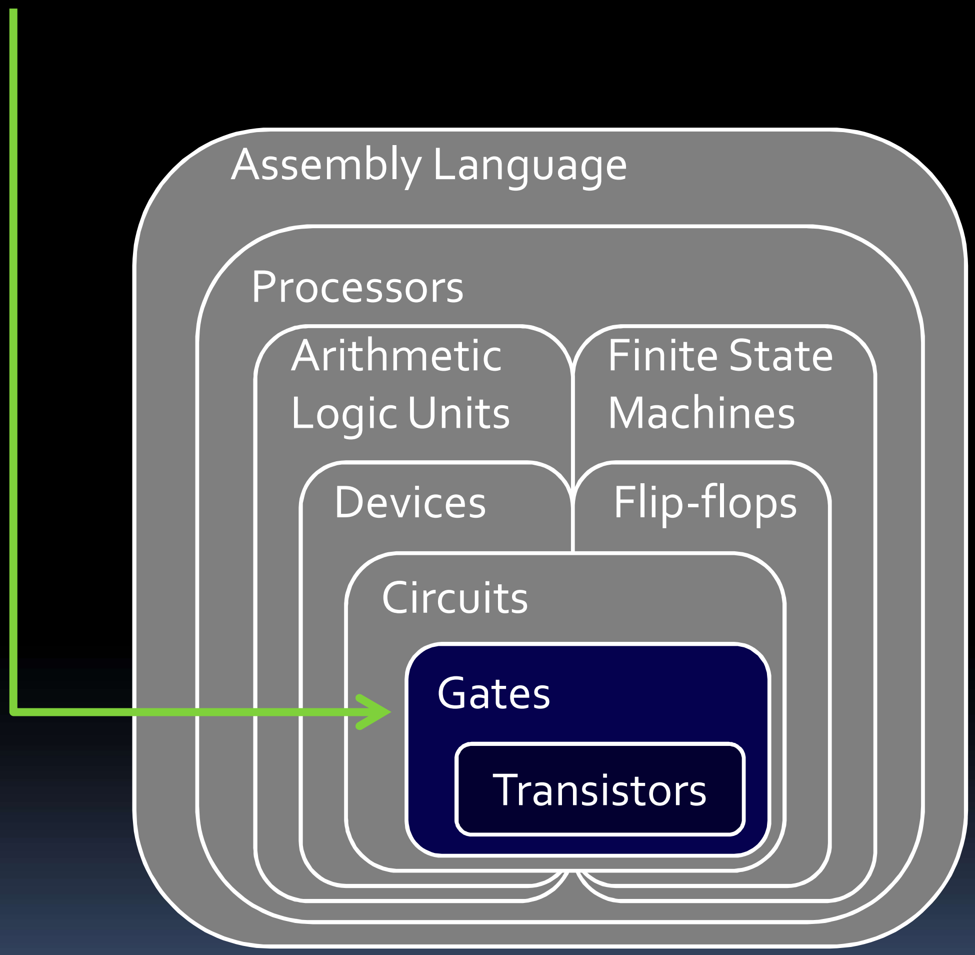
Circuit Creation

You are here


Assembly Language
Processors
Arithmetic Logic Units
Finite State Machines
Devices
Flip-flops
Circuits
Gates
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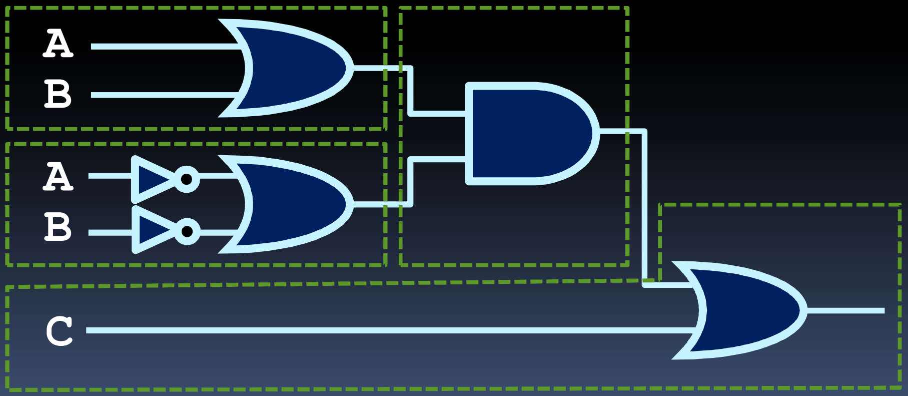
Making boolean expressions

- So how would you represent boolean expressions using logic gates?

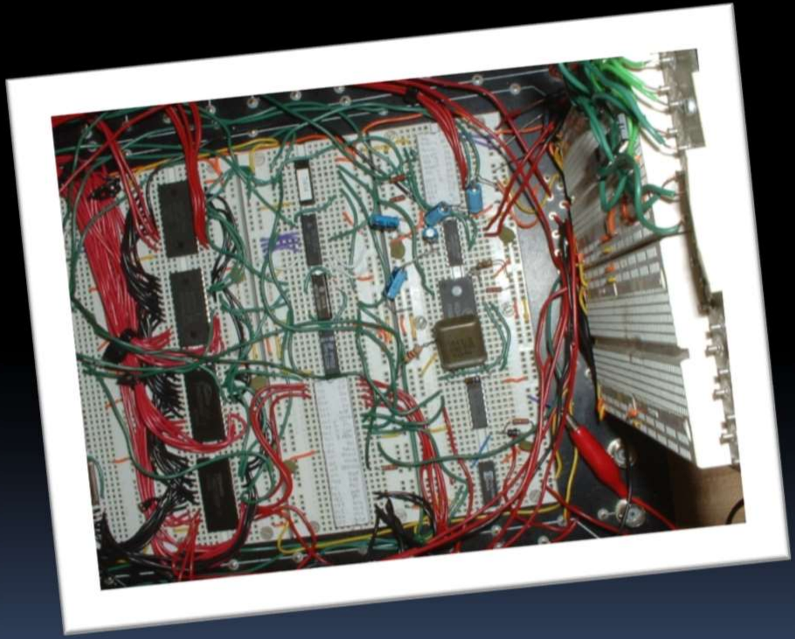
```
Y = (A or B) and (not A or not B) or C
```

