Chapter 6 Inner Product Spaces

6.1 Inner Products and Norms

Definition. ***Inner Product*** *Let V be a vector space over F. An inner product on V is a function that assigns, to every ordered pair of vectors x and y in V, a scalar in F, denoted $\langle x, y \rangle$, such that for all $x, y, z \in VE$ and all $c \in F$,*

1. $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$
2. $\langle cx, y \rangle = c\langle x, y \rangle$
3. $\overline{\langle x, y \rangle} = \langle y, x \rangle$
4. $\langle x, x \rangle > 0$ *if* $x \neq 0$

First two condition requires inner product be linear in the first component. Also

$$\langle \sum_i a_i v_i, y \rangle = \sum_i a_i \langle v_i, y \rangle$$

Definition. ***Conjugate Transpose or Adjoint of a Matrix*** *Let $A \in M_{m\times n}(F)$, the conjugate transpose or adjoint of A is an $n \times m$ matrix A^* such that $(A^*)_{ij} = \overline{A_{ji}}$ for all i, j. For $F = \mathbb{R}$, $A^* = A^T$*

Definition. ***Inner Product Definition Example***

1. ***Standard Inner Product on F^n*** *For $x = (a_1, a_2, \cdots, a_n)$ and $y = (b_1, b_2, \cdots, b_n)$ in F^n, the standard inner product on F^n is given by*

$$\langle x, y \rangle = \sum_{i=1}^{n} a_i \bar{b}_i$$

2. *Inner Product for* ***Real-valued Continuous Functions*** *on* $[0, 1]$*. Let $V = C([0, 1])$, $f, g \in V$, define*

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt$$

3. ***Frobenius Inner Product for Matrices*** *Let $V = M_{n\times n}(F)$, $A, B \in V$, then*

$$\langle A, B \rangle = tr(B^* A) = \sum_{i=1}^{n} (B^* A)_{ii}$$

Definition. ***Inner Product Space*** *A vector space over F endowed with a specific inner product is called an inner product space. If $F = C$, V is a complex inner product space; if $F = \mathbb{R}$, then V is a real inner product space*

Theorem. ***6.1 Properties From Inner Product Conditions*** *Let V be an inner product space. Then for $x, y, z \in V$ and $c \in F$, the following statements are true*

1. $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$
2. $\langle x, cy \rangle = \overline{c}\langle x, y \rangle$
3. $\langle x, 0 \rangle = \langle 0, x \rangle = 0$
4. $\langle x, x \rangle = 0$ *if and only if* $x = 0$
5. *If* $\langle x, y \rangle = \langle x, z \rangle$ *for all* $x \in V$, *then* $y = z$

The inner product is conjugate linear in the second argument

Definition. ***Norm/Length*** *Let V be an inner product space. For $x \in V$, define norm or length of x by*

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Definition. ***6.2 Properties of Norm*** *Let V be an inner product space over F. Then for all $x, y \in V$ and $c \in F$, the following statements are true*

1. $\|cx\| = |c| \cdot \|x\|$
2. $\|x\| = 0$ *if and only if* $x = 0$*. In any case,* $\|x\| \geq 0$
3. ***Cauchy-Schwarz Inequality*** $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$
4. ***Triangular Inequality*** $\|x+y\| \leq \|x\| + \|y\|$

Definition. ***Angle*** *For $F = \mathbb{R}$, $x, y \neq 0$, and θ be angle between x and y*

$$\cos\theta = \frac{\langle x, y \rangle}{\|x\| \, \|y\|} \qquad \theta = \cos^{-1}\left(\frac{\langle x, y \rangle}{\|x\| \, \|y\|}\right)$$

Note

$$\left| \frac{\langle x, y \rangle}{\|x\| \, \|y\|} \right| \leq 1$$

So valid input to arccos function

Definition. ***Orthogonal Vectors*** *Let V be an inner product space. Vectors x and y in V are orthogonal (perpendicular) if $\langle x, y \rangle = 0$.*

Definition. ***Orthogonal Sets and Orthonormal Sets*** *A subset S of V is orthogonal if any two distinct vectors in S are orthogonal. A vector x in V is a unit vctor if $\|x\| = 1$. A subset S of V is orthonormal if S is orthogonal and consists entirely of unit vectors.*

