Lecture 3: Large Sample Theory

STA261 – Probability & Statistics II

Ofir Harari

Department of Statistical Sciences

University of Toronto

Outline

UNIVERSITY OF TORONTO

Example: the exponential distribution

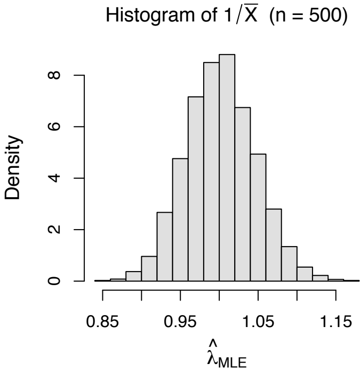
- Suppose now that $X_1 \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Exp}(\lambda)$ (with pdf $f(x|\lambda) = \lambda e^{-\lambda x}$, $x \geq 0$)
- Let us find the MLE of λ –
  - ⋆ $\mathcal{L}(\lambda) = f(x_1, \ldots, x_n|\lambda) = \prod_{i=1}^{n} f(x_i|\lambda) = \lambda^n \exp\left\{-\lambda \sum_{i=1}^{n} x_i\right\}$
  - ⋆ $\ell(\lambda) = n \log \lambda - \lambda \sum_{i=1}^{n} x_i$
  - ⋆ $\ell'(\lambda) = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0 \Longrightarrow \widehat{\lambda} = \frac{1}{\overline{X}}$ (surprising?)

    yes; because MME for exponential is X instead
  - ⋆ $\ell''(\hat{\lambda}) = -\frac{n}{\widehat{\lambda}^2} < 0 \Longrightarrow \max$
- Let us now explore the sampling distribution of the MLE
- Assume $\lambda = 1$ and draw a histogram of $\widehat{\lambda}_{\text{MLE}} = \frac{1}{\overline{X}}$

Sampling distribution of $\widehat{\lambda}_{\mathrm{MLE}}$ by simulation

distribution of estimator (for ~Exp)

```
> lambda <- 1
> n <- 20 #sample size
>
> X <- matrix(rexp(n*10000 lambda), ncol=n) # a 10,000 by n matrix of random exponentials
> lambdaHat <- 1/apply(X, 1, mean) # a sample of 10,000 MLEs for lambda
> hist(lambdaHat, freq=FALSE) # plotting the histogram
```



UNIVERSITY OF TORONTO

Asymptotic normality

Definition

Let $X_i, \ldots, X_n \sim f_\theta$. We say that $\widehat{\theta}_n = \widehat{\theta}_n(X_1, \ldots, X_n)$ is *asymptotically normal* with mean θ and variance $\dfrac{\sigma^2}{n}$ if for all $z \in \mathbb{R}$

$$F_{Z_n}(z) \overset{n\to\infty}{\longrightarrow} \Phi(z),$$

where $F_{Z_n}(\cdot)$ is the cdf of $Z_n = \dfrac{\sqrt{n}}{\sigma}\left(\widehat{\theta}_n - \theta\right)$ and $\Phi(\cdot)$ is the standard normal cdf.

- Alternatively, write

  equivalent to convergence in cdf

  $$\sqrt{n}\left(\widehat{\theta}_n - \theta\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2) \text{ (converges } \textit{in distribution}\text{)}$$

- Not a new concept: if $\{X_i\}$ are i.i.d r.v.'s with mean μ and variance σ^2, we know (from the CLT) that

  $$Z_n = \sqrt{n}\left(\overline{X}_n - \mu\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$$

- We expect $\overline{X}$ to be asymptotically normal then – but why would $\dfrac{1}{\overline{X}}$...?

  i.e. estimator is asymptotically normal too?

UNIVERSITY OF TORONTO

The Score statistic and the Fisher Information

Definition

Let $X_i, \ldots, X_n \sim f_\theta$ with $\ell(\theta) = \log f(x_1, \ldots, x_n | \theta)$.