- Like so:



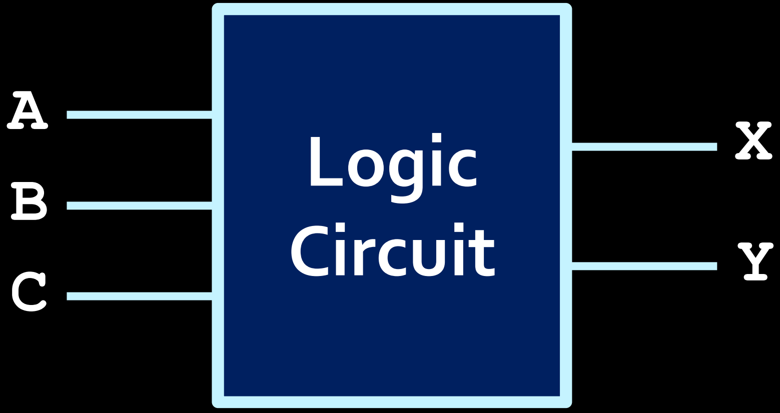
Creating complex circuits

- What do we do in the case of more complex circuits, with several inputs and more than one output?
  - If you're lucky, a truth table is provided to express the circuit.
  - Usually the behaviour of the circuit is expressed in words, and the first step involves creating a truth table that represents the described behaviour.



Circuit example

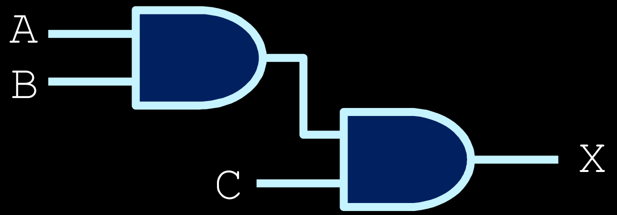
- The circuit on the right has three inputs (A, B and C) and two outputs (X and Y).

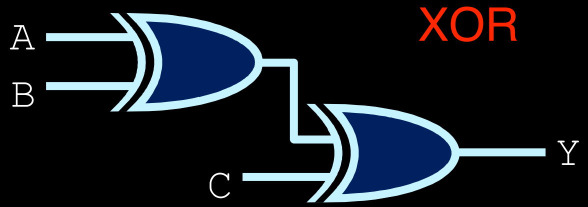


- *What logic is needed to set X high when all three inputs are high?*
- *What logic is needed to set Y high when the number of high inputs is odd?*

Combinational circuits

- Small problems can be solved easily.





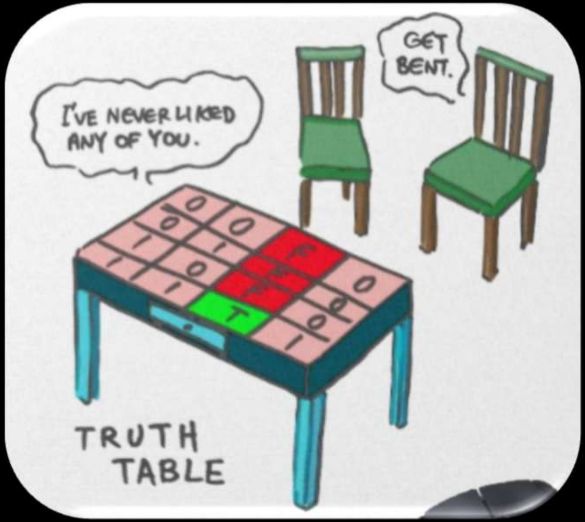
- Larger problems require a more systematic approach.
  - Example: Given three inputs A, B, and C, make output Y high in the case where all of the inputs are low, or when A and B are low and C is high, or when A and C are low but B is high, or when A is low and B and C are high.

A	B	C	Y
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1

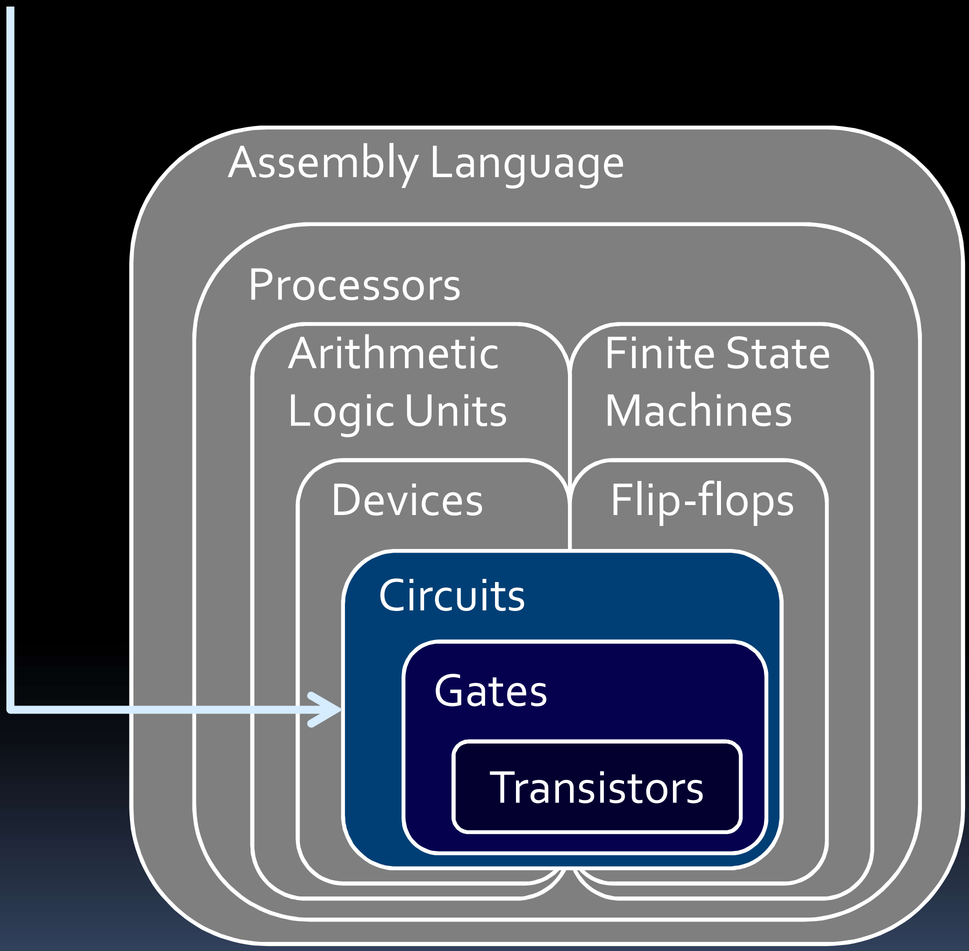
Y = ~A

Creating complex logic

- How do we approach problems like these (and circuit problems in general)?
- Basic steps:
  1. Create truth tables.
  2. Express as boolean expression.
  3. Convert to gates.
- The key to an efficient design?
  - Spending extra time on Step #2.