1. $S = \{v_1, v_2, \cdots\}$, *then S is orthonormal if and only if* $\langle v_i, v_j \rangle = \delta_{ij}$

2. *We can* ***normalize*** *an orthogonal set* S, *by multiplying* $1/\|x\|$ *for each* $x \in S$

Definition. ***Orthonormal Set Property*** *Let* V *be inner product space and* $S = \{s_1, s_2, \cdots\} \subseteq V$ *be an orthonormal set. Let* $v \in span(S)$, *then* $v = a_1 s_1 + \cdots + a_k s_k$. *Then*

$$\langle v, s_j \rangle = a_j$$

by

$$\langle v, s_j \rangle = \langle \sum_i a_i s_i, s_j \rangle = \sum_i a_i \langle s_i, s_j \rangle = \sum_i a_i \delta_{ij} = a_j$$

Gram-Schmidt Orthogonalization Process and Orthogonal Complements

Definition. ***Orthonormal Basis*** *Let* V *be an inner product space. A subset of* V *is an orthonormal basis for* V *if it is an ordered basis that is orthonormal*

Definition. ***Every Inner Product Space has*** n ***Orthogonal Basis*** *Let* V *be an inner product space and* $S = \{v_1, v_2, \cdots, v_k\}$ *be an orthogonal subset of* V *consisting of nonzero vectors. If* $y \in span(S)$, *then*

$$y = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i$$

Corollary. ***Special case for Orthonormal Set*** *If, in addition to hypotheses of previous theorem,* S *is orthonormal and* $y \in span(S)$, *then*

$$y = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$$

Corollary. ***Nonzero Orthonormal Set is Linearly Independent*** *Let* V *be an inner product space, andf let* S *be an orthogonal subset of* V *consisting of nonzero vectors. Then* S *is linearly independent*

Theorem. ***6.4 Gram-Schmidt Process*** *Let* V *be an inner product space and* $S = \{w_1, w_2, \cdots, w_n\}$ *be a linearly independent subset of* V. *Define* $S' = \{v_1, v_2, \cdots, v_n\}$, *where* $v_1 = w_1$ *and*

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j \qquad 2 \leq k \leq n$$

Then S' *is an orthogonal set of nonzero vectors such that* $span(S') = span(S)$

Theorem. ***6.5 Every Finite Dimensional I.P.S has an Orthonormal Basis*** *Let* V *be a nonzero finite-dimensional inner product space. Then* V *has an orthonormal basis* β. *Furthermore, if* $\beta = \{v_1, v_2, \cdots, v_n\}$ *and* $x \in V$, *then*

$$x = \sum_{x=1}^{n} \langle x, v_i \rangle v_i$$

Corollary. ***Expression for Matrix Representation of Transformation on Orthonormal Basis*** *Let V be a finite-dimensional inner product space with an orthonormal basis $\beta = \{v_1, v_2, \cdots, v_n\}$. Let T be a linear operator on V, and let $A = [T]_\beta$. Then for any i and j, $A_{ij} = \langle T(v_j), v_i \rangle$, i.e.*

$$T(v_j) = \sum_{i=1}^{n} \langle T(v_j), v_i \rangle v_i$$

Definition. ***Fourier Coefficients*** *Let β be an orthonormal subset (possibly infinite) of an inner product space V, and let $x \in V$. We define the Fourier coefficients of x relative to β to be the scalars $\langle x, y \rangle$, where $y \in \beta$*

Orthogonal Complements

Definition. ***Orthogonal Complements*** *Let S be a nonempty subset of an inner product space V. We define $S^\perp = \{x \in V : \langle x, y \rangle = 0 \text{ for all } y \in S\}$. The set $S^\perp$ is called the orthogonal complement of S*

1. *$\{0\}^\perp = V$ and $V^\perp = \{0\}$*

Theorem. ***6.6 Finding Projection of a Vector onto a Subspace*** *Let W be a finite-dimensional subspace of an inner product space V, and let $y \in V$. Then there exist unique vectors $u \in W$ and $z \in W^\perp$ such that $y = u + z$. Furthermore, if $\{v_1, v_2, \cdots, v_k\}$ is an orthonormal basis for W, then*

$$u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$$

where u is the orthogonal projection of y on W.

