

# Chapter 1 Introduction to Groups

## Contents

<b>1</b>	<b>Basic Axioms and Examples</b>	<b>2</b>
<b>2</b>	<b>Dihedral Groups</b>	<b>3</b>
<b>3</b>	<b>Symmetric Groups</b>	<b>4</b>
<b>4</b>	<b>Matrix Groups</b>	<b>4</b>
<b>5</b>	<b>Quaternion Group</b>	<b>5</b>
<b>6</b>	<b>Homomorphisms and Isomorphisms</b>	<b>5</b>
<b>7</b>	<b>Group Actions</b>	<b>7</b>

# 1 Basic Axioms and Examples

## Definition. (Binary Operation)

1. **(binary operation)**  $\star$  on a set  $G$  is a function  $\star : G \rightarrow G$ . write  $a \star b$  instead of  $\star(a, b)$
  2. **(associative  $\star$ )** A binary operation on  $G$  is associative if for all  $a, b, c \in G$   $a \star (b \star c) = (a \star b) \star c$
  3. **(commutative  $\star$ )** A binary operation on  $G$  is commutative if for all  $a, b \in G$ ,  $a \star b = b \star a$
  4. **(closed under  $\star$ )**  $\star$  is a binary operation on  $G$  and  $H \subset G$ , if  $\star|_H$  is a binary operation on  $H$ , i.e. for all  $a, b \in H$ ,  $a \star b \in H$ , then  $H$  is closed under  $\star$ . Associativity/Commutativity of  $\star$  is inherited on  $H$
- (examples)
    1.  $+$  on  $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$  is a commutative binary operation
    2.  $\times$  on  $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$  is a commutative binary operation
    3.  $-$  is not commutative on  $\mathbb{Z}$  ( $a - b \neq b - a$  usually)
    4.  $-$  is not commutative on  $\mathbb{Z}^+$  ( $1, 2 \in \mathbb{Z}^+$ , but  $1 - 2 = -1 \notin \mathbb{Z}^+$ )

## Definition. (Group)

1. **(group)** A group is an ordered pair  $(G, \star)$  where  $G$  is a set and  $\star$  is a binary operation on  $G$  satisfying
  - (a) (associative)  $\forall a, b, c \in G$ ,  $(a \star b) \star c = a \star (b \star c)$
  - (b) (identity)  $\exists e \in G \forall a \in G$   $a \star e = e \star a = a$  ( $e$  is an identity of  $G$ , alternatively denoted by  $1$ )
  - (c) (inverse)  $\forall a \in G \exists a^{-1} \in G$ ,  $a \star a^{-1} = a^{-1} \star a = e$  ( $a^{-1}$  is an inverse of  $a$ )
2. **(abelian group)** A group is abelian/commutative if  $a \star b = b \star a$  for all  $a, b \in G$
3. **(finite group)**  $G$  is a finite group if  $G$  is a finite set
4. **(direct product)** If  $(A, \star)$  and  $(B, \circ)$  are groups, a new group  $A \times B$  called direct product are defined as

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

with binary operation defined component-wise

$$(a_1, b_1)(a_2, b_2) = (a_1 \star a_2, b_1 \circ b_2)$$

- (examples)
  - $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$  are groups under  $+$  ( $e = 0$ ,  $a^{-1} = -a$ , associativity by axioms of  $+$ )
  - $\mathbb{Q} - \{0\}, \mathbb{R} - \{0\}, \mathbb{C} - \{0\}, \mathbb{Q}^+, \mathbb{R}^+$  are groups under  $\times$  ( $e = 1$ ,  $a^{-1} = 1/a$ , associativity by  $\times$ )
  - $(\mathbb{Z} - \{0\}, \times)$  is not a group ( $2^{-1} = 1/2 \notin \mathbb{Z} - \{0\}$ )
  - $(V, +)$  is an abelian group, where  $V$  is a vector space (commutativity by axioms of a vector space)
  - $(\mathbb{Z}/n\mathbb{Z}, +)$  is an abelian group ( $e = \bar{1}$ ,  $a^{-1} = \overline{-a}$ )
  - $((\mathbb{Z}/n\mathbb{Z})^\times, \times)$  is abelian group ( $e = \bar{1}$ ,  $a^{-1}$  exists by definition of  $(\mathbb{Z}/n\mathbb{Z})^\times$ )
- **(theorem)** direct product of two groups is a group
- **(proposition)**
  1. (identity unique) identity of  $G$  is unique
  2. (inverse unique) inverse  $a^{-1}$  of any  $a$  in  $G$  is unique
  3.  $(a^{-1})^{-1} = a$  for all  $a$  in  $G$
  4.  $(a \star b)^{-1} = b^{-1} \star a^{-1}$
  5. (generalized associativity law) value of  $a_1 \star a_2 \star \dots \star a_n$  independent of how its bracketed
- (notation)