Now you are here



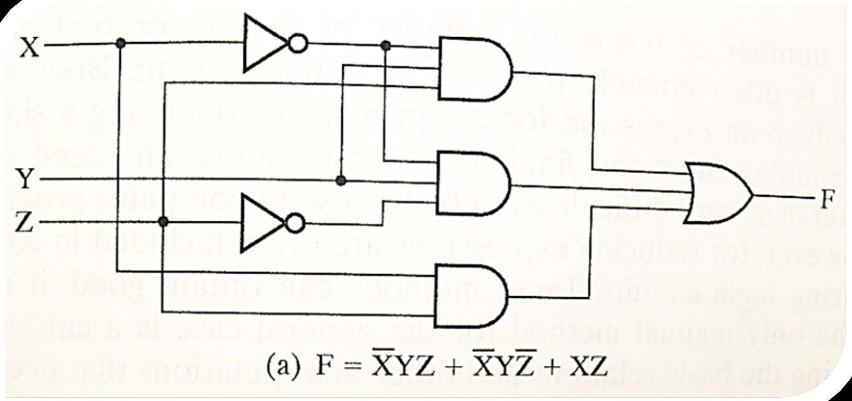
Lecture Goals

- After this lecture, you should be able to:
  - Create a truth table that represents the behaviour of a circuit you want to create.
  - Translate the minterms from a truth table into gates that implement that circuit.
  - Use Karnaugh maps to reduce the circuit to the minimal number of gates.

Lecture Goals

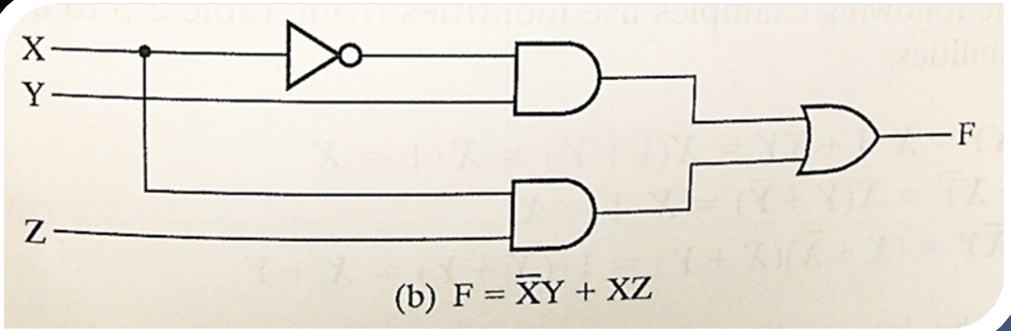
- Which implementation do you prefer? Why?

A.



(a) $F = \overline{X}YZ + \overline{X}Y\overline{Z} + XZ$

B.



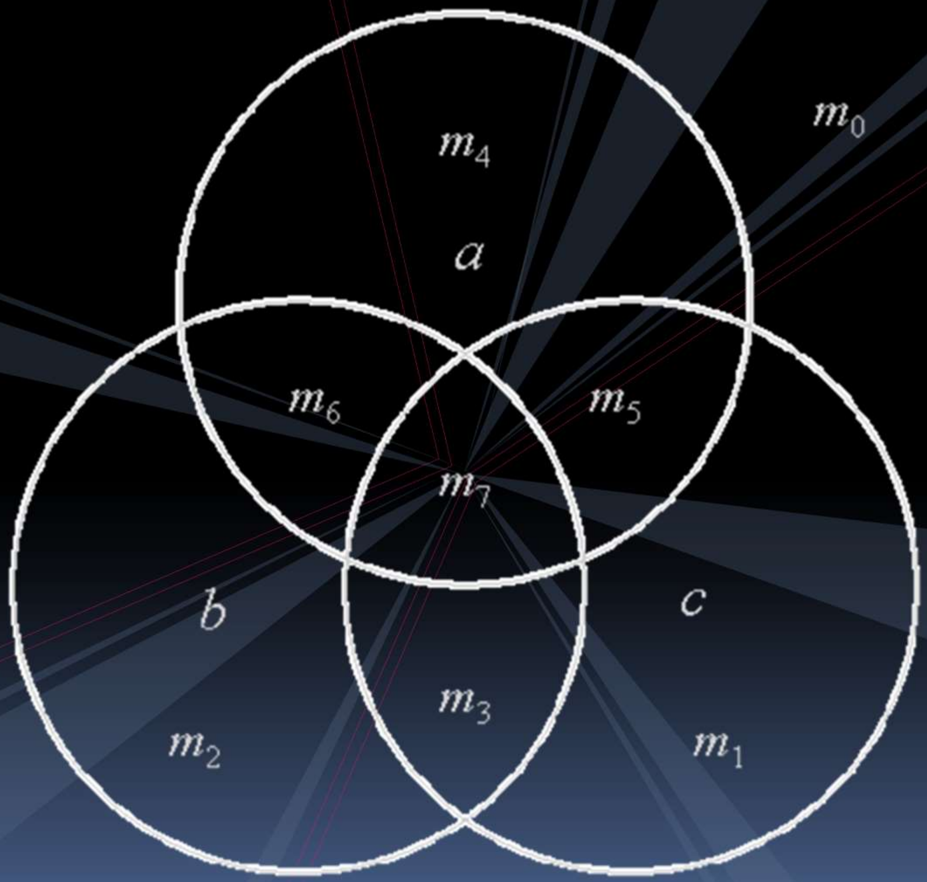
(b) $F = \overline{X}Y + XZ$

Example truth table

- Consider the following example:
  - *"Given three inputs A, B, and C, make output Y high wherever any of the inputs are low, except when all three are low or when A and C are high."*
- This leads to the truth table on the right.
  - Is there a better way to describe the cases when the circuit's output is high?

