal densities corresponding to the cdf:
$F(x,y)=(1-e^{-\alpha x})(1-e^{-\beta y}), \quad x\geqslant 0, \quad y\geqslant 0, \quad \alpha>0, \quad \beta>0$

**Ans:**
Joint density (joint pdf, jpdf):

$$f_{X,Y}(x,y)=\frac{d^2}{dxdy}F_{X,Y}(x,y)=\frac{d^2}{dxdy}(1-e^{-\alpha x})(1-e^{-\beta y})=\frac{d}{dx}(1-e^{-\alpha x})\beta e^{-\beta y}=\alpha e^{-\alpha x}\beta e^{-\beta y}$$

Marginal densities:

Marginal density of X: $f_X(x)=\int_0^\infty f_{X,Y}(x,y)\,dy=\int_0^\infty \alpha e^{-\alpha x}\beta e^{-\beta y}\,dy=\alpha e^{-\alpha x}\int_0^\infty \beta e^{-\beta y}\,dy=\alpha e^{-\alpha x}$

Marginal density of Y: $f_Y(y)=\int_0^\infty f_{X,Y}(x,y)\,dx=\int_0^\infty \alpha e^{-\alpha x}\beta e^{-\beta y}\,dx=\beta e^{-\beta y}\int_0^\infty \alpha e^{-\alpha x}\,dx=\beta e^{-\beta y}$

**14.** Suppose that $f(x,y)=xe^{-x(y+1)}, \quad 0\leqslant x<\infty, \quad 0\leqslant y<\infty$

a. Find the marginal densities of X and Y. Are X and Y independent?
b. Find the conditional densities of X and Y.

**Ans:**

a. Marginal densities:

Marginal density of X:

$$f_X(x)=\int_0^\infty f(x,y)dy=\int_0^\infty xe^{-x(y+1)}dy=\int_0^\infty xe^{-xy}e^{-x}dy=e^{-x}\int_0^\infty xe^{-xy}dy=e^{-x}$$

Marginal density of Y:

$$f_Y(y)=\int_0^\infty f(x,y)\,dx=\int_0^\infty xe^{-x(y+1)}dx$$

Integration by parts: Let $u=x,\, dv=e^{-x(y+1)}dx$ then $du=dx,\, v=\frac{-1}{y+1}e^{-x(y+1)}dx$ and

$$f_Y(y)=\int_0^\infty udv=[uv]_0^\infty-\int_0^\infty udv=[x(\frac{-1}{y+1}e^{-x(y+1)})]_{x=0}^{\infty}-\int_0^\infty \frac{-1}{y+1}e^{-x(y+1)}dx=0+\frac{1}{y+1}[\frac{-1}{y+1}e^{-x(y+1)}]_{x=0}^{\infty}$$

So $f_Y(y)=\frac{1}{(y+1)^2}$

Since $f_X(x)f_Y(y)=e^{-x}\frac{1}{(y+1)^2}\neq f_{X,Y}(x,y)$ , X and Y are not independent.

b. Conditional densities:

Conditional density of X given Y: $f_{X|Y}(x|y)=\frac{f_{X,Y}(x,y)}{f_Y(y)}=\frac{xe^{-x(y+1)}}{\frac{1}{(y+1)^2}}=(y+1)^2xe^{-x(y+1)}$

Conditional density of Y given X: $f_{Y|X}(y|x)=\frac{f_{X,Y}(x,y)}{f_X(x)}=\frac{xe^{-x(y+1)}}{e^{-x}}=xe^{-xy}$

**19.** $T_1, T_2$ have independent exponential distributions with parameters $\alpha$ and $\beta$ respectively.
Find: a. $P(T_1 > T_2)$
b. $P(T_1 > 2T_2)$

**Ans:**
Densities of $T_1, T_2$ : $f_{T_1}(t_1) = \alpha e^{-\alpha t_1}$ , $f_{T_2}(t_2) = \beta e^{-\beta t_2}$ . Since they are independent, we have joint density: $f_{T_1,T_2}(t_1, t_2) = f_{T_1}(t_1) f_{T_2}(t_2) = \alpha\beta e^{-\alpha t_1} e^{-\beta t_2}$

a. $P(T_1 > T_2)$ is calculated by taking the integration of joint density in the region $T_1 > T_2$

$$P(T_1 > T_2) = \iint_{T_1 > T_2} f_{T_1,T_2}(t_1, t_2) = \int_0^\infty \int_0^{t_1} f_{T_1,T_2}(t_1, t_2)\, dt_2\, dt_1 = \int_0^\infty \int_0^{t_1} \alpha\beta e^{-\alpha t_1} e^{-\beta t_2}\, dt_2\, dt_1$$