- $(\times)$  denote  $x^n = xx \cdots x$  by  $x^n$  and  $x^{-n} = x^{-1}x^{-1} \cdots x^{-1}$  and  $x^0 = 1$  the identity
- $(+)$  denote  $na = a + a + \cdots + a$  and  $-na = -a - a - \cdots - a$  and  $0a = 0$  the identity

• **(proposition)** Let  $a, b, u, v \in G$

1. (left cancellation law holds) if  $au = av$ , then  $u = v$
2. (right cancellation law holds) if  $ub = vb$ , then  $u = v$

**Definition. (order for an element  $x \in G$ )** is the smallest positive integer  $n \in \mathbb{Z}^+$  such that  $x^n = 1$ , denoted by  $|x|$ . If no positive power of  $x$  is the identity, the order of  $x$  is defined to be infinity

• (examples)

- if  $|x| = 1$ , then  $x = 1$  the identity
- In  $(\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, +)$ , every nonzero elements has infinite order
- In  $(\mathbb{R} - \{0\}, \mathbb{Q} - \{0\}, \times)$ ,  $|-1| = 2$  and all other nonidentity elements have infinite order
- In  $\mathbb{Z}/9\mathbb{Z}$ ,  $|\bar{5}| = 9$  since 9 is the smallest integer multiple of 5 that is congruent to 0 (mod 9)
- In  $(\mathbb{Z}/7\mathbb{Z})^\times$ ,  $|\bar{3}| = 6$  since  $3^6$  is smallest positive power of 3 that is congruent to 1 (mod 7)

**Definition. (multiplication/group table)** Let  $G = \{g_1, g_2, \dots, g_n\}$  be a finite group where  $g_1 = 1$ . The multiplication or group table of  $G$  is a  $n \times n$  matrix  $A$  where  $A_{ij} = g_i g_j$ .

• (fact) For finite groups, the group table contains all information about the group

## 2 Dihedral Groups

**Definition. (Dihedral Groups)**

1. **(symmetry of  $n$ -gon)** is any rigid motion of the  $n$ -gon. We can describe symmetry by choosing a labelling of vertices  $\{1, 2, \dots, n\}$  and let the corresponding permutation  $\sigma$  over the set as symmetry  $s$
2. **(order of  $D_{2n}$ )** is  $2n$ . (lower bound: vertex 1 can be sent to any vertex  $i$ , and vertex 2 can be sent to either  $i - 1$  or  $i + 1$ . Knowing position of 1, 2 determines position of all other vertices; upper bound: by reasoning that any element of  $D_{2n}$  can be written as  $r^i s^j$  where  $0 \leq i \leq n - 1$  and  $0 \leq j \leq 1$ )
3. **(dihedral group  $D_{2n}$ )** Fix a regular  $n$ -gon at origin and label vertices through from 1 to  $n$  in a clockwise manner. Let  $r$  be rotation clockwise about the origin through  $2\pi/n$  radian and let  $s$  be reflection about line of symmetry through vertex 1 and the origin.

$$D_{2n} = \{r, s \mid r^n = s^2 = 1, sr^k = r^{-k}s\} = \{1, r, r^2, \dots, r^{n-1}, s, rs, r^2s, \dots, r^{n-1}s\}$$