A	B	C	Y
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

Minterms and Maxterms



Minterm: product (AND) of all variable in function, for every row in truth table

Minterms

The function is then given by the sum (OR) of those minterms corresponding to rows where the function is 1

- An easier way to express circuit behaviour is to assume the standard truth table format, and then list which input rows cause high output.
  - These rows are referred to as minterms.

row numbers obtained by reading values of the variable on the row as a binary

A	B	C	Y
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

$\longleftrightarrow$

Minterm	Y
m_0	0
m_1	1
m_2	1
m_3	1
m_4	1
m_5	0
m_6	1
m_7	0

Minterms and maxterms

- A more formal description:
  - Minterm = an AND expression with every input present in true or complemented form.
  - Maxterm = an OR expression with every input present in true or complemented form.
  - For example, given four inputs (A, B, C, D):
    - Valid minterms:
      - $A\cdot\overline{B}\cdot C\cdot D$, $\overline{A}\cdot B\cdot\overline{C}\cdot D$, $A\cdot B\cdot C\cdot D$
    - Valid maxterms: most of time on.. maxterm describes few cases where it is off
      - $A+\overline{B}+C+D$, $\overline{A}+B+\overline{C}+D$, $A+B+C+D$
    - Neither minterm nor maxterm:
      - $A\cdot B+C\cdot D$, $A\cdot B\cdot D$, $A+B$

has to include every term

Creating boolean expressions

- While we’re talking about notation…
  - AND operations are denoted in these expressions by the multiplication symbol.
    - e.g. $A \cdot B \cdot C$ or $A*B*C \approx A \wedge B \wedge C$
  - OR operations are denoted by the addition symbol.
    - e.g. $A+B+C \approx A \vee B \vee C$
  - NOT is denoted by multiple symbols.
    - e.g. $\neg A$ or A' or $\overline{A}$
  - XOR occurs rarely in circuit expressions.
    - e.g. $A \oplus B$

Back to minterms

- Circuits are often described using minterms or maxterms, as a form of logic shorthand.
  - Given n inputs, there are 2^n minterms and maxterms possible (same as rows in a truth table).
  - Naming scheme:
    - Minterms are labeled as m_x, maxterms are labeled as M_x
      - The x subscript indicates the row in the truth table.
      - x starts at 0 (when all inputs are low), and ends with $2^n - 1$.
  - Example: Given 3 inputs –
    - Minterms are m_0 $(\overline{A} \cdot \overline{B} \cdot \overline{C})$ to m_7 $(A \cdot B \cdot C)$
    - Maxterms are M_0 $(A+B+C)$ to M_7 $(\overline{A}+\overline{B}+\overline{C})$

Quick Exercises

- Given 4 inputs A, B, C and D write:
  - m_9
  - m_{15}
  - m_{16}
  - M_2

1001 -> A and not B and not C and D
1111 -> A and B and C and D
N//A for 4 inputs since we go up to m15
0010   A or B or not C or D

- Which minterm is this?
  - $\overline{A} \cdot B \cdot \overline{C} \cdot \overline{D}$

m4

- Which maxterm is this?
  - $A+B+C+\overline{D}$

M1

can just convert Maxterm to minterm by taking negation

Text

Using minterms and maxterms

- What are minterms used for?
  - A single minterm indicates a set of inputs that will make the output go high.
  - Example: m_2
    - Output only goes high in third line of truth table.

A	B	C	D	m_2
0	0	0	0	0
0	0	0	1	0
0	0	1	0	1
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

use min term because usually circuit is off and only a few case with on…

Using minterms and maxterms

OR

- What happens when you OR two minterms?
  - Result is output that goes high in both minterm cases.
  - For m_2+m_8, both third and ninth lines of truth table result in high output.

A	B	C	D	m_2	m_8	m_2+m_8
0	0	0	0	0	0	0
0	0	0	1	0	0	0
0	0	1	0	1	0	1
0	0	1	1	0	0	0
0	1	0	0	0	0	0
0	1	0	1	0	0	0
0	1	1	0	0	0	0
0	1	1	1	0	0	0
1	0	0	0	0	1	1
1	0	0	1	0	0	0
1	0	1	0	0	0	0
1	0	1	1	0	0	0
1	1	0	0	0	0	0
1	1	0	1	0	0	0
1	1	1	0	0	0	0
1	1	1	1	0	0	0

Creating boolean expressions

- Two canonical forms of boolean expressions:
  - Sum-of-Minterms (SOM):
    - Since each minterm corresponds to a single high output in the truth table, the combined high outputs are a union of these minterm expressions.
    - Expressed in "Sum-of-Products" form.
  - Product-of-Maxterms (POM):
    - Since each maxterm only produces a single low output in the truth table, the combined low outputs are an intersection of these maxterm expressions.
    - Expressed in "Product-of-Sums" form.

$Y = m_2 + m_6 + m_7 + m_{10}$ (SOM)

A	B	C	D	m_2	m_6	m_7	m_{10}	Y
0	0	0	0					
0	0	0	1					
0	0	1	0					
0	0	1	1					
0	1	0	0					
0	1	0	1					
0	1	1	0					
0	1	1	1					
1	0	0	0					
1	0	0	1					
1	0	1	0					
1	0	1	1					
1	1	0	0					
1	1	0	1					
1	1	1	0					
1	1	1	1					

since OR operation is performed on minterms
as long as there is a 1, Y = 1

$$Y = m_2 + m_6 + m_7 + m_{10} \quad \text{(SOM)}$$

A	B	C	D	m_2	m_6	m_7	m_{10}	Y
0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0
0	0	1	0	1	0	0	0	1
0	0	1	1	0	0	0	0	0
0	1	0	0	0	0	0	0	0
0	1	0	1	0	0	0	0	0
0	1	1	0	0	1	0	0	1
0	1	1	1	0	0	1	0	1
1	0	0	0	0	0	0	0	0
1	0	0	1	0	0	0	0	0
1	0	1	0	0	0	0	1	1
1	0	1	1	0	0	0	0	0
1	1	0	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0
1	1	1	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0