1. The *Score* with respect to θ is

$$u(\theta) := \ell'(\theta).$$ statistics: a random variable

2. The *Fisher Information* for θ is

$$\mathcal{I}(\theta) := -\mathbb{E}\left\{\ell''(\theta)\right\}.$$ a scalar

- In the exponential example, we have already calculated

$$u(\lambda) = \ell'(\lambda) = \frac{n}{\lambda} - \sum_{i=1}^{n} X_i \quad \text{and} \quad \ell''(\lambda) = -\frac{n}{\lambda^2}$$

- The Fisher information is therefore $\mathcal{I}(\lambda) = -\mathbb{E}\{\ell''(\lambda)\} = \dfrac{n}{\lambda^2}$

note n and lambda are both fixed

UNIVERSITY OF TORONTO

The Score and the Information (cont.)

- For $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} f_\theta$, denote

$$\mathcal{I}^*(\theta) := -\mathbb{E}\left\{\frac{\partial^2 \log f(x|\theta)}{\partial\theta^2}\right\}$$

  - the Fisher information of θ based on a single observation

- Note that

$$\mathcal{I}(\theta) = -\mathbb{E}\left\{\frac{\partial^2 \log f(x_1,\ldots,x_n|\theta)}{\partial\theta^2}\right\} = -\mathbb{E}\left\{\frac{\partial^2 \log \prod_{i=1}^n f(x_i|\theta)}{\partial\theta^2}\right\}$$

$$= -\mathbb{E}\left\{\frac{\partial^2 \sum_{i=1}^n \log f(x_i|\theta)}{\partial\theta^2}\right\} = -\sum_{i=1}^n \mathbb{E}\left\{\frac{\partial^2 \log f(x_i|\theta)}{\partial\theta^2}\right\}$$

linearity of derivative

linearity of expected value

$$= n\mathcal{I}^*(\theta)$$

The Score and the Information (cont.)

Proposition

Under some regularity conditions

1. $\mathcal{I}(\theta) = \mathbb{E}\left[u^2(\theta)\right]$
2. $\dfrac{u(\theta)}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \mathcal{I}^*(\theta)\right)$ (asymptotically normal)
3. $-\dfrac{1}{n}\dfrac{\partial^2 \ell(\theta)}{\partial \theta^2} \xrightarrow{\mathrm{P}} \mathcal{I}^*(\theta)$

Proof:

1. Denoting $\underline{\boldsymbol{x}} = (x_1, \dots, x_n)$, we have

$$\mathbb{E}[u(\theta)] = \int u(\theta) f(\underline{\boldsymbol{x}}|\theta)\mathrm{d}\underline{\boldsymbol{x}} = \int \frac{\partial \log f(\underline{\boldsymbol{x}}|\theta)}{\partial \theta} f(\underline{\boldsymbol{x}}|\theta)\mathrm{d}\underline{\boldsymbol{x}} = \int \frac{\partial f(\underline{\boldsymbol{x}}|\theta)}{\partial \theta}\mathrm{d}\underline{\boldsymbol{x}}$$

$$= \frac{\partial}{\partial \theta}\int f(\underline{\boldsymbol{x}}|\theta)\mathrm{d}\underline{\boldsymbol{x}} = \frac{\partial}{\partial \theta}(1) = 0.$$

UNIVERSITY OF TORONTO

The Score and the Information (cont.)

We have shown that $\mathbb{E}[u(\theta)] = 0$. Differentiating once again we have

$$0 = \frac{\partial}{\partial\theta}\mathbb{E}[u(\theta)] = \frac{\partial}{\partial\theta}\int \frac{\partial \log f(\underline{\boldsymbol{x}}|\theta)}{\partial\theta} f(\underline{\boldsymbol{x}}|\theta)\mathrm{d}\underline{\boldsymbol{x}}$$