Corollary. ***Orthogonal Projection is Unique and Closest to Projected Vector*** *In the notation of previous theorem, the vector u the unique vector in W that is closest to y; that is, for any $x \in W$, $\|y - x\| \geq \|y - u\|$, and this inequality is an equality if and only if $x = u$*

Theorem. ***6.7 Orthonormal Basis and Subspaces*** *Suppose that $S = \{v_1, v_2, \cdots, v_k\}$ is an orthonormal set in an n-dimensional inner product space V.Then*

1. *S can be extended to an orthonormal basis $\{v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_n\}$ for V.*
2. *If $W = span(S)$, then $S_1 = \{v_{k+1}, \cdots, v_n\}$ is an orthonormal basis for $W^\perp$*
3. *If W is any subspace of V, then $dim(V) = dim(W) + dim(W^\perp)$*

6.3 The Adjoint of a Linear Operator

Definition. ***Dual Space*** *is a space of all linear transformations from a vector space V to its field F.*

Theorem. ***6.8 Every Linear Transformation from V to F Can Be Written as a Inner Product*** *Let V be a finite-dimensional inner product space over F, and let $g : V \to F$ be a linear transformation. Then there exists a unique vector $y \in V$ such that $g(x) = \langle x, y\rangle$ for all $x \in V$, where*

$$y = \sum_i \overline{g(v_i)} v_i \qquad \beta = \{v_1, \cdots, v_n\} \text{ is orthonormal basis}$$

Definition. ***Adjoint Linear Operator*** *Given inner product space V, let T be a linear operator on V. The adjoint of operator T, T^*, is the unique operator on V satisfying*

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle \qquad \text{for all } x, y \in V$$

Theorem. ***6.9 Adjoint of an Linear Operator Exist for f.d. Inner Product Space*** *Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Then there exists a unique function, called the adjoint of T, $T^* : V \to V$ such that*

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle$$

for all $x, y \in V$. Furthermore, T^* is linear. We can view the equation symbolically as adding an asterik $*$ to T when shifting position inside the inner product symbol

Theorem. ***6.10 Adjoint of a Linear Operator in Matrix Form is the Adjoint of Matrix Form of that Linear Operator*** *Let v be a finite-dimensional inner product space. Let β be an orthonormal basis for V. If T is a linear operator on V, thern*

$$[T^*]_\beta = [T]^*_\beta$$

Corollary. ***For Left-Matrix Transformation*** *Let A be $n \times n$ matrix, then $L_{A^*} = (L_A)^*$.* *(theorem 2.16)*

Theorem. ***6.11 Properties of Adjoint of Linear Operators***
Let V bewr an inner product space, and let T, U be linear operators on V, then

1. $(T + U)^* = T^* + U^*$
2. $(cT)^* = \overline{c}T^*$ *for any* $c \in F$
3. $(TU)^* = U^*T^*$
4. $T^{**} = T$
5. $I^* = I$

assuming adjoints always exists.

Corollary. ***For Matrix***
Let A and B be $n \times n$ matrix, then

1. $(A+B)^* = A^* + B^*$
2. $(cA)^* = \overline{c}A^*$ *for all* $c \in F$
3. $(AB)^* = B^*A^*$
4. $A^{**} = A$
5. $I^* = I$

Least Squares Approximation

Definition. ***Some notation*** *Fort $x, y \in F^n$*

1. $\langle x, y \rangle_n$ *is the standard inner product of x and y in F^n*
2. *If x and y are column vectors, then $\langle x, y \rangle_n = y^* x$*

Lemma. *Let $A \in M_{m \times n}(F)$, $x \in F^n$ and $y \in F^m$, then*

$$\langle Ax, y \rangle_m = \langle x, A^* y \rangle_n$$

Lemma. *Let $A \in M_{m \times n}(F)$. Then $rank(A^*A) = rank(A)$*

Corollary. *If A is $m \times n$ matrix such that $rank(A) = n$, then A^*A is invertible*