(a)  $|r| = n$  and  $|s| = 2$

(b)  $s \neq r^i$  for any  $i$  and  $sr^i \neq sr^j$  for all  $i \neq j$

(c)  $r^k s = sr^{-k}$  for all  $0 \leq i \leq n$

4. **(interpreting presentation for  $D_{2n}$ )**  $r^n = 1$  means any power of  $r$  can be reduced so that the power lie between 0 and  $n - 1$ . Similarly, any power of  $s$  can be reduced so that the power is either 0 or 1.  $sr^k = r^{-k}s$  means every element in the group can be written as  $r^i s^j$  for some  $i, j$

• (fact)  $D_{2n}$  for  $n \geq 3$  is non-abelian

**Definition. (generators and relations)**

1. **(generators of  $G$ )** is the set  $S \subset G$  where every element of  $G$  can be written as a (finite) product of elements of  $S$  and their inverses. Denote  $G = \langle S \rangle$  and say  $G$  is generated by  $S$  and  $S$  generates  $G$
2. **(relations in  $G$ )** any equation in a general group  $G$  that the generator satisfies
3. **(presentation of  $G$ )** If  $G = \langle S \rangle$  and  $R_1, R_2, \dots, R_m$  are relations in  $G$  such that any relation among  $S$  can be deduced from these, the generators and relations are called presentations

$$G = \langle S \mid R_1, R_2, \dots, R_m \rangle$$

• (example)  $\mathbb{Z} = \langle 1 \rangle$

• (example)  $D_{2n} = \langle r, s \rangle$

### 3 Symmetric Groups

**Definition. (Symmetric Group)**

1. (**symmetric group**  $S_\Omega$  **on set**  $\Omega$ ) Let  $\Omega$  be nonempty set,  $S_\Omega = \{\sigma : \Omega \rightarrow \Omega \mid \sigma \text{ is a bijection}\}$ , the set of all permutations of  $\Omega$ .  $(S_\Omega, \circ)$  is the symmetric group on  $\Omega$ .
2. (**symmetric group of degree**  $n$ ) If  $\Omega = \{1, 2, \dots, n\}$ ,  $S_n$  is the symmetric group of degree  $n$
3. ( $|S_n| = n!$ ) (by counting number of possible permutations using the constraint that  $\sigma$  is injective)
4. (**cycle**) a string of integers representing elements of  $S_n$ , which cyclically permutes them.  $(a_1 a_2 \dots a_m)$  is the permutation sending  $a_i$  to  $a_{i+1}$ .  $1 \leq i \leq m-1$  and sends  $a_m$  to  $a_1$
5. (**length of cycle**) is the number of integers which appear in it
6. (**t-cycle**) is a cycle with length  $t$
7. (**disjoint cycle**) A cycle is disjoint if they have no numbers in common
8. (**k cycles**) Any  $\sigma \in S_n$ , we can represent  $\sigma$  with  $k$  cycles of the form

$$(a_1 a_2 \dots a_{m_1})(a_{m_1+1} a_{m_1+2} \dots a_{m_2}) \dots (a_{m_{k-1}+1} a_{m_{k-1}+2} \dots a_{m_k})$$

9. (**cycle-decomposition of**  $\sigma$ ) is the product of  $k$ -cycles that representing  $\sigma$
10. (**transposition**) is a permutation which exchanges two element while leaving other elements fixed, i.e. a 2-cycle
  - (convention) 1-cycle not written during cycle-decomposition. This convention ensures that cycle decomposition of  $\tau \in S_n$  is exactly the same as cycle decomposition of permutation in  $S_m$  where  $m > n$ , which acts as  $\tau$  on  $\{1, 2, \dots, n\}$  and fixes elements in  $\{n+1, n+2, \dots, m\}$
  - (computing inverse) Let  $\sigma \in S_n$ , cycle decomposition of  $\sigma^{-1}$  can be obtained by writing numbers in each cycle of the cycle decomposition of  $\sigma$  in reverse order
  - (computing product) by following elements in successive permutations
  - (example)  $S_n$  is non-abelian for  $n \geq 3$  (counterexample:  $(12) \circ (13) = (1\ 3\ 2)$  but  $(13) \circ (12) = (1\ 2\ 3)$ )
  - (**proposition**) disjoint cycle commutes
  - (**proposition**) cycle-decomposition uniquely expresses a permutation as a product of disjoint cycles
  - (**proposition**) The order of a permutation is the l.c.m. of the lengths of cycles in its cycle decomposition
  - (**proposition**) transpositions generate finite symmetric group