product rule

$$= \int \frac{\partial^2 \log f(\underline{\boldsymbol{x}}|\theta)}{\partial\theta^2} f(\underline{\boldsymbol{x}}|\theta)\mathrm{d}\underline{\boldsymbol{x}} + \int \frac{\partial \log f(\underline{\boldsymbol{x}}|\theta)}{\partial\theta}\frac{\partial f(\underline{\boldsymbol{x}}|\theta)}{\partial\theta}\mathrm{d}\underline{\boldsymbol{x}}$$

chain rule

$$= \mathbb{E}\left\{\frac{\partial^2 \log f(\underline{\boldsymbol{X}}|\theta)}{\partial\theta^2}\right\} + \int \left[\frac{\partial \log f(\underline{\boldsymbol{x}}|\theta)}{\partial\theta}\right]^2 f(\underline{\boldsymbol{x}}|\theta)\mathrm{d}\underline{\boldsymbol{x}}$$

$$= -\mathcal{I}(\theta) + \mathbb{E}\left\{\left[\frac{\partial \ell(\theta)}{\partial\theta}\right]^2\right\} = -\mathcal{I}(\theta) + \mathbb{E}\left[u^2(\theta)\right].$$

UNIVERSITY OF TORONTO

The Score and the Information (cont.)

2. Since note the score is a RV i.e. sqrt{n} * u / n ~ N(0, \sigma^2)

$$\frac{u(\theta)}{\sqrt{n}} = \frac{1}{\sqrt{n}} \frac{\partial \log f(x_1, \ldots, x_n | \theta)}{\partial \theta} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\partial \log f(x_i | \theta)}{\partial \theta},$$

it is asymptotically normal (from the CLT). We have already shown that $\mathbb{E}[u] = 0$, and thus

$$\mathrm{Var}[u(\theta)] = \mathbb{E}[u^2(\theta)] = \mathcal{I}(\theta),$$

hence $\mathrm{Var}\left[\frac{u(\theta)}{\sqrt{n}}\right] = \frac{1}{n}\mathcal{I}(\theta) = \mathcal{I}^*(\theta)$.

fisher info based on single observation

3. Simple application of the Weak Law of Large Numbers yields

$$-\frac{1}{n}\frac{\partial^2 \ell(\theta)}{\partial \theta^2} = -\frac{1}{n}\sum_{i=1}^{n} \frac{\partial^2 \log f(x_i | \theta)}{\partial \theta^2} \xrightarrow{\mathrm{P}} -\mathbb{E}\left\{\frac{\partial^2 \log f(x | \theta)}{\partial \theta^2}\right\} = \mathcal{I}^*(\theta).$$

UNIVERSITY OF TORONTO

Slutsky's Theorem

- Before we proceed to prove the asymptotic normality of MLEs, we bring (without a proof) the following result:

Slutsky's Theorem

Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of r.v.'s such that $X_n \xrightarrow{\mathcal{D}} X$ and $Y_n \xrightarrow{\mathcal{P}} c$ (for some constant c), and let $g(\cdot, \cdot)$ be a continuous function. Then

$$g(X_n, Y_n) \xrightarrow{\mathcal{D}} g(X, c).$$

- In particular, if $X_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$ and $Y_n \xrightarrow{\mathcal{P}} c$,

$$X_n Y_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, c^2\sigma^2).$$

X_n Y_n —> cX

UNIVERSITY OF TORONTO

Asymptotic normality of MLEs

Theorem

Let $X_1, \ldots, X_n$ be a random sample from f_θ and let $\widehat{\theta}_n$ denote the maximum likelihood estimator of θ. Under the same regularity conditions as before, $\widehat{\theta}_n$ is asymptotically normal with mean θ and variance $\mathcal{I}^{-1}(\theta)$.

"Proof":

- When deriving the Newton-Raphson iterations, we showed that

$$\widehat{\theta}_{\text{MLE}} - \theta \approx -\frac{\ell'(\theta)}{\ell''(\theta)} = -\frac{u(\theta)}{\ell''(\theta)}.$$

- Write

$$\sqrt{n}\left(\widehat{\theta}_{\text{MLE}} - \theta\right) = -\sqrt{n} \cdot \frac{u(\theta)}{\ell''(\theta)} = \frac{\frac{1}{\sqrt{n}} u(\theta)}{-\frac{1}{n}\ell''(\theta)}$$

UNIVERSITY OF TORONTO

Asymptotic normality of MLEs (cont.)