Theorem. ***6.12 Close Form Solution for Least Squared Problem*** *Let $A \in M_{m \times n}(F)$ and $y \in F^m$. Then there exists $x_0 \in F^n$ such that $(A^*A)x_0 = A^*y$ and $\|Ax_0 - y\| \leq \|Ax - y\|$ for all $x \in F^n$. Furthermore, if $rank(A) = n$, then $x_0 = (A^*A)^{-1}A^*y$*

6.4 Normal and Self-Adjoint Operators

Definition. ***Motivation*** *Condition for orthonormal basis of eigenvectors in*

1. $F = \mathbb{C}$, *T normal*
2. $F = \mathbb{R}$, *T self-adjoint*

Lemma. ***Condition on Existence of Eigenvector for Adjoint Linear Operators*** *Let T be a linear operator on a finite-dimensional inner product space V. If T has an eigenvector, then so does T^*. If λ is an eigenvalue of T, then $\overline{\lambda}$ is an eigenvalue of T^**

Proof. Let v be eigenvector of T with corresponding eigenvalue λ, then for any $x \in V$,

$$0 = \langle 0, x \rangle = \langle (T - \lambda I)(v), x \rangle = \langle v, (T - \lambda I)^*(x) \rangle = \langle v, (T^* - \overline{\lambda} I)(x) \rangle$$

Let $W = span(\{v\})$, so $R(T^* - \overline{\lambda} I) \subseteq W^\perp$. Note $rank(T^* - \overline{\lambda} I) \leq dim(W^\perp) = n - 1$, then $N(T^* - \overline{\lambda} I) \neq \{0\}$. So exists $u \in N(T^* - \overline{\lambda} I)$ such that $T^*(u) = \overline{\lambda} u$ □

Theorem. ***6.14 (Schur's Theorem)***

$P_T(t)$ ***Splits Implies Exists O.N. Basis st.*** $[T]_\beta$ ***is Upper Triangular***

Let T be a linear operator on a finite-dimensional inner product space V. Suppose that the characteristic polybnomial of T splits. Then there exists an orthonormal basis β for V such that the matrix $[T]_\beta$ is upper triangular

Proof. With induction, idea is to construct an orthonormal basis $\beta = \gamma \cup \{z\}$, where γ is an orthonormal basis for $W^\perp$ and $z \in W = span(z)$, where z is unit eigenvector for T^* whose existence ensured by previous lemma. The induction hypothesis mandates

1. $W^\perp$ is a T-invariant subspace as an assumption, i.e. if $y \in W^\perp, x \in W$, then $\langle T(y), x \rangle = 0$
2. $P_{T_{W^\perp}}(t) | P_T(t)$, so characteristic polyonmial of $T_{W^\perp}$ splits

to get the orthonormal basis γ, for which $[T_{W^\perp}]_\gamma$ is upper triangular. □

Definition. ***Normal Linear Operator*** *Let V be an inner product space, and let T be a linear operator on V. We say that T is normal if $TT^* = T^*T$. An $n \times n$ real or complex matrix A is normal if $AA^* = A^*A$ (Commutivity).*

1. *Motivation is that if $[T]_\beta$ is diagonal, then T^* also diagonal, hence T and T^* commutes*
2. *T is normal if and only if $[T]_\beta$ is normal, where β is an orthonormal basis*
3. *Skew-symmetric matrix ($A^t = -A$) is normal by $A^tA = -A^2 = AA^t$*
4. *Normality not sufficient to guarantee an orthonormal basis of eigenvectors. However, normality suffices if V is a complex inner product space*

Theorem. ***6.15 Properties of Normal Operator***
Let V be an inner product space, and let T be a normal operator on V. Then the following are true

1. *$\|T(x)\| = \|T^*(x)\|$ for all $x \in V$*
2. *$T - cI$ is normal for every $c \in F$*
3. *If x is an eigenvector of T, then x is also an eigenvector of T^*. In fact,if $T(x) = \lambda x$, then $T^*(x) = \overline{\lambda} x$*