So $P(T_1 > T_2) = \int_0^\infty \alpha e^{-\alpha t_1} \int_0^{t_1} \beta e^{-\beta t_2}\, dt_2\, dt_1 = \int_0^\infty \alpha e^{-\alpha t_1} [-e^{-\beta t_2}]_{t_2=0}^{t_1}\, dt_1 = \int_0^\infty \alpha e^{-\alpha t_1}(-e^{-\beta t_1} + 1)\, dt_1$

Then $P(T_1 > T_2) = \int_0^\infty -\alpha e^{-\alpha t_1 - \beta t_1} + \alpha e^{-\alpha t_1}\, dt_1 = \left[\frac{\alpha}{\alpha+\beta} e^{-(\alpha+\beta) t_1}\right]_{t_1=0}^{\infty} + 1 = 1 - \frac{\alpha}{\alpha+\beta} = \frac{\beta}{\alpha+\beta}$

b. $P(T_1 > 2T_2)$ is calculated by taking the integration of joint density in the region $T_1 > 2T_2$

$$P(T_1 > 2T_2) = \iint_{T_1 > 2T_2} f_{T_1,T_2}(t_1, t_2) = \int_0^\infty \int_0^{t_1/2} f_{T_1,T_2}(t_1, t_2)\, dt_2\, dt_1 = \int_0^\infty \int_0^{t_1/2} \alpha\beta e^{-\alpha t_1} e^{-\beta t_2}\, dt_2\, dt_1$$

So $P(T_1 > 2T_2) = \int_0^\infty \alpha e^{-\alpha t_1} \int_0^{t_1/2} \beta e^{-\beta t_2}\, dt_2\, dt_1 = \int_0^\infty \alpha e^{-\alpha t_1} [-e^{-\beta t_2}]_{t_2=0}^{t_1/2}\, dt_1 = \int_0^\infty \alpha e^{-\alpha t_1}(-e^{-\beta t_1/2} + 1)\, dt_1$

Then

$$P(T_1 > 2T_2) = \int_0^\infty -\alpha e^{-\alpha t_1 - \beta t_1/2} + \alpha e^{-\alpha t_1}\, dt_1 = \left[\frac{\alpha}{\alpha+\beta/2} e^{-(\alpha+\beta/2) t_1}\right]_{t_1=0}^{\infty} + 1 = 1 - \frac{\alpha}{\alpha+\beta/2} = \frac{\beta/2}{\alpha+\beta/2} = \frac{\beta}{2\alpha+\beta}$$

**20.** $X_1$ is uniform on [0,1], and conditional on $X_1$ , $X_2$ is uniform on $[0, X_1]$ , find joint and marginal distribution of $X_1$ , $X_2$

**Ans:**
By definition of uniform distribution we have:
Marginal density of $X_1$ , $f_{X_1}(x_1) = 1$ when $0 \leqslant x_1 \leqslant 1$ and is 0 otherwise.
Conditional density of $X_2$ given $X_1$ : $f_{X_2|X_1}(x_2|x_1) = \frac{1}{x_1}$ when $0 \leqslant x_2 \leqslant x_1$ and is 0 otherwise.
So joint density (jpdf) of $X_1$ , $X_2$ : $f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2|X_1}(x_2|x_1) = \frac{1}{x_1}$ when $0 \leqslant x_2 \leqslant x_1 \leqslant 1$ and is 0 otherwise.
Hence marginal density of $X_2$ : $f_{X_2}(x_2) = \int_0^1 f_{X_1,X_2}(x_1, x_2)\, dx_1 = \int_{x_2}^1 \frac{1}{x_1}\, dx_1 = [\ln(x_1)]_{x_2}^1 = -\ln(x_2)$
when $0 < x_2 \leqslant 1$ and is 0 otherwise.