"Proof" (cont.):

- So far we have

$$\sqrt{n}\left(\widehat{\theta}_{\text{MLE}} - \theta\right) = \frac{\frac{1}{\sqrt{n}}u(\theta)}{-\frac{1}{n}\ell''(\theta)}$$

- In the proposition we proved, we showed that
  1. The numerator $\xrightarrow{\mathcal{D}} \mathcal{N}\big(0, \mathcal{I}^*(\theta)\big)$
  2. The denominator $\xrightarrow{\mathcal{P}} \mathcal{I}^*(\theta)$
- Now seems like the right time to apply Slutsky's Theorem:

$$\sqrt{n}\left(\widehat{\theta}_{\text{MLE}} - \theta\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, (\mathcal{I}^*(\theta))^{-2}\,\mathcal{I}^*(\theta)\right) = \mathcal{N}\left(0, \frac{1}{\mathcal{I}^*(\theta)}\right),$$

- Alternatively, $\widehat{\theta}_{\text{MLE}} \sim AN\left(\theta, \dfrac{1}{n\mathcal{I}^*(\theta)}\right) = AN\left(\theta, \mathcal{I}^{-1}(\theta)\right).$

Back to the exponential example

- We just proved:

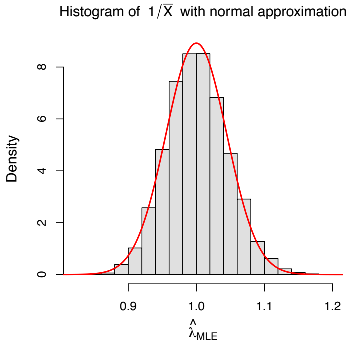
$$\widehat{\theta}_{\text{MLE}} \sim AN\left(\theta, \mathcal{I}^{-1}(\theta)\right) \quad \text{(asymptotically normal)}$$

- We have already calculated $\mathcal{I}(\lambda) = \dfrac{n}{\lambda^2}$

- From the latest Theorem, the distribution of $\widehat{\lambda}_{\text{MLE}} = \dfrac{1}{\overline{X}}$ should be approximately normal, with mean λ and variance $\dfrac{1}{\mathcal{I}(\lambda)} = \dfrac{\lambda^2}{n}$

Back to the R simulation

```
> lambda <- 1
> n <- 500
> X <- matrix(rexp(n*10000, lambda), ncol=n)
> lambdaHat <- 1/apply(X, 1, mean)
> hist(lambdaHat, freq=FALSE)
> z <- seq(-10, 10, by=.01)
> lines(z, dnorm(z, mean=lambda, sd=lambda/sqrt(n)))
```



Example: Bernoulli distribution

Example

Let $X_1 \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Binom}(1, p)$. Find the MLE of p and derive its normal approximation.

Solution:

Let us write the likelihood first –

Note that we are ignoring nCx here since constant does not contribute to likelihood

$$\mathcal{L}(p) = \prod_{i=1}^{n} f(x_i | p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{\sum_{i=1}^{n} x_i}(1-p)^{n-\sum_{i=1}^{n} x_i},$$

and the log-likelihood –

$$\ell(p) = \sum_{i=1}^{n} x_i \log p + \left(n - \sum_{i=1}^{n} x_i\right) \log(1-p).$$

Bernoulli distribution (cont.)

Solution (cont.):

$$\ell(p) = \sum_{i=1}^{n} x_i \log p + \left(n - \sum_{i=1}^{n} x_i\right) \log(1-p).$$

Solving

$$\ell'(p) = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{n - \sum_{i=1}^{n} x_i}{1-p} = 0 \Longrightarrow (1-p)\sum_{i=1}^{n} x_i = \left(n - \sum_{i=1}^{n} x_i\right) p$$

$$\Longrightarrow \sum_{i=1}^{n} x_i = np \Longrightarrow \widehat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}.$$

⋆ Note that

$$\ell''(p) = -\frac{\sum_{i=1}^{n} x_i}{p^2} - \frac{n - \sum_{i=1}^{n} x_i}{(1-p)^2} < 0,$$

therefore it is a maximum.

