

Chapter 2 Subgroups

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1 Definition and Examples

Definition. (Subgroup)

1. **(subgroup)** Let G be a group. The subset H of G is a subgroup of G , denoted as $H \leq G$ if

(a) H is nonempty

(b) H is closed under products and inverses, i.e. $x, y \in H$ implies $x^{-1}, xy \in H$

If $H \leq G$ and $H \neq G$, then $H < G$. $H \leq G$ implies operation on H is the operation on G restricted to H . So any equation in H can also be viewed as equation in G

2. **(The Subgroup Criterion)** $H \subset G$ is a subgroup if and only if

(a) $H \neq \emptyset$

(b) for all $x, y \in H$, $xy^{-1} \in H$

Furthermore, if H is finite, then suffice to check H is nonempty and closed under multiplication

• (examples)

– $G \leq G$ and $\{1\} \leq G$ (latter is called the trivial subgroup)

– $\mathbb{Z} \leq \mathbb{Q} \leq \mathbb{R}$ under operation of addition

– $\{1, r, r^2, \dots, r^{n-1}\} \leq D_{2n}$

– $2\mathbb{Z} \leq \mathbb{Z}$

– $(\mathbb{Q}^\times, \times) \not\leq (\mathbb{R}, +)$ (operation are different)

– $\mathbb{Z}^+ \leq \mathbb{Z}$ and $(\mathbb{Z}^+)^\times \not\leq \mathbb{Q}^\times$ (not closed under inverses and does not contain identity)

– $D_6 \not\leq D_8$ ($D_6 \not\subset D_8$)

• (theorem) subgroup is a transitive relation, i.e. $K \leq H, H \leq G$, then $K \leq G$

2 Centralizers and Normalizers, Stabilizers and Kernels

Definition. (Centralizers and Normalizers) Let G be a group and $A \subset G$ be nonempty

1. **(centralizer)** The centralizer of A in G is a subset of G which commute with every element of A

$$C_G(A) = \{g \in G \mid gag^{-1} = a \text{ for all } a \in A\}$$

$$\bullet \quad ga = ag \iff gag^{-1} = a$$

2. **(center)** The center of G is a subset of G which commutes with all the elements of G

$$Z(G) = C_G(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$$

3. **(normalizer)** The normalizer of A in G are subsets of G that fixes A by conjugation

$$N_G(A) = \{g \in G \mid gAg^{-1} = A\}$$

$$\text{where } gAg^{-1} = \{gag^{-1} \mid a \in A\}$$

• (convention) For $A = \{a\}$, write $C_G(a)$ instead of $C_G(\{a\})$. Note $a^n \in C_G(a)$ for all $n \in \mathbb{Z}^+$

• (theorem) $Z(G) \leq C_G(A) \leq N_G(A) \leq G$

Proof. proofs for $C_G(A)$ and $N_G(A)$ are subgroups of G are similar. For now want to show $N_G(A) \leq G$. Note $1 \in N_G(A)$ so $N_G(A) \neq \emptyset$. Let $g_1, g_2 \in N_G(A)$, then $g_1Ag_1^{-1} = A$ and $g_2Ag_2^{-1} = A$. therefore

$$g_1g_2^{-1}A(g_1g_2^{-1})^{-1} = g_1g_2^{-1}(g_2Ag_2^{-1})g_2g_1^{-1} = g_1Ag_1^{-1} = A$$

hence $g_1g_2^{-1} \in N_G(A)$. So $N_G(A) \leq G$. □

• (examples)

- If G is abelian
 - * $Z(G) = G$
 - * $C_G(A) = N_G(A) = G$ for any subset A ($gag^{-1} = gg^{-1}a = a$ for all $a \in A, g \in G$)
- $C_{Q_8}(i) = \{\pm 1, \pm i\}$
- Let $G = D_8$ and $A = \{1, r, r^2, r^3\} \leq G$ be subgroup of rotations
 - * $C_{D_8}(A) = A$
 - * $N_{D_8}(A) = D_8$
 - * $Z(D_8) = \{1, r^2\}$

Proof. **(1)** Since all powers of r commutes with each other, $A \leq C_{D_8}(A)$. since $sr^i = r^{-i}s \neq r^is$, s does not commute with any rotation, so $s \notin C_{D_8}(A)$. In fact, any $sr^i \notin C_{D_8}(A)$ where $i \in \{0, 1, 2, 3\}$. If assume for contradiction, $s = (sr^i)(r^{-i}) \in C_{D_8}(A)$, a contradiction. Hence $C_{D_8} = A$. **(2)** Note, $A \leq N_{D_8}(A)$ by fact that centralizer is contained in normalizer. Now consider

$$sAs^{-1} = \{s1s^{-1}, srs^{-1}, sr^2s^{-1}, sr^3s^{-1}\} = \{1, r^3, r^2, r\} = A$$

so that $s \in N_{D_8}$. Since $r, s \in N_{D_8}(A)$ and N_{D_8} is closed under multiplication (its a subgroup!), $s^i r^j \in N_{D_8}$ for all i, j . $D_8 \leq N_{D_8}$, hence $N_{D_8}(A) = D_8$ **(3)** Note, $Z(D_8) \leq A$ by fact that center is contained in the centralizer. Note $sr = r^{-1}s = r^3s \neq rs$ and $sr^3 = r^{-3}s = rs \neq r^3s$ hence $r, r^3 \notin Z(D_8)$ (but $sr^2 = r^{-2}s = r^2s$). Therefore $Z(D_8) \leq \{1, r^2\}$. The reverse inclusion holds by 1 (and r^2) commutes with r and s . Since r, s generates D_8 , every element of D_8 commutes with 1 (and r^2) hence $\{1, r^2\} \leq Z(D_8)$ and so equality holds. \square