Using Sum-of-Minterms

- Sum-of-Minterms is a way of expressing which inputs cause the output to go high.
  - Assumes that the truth table columns list the inputs according to some logical or natural order.
- Minterm and maxterm expressions are used for efficiency reasons:
  - More compact that displaying entire truth tables.
  - Sum-of-minterms are useful in cases with very few input combinations that produce high output.
    - Product-of-maxterms useful when expressing truth tables that have very few low output cases…

$$Y = M_3 \cdot M_5 \cdot M_7 \cdot M_{10} \cdot M_{14} \text{ (POM)}$$

A	B	C	D	M_3	M_5	M_7	M_{10}	M_{14}	Y
0	0	0	0						
0	0	0	1						
0	0	1	0						
0	0	1	1						
0	1	0	0						
0	1	0	1						
0	1	1	0						
0	1	1	1						
1	0	0	0						
1	0	0	1						
1	0	1	0						
1	0	1	1						
1	1	0	0						
1	1	0	1						
1	1	1	0						
1	1	1	1						

since AND operation is performed on minterms
as long as there is a 0, Y = 0

$$Z = M_3 \cdot M_5 \cdot M_7 \cdot M_{10} \cdot M_{14} \text{ (POM)}$$

A	B	C	D	M_3	M_5	M_7	M_{10}	M_{14}	Z
0	0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1	1
0	0	1	0	1	1	1	1	1	1
0	0	1	1	0	1	1	1	1	0
0	1	0	0	1	1	1	1	1	1
0	1	0	1	1	0	1	1	1	0
0	1	1	0	1	1	1	1	1	1
0	1	1	1	1	1	0	1	1	0
1	0	0	0	1	1	1	1	1	1
1	0	0	1	1	1	1	1	1	1
1	0	1	0	1	1	1	0	1	0
1	0	1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1	1	1
1	1	0	1	1	1	1	1	1	1
1	1	1	0	1	1	1	1	0	0
1	1	1	1	1	1	1	1	1	1

Converting SOM to gates

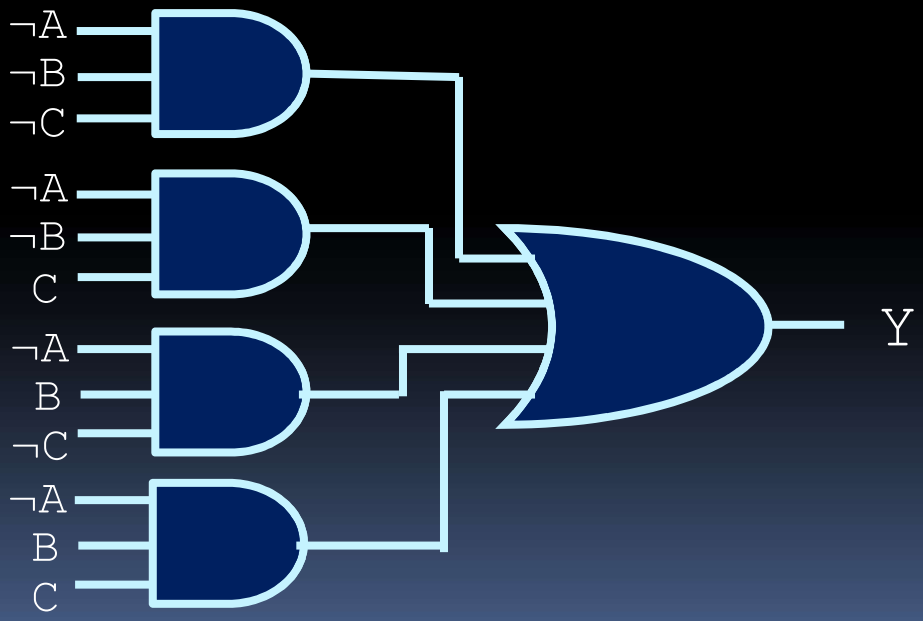
- Once you have a Sum-of-Minterms expression, it is easy to convert this to the equivalent combination of gates:

m_0 -> 0 and 0 and 0 produce a high output

$$m_0 + m_1 + m_2 + m_3 =$$

$$\overline{A}\cdot\overline{B}\cdot\overline{C} + \overline{A}\cdot\overline{B}\cdot C + \overline{A}\cdot B\cdot\overline{C} + \overline{A}\cdot B\cdot C =$$

assume all other cases m_4 … are low



2-input XOR gate (SOM,POM)

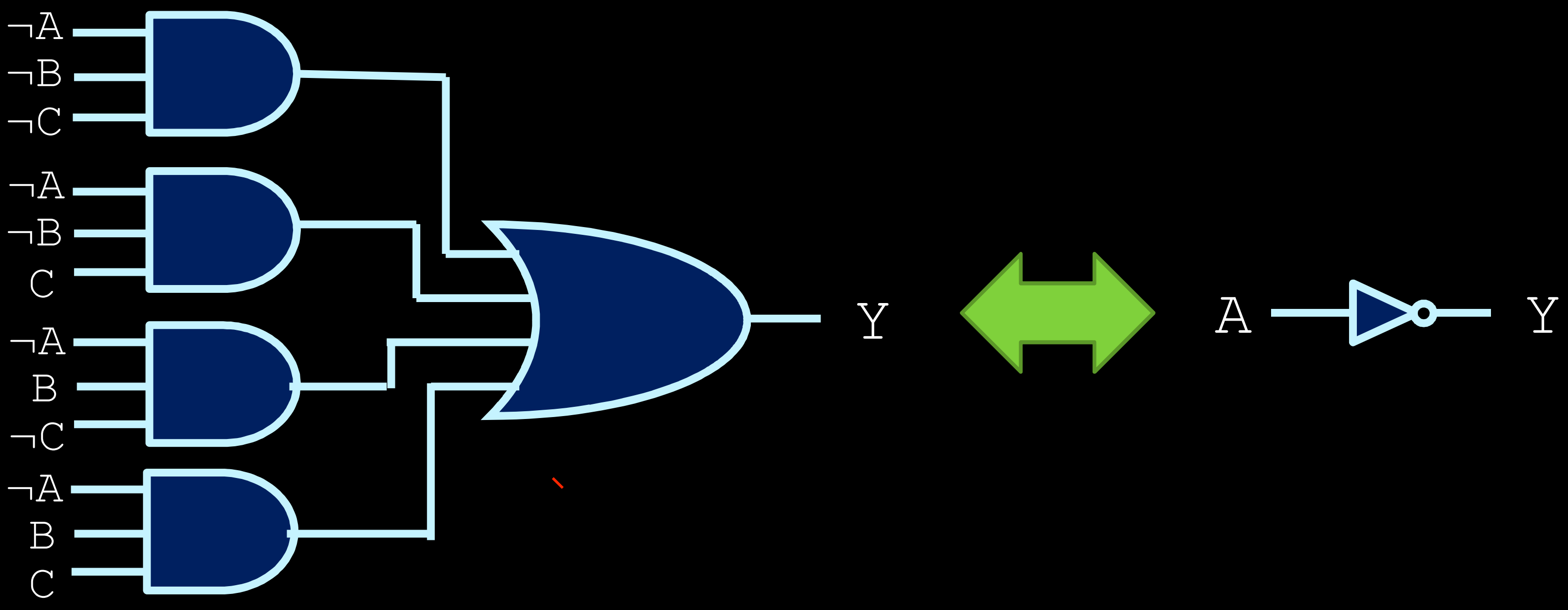
- $m_x = M_x'$
  - Minterm x is the complement of maxterm x.
  - e.g., $m_0 = A'B'$ while $M_0 = A + B$
- 2-input XOR gate in SOM and POM form.
  - Sum-Of-Minterms: $F = m_1 + m_2$
  - Product-Of-Maxterms : $F = M_0 \cdot M_3$
- Write F' in Sum-Of-Minterms form:
  - We need to include the minterms not present in F.
  - $F' = m0 + m3$

