UNIVERSITY OF TORONTO
Faculty of Arts and Sciences

APRIL 2015 EXAMINATIONS

MAT237Y1Y

Duration - 3 hours
No Aids Allowed

Instructions: There are two parts to this exam: PART A consists of 7 written answer questions, and part B is the multiple choice section containing 9 questions; each correct answer is worth 3 marks and no marks will be given to incorrect or multiple answers. There are 16 pages (numbered on top left corner of the pages) including the cover page and the scrap page at the back. Please do not remove the last page. There is a total of 156 marks which include 31 bonus mark. Please write your answers within the space provided, (and clearly specify if you use back of a sheet to answer a question.) The marking scheme allocates part marks to the details which are inherent in the process, so please use your judgment to include as much details as you know and you consider essential. In the process of writing a proof please carefully mention the results that you may be using in your proofs.

NAME: (last, first) ______________________

STUDENT NUMBER: ______________________

SIGNATURE: ______________________

MARKER'S REPORT:

Question	MARK
Q1	/20
Q2	/20
Q3	/20
Q4	/14
Q5	/22
Q6	/8
Q7	/25
MC	/27
TOTAL	/125

PART A Provide your complete answers in the spaces provided. Please note, part marks will be given to the justifications. Use your judgment and include as much justifications for your solutions with reference to the theorems, definitions and properties from the textbook.

1. Mean Value Theorem for integrals

   a) (6 marks) Present precise statements of theorems and definition:

   - Extreme Value Theorem

   - Intermediate Value Theorem

   - A subset $S \subset \mathbb{R}^2$ is a content zero set:

   b) (6 marks) Use ϵ characterization of integrability for a bounded function g, to prove that if $g(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in S \subset \mathbb{R}^2$ except on a set Z of content zero, then $\iint_S g dA = 0$. (Note: make sure to show both: g is integrable and the integral is zero.)

c) (8 marks) Let S be a compact connected measurable subset of $\mathbb{R}^2$. Let f and g be two continuous functions on S with $g \geq 0$. Prove that there is a point $\boldsymbol{a} \in S$ such that

$$\iint_S f(\boldsymbol{x})g(\boldsymbol{x})dA = f(\boldsymbol{a}) \iint_S g(\boldsymbol{x})dA.$$

Then conclude that under the conditions above there is a point $\boldsymbol{a} \in S$ such that $\iint_S f dA = f(\boldsymbol{a})|S|$. (Note: please make sure to clearly indicate how all the assumptions are used. Also clearly quote other theorems used in the proof.)

2. solving $\nabla f = \boldsymbol{G}$:

a) (4 marks) Present definition and statement:

- Green's theorem: for a C^1 vector field $\boldsymbol{G}(x, y) = (P(x, y), Q(x, y))$ on a regular region S enclosed by a simple closed curve C, oriented counterclockwise, we have ...

- A set S is convex if ....

b) (10 marks) Assume $\boldsymbol{G}(x, y) = (P(x, y), Q(x, y))$ is C^1 on an open convex set $S \subset \mathbb{R}^2$. Also assume that $\partial_x Q = \partial_y P$. Let $\boldsymbol{a}$ be a fixed point in S, and for any given $\boldsymbol{x} \in S$ define $f(\boldsymbol{x}) = \int_{L(\boldsymbol{a},\boldsymbol{x})} \boldsymbol{G} \cdot d\boldsymbol{x}$, where $L(\boldsymbol{a}, \boldsymbol{x})$ is the straight line connecting $\boldsymbol{a}$ to $\boldsymbol{x}$. Explain why this definition is possible, and why it uniquely defines $f(\boldsymbol{x})$, as a function on S. Then prove that $\nabla f = \boldsymbol{G}$. (Note: You need to just prove $\partial_1 f = P(x, y)$. In your proof clearly indicate where you use Green's theorem and where you use the Mean Value Theorem for Integrals.)

5/16

c) (6 marks) Determine whether $\boldsymbol{G}(x,y) = (2xy - \cos(y))\hat{\boldsymbol{\imath}} + (x^2 + x sin(y))\hat{\boldsymbol{\jmath}}$ qualifies to be ∇f for some scalar valued function f. If so compute this function f (be sure to demonstrate how you compute f.) Then, using this information, evaluate $\int_C \boldsymbol{G} \cdot d\boldsymbol{x}$ where C is the curve described by $\boldsymbol{g}(t) = (t, \pi t^2/2), \; -1 \le t \le 2.$

6/16

3. Integration

a) (6 marks) Use a change of variable (a transformation) to evaluate the double integral $\iint_S \frac{dA}{1-3x-2y}$, where S is the region of the third quadrant of the xy plane bounded by the lines $-3x-2y=1$ and $-3x-2y=4$.

b) (6 marks) Calculate the iterated integral: $\int_0^1 \int_{\sqrt{x}}^1 \cos(y^3+1)dydx$ (note: a portion of the mark is given to the proper use of Fubini's theorem.)

c) (8 marks) Consider the iterated integral $\int_0^1 \int_0^{1-x^2} \int_0^{2-z} dydzdx$. Present the iterated integral equivalent to this integral in the form $\iint_D \int dzdA$. (Note: D is a region of the xy plane and may consist of two subregions.) Make sure to clearly describe (and draw) the regions D and its subregions as well as the bounds of integration for the single integral for z.

