UNIVERSITY OF TORONTO
Faculty of Arts and Science

Midterm 2, Version 1
CSC263H1F

Friday November 11, 2016, 10:10-11:00am (**50 min.**)
Examination Aids: No aids allowed

Name:

Student Number:

Please read the following guidelines carefully!

- Please write your name on the front **and back** of the exam.
- This examination has **3** questions. There are a total of **10 pages, DOUBLE-SIDED**.
- Answer questions clearly and completely. Give complete justifications for all answers unless explicitly asked not to. You may use any claim/result from class, unless you are being asked to prove that claim/result, or explicitly told not to.

Take a deep breath.

This is your chance to show us

How much you've learned.

We **WANT** to give you the credit

That you've earned.

A number does not define you.

Good luck!

1. **Hashing**. The three parts can be done in any order.

   (a) **[2 marks]** Suppose we have an empty hash table of length $m > 4$ that resolves collisions using closed addressing (linked list chaining), and we randomly choose **four** distinct keys to insert into the hash table. Assume that for *each* of the four choices, and every index $0 \leq i < m$, the probability that the chosen key hashes to i is $\frac{1}{m}$. **Find the exact probability** that all linked lists in the hash table are length 0 or 1 after four keys are inserted.

Solution

The four keys must have four distinct hash values. m choices for the first key, $m-1$ for the second, etc:

$$\frac{m}{m} \cdot \frac{m-1}{m} \cdot \frac{m-2}{m} \cdot \frac{m-3}{m} = \frac{m(m-1)(m-2)(m-3)}{m^4}.$$

   (b) **[2 marks]** David says that for a hash table of size m using chaining, "the worst-case running time for SEARCH when there are n keys is $\mathcal{O}(1+\alpha) = \mathcal{O}(1+\frac{n}{m})$."
   If you believe he is correct, perform a *upper bound* analysis on the worst-case running time of this operation. If not, perform a *lower bound* analysis on the worst-case running time.

Solution

This is incorrect (that's the bound for an average-case analysis, under the simple uniform hashing assumption).

For a lower bound, consider a hash table with n keys that all have the same hash value, and so are all in the same linked list. Searching for the last key in the chain takes $\Omega(n)$ time, as the entire linked list (of length n) must be traversed.

(c) **[3 marks]** Suppose we have a hash table of length m that resolves collisions using open addressing, and that currently stores n key-value pairs.

Suppose we want to support the operation InverseSearch, which is given a hash table and value, and returns a list of all keys in the hash table which correspond to that value.

Give both the **pseudocode** for an implementation of this algorithm, as well as a **worst-case upper bound analysis** for your algorithm. Your upper bound should be tight, but you do not need to prove this.

Your solution will be graded on both correctness and efficiency.

Solution

This algorithm relies on two ideas: looping through array elements, and the fact that hash tables store both the key and value when pairs are inserted.

```
1 def InverseSearch(H, value):
2     keys = []
3     for i from 0 to H.length - 1:
4         if H[i].value == value:
5             add H[i].key to keys
6     return keys
```

For running time, the loop runs m times, and each iteration takes constant time, for a total running time of $\mathcal{O}(m)$. [Comment: in fact, the loop always runs m times, regardless of the contents of the hash table or value being searched for. So the running time of this algorithm is always $\Theta(m)$.]

2. **Graph Searches, Randomization**. Part (a) is independent of the other three parts.

   (a) **[2 marks]** Consider this recursive version of Depth-First Search:

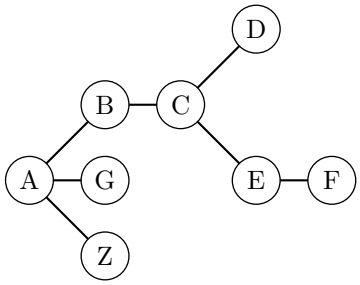
```
def DFS(graph, s):
    initialize all vertices in the graph to not started
    DFS_helper(graph, s)

def DFS_helper(graph, v):
    v.started = True
    Visit(v)  # do something with v, like print out its label
    for each neighbour u of v:
        if not u.started:
            DFS_helper(graph, u)
```

   Find an exact upper bound on the maximum possible *number* of calls to `DFS_helper` when we run DFS. Your answer should be in terms of the number of vertices, $|V|$, and/or edges, $|E|$, in the graph. You do not need to prove that this bound is tight.

Solution

The most important observation is that `DFS_helper` is not called on the same vertex twice. When this function is called on a vertex, its `started` attribute is set to True, and `DFS_helper` is only called on vertices that have not been started. So `DFS_helper` is called a maximum of $|V|$ times.

   (b) **[1 mark]** Consider the following graph.



   Suppose we *randomize* BFS and DFS so that for each vertex v, we randomly permute its neighbours before looping through them:

```
for each neighbour u of v, in a random order:
```

   Consider running this randomized BFS on the above graph, starting at vertex A. **State one possible order** of vertex visits made by this BFS, and **the number of vertices** visited before Z is visited in this order. No justification is required.

Solution

Many possible answers, including A, B, G, Z, C, D, E, F. 3 vertices are visited before Z in this order.

(c) **[2 marks]** Let T_{BFS} be a random variable representing the number of vertices visited before vertex Z when we run a randomized BFS on this graph starting at A.