*Proof.* By cycle-decomposition, we can express every permutation with products of disjoint cycles. It suffices to show that every cycle can be expressed as a product of transpositions. Let  $(a_1 a_2 \dots a_n)$  be any  $r$ -cycle. We can show that the  $r$ -cycle can be written with a  $r-1$ -cycle  $(a_1 a_r)(a_1 a_{r-1}) \dots (a_1 a_2)$ .  $\square$

### 4 Matrix Groups

**Definition. (Field and Matrix Group)**

1. (**field**) A field is a set  $F$  with two binary operations  $+$  and  $\cdot$  such that  $(F, +)$  is an abelian group and  $(F - \{0\}, \cdot)$  is also an abelian group, and follows distributive law

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

Denote  $F^\times = F - \{0\}$ .

2. (**general linear group**) For each  $n \in \mathbb{Z}^+$ , let  $GL_n(F)$  be the set of all  $n \times n$  matrices whose entries come from  $F$  and whose determinant is nonzero

$$GL_n(F) = \{A \mid A \text{ is } n \times n \text{ matrix with entries from } F \text{ and } \det A \neq 0\}$$

with matrix multiplication as the binary operation.  $GL_n(F)$  is a group under matrix multiplication, called **general linear group of degree  $n$** : since its closed under matrix multiplication, and satisfies inverse/identity axioms

- (example)  $\mathbb{Q}, \mathbb{R}$ , and  $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$  where  $p$  is prime are fields
- (fact)  $GL_n(F)$  for  $n \geq 2$  is nonabelian (matrix multiplication does not commute)
- (**theorem**) If  $F$  is a field and  $|F| < \infty$ , then  $|F| = p^m$  for some prime  $p$  and integer  $m$
- (**theorem**) If  $|F| = q < \infty$ , then  $|GL_n(F)| = (q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1})$

## 5 Quaternion Group

**Definition. (quaternion group)** The quaternion group  $Q_8$  is defined by

$$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$$

with product  $\cdot$  defined as

$$\begin{aligned} 1 \cdot a &= a \cdot 1 = a \quad \forall a \in Q_8 \\ (-1) \cdot (-1) &= 1 \\ (-1) \cdot a &= a \cdot (-1) = -a \quad \forall a \in Q_8 \\ i \cdot i &= j \cdot j = k \cdot k = -1 \\ i \cdot j &= k \quad j \cdot k = i \quad k \cdot i = j \\ j \cdot i &= -k \quad k \cdot j = -i \quad i \cdot k = -j \end{aligned}$$

- (fact)  $Q_8$  is non-abelian
- (fact) order of elements in  $Q_8$

element	order
1	1
-1	2
i, -i, j, -j, k, -k	4

## 6 Homomorphisms and Isomorphisms

**Definition. (homomorphisms)** Let  $(G, \star)$  and  $(H, \diamond)$  be groups. A map  $\varphi : G \rightarrow H$  such that

$$\varphi(x \star y) = \varphi(x) \diamond \varphi(y) \quad \forall x, y \in G$$

is called a **homomorphism**. Intuitively,  $\varphi$  respects the group structures of its domain and codomain

- (**theorem**) If  $\phi : G \rightarrow h$  is a homomorphism, then

1.  $\varphi(e_G) = e_H$
2.  $\varphi(x^{-1}) = \varphi(x)^{-1}$
3.  $\varphi(x^n) = \varphi(x)^n$  for all  $n \in \mathbb{Z}$

**Definition. (Isomorphisms)**

1. (**isomorphisms**) The map  $\varphi : G \rightarrow H$  is called an **isomorphism** and  $G$  and  $H$  are said to be **isomorphic** or of the same **isomorphic type**, write  $G \cong H$ , if

(a)  $\varphi$  is a homomorphism

(b)  $\varphi$  is a bijection

$G$  and  $H$  are the same group, except that elements/operations are written differently.