UNIVERSITY OF TORONTO

Bernoulli distribution (cont.)

Solution (cont.):

$$\ell''(p) = -\frac{\sum_{i=1}^n x_i}{p^2} - \frac{n - \sum_{i=1}^n x_i}{(1-p)^2}$$

note fisher information is a function of true population parameter

- The Fisher information is

$$\mathcal{I}(p) = -\mathbb{E}\left[\ell''(p)\right] = \frac{\sum_{i=1}^n \mathbb{E}[X_i]}{p^2} + \frac{n - \sum_{i=1}^n \mathbb{E}[X_i]}{(1-p)^2}$$

$$= \frac{np}{p^2} + \frac{n-np}{(1-p)^2} = \frac{n}{p} + \frac{n}{(1-p)} = \frac{n}{p(1-p)},$$

hence

$$\widehat{p} \sim AN\left(p, \frac{p(1-p)}{n}\right).$$

⋆ Old news – this is the CLT for Bernoulli r.v.'s

by asymptotic normality of MLE

Invariance of MLEs and transformations

Theorem

Let $X_1 \ldots, X_n$ be a sample from f_θ and let $\eta = g(\theta)$ for some function $g(\cdot)$. Then –

1. $\widehat{\eta}_{\text{MLE}} = g\left(\widehat{\theta}_{\text{MLE}}\right)$, and
2. If $g(\cdot)$ is differentiable then $\widehat{\eta}_{\text{MLE}} \sim AN\left(\eta, [g'(\theta)]^2 \mathcal{I}^{-1}(\theta)\right)$

Proof:

1. Denote $\widehat{\eta} = g\left(\widehat{\theta}_{\text{MLE}}\right)$. We need to show that $\widehat{\eta} = \widehat{\eta}_{\text{MLE}}$, that is: $f(x_1, \ldots, x_n | \eta) \le f\left(x_1, \ldots, x_n \middle| \widehat{\eta}\right)$ For any η. To show that,

$$f(x_1, \ldots, x_n | \eta) = \max_{\theta : g(\theta) = \eta} f(x_1, \ldots, x_n | \theta) \le \max_{\theta} f(x_1, \ldots, x_n | \theta)$$

how does this work

$$= f\left(x_1, \ldots, x_n \middle| \widehat{\theta}_{\text{MLE}}\right) = \max_{\theta : g(\theta) = \widehat{\eta}} f(x_1, \ldots, x_n | \theta)$$

$$= f\left(x_1, \ldots, x_n \middle| \widehat{\eta}\right).$$

Invariance of MLEs and transformations (cont.)

Proof (cont.):

2. From the last Theorem, $\widehat{\eta}_{\mathrm{MLE}} \sim AN\left(\eta, \mathcal{I}^{-1}(\eta)\right)$. Now,

$$u(\theta) = \ell'(\theta) = \ell'(\eta) g'(\theta),$$

chain rule: l is a function a function of eta

hence

$$\mathcal{I}(\theta) = \mathbb{E}\left[u^2(\theta)\right] = \left[g'(\theta)\right]^2 \mathbb{E}\left\{\left[\ell'(\eta)\right]^2\right\} = \left[g'(\theta)\right]^2 \mathcal{I}(\eta),$$

thus

$$\frac{1}{\mathcal{I}(\eta)} = \frac{\left[g'(\theta)\right]^2}{\mathcal{I}(\theta)}.$$

Example: MLE for the log-odds

Theorem

Let $X_1 \dots, X_n \overset{\text{i.i.d.}}{\sim} \text{Binom}(1, p)$. Find the MLE of the log-odds

$$\psi = \log \frac{p}{1-p}$$

and derive its asymptotic sampling distribution.