$\frac{8}{16}$

4. Divergence theorem

a) (10 marks) Use divergence theorem to calculate the flux of the vector field $\boldsymbol{F} = z^2 y\hat{\boldsymbol{i}} + xz\hat{\boldsymbol{j}} + (1+z^2)\hat{\boldsymbol{k}}$ across the surface S consisting of the surface $z = 2 - x^2 - y^2$ above the xy plane. oriented outward.

b) (4 marks) Show that for any vector field $\boldsymbol{F}$ of class C^2, the total flux of $\nabla \times \boldsymbol{F}$, across any closed surface S is zero; that is, prove $\iint_S \nabla \times \boldsymbol{F} \cdot \boldsymbol{n} dA = 0$.

5.

a) (2 mark) present definition: $f : S \longrightarrow \mathbb{R}^n$ is uniformly continuous on S ...

b) (6 marks) Use an appropriate inequality to prove that if for a function $\boldsymbol{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^2$, denoted by $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}))$, is uniformly continuous on a set S if and only if both f_1 and f_2 are uniformly continuous on S.

c) (8 marks) Assume that $S \subset \mathbb{R}^2$ is compact and measurable, and $f(x, y, z)$ is continuous on $[a, b] \times S$. Prove that the function $F(x) = \iint_S f(x, y, z)dA$ is uniformly continuous on $[a, b]$.

d) (6 marks) Use the chain rule and the result about differentiating under the integral sign to determine $F'(1)$ if

$$F(x) = \int_{\sqrt{x}}^{1} \frac{\sqrt{x}}{1 + x^2 + y} dy.$$

6. Bolzano-Weierstrass

a) (2 marks) State the Bolzano-Weierstrass theorem (characterization of compactness):

b) (6 marks) Use part (a) to prove that the continuous image of a compact set is compact; that is, let $\boldsymbol{f}$ be continuous, and S be compact, then the set $T = \boldsymbol{f}(S) = \{\boldsymbol{f}(\boldsymbol{x}) : \boldsymbol{x} \in S\}$ is also compact.

7. Chain Rule and the Mean Value Theorem

a) (8 marks) Definitions:

- Statement of Chain rule (I) for functions $F(\boldsymbol{x})$ and $\boldsymbol{g}(t)$:

- Statement of the Mean Value Theorem for $f(\boldsymbol{x})$ and a set $S \subset \mathbb{R}^n$:

- Taylor's theorem with lagrange remainder: Suppose the function $f(x)$ is $k+1$ times differentiable on an interval $I \subset \mathbb{R}$ and $a \in I$. Then for any h such that $a+h \in$ _______, then

  $f(a+h) = f(a)+$ ______________________________, and there is some point c between ________________

  such that $R_{a,k}(h) =$ ______________________________.

b) (8 marks) Suppose that $F : U \longrightarrow \mathbb{R}$ (with U an open subset of $\mathbb{R}^3$,) is differentiable, and the set $S = \{(x,y,z) \in U : F(x,y,z) = 0\}$ is a smooth surface. If $\boldsymbol{a} \in S$ and $\nabla F(\boldsymbol{a}) \neq \boldsymbol{0}$ then the vector $\nabla F(\boldsymbol{a})$ is perpendicular to the surface S at $\boldsymbol{a}$. Use this fact to find the formula for the tangent plane to the surface $z = \sqrt{x} + arctan(y)$ at the point $\boldsymbol{a} = (9,0,3)$.

c) (3 marks) Remember that Taylor's theorem as in part (a), is a generalization of the MVT. Demonstrate that in case $k = 0$ Taylor's theorem is the MVT.

d) (6 marks) To derive the Taylor's formula for multivariate case from the one variable case: given $f(\boldsymbol{x})$ and a point $\boldsymbol{a}$ and the increment vector $\boldsymbol{h} = \boldsymbol{x} - \boldsymbol{a}$, we consider applying the one variable version of Taylor's to the function $g(t) = f(\boldsymbol{a} + t\boldsymbol{h})$. Do this for $k = 2$ to find the second order Taylor for two variables $\boldsymbol{x} = (x, y)$ and $\boldsymbol{h} = (h_1, h_2)$.

PART B: multiple choice questions Only one answer is correct, and it receives 3 marks. There is no mark for multiple selections or incorrect answers.

1. Let $f(x,y) = xe^y$. With $\boldsymbol{u}$ the unit vector in the direction of $(1,2)$ evaluate $\partial_{\boldsymbol{u}} f(2,0)$.

   A 0, **B** 5, **C** $\sqrt{5}$, **D** $2/\sqrt{5}$, **E** none of the answers A-D.

2. How many of the following implications are correct?

   a) If partial derivatives of f exist then f is differentiable.
   b) If partial derivatives of f exist then f is continuous.
   c) If f is C^1 then f is continuous.
   d) If f has partial derivatives and f is continuous, then f is differentiable.
   e) If $\nabla f(\boldsymbol{x}) = \boldsymbol{0}$ on a set S then f is constant on S.