2. **(isomorphism classes)** Let  $\mathcal{G}$  be nonempty collection of groups. Then  $\cong$  is an equivalence relation on  $\mathcal{G}$ . the equivalence classes are called isomorphism classes

3. **(classification theorems)** determine what properties of a structure specify its isomorphic types, i.e.

any non-abelian group of order 6 is isomorphic to  $S_3$

from which we know  $D_6 \cong S_3$  and  $GL_2(\mathbb{F}_2) \cong S_3$

- **(theorem)**  $G$  and  $H$  share properties which rely on group structures (i.e. commutativity)
- **(theorem)** Isomorphic type of a symmetric group depends on cardinality only  $S_\Delta \cong S_\Omega \iff |\Delta| = |\Omega|$
- **(theorem)** If  $\varphi : G \rightarrow H$  is an isomorphism, then
  1.  $|G| = |H|$
  2.  $G$  is abelian iff  $H$  is abelian
  3. for all  $x \in G$ ,  $|x| = |\varphi(x)|$
- (examples)
  - $G \cong G$  by the identity map or conjugation  $g \mapsto xgx^{-1}$  for some  $x \in G$
  - $(\mathbb{R}, +) \cong (\mathbb{R}^+, \times)$  by the exponential map  $\exp : \mathbb{R} \rightarrow \mathbb{R}^+; x \mapsto e^x$
  - $S_3 \cong D_6$  by the example classification theorem
  - $GL_n(F) \cong F^\times$  by  $\det : GL_n(F) \rightarrow F^\times$ , i.e.  $\det AB = \det A \det B$
  - $S_3 \not\cong \mathbb{Z}/6\mathbb{Z}$  ( $S_3$  is non-abelian;  $\mathbb{Z}/6\mathbb{Z}$  is abelian)
  - $(\mathbb{R}, +) \not\cong (\mathbb{R}^\times, \times)$  ( $-1 \in \mathbb{R}$  has order 2;  $\mathbb{R}^\times$  has no element of order 2)

**Definition. (homomorphism/isomorphism and presentations)** Let  $G$  be a finit group of order  $n$  with a presentation. Let  $S = \{s_1, \dots, s_m\}$  be the generators let  $H$  be another group and  $R = \{r_1, \dots, r_m\}$  be elements of  $H$ . If any relation satisfied in  $G$  by  $s_i$  is also satisfied in  $H$  when each  $s_i$  is replaced with  $r_i$ . Then there exists unique homomorphism  $\varphi : G \rightarrow H; s_i \mapsto r_i$

1. if  $H$  is generated by  $\{r_1, \dots, r_m\}$ , then  $\varphi$  is surjective (any  $r_{i_1}r_{i_2}\dots \in H$  is  $\varphi(s_{i_1}s_{i_2}\dots)$ )
2. if in addition,  $|H| = |G|$ , then surjective  $\varphi$  is necessarily injective,  $\varphi$  is a bijection and  $G \cong H$

(examples)

- Let  $D_{2n} = \{r, s \mid r^n = s^2 = 1 \text{ } sr = r^{-1}s\}$ . Let  $D_{2k} = \{r_1, s_1 \mid r_1^n = s_1^2 = 1 \text{ } s_1r_1 = r_1^{-1}s_1\}$  where  $k \mid n$ , specifically  $n = km$ . Then

$$\varphi : D_{2n} \rightarrow D_{2k} \quad \text{by} \quad \varphi(r) = r_1 \quad \varphi(s) = s_1$$

is a homomorphism by the previous theorem as  $r_1, s_1$  satisfies the relation of  $D_{2n}$ , specifically

$$r_1^n = (r_1^k)^m = 1^m = 1$$

Since  $r_1, s_1$  generates  $D_{2k}$ ,  $\varphi$  is surjective. However for any  $k < n$ ,  $|D_{2n}| \neq |D_{2k}|$  so  $D_{2n} \not\cong D_{2k}$

- $D_6 \cong S_3$

*Proof.* Let  $G = D_6$  and  $H = S_3$ .  $a = (1 \ 2 \ 3), b = (1 \ 2) \in H$  satisfies  $a^3 = b^2 = 1$  and  $ba = ab^{-1}$ . Hence exists unique homomorphism  $\varphi$  by  $\varphi(r) \mapsto a$  and  $\varphi(s) \mapsto b$ . Note  $a, b$  generates  $S_3$ . Therefore  $\varphi$  surjective. Since  $|D_6| = |S_3|$ ,  $\varphi$  is an isomorphism  $\square$