Solution:

use invariance of MLE to calcualte

- We have already calculated $\widehat{p}_{\text{MLE}} = \overline{X}$ and $\mathcal{I}(p) = \dfrac{n}{p(1-p)}$.
- Denote $\theta := g(p) = \log \dfrac{p}{1-p}$. From the invariance of the MLE,

$$\widehat{\theta}_{\text{MLE}} = g(\hat{p}_{\text{MLE}}) = \log \frac{\overline{X}}{1-\overline{X}}.$$

Example: MLE for the log-odds (cont.)

Solution (cont.):

$$g(p) = \log \frac{p}{1-p}$$

- Now,

$$g'(p) = \frac{1-p}{p} \cdot \frac{1 \cdot (1-p) - (-1) \cdot p}{(1-p)^2} = \frac{1}{p(1-p)},$$

  and the asymptotic variance of $\widehat{\theta}_{\text{MLE}}$ is given by

$$\left[g'(p)\right]^2 \mathcal{I}^{-1}(p) = \frac{1}{p^2(1-p)^2} \cdot \frac{p(1-p)}{n} = \frac{1}{np(1-p)}.$$

- In conclusion,

$$\log \frac{\overline{X}}{1-\overline{X}} \sim AN\left(\log \frac{p}{1-p}, \frac{1}{np(1-p)}\right).$$

Consistency of MLEs

intuition: MLE already converges in distribution to normal

- We have shown that $\mathcal{I}(\theta) = n\mathcal{I}^*(\theta)$. From this we gather that –
  1. the Fisher information grows along with the sample size,
  2. the variance of $\widehat{\theta}_{\text{MLE}}$ tends to 0 as $n \to \infty$, and that
  3. $\sqrt{n}(\widehat{\theta}_{\text{MLE}} - \theta) \sim AN(0, \mathcal{I}^{*-1}(\theta))$ no this is right!!! have to be star

Theorem

If the regularity conditions for the asymptotic normality are satisfied, the MLE is consistent (in probability).

Proof: Recall that the meaning of asymptotic normality is that

$$F_{Z_n}(z) \overset{n\to\infty}{\longrightarrow} \Phi(z) \quad \text{for} \quad Z_n = \sqrt{n\mathcal{I}^*(\theta)}(\widehat{\theta}_{\text{MLE}} - \theta).$$

Consistency of MLEs (cont.)

- We need to show that for any $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(|\widehat{\theta}_{\text{MLE}} - \theta| > \varepsilon\right) = 0,$$

or conversely –

$$\lim_{n\to\infty} \mathbb{P}\left(|\widehat{\theta}_{\text{MLE}} - \theta| \le \varepsilon\right) = 1.$$

- But

should be I not I* here

$$\mathbb{P}\left(|\widehat{\theta}_{\text{MLE}} - \theta| \le \varepsilon\right) = \mathbb{P}\left(\left|\sqrt{n\mathcal{I}^*(\theta)}(\widehat{\theta}_{\text{MLE}} - \theta)\right| \le \varepsilon\sqrt{n\mathcal{I}^*(\theta)}\right)$$

$$= 2\Phi\left(\varepsilon\sqrt{n\mathcal{I}^*(\theta)}\right) - 1 + \underline{\delta_n},$$

where $\delta_n \to 0$ as $n \to \infty$ (why?)

deviation from normality

- Hence

$$\lim_{n\to\infty} \mathbb{P}\left(|\widehat{\theta}_{\text{MLE}} - \theta| \le \varepsilon\right) = 2 - 1 + 0 = 1.$$

The plug-in principle

- In the Poisson example, we concluded that $\widehat{\lambda}_{\text{MLE}} \sim AN\left(\lambda, \frac{\lambda}{n}\right)$
- Likewise, in the Bernoulli example $\widehat{p}_{\text{MLE}} \sim AN\left(p, \frac{p(1-p)}{n}\right)$
- In general $\widehat{\theta}_{\text{MLE}} \sim AN\left(\theta, \sigma^2(\theta)\right)$ is this always true
- We wish to provide some measure of uncertainty about our estimate of θ
  - A natural candidate: the standard deviation
  - Useless if dependent on θ – the same parameter we wish to estimate

Definition

The standard deviation of an estimator $\widehat{\theta}$ of a parameter θ is called the *standard error* of $\widehat{\theta}$.