2-input XOR gate (SOM,POM)-cont'd

- Write F' in Sum-Of-Minterms form:
  - We need to include the minterms not present in F.
  - $F' = m_0 + m_3$
- Now let's take the complement of F'.
  - $(F')' = F = (m_0 + m_3)' = m_0'm_3'$
  - But m_0' is M_0 and m_3' is M_3
  - Therefore, $F = M_0 \cdot M_3$
- The canonical representations SOM and POM for a given function are equivalent! ☺

Reducing circuits

Reasons for reducing circuits



- Note example of Sum-of-Minterms circuit design.
- To minimize the number of gates, we want to reduce the boolean expression as much as possible from a collection of minterms to something smaller.
- This is where CSC165 skills come in handy ☺

Boolean algebra review

- Axioms:

$$0 \cdot 0 = 0 \qquad 0 \cdot 1 = 1 \cdot 0 = 0$$
$$1 \cdot 1 = 1 \qquad \text{if } x = 1,\ \overline{x} = 0$$

- From this, we can extrapolate:

If one input of a 2-input AND gate is 1, then the output is whatever value the other input is.

$$x \cdot 0 = 0 \qquad x+1 = 1$$
$$x \cdot 1 = 1 \qquad x+0 = X$$
$$x \cdot x = X \qquad x+x = X$$
$$x \cdot \overline{x} = 0 \qquad x+\overline{x} = 1$$
$$\overline{\overline{x}} = x$$

If one input of a 2-input OR gate is 0, then the output is whatever value the other input is.

Boolean algebra review

- Axioms:

$$0 \cdot 0 = 0 \qquad 0 \cdot 1 = 1 \cdot 0 = 0$$
$$1 \cdot 1 = 1 \qquad \text{if } x = 1,\ \overline{x} = 0$$

- From this, we can extrapolate:

$$x \cdot 0 = 0 \qquad x+1 = 1$$
$$x \cdot 1 = x \qquad x+0 = x$$
$$x \cdot x = x \qquad x+x = x$$
$$x \cdot \overline{x} = 0 \qquad x+\overline{x} = 1$$
$$\overline{\overline{x}} = x$$

Other Boolean identities

- Commutative Law:

$$x \cdot y = y \cdot x \qquad x + y = y + x$$

- Associative Law:

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$
$$x + (y + z) = (x + y) + z$$

- Distributive Law:

$$x \cdot (y + z) = x \cdot y + x \cdot z$$
$$x + (y \cdot z) = (x + y) \cdot (x + z)$$

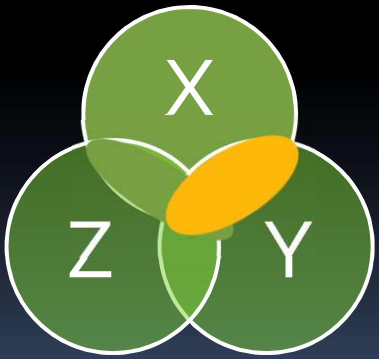
Does this hold in conventional algebra?

Consensus Law Proof -Venn diagram

- Consensus Law:

$$x \cdot y + \overline{x} \cdot z + y \cdot z = x \cdot y + \overline{x} \cdot z$$

- Proof by Venn diagram:
  - $x \cdot y$
  - $\overline{x} \cdot z$
  - $y \cdot z$
    - Already covered!

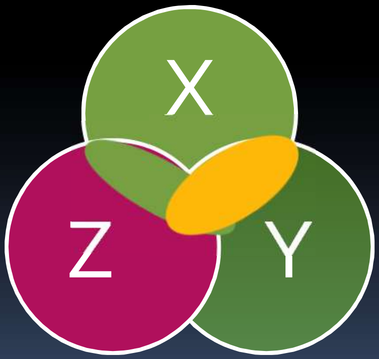


Consensus Law Proof -Venn diagram

- Consensus Law:

$$x \cdot y + \overline{x} \cdot z + y \cdot z = x \cdot y + \overline{x} \cdot z$$

- Proof by Venn diagram:
  - $x \cdot y$
  - $\overline{x} \cdot z$
  - $y \cdot z$
    - Already covered!