- Let $G = S_3$ and $A = \{1, (1\ 2)\}$,
 - * $C_{S_3}(A) = A$
 - * $N_{S_3}(A) = A$
 - * $Z(S_3) = \{1\}$

Proof. **(1)** Both 1 and $(1\ 2)$ commutes with all of A hence $A \leq C_{S_3}(A)$ (for $a, g = (1\ 2)$, $gag^{-1} = (1\ 2)(1\ 2)(2\ 1) = (1\ 2)$, commutativity with $a = 1$ or $g = 1$ is trivial). To show $C_{S_3}(A) \leq A$, enough to show that both $(2\ 3)$ and $(1\ 3)$ do not commute with all elements of A , specifically $(1\ 2)$ (by fact that transpositions generates S_3). $(2\ 3)(1\ 2) = (1\ 3\ 2) \neq (1\ 2\ 3) = (1\ 2)(2\ 3)$ similarly for $(1\ 3)$. Alternatively, by Lagrange theorem, $|C_{S_3}(A)| \mid |S_3| = 6$ and $2 = |A| \mid |C_{S_3}(A)|$. Possible values for $|C_{S_3}(A)|$ are 2 or 6. If latter is true, then $C_{S_3}(A) = S_3$ but this is a contradiction since $(2\ 3)$ does not commute with $(1\ 2)$. So $|C_{S_3}(A)| = 2$ hence $C_{S_3}(A) = A$. **(2)** Note $N_{S_3}(A) = A$ because $\sigma \in N_{S_3}(A)$ if and only if

$$\sigma A \sigma^{-1} = \{\sigma 1 \sigma^{-1}, \sigma(1\ 2) \sigma^{-1}\} = \{1, (1\ 2)\} = A$$

if and only if $\sigma(1\ 2) \sigma^{-1} = (1\ 2)$, i.e. $\sigma \in C_{S_3}(A) = A$. **(3)** $Z(S_3) \leq C_{S_3}(A) = A$ and $(1\ 2) \notin Z(S_3)$ \square

Definition. (Stabilizers and Kernels of Group Actions)

1. (**stabilizer**) If G is a group acting on a set S and $s \in S$ is a fixed element, the stabilizer of s in G is

$$G_s = \{g \in G \mid g \cdot s = s\}$$

2. (**kernel**) of action of G on S is defined as

$$\ker \varphi = \{g \in G \mid g \cdot s = s \text{ for all } s \in S\}$$

3. (**centralizers and normalizers as kernels of some group action**)

(a) (normalizer) Let $G \curvearrowright \mathcal{P}(G)$ by conjugation, i.e. for any $g \in G$ and $B \subset G$

$$g : B \rightarrow gBg^{-1} \quad \text{where} \quad gBg^{-1} = \{gbg^{-1} \mid b \in B\}$$

This is a group action.

The normalizer of G on A is the stabilizer of A when G acts on $\mathcal{P}(G)$ by conjugation, i.e. $N_G(A) = G_s$ where $s = A \subset \mathcal{P}(G)$. Therefore $N_G(A) \leq G$

(b) (centralizer) Let $N_G(A) \curvearrowright A$ by conjugation, i.e. for any $g \in N_G(A)$ and $a \in A$

$$g : a \rightarrow gag^{-1}$$

which maps A to A by definition of $N_G(A)$ fixing A and so gives an action on A .

The centralizer of G on A is simply the kernel of $N_G(A)$ acting on A by conjugation.

Since $C_G(A) \leq N_G(A)$ and $N_G(A) \leq G$, we have $C_G(A) \leq G$

(c) (center) The center of G is the kernel of G acting on $S = G$ by conjugation

- **(theorem)** $\ker(G \curvearrowright S) \leq G$
- **(theorem)** $G_s \leq G$ ($1 \in G_s$ and $(xy^{-1}) \cdot s = (xy^{-1}) \cdot (y \cdot s) = x \cdot s = s$ for any $x, y \in G_s$)
- (examples)
 - Let $G = D_8$ and $A = \{1, 2, 3, 4\}$ the vertices of a square. Then the stabilizer of any vertex $a \in A$ is the subgroup $\{1, t\} \leq D_8$, where t is the reflection about line of symmetry passing through a and center of the square. The kernel of the action is just the identity
 - Let $G = D_8$ and $A = \{\{1, 3\}, \{2, 4\}\}$ be the two unordered pairs of opposite vertices. The kernel of the action of G on A is the subgroup $\{1, s, r^2, sr^2\}$ and for any $a \in A$, the stabilizer of a in D_8 is equal to the kernel of the action