4. *If λ_1 and λ_2 are distinct eigenvalues of T with corresponding eigenvectors x_1 and x_2, then x_1* ***and*** *x_2* ***are orthogonal****, i.e. $\langle x_1, x_2 \rangle = 0$*

Theorem. ***6.16 Normal Operator iff Diagonalizable*** *($F = \mathbb{C}$)*
Let T be a linear operator on a finite-dimensional **complex** *inner product space V. Then T is normal if and only if there exists an orthonormal basis for V consisting of eigenvectors of T.*

Proof. Idea is the orthonormal basis that makes T an upper triangular matrix (using Schur's Theorem) happens to be a set of eigenvectors □

1. *example showing theorem does not work on infinite dimension vector spaces with problem definition here . Specifically an example where T is normal and that T has no eigenvectors*

2. *Normality not sufficient for existence of orthonormal basis of eigenvectors for real inner product spaces*

Definition. ***Self-Adjoint (Hermitian)*** *Let T be a linear operator on an inner product space V. We say that T is self-adjoint (Hermitian) if $T = T^*$. An $n \times n$ real or complex matrix A is self-adjoint (Hermitian) if $A = A^*$*

1. *If β is orthonormal basis, then T is self-adjoint if and only if $[T]_\beta$ is self-adjoint (symmetric matrix for $F = \mathbb{R}$)*

2. *If T is self-adjoint, then T is normal*

Lemma. ***Properties of Self-Adjoints***
Let T be a self-adjoint operator on a finite-dimensional inner product space V. Then

1. *Every eigenvalue of T is real*

2. *Suppose that V is a real inner product space ($F = \mathbb{R}$). Then the characteristic polynomial of T splits*

Theorem. ***6.17 Self-Adjoints iff Diagonalizable*** *($F = \mathbb{R}$)*
Let T be a linear operator on a finite-dimensional real inner product space V. Then T is self-adjoint if and only if there exists an orthonormal basis β for v consisting of eigenvectors of T.

Definition. ***Computing Squared Root of Imaginary Number***
Relies on Euler's formula

$$e^{ix} = \cos x + i \sin x$$

Therefore we have

$$e^{i\pi} = -1 \qquad i = \sqrt{-1} = e^{i\pi/2} \qquad \sqrt{i} = e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

Definition. ***Positive Definite/Semidefinite***
A linear operator T on a finite-dimensional inner product space is called positive definite (positive semidefinite) if T is

1. *self-adjoint, and*
2. $\langle T(x), x\rangle > 0$ *(*$\langle T(x), x\rangle \geq 0$*) for all* $x \neq 0$

An $n \times n$ matrix A with entries from $\mathbb{R}$ or $\mathbb{C}$ is positive definite (positive semidefinite) if L_A is positive definite (positive semidefinite)

5.2 Direct Sums

Definition. ***Sum*** *Let $W_1, W_2, \cdots, W_k$ be subspaces of a vector space V. The sum of these subspacecs is the set*

$$\{v_1 + \cdots + v_k : v_i \in W_i \text{ for } 1 \leq i \leq k\}$$

which we denote by $W_1 + \cdots + W_k$ or $\sum_{i=1}^{k} W_i$

Definition. ***Direct Sum*** *Let $W_1, W_2, \cdots, W_k$ be subspaces of a vector space V. We call V the direct sum of the subspaces $W_1, W_2, \cdots, W_k$ and write $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ if*

$$V = \sum_{i=1}^{k} W_i \qquad \text{and} \qquad W_j \cap \sum_{i \neq j} W_i = \{0\} \text{ for each } 1 \leq j \leq k$$

1. *dimension of direct sum is sum of dimension of the subspaces in the sum*

$$dim(V) = dim(W_1) + \cdots + dim(W_k)$$

Theorem. ***5.10 Equivalence Condition for Direct Sum***
Let $W_1, \cdots W_k$ be subspaces of finite-dimensional vector space V. The following results are equivalent