   A 0; **B** 1; **C** 2; **D** 3; **E** 4.

3. How many of the following ideas are correct?

   a) For any two vectors in $\boldsymbol{R}^3$ we have the Euclidean identity: $|\boldsymbol{a}+\boldsymbol{b}|^2 = |\boldsymbol{a}|^2 + 2\boldsymbol{a}\cdot\boldsymbol{b} + |\boldsymbol{b}|^2$.
   b) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ has a local min at the point a then $f'(a) = 0$.
   c) A curve $y = f(x)$ is smooth at a point $(a, f(a))$ if f is C^1 at a.
   d) A curve $F(\boldsymbol{x}) = 0 = G(\boldsymbol{x})$ is smooth at a point $\boldsymbol{a}$ on the curve if $\nabla F(\boldsymbol{a}) \times \nabla G(\boldsymbol{a}) \neq \boldsymbol{0}$.
   e) For any two vectors in $\boldsymbol{R}^3$ we have: $|\boldsymbol{a}\times\boldsymbol{b}|^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}\cdot\boldsymbol{b})^2$.

   A 0; **B** 1; **C** 2; **D** 3; **E** more than 3.

4. The critical point $\boldsymbol{a}$ of a function f with the Hessian matrix

$$H(\boldsymbol{a}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

   A is a saddle point
   B is a local max
   C is a local min
   D could be local min
   E this matrix cannot be a Hessian at a critical point.

5. Consider the inequalities about the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^3$:

a) $|\boldsymbol{a} \times \boldsymbol{b}| \leq |\boldsymbol{a}||\boldsymbol{b}|$
b) $|(x, y, z)| \leq \sqrt{3}(|x| + |y| + |z|)$
c) $\boldsymbol{a} \cdot \boldsymbol{b} \leq |\boldsymbol{a}||\boldsymbol{b}|$
d) $\boldsymbol{a} + \boldsymbol{b} \leq |\boldsymbol{a}| + |\boldsymbol{b}|$
e) $|\boldsymbol{a} - \boldsymbol{b}| \leq |\boldsymbol{a}| + |\boldsymbol{b}|$
f) $|\boldsymbol{a} - \boldsymbol{b}| \leq |\boldsymbol{a}| - |\boldsymbol{b}|$

How many of these inequalities are INCORRECT?

A 0, **B** 1, **C** 2, **D** 3, **E** more than 3.

6. Let S be the portion of the sphere centered at the origin with radius $\sqrt{2}$ above the plane $z = -1$, oriented outward. Consider the vector field $\boldsymbol{F}(x, y, z)$ satisfying $\nabla \times \boldsymbol{F} = (0, 0, 1)$. Then $\int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{x} =$

A π **B** $-\pi$ **C** -2π **D** $\sqrt{2}\pi$ **E** none of the above answers.

7. Consider the region S of the xy plane defined by

$$S = \{(x, y) : 0 \leq x \leq 1; \text{ and } 0 < y \leq 2 \text{ if } x \in \mathbb{Q}, \text{ and } 1 \leq y \leq 2 \text{ if } x \notin \mathbb{Q}\}.$$

Determine which is a correct attribute of S:

A closed **B** convex **C** zero content and the area is 1 **D** measurable and the area is 1 **E** no correct attribute.

8. Consider the vector field $\boldsymbol{F}(x, y, z) = (x - yz, y, y - y^2)$ defined on the surface S parametrized as follows: $\boldsymbol{f}(u, v) = (u^2 + 1, v^3 + 1, u + v)$. At the point $(2, 1, 1)$ the $\boldsymbol{F} \cdot \boldsymbol{n}$ is

A -1 **B** $\frac{-1}{\sqrt{2}}$ **C** -2 **D** $(2u, 3v^2, 2)$ **E** none of the above.

9. Which of the following is the correct description and regularity condition for a 2-dimensional smooth manifold in $\mathbb{R}^5$ defined by $S = \{\boldsymbol{x} \in \mathbb{R}^5 : \boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{0}\}$, where $\boldsymbol{F} = (F_1, F_2, F_3)$ is C^1.

A rank $D\boldsymbol{F}(\boldsymbol{x}) = 2$ for each $\boldsymbol{x} \in S$.

B $\nabla F_1(\boldsymbol{x}), \nabla F_2(\boldsymbol{x})$ and $\nabla F_3(\boldsymbol{x})$ are linearly independent for each $\boldsymbol{x} \in S$.

C S must be written as a $x_3 = f_1(x_1, x_2), x_4 = f_2(x_1, x_2)$, and $x_5 = f_3(x_1, x_2)$ for some functions f_1, f_2, f_3.

D S must be written as the image of a C^1 function $\boldsymbol{f}(u, v) = (f_1(u, v), f_2(u, v), f_3(u, v))$ with $\boldsymbol{f}' \neq \boldsymbol{0}$.

E None of the of the options A-D is the correct regularity condition for the given presentation.

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