Find the expected value of T_{BFS}. Make sure to explain your work here.

Hint: There are only a few possible values for T_{BFS}; you don't need to enumerate all possible choices made by BFS to answer this question.

Solution

The key insight is that only the order in which B, G, and Z are visited matters; Z is guaranteed to be visited before all other vertices.

There is a $\frac{1}{3}$ chance that Z is picked first among these three, a $\frac{1}{3}$ chance Z is picked second, and a $\frac{1}{3}$ chance Z is picked third. The expected number of vertices visited before Z is

$$\mathbb{E}[T_{BFS}] = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 2 + \frac{1}{3} \cdot 3 = 2.$$

(d) **[2 marks]** Define T_{DFS} as analogous to T_{BFS}, except that we run a randomized DFS instead of BFS, but still starting at A.

Find the expected value of T_{DFS}. Make sure to explain your work here.

Solution

This again depends only on the order in which B, G, and Z are chosen, with the difference being that when B is visited, all of {C, D, E, F} are also visited before the next neighbour of A. Skipping some justification:

$$\begin{aligned}\mathbb{E}[T_{DFS}] &= 1 \cdot \Pr[T_{DFS} = 1] + 2 \cdot \Pr[T_{DFS} = 2] + 6 \cdot \Pr[T_{DFS} = 6] + 7 \cdot \Pr[T_{DFS} = 7] \\ &= 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} + 7 \cdot \frac{1}{3}.\end{aligned}$$

3. **Graph Search Applications**. Part (a) can be after (b) and (c).

A **triangle** in a graph is a set of three vertices that are all neighbours of each other (i.e., a cycle of length 3). Suppose we are given a graph $G = (V, E)$ and vertex $v \in V$, and want to determine whether v is part of a triangle. Here is a brute force way of doing this:

```
def FindTriangle(graph, v):
    for each neighbour u of v:
        for each neighbour w of v:
            if u != w and (u, w) is an edge:
                return True
    return False
```

(a) **[3 marks]** Prove that FindTriangle takes $\Omega(|V|^2)$ time in the worst case, assuming $|E| = \Omega(|V|)$. Assume that checking whether (u, w) is an edge always takes constant time (so this isn't the slow part).

Solution

For an input family, consider a graph with vertex v that is connected to every other vertex, and there are no other edges. So v is not part of a triangle, but running FindTriangle on this graph and v will require $|V| - 1$ iterations of outer loop, and $|V| - 1$ iterations of the inner loop, for a total running time of $\Omega(|V|^2)$. [Comment: this is where we use the fact that there are $\Omega(|V|)$ edges. If there were fewer, e.g., 10, then this algorithm would take $\Omega(|E|^2)$ time instead.]

BFS Algorithm (reference for part (b)).

```
def BFS(graph, s):
    queue = new empty queue
    initialize all vertices in the graph to not enqueued

    queue.enqueue(s)
    s.enqueued = True

    while queue is not empty:
        v = queue.dequeue()
        Visit(v)  # do something with v, like print out its label

        for each neighbour u of v:
            if not u.enqueued:
                queue.enqueue(u)
                u.enqueued = True
```

(b) **[3 marks]** Show how to implement FindTriangle in worst-case time $\mathcal{O}(|V| + |E|)$. Give both **pseudocode** as well as **brief English justification** about why your algorithm is correct. You may again assume that it is always possible to check whether a given tuple (v_1, v_2) is an edge in constant time.

Hint: Recall how we analysed BFS in lecture. You can use the main idea from the previous algorithm, and/or from BFS itself.

Solution

One solution would be to take the idea of BFS, except explicitly only explore up to distance 2. In this case, if a vertex has an "enqueued" neighbour not equal to v, this must form a triangle.

```
def FindTriangle(graph, v):
    queue = new empty queue
    initialize all vertices in the graph to not enqueued

    for each neighbour u of v:
        queue.enqueue(u)
        u.enqueued = True

    # Run BFS on the neighbours
    while queue is not empty:
        w = queue.dequeue()

        for each neighbour u of w:
            # Rather than enqueue more vertices,
            # simply check if it's been enqueued.
            if u.enqueued:
                return True

    return False
```

A cute simplification of this idea makes it obvious that it is a rather basic variation of the given algorithm:

```
def FindTriangle(graph, v):
    for each neighbour u of v:
        for each neighbour w of u:
            if v != w and (v, w) is an edge:
                return True
    return False
```

(c) **[2 marks]** Show that your algorithm runs in $\mathcal{O}(|V| + |E|)$ time in the worst case.
You may **not** assume that BFS runs in $\mathcal{O}(|V|+|E|)$ time. You should analyse your algorithm directly, although you may repeat the ideas in our BFS analysis.

Solution

Since we can't assume the running time of BFS, we have to remember how we analysed it. Initializing attributes takes $\mathcal{O}(|V|)$ time. The inner loop runs in time d_u, the degree of the current vertex u. So the total number of iterations of the inner loop is less than or equal to the sum of all degrees in the graph, which is $\mathcal{O}(|E|)$. Putting these together lead to a $\mathcal{O}(|V| + |E|)$ running time.

Use this page for rough work.

Use this page for rough work.

Name:

	Q1	Q2	Q3	Total
Grade				
Out Of	7	7	8	22