**Definition. (automorphisms)** Let  $G$  be a group and define

$$\text{Aut}(G) = \{\varphi : G \rightarrow G \mid \varphi \text{ is an isomorphism}\}$$

Then  $(\text{Aut}(G), \circ)$  is a group under function composition, called **automorphism group of  $G$** . Any element of  $\text{Aut}(G)$  is an **automorphism** of  $G$

## 7 Group Actions

**Definition. (Group Action)**

1. **(group action)** A group action of a group  $G$  on a set  $A$  is a map satisfying

$$G \curvearrowright A : G \times A \rightarrow A$$

$$(g, a) \mapsto g \cdot a$$

- (a)  $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$  for all  $g_1, g_2 \in G$  and  $a \in A$
- (b)  $1 \cdot a = a$  for all  $a \in A$

2. **(permutation representation)** For each fixed  $g \in G$ , define a map  $\sigma_g$  by

$$\sigma_g : A \rightarrow A$$

$$a \mapsto g \cdot a$$

is a permutation, i.e.  $\sigma_g \in S_A$ . The permutation representation of  $G \curvearrowright A$

$$\varphi : G \rightarrow S_A$$

$$g \mapsto \sigma_g$$

is a homomorphism. Intuitively, a group action of  $G$  on a set  $A$  means every element  $g \in G$  acts as a permutation on  $A$  in a manner consistent with the group operation in  $G$ .

3. **(faithful)** If  $G$  acts on  $B$ , then the action is faithful if

- distinct elements of  $G$  induce distinct permutations of  $B$
- permutation representation is injective

4. **(kernel)** The kernel of action of  $G$  on  $B$  is defined by to

$$\{g \in G \mid gb = b \text{ for all } b \in B\}$$

i.e. elements of  $G$  that fix all elements of  $B$

5. **(trivial action)**  $g \cdot a = a$  is the trivial action and the permutation representation  $\varphi$  is the trivial homomorphism.  $\ker \varphi = G$  and action not faithful when  $|G| > 1$
6. **(left regular action)**  $G \curvearrowright G$  by left multiplication (translation)  $g \cdot a = ga$  ( $g \cdot a = g + a$ ) for all  $g, a \in G$ . Then this action is called left regular action and is faithful by cancellation laws.
- **(theorem)** The actions of a group  $G$  on a set  $A$  and the homomorphisms from  $G$  into  $S_A$  are in bijective correspondence, i.e. the same thing.

*Proof.* ( $\rightarrow$ ) from construction of permutation representation ( $\leftarrow$ ) Let  $\varphi : G \rightarrow S_A$  be any homomorphism, then the map

$$g : G \times A \rightarrow A$$

$$(g, a) \mapsto g \cdot a = \varphi(g)(a)$$

satisfies properties of a group action  $G \curvearrowright A$  □

- (examples)

- $F^\times \curvearrowright V$  where  $F^\times$  is a field and  $V$  is a vector space. For example, action for  $\mathbb{R} \curvearrowright \mathbb{R}^n$  specified by

$$\alpha \cdot (x_1, x_2, \dots, x_n) = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$$

- $S_A \curvearrowright A$  by  $\sigma \cdot a = \sigma(a)$ . The permutation representation  $\varphi : S_A \rightarrow S_A$  is the identity map
- $D_{2n} \curvearrowright \{1, 2, \dots, n\}$  where  $\{1, 2, \dots, n\}$  is a labelling of vertices of a regular  $n$ -gon. The action is faithful or the associated permutation representation  $\varphi : D_{2n} \rightarrow S_n$  is injective since distinct symmetries of a regular  $n$ -gon induce distinct permutations of the vertices. Since  $|D_{2n}| = |S_n|$ ,  $\varphi$  is surjective and so  $\varphi$  is an isomorphism and  $D_6 \cong S_3$ . Geometrically, this means any permutation of vertices of a triangle is a symmetry; however this is not true for any  $n$ -gon with  $n \geq 4$