Other boolean identities

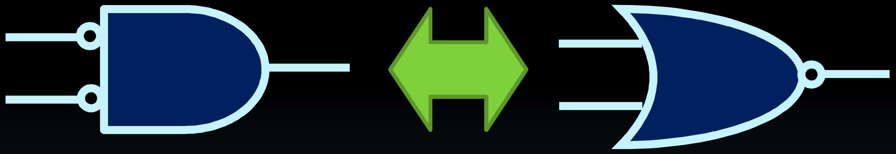
- Absorption Law:

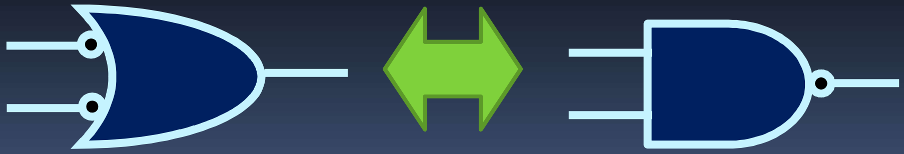
$$x \cdot (x+y) = x \qquad x + (x \cdot y) = x$$

- De Morgan's Laws:

$$\overline{x} \cdot \overline{y} = \overline{x+y}$$
$$\overline{x} + \overline{y} = \overline{x \cdot y}$$

AND gate with negation of input ==> NOR gate

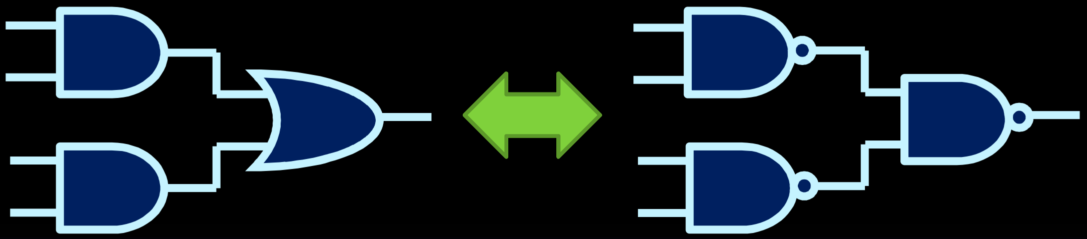




OR gate with negation of input ==> NAND gate

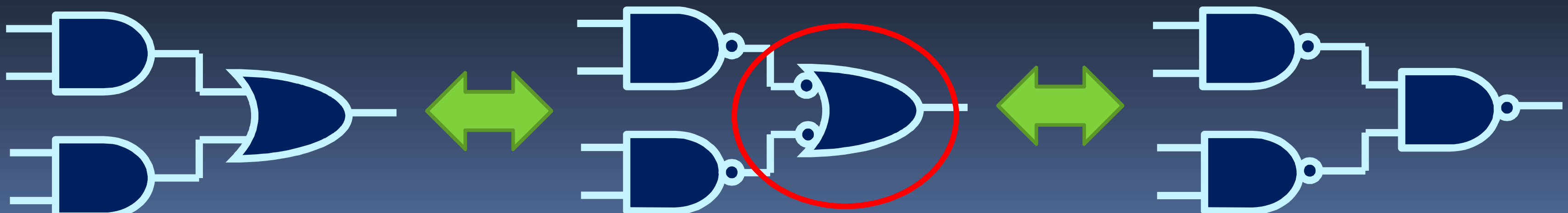
Converting to NAND gates

- De Morgan's Law is important because out of all the gates, NANDs are the cheapest to fabricate.
  - a Sum-of-Products circuit could be converted into an equivalent circuit of NAND gates:



AB + CD = ~(AB) ~(CD) = (~A + ~B) (~C + ~D)

- This is all based on de Morgan's Law:



Reducing boolean expressions

A	B	C	Y
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

- Assuming logic specs at left, we get the following:

$$Y = \overline{A}\cdot B\cdot C + A\cdot\overline{B}\cdot\overline{C} + A\cdot B\cdot\overline{C} + A\cdot B\cdot C$$

- Now start combining terms, like the last two:

$$Y = \overline{A}\cdot B\cdot C + A\cdot\overline{B}\cdot\overline{C} \mathbf{+ A\cdot B}$$

Reducing boolean expressions

- Different final expressions possible, depending on what terms you combine.
- For instance, given the previous example:

$$Y = \overline{A}\cdot B\cdot C + A\cdot\overline{B}\cdot\overline{C} + A\cdot B\cdot\overline{C} + A\cdot B\cdot C$$

- If you combine the end and middle terms…

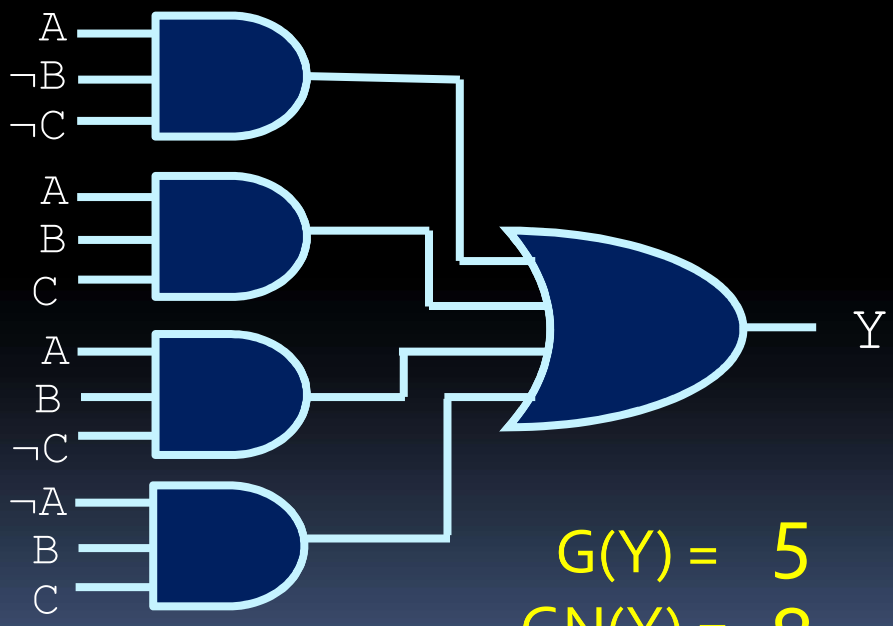
$$Y = B\cdot C + A\cdot\overline{C}$$

- Which reduces the number of gates and inputs!