The plug-in principle (cont.)

- The following result is a further application of Slutsky's Theorem:

Theorem

Let $\widehat{\theta}_{\text{MLE}}$ be the MLE of θ satisfying the regularity conditions for asymptotic normality, i.e.

$$\widehat{\theta}_{\text{MLE}} \sim AN\big(\theta, \mathcal{I}^{-1}(\theta)\big).$$

Then for any consistent estimator $\widehat{\theta}$ of θ

$$\widehat{\theta}_{\text{MLE}} \sim AN\big(\theta, \mathcal{I}^{-1}(\widehat{\theta})\big).$$

In particular,

$$\widehat{\theta}_{\text{MLE}} \sim AN\big(\theta, \mathcal{I}^{-1}(\widehat{\theta}_{\text{MLE}})\big).$$

pluggin in consistent estimators to variance does not affect asymptotic normality of MLE sampling distribution

Example: Bernoulli sample

- Suppose that we flip a coin 100 times and it comes up heads 80 times
- p - the probability of coming up heads
- We have found that $\widehat{p}_{\mathrm{MLE}} = \overline{X} = 80/100 = 0.8$
- Moreover, we calculated that $\widehat{p}_{\mathrm{MLE}} \sim AN\left(p, \frac{p(1-p)}{n}\right)$
- The latest result states that

$$\widehat{p}_{\mathrm{MLE}} \sim AN\left(p, \frac{\overline{X}(1-\overline{X})}{n}\right) = AN\left(p, \frac{0.8 \times 0.2}{100}\right) = AN\left(p, 0.04^2\right)$$

plugin estimators to population param

- We say that "the maximum likelihood estimate of p is 0.8, with an estimated standard error of 0.04".

Summary of Maximum Likelihood Estimation

- Under some regularity conditions

$$\widehat{\theta}_{\mathrm{MLE}} \sim AN\left(\theta, \mathcal{I}^{-1}(\theta)\right), \quad \text{where} \quad \mathcal{I}(\theta) = -\mathbb{E}\left[\ell''(\theta)\right].$$

- If said conditions are met, the MLE is also consistent in probability.
- The MLE of $g(\theta)$ is $g(\widehat{\theta}_{\mathrm{MLE}})$. If $g(\cdot)$ is differentiable then

$$g(\widehat{\theta}_{\mathrm{MLE}}) \sim AN\left(g(\theta), \left[g'(\theta)\right]^2 \mathcal{I}^{-1}(\theta)\right).$$

- When the asymptotic variance of $\widehat{\theta}_{\mathrm{MLE}}$ is a function of θ itself, we can substitute $\widehat{\theta}_{\mathrm{MLE}}$ for θ. The *estimated standard error* is then

$$\widehat{\sigma}_{\widehat{\theta}_{\mathrm{MLE}}} = \mathcal{I}^{-1/2}(\widehat{\theta}_{\mathrm{MLE}}).$$

1. MLE sampling distribution
2. MLE is consistent
3. MLE is functional invariant
4. can derive estimated MLE standard error by plug in principle

A note on regularity conditions

- We repeatedly mentioned “regularity conditions” that must be satisfied for MLEs to be asymptotically normal
- A long list of conditions, allowing for the full proof to be carried out
- For example, differentiation and integration need be interchangeable
- $\widehat{\theta}_{\text{MLE}}$ must not be a boundary point of the parameter space
- Other conditions concern the differentiability of the likelihood
- When these conditions are violated, $\sqrt{n}\left(\widehat{\theta}_{\text{MLE}} - \theta\right)$ will not necessarily converge to a normal distribution
- See handout for an example