1. $V = W_1 \oplus \cdots \oplus W_k$
2. $V = \sum_{i=1}^{k} W_i$ *and, for any vector $v_1, \cdots, v_k$ such that $v_i \in W_i$ ($1 \leq i \leq k$), if $v_1 + \cdots v_k = 0$, then $v_i = 0$ for all i.*
3. *Each vector $v \in V$ can be uniquely written as $v = v_1 + v_2 + \cdots + v_k$ where $v_i \in W_i$*
4. *If γ_i is an ordered basis for W_i ($1 \leq i \leq k$), then $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is an ordered basis for V*
5. *For each $i = 1, 2, \cdots, k$, there exists an ordered basis γ_i for W_i such that $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is an ordered basis for V*

Theorem. ***5.11*** *A linear operator T on a finite-dimensional vector space V is diagonalizable if and only if V is the direct sum of the eigenspaces of T*

6.6 Orthogonal Projection and the Spectral Theorem

Definition. ***Projection on a Subspace***
Let V be a vector space and W_1 and W_2 be subspaces of V such taht $V = W_1 \oplus W_2$. A function $T : V \to V$ is called the projection on W_1 along W_2 if, for $x = x_1 + x_2$ with $x_1 \in W_1$ and $x_2 \in W_2$, we have $T(x) = x_1$

1. *$R(T) = W_1 = \{x \in V : T(x) = x\}$ and $N(T) = W_2$*
2. *$V = R(T) \oplus N(T)$, i.e.*
   (a) *every projection uniquely determined by its range and nullspace*
   (b) *W_1 does not uniquely determine T*
3. *T* ***is a projection if and only if*** *$T = T^2$*

Proof. Proving T is a projection iff $T^2 = T$. Forward direction,

$$T^2(x) = T(T(x)) = T(x_1) = x_1 = T(x)$$

For reverse direction, assume $T^2 = T$, then $T(I-T) = 0_V$. Let $W_1 = R(T)$ and $W_2 = N(T)$, now claim $V = W_1 \oplus W_2$. We first prove $N(T) = R(I-T)$. Since $T(I-T) = 0$, then $R(I-T) \subseteq N(T)$. Conversely, if $x \in N(T)$, then $(I-T)(x) = x - T(x) = x$, i.e. $x \in R(I-T)$. Now write $I = T + I - T$, then $x = T(x) + (I-T)(x)$ for any $x \in V$. Then $v = w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$. Now we prove uniqueness, i.e. $\{0\} = R(T) \cap N(T)$. Let $T(x) \in R(T) \cap N(T)$ as $T(x) \in R(T)$ by default and let $T(x) \in N(T)$. Then $T(x) = 0$. Proved $V = W_1 \oplus W_2$. Then we can write $x = x_1 + x_2$ where $x_1 \in R(T)$ and $x_2 \in N(T)$, hence

$$T(x) = T(x_1 + x_2) = T(x_1) + T(x_2) = T(x_1) = x_1$$

where the last equality given by letting $x_1 = T(y)$, then $T(x_1) = T^2(y) = T(y) = x_1$ □

Definition. ***Orthogonal Projection*** *Let V be an inner product space, and let $T : V \to V$ be a a linear operator. We say that T is an orthogonal projection if*

1. *T is a projection, and*
2. *$R(T)^{\perp} = N(T)$ and $N(T)^{\perp} = R(T)$*

Note

1. *If V finite-dimensional, need to assume either condition in 2. hold.*
2. *Orthogonal projection T is uniquely determmined by its range W, so instead call T the orthogonal projection of V on W*

Proposition. ***Projection of Vector to a Subspace is an Orthogonal Projection*** *Let W be subspace of V. there exists $u \in W$ and $z \in W^{\perp}$ and $y \in V$ such that $y = u + z$. If we define linear operator $T : V \to V$ by*

$$T(y) = u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$$

where $\{v_1, \cdots, v_n\}$ is an orthonormal basis for W. Then T is an orthogonal projection.

