Ver.A

University of Toronto

Faculty of Arts and Science
April 2016 Examination
MAT237Y1Y - Advanced Calculus
Duration - 3 hours
Examiners: T. Holden, D. Le, E. Mazzeo
No Aids Permitted

Last Name: ______________________________

First Name: ______________________________

Student Number: ______________________________

This exam contains 12 pages (including this cover page) and 9 problems. Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in.

No aids are permitted on this examination. Examples of aids that are not permitted include, but are not limited to, textbooks, notes, calculators, or any electronic device.

Unless otherwise indicated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work,** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **The last two pages of the test are for rough work, and will not be marked.**

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total:	90	

1. **In each case below, determine whether the set S is closed. Answer one of "yes", "no", or "cannot be determined'. Only the answer in the box will be marked.**

(a) (2 points) $S = \mathbb{Z}$, the integers.

Yes	No	Cannot be determined

(b) (2 points) $S = \{a_n : n \in \mathbb{N}\}$. where $(a_n)_{n=1}^{\infty}$ is a convergent sequence in $\mathbb{R}$.

Yes	No	Cannot be determined

(c) (2 points) $S =$ The union of a convergent sequence in $\mathbb{R}$ and its limit point.

Yes	No	Cannot be determined

(d) (2 points) $S = \{x \in \mathbb{R} : e^{\cos(x)} = \frac{1}{2}\}$

Yes	No	Cannot be determined

(e) (2 points) $S = \{x \in \mathbb{R} : \sin(\frac{1}{x}) = 0\}$

Yes	No	Cannot be determined

2. (10 points) Prove **one** of the following results. Clearly indicate which result you are proving by indicating as such in the given box.

   a) Let $K \subseteq \mathbb{R}^n$ be a compact set, and $f : \mathbb{R}^n \to \mathbb{R}^m$ a continuous function. Show that $f(K) \subseteq \mathbb{R}^m$ is compact.

   b) Let $K \subseteq \mathbb{R}^n$ be a path connected set, and $f : \mathbb{R}^n \to \mathbb{R}^m$ a continuous function. Show that $f(K) \subseteq \mathbb{R}^m$ is path connected.

Choice

3. (a) (6 points) Let $f, g, h : \mathbb{R}^2 \to \mathbb{R}$ be C^1-functions with $z = f(x, y), x = g(s, t)$, and $y = h(s, t)$. Using the following table, determine $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ at $(s, t) = (0, 0)$.

$$\begin{array}{lll} f(1,2) = 4 & f_x(1,2) = -1 & f_y(1,2) = 2 \\ g(0,0) = 1 & g_s(0,0) = 2 & g_t(0,0) = 1 \\ h(0,0) = 2 & h_s(0,0) = 0 & h_t(0,0) = -3. \end{array}$$

(b) (4 points) Define the function $F : \mathbb{R}^2 \to \mathbb{R}$ by $F(s, t) = f(g(s, t), h(s, t))$ as above. Write down the first order Taylor polynomial for F at $(s, t) = (0, 0)$.

4. (10 points) Find the maximum and minimum of the function $f(x, y) = xy$ on the set

$$S = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{8} + \frac{y^2}{2} \leq 1 \right\}.$$

5. Let S be the set $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq z \leq 1\} \subset \mathbb{R}^3$.

(a) (3 points) Find a parametrization for S, and state its domain.

(b) (3 points) Find the determinant of the jacobian matrix of the parametrization in your answer to part (a).

(c) (4 points) Calculate $\int\int\int_S e^{x^2+y^2+z}\,dxdydz$ in terms of the constants e and π.

6. Let $D \subset \mathbb{R}^2$ be the region in the first quadrant bounded by the curves:

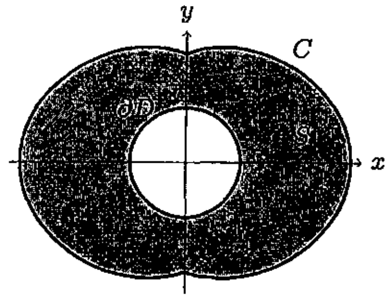
$$y = x^2, \quad y = x^2/5, \quad xy = 2, \quad xy = 4.$$

(a) (4 points) Define the variables $u = x^2/y$ and $v = xy$. Compute $dxdy$ in terms of $dudv$.

(b) (6 points) Compute the area of D by changing variables from (x, y) to (u, v).

7. Let $D = \{(x, y) : x^2 + y^2 < 1\}$ be the open unit disk in $\mathbb{R}^2$ and C be an arbitrary simple closed curve enclosing D. Let S be the area enclosed by C without D. shown in the figure. Consider the vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(x, y) = \frac{7}{x^2 + y^2}(-y, x).$$



(a) (2 points) Using the given figure, clearly label the Stokes' orientation on ∂S.

(b) (4 points) Use Green's Theorem to show that, under the Stokes' orientation

$$\int_C F \cdot dx = -\int_{\partial D} F \cdot dx.$$

(c) (4 points) By explicitly parameterizing ∂D, compute $\int_C F \cdot dx$.

8. Let $S \subset \mathbb{R}^3$ be the part of the surface of the cyclinder $x^2 + y^2 = 4$ between the two planes $z = 0$ and $z = y + 3$.

   (a) (2 points) Find a parameterization $\phi : D \to S$ for the surface S.

   (b) (2 points) Draw a picture of the domain D of the parametrization ϕ of S.

   (c) (3 points) Compute the vector surface area element $\hat{\mathbf{n}}\ dA$ in terms of the parameters. (Note: dA represents the scalar surface area element. It can also be denoted dS.)

   (d) (3 points) Let $\mathbf{F}(x, y, z) = (x, y, z)$. Compute the surface integral $\displaystyle\int\int_S \mathbf{F} \cdot \hat{\mathbf{n}}\ dA$.

9. (a) (3 points) State Stokes' Theorem.

(b) (2 points) Let S_1 and S_2 be smooth surfaces in $\mathbb{R}^3$ with piecewise smooth boundary satisfying $\partial S_1 = \partial S_2$. If $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1 vector field, and S_1 and S_2 are oriented so that the Stokes orientation on ∂S_1 is the same as that of ∂S_2, show that

$$\iint_{S_1} (\operatorname{curl} \mathbf{F}) \cdot \hat{\mathbf{n}}\, dA = \iint_{S_2} (\operatorname{curl} \mathbf{F}) \cdot \hat{\mathbf{n}}\, dA.$$

(c) (5 points) Assume that $S \subseteq \mathbb{R}^3$ is an oriented smooth surface with boundary

$$\partial S = \{(x, y, 0) : x^2 + y^2 = 1\}.$$

If $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1 vector field such that $(\operatorname{curl} \mathbf{F}) \cdot (0, 0, 1) = 0$, show that

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot \hat{\mathbf{n}}\, dA = 0.$$

Rough Work

This page will not be marked

Rough Work

This page will not be marked