alternative ways of reduction

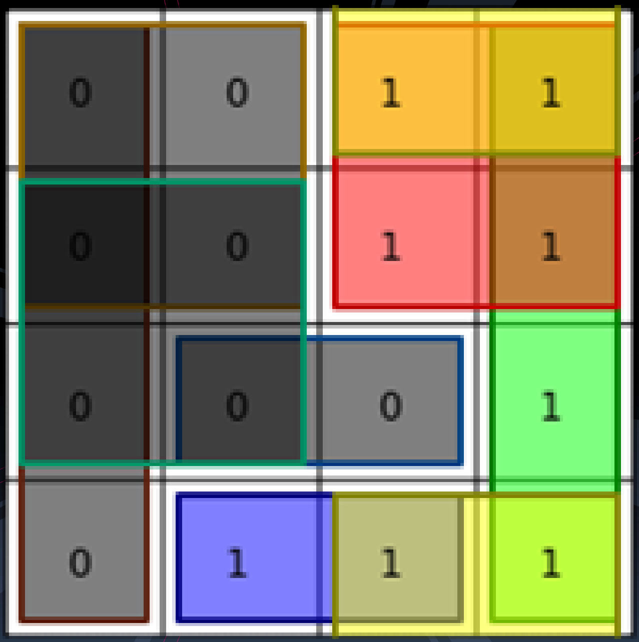
Reducing boolean expressions

- What is considered the "simplest" expression?
  - In this case, "simple" denotes the lowest gate cost (G) or the lowest gate cost with NOTs (GN).
  - To calculate the gate cost, simply add all the gates together (as well as the cost of the NOT gates, in the case of the GN cost).



G(Y) = 5
GN(Y) = 8

Karnaugh maps



0	0	1	1
0	0	1	1
0	0	0	1
0	1	1	1

Reducing boolean expressions

- How do we find the "simplest" expression for a circuit?
  - Technique called Karnaugh maps (or K-maps).
  - Karnaugh maps are a 2D grid of minterms, where adjacent minterm locations in the grid differ by a single literal.
  - Values of the grid are the output for that minterm.

	$\overline{B}\cdot\overline{C}$	$\overline{B}\cdot C$	$B\cdot C$	$B\cdot\overline{C}$
$\overline{A}$	0	0	1	0
A	1	0	1	1

can toss c out for this pair. neighbours with all 1 can combine them

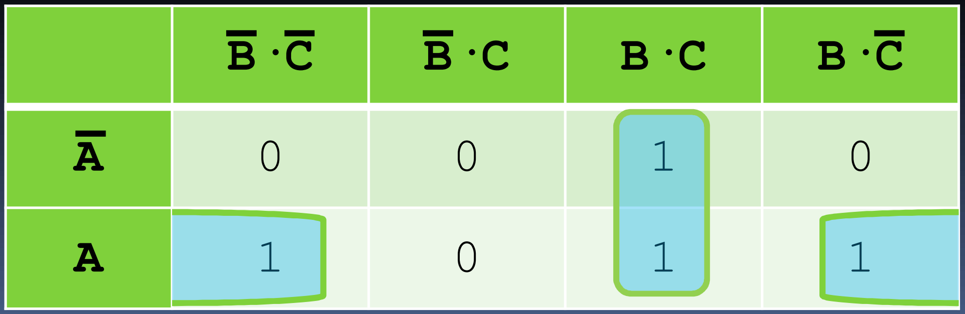
Karnaugh maps

- Karnaugh maps can be of any size, and have any number of inputs.
  - i.e. the 4-input example here.
- Since adjacent minterms only differ by a single value, they can be grouped into a single term that omits that value.

	$\overline{C}\cdot\overline{D}$	$\overline{C}\cdot D$	$C\cdot D$	$C\cdot\overline{D}$
$\overline{A}\cdot\overline{B}$	m_0	m_1	m_3	m_2
$\overline{A}\cdot B$	m_4	m_5	m_7	m_6
$A\cdot B$	m_{12}	m_{13}	m_{15}	m_{14}
$A\cdot\overline{B}$	m_8	m_9	m_{11}	m_{10}

Using Karnaugh maps

- Once Karnaugh maps are created, draw boxes over groups of high output values.
  - Boxes must be rectangular, and aligned with map.
  - Number of values contained within each box must be a power of 2.
  - Boxes may overlap with each other.
  - Boxes may wrap across edges of map.

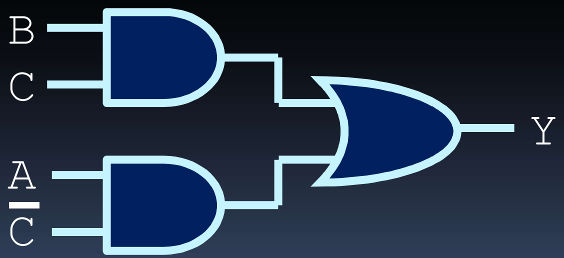


	$\bar{B}\cdot\bar{C}$	$\bar{B}\cdot C$	$B\cdot C$	$B\cdot\bar{C}$
$\bar{A}$	0	0	1	0
A	1	0	1	1

Using Karnaugh maps

	$\overline{B} \cdot \overline{C}$	$\overline{B} \cdot C$	$B \cdot C$	$B \cdot \overline{C}$
$\overline{A}$	0	0	1	0
A	1	0	1	1

- Once you find the minimal number of boxes that cover all the high outputs, create boolean expressions from the inputs that are common to all elements in the box.
- For this example:
  - Vertical box: $B \cdot C$
  - Horizontal box: $A \cdot \overline{C}$
  - Overall equation: $Y = B \cdot C + A \cdot \overline{C}$



Karnaugh maps and maxterms

- Can also use this technique to group maxterms together as well.
- Karnaugh maps with maxterms involves grouping the zero entries together, instead of grouping the entries with one values.

	$C+D$	$C+\overline{D}$	$\overline{C}+\overline{D}$	$\overline{C}+D$
$A+B$	M_0	M_1	M_3	M_2
$A+\overline{B}$	M_4	M_5	M_7	M_6
$\overline{A}+\overline{B}$	M_{12}	M_{13}	M_{15}	M_{14}
$\overline{A}+B$	M_8	M_9	M_{11}	M_{10}

Quick Exercise

	C′D′	C′D	CD	CD′
A′B′	0	0	1	1
A′B	1	1	0	0
AB	1	1	0	0
AB′	0	0	0	0

$$F = B \cdot C' + A' \cdot B' \cdot C$$