Proof. Prove that $T^2 = T = T^*$. For any $v_j \in \beta$, we have

$$T(T(v_j)) = T(\sum_{i=1}^{k} \langle v_j, v_i \rangle v_i) = T(v_j)$$

therefore $T^2 = T$ since linear operator characterized by basis entirely. Let $x, y \in V$,

$$\langle T(x), y \rangle = \langle \sum_i \langle x, v_i \rangle v_i, y \rangle = \sum_i \langle x, v_i \rangle \langle v_i, y \rangle = \sum_i \overline{\langle y, v_i \rangle} \langle x, v_i \rangle = \langle x, \sum_i \langle y, v_i \rangle v_i \rangle = \langle x, T(y) \rangle$$

therefore $T = T^*$ by theorem 6.24, T is orthogonal projection. □

Proof. Alternatively we can prove that $N(T)$ and $R(T)$ are reciprocally orthogonal sets. Since T is a projection, we have $V = R(T) \oplus N(T)$, where $N(T) = \{x \in V : \langle x, v_i \rangle = 0 \text{ for all } i\} = W^{\perp}$ and $R(T) = W$. Therefore $N(T) = R(T)^{\perp}$ and $N(T)^{\perp} = (R(T)^{\perp})^{\perp} \supseteq R(T)$. Now we show if $x \in N(T)^{\perp}$, then $x \in R(T) = W$. Now by direct sum, we can write $x = y + w$ where $y \in N(T)$ and $w \in R(T) = W$, then

$$0 = \langle x, y \rangle = \langle y, y \rangle + \langle w, y \rangle \quad \to \quad \langle y, y \rangle = 0 \quad \to \quad y = 0$$

therefore $x = w \in W$. □

Theorem. ***6.24 $T^2 = T = T^*$ iff Orthogonal Projection***
Let V be an inner product space, and let T be a linear operator on V. Then T is an orthogonal projection if and only if T has an adjoint T^* and $T^2 = T = T^*$

1. *Let T be orthogonal projection of V on W, and $\beta = \{v_1, \cdots, v_n\}$ is an orthonormal basis for V, and $\{v_1, \cdots, v_k\}$ is a basis for W, then*

$$[T]_\beta = \begin{pmatrix} I_k & O \\ O & O \end{pmatrix}$$

2. *Let U be any projection on W, then exists γ such that $[U]_\gamma$ has same for as above, but γ need not be orthonormal*

Theorem. ***6.25 The Spectral Theorem***
Suppose T a linear operator on a finite-dimensional inner product space V over F with distinct eigenvalues $\lambda_1, \cdots, \lambda_k$. Assume T is normal if $F = \mathbb{C}$ and that T is self-adjoint if $F = \mathbb{R}$. (i.e. guarantees orthonormal basis of eigenvectors). For each i ($1 \leq i \leq k$), let W_i be the eigenspace of T corresponding to the eigenvalue λ_i, and let T_i be the orthogonal projection of V on W_i. Then the following statements are true

1. $V = W_1 \oplus \cdots \oplus W_k$
2. *If W_i' denotes the direct sum of subspaces W_j for $j \neq i$, then $W_i^\perp = W_i'$*
3. $T_i T_j = \delta_{ij} T_i$ *for* $1 \leq i, j \leq k$
4. $I = T_1 + T_2 + \cdots + T_k$ ***(resolution of identity operator induced by** T)*
5. $T = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k$ ***(spectral decomposition)***

note

1. *Note since T_i orthogonal projection, we have $N(T_i) = R(T_i)^\perp = W_i^\perp = W_i'$*
2. ***Spectrum*** *The set $\{\lambda_1, \cdots, \lambda_k\}$ is called spectrum of T*
3. *Let β be union of orthonormal basis of W_i's and let $m_i = dim(W_i)$, then*

$$[T]_\beta = \begin{pmatrix} \lambda_1 I_{m_1} & O & \cdots & O \\ O & \lambda_2 I_{m_2} & \cdots & O \\ \vdots & \vdots & & \vdots \\ O & O & \cdots & \lambda_k I_{m_k} \end{pmatrix}$$

Corollary. ***Condition for Normal*** *If $F = \mathbb{C}$, then T is normal if and only if $T^* = g(T)$ for some polynomial*

Corollary. ***Condition of Unitary*** *If $F = \mathbb{C}$, then T is unitary if and only if T is normal and $|\lambda| = 1$ for every eigenvalue λ of T*

Corollary. ***Condition for Self-Adjoint*** *If $F = \mathbb{C}$ and T is normal, then T is self-adjoint if and only if every eigenvalue of T is real*