1 Multi-indices and higher order partials

1.1 Second-Order Partial Derivatives

Theorem 1.1. ***Clairut's Theorem*** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and $a \in \mathbb{R}^n$ a point. Let $i, j \in \{1, ..., n\}$ with $i \neq j$. If $\partial_{ij} f(a)$ and $\partial_{ji} f(a)$ both exist and are continuous in a neighbourhood of a, then $\partial_{ij} f(a) = \partial_{ji} f(a)$*

Definition 1.1. C^2 ***Functions*** *Let $U \in \mathbb{R}^n$ be an open set. We define $C^2(U, \mathbb{R})$ to be the collection of $f : \mathbb{R}^n \to \mathbb{R}$ whose second partial derivatives exist and are continuous at every point in U*

Remark. Therefore, if f is a C^2 function, Clairut's theorem immediately imply that it's mixed partials exists, continuous, and hence are equal.

An example in using high-order partial derivatives in conjunction with the chain rule. Let $u = f(x, y)$ and suppose x, y are functions of (s, t), i.e. $x(s, t)$, $y(s, t)$. Compute $\dfrac{\partial^2 y}{\partial s^2}$

Solution.
Using the chain rule we have first order partials

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

Then we take partials again with respect to s

$$\frac{\partial^2 u}{\partial s^2} = \frac{\partial}{\partial s}\left[\frac{\partial u}{\partial s}\right] = \frac{\partial}{\partial s}\left[\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}\right] + \frac{\partial}{\partial s}\left[\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}\right]$$

Note here $\dfrac{\partial u}{\partial s}$ is a function of (x, y). Thus to differentiate this function with respect to s, we must once again use the chain rule.

$$\begin{aligned}
\frac{\partial}{\partial s}\left[\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}\right] &= \left[\frac{\partial}{\partial s}\frac{\partial u}{\partial x}\right]\frac{\partial x}{\partial s} + \frac{\partial u}{\partial x}\frac{\partial^2 x}{\partial s^2} && \text{(product rule)} \\
&= \left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial y}{\partial s}\right]\frac{\partial x}{\partial s} + \frac{\partial u}{\partial x}\frac{\partial^2 x}{\partial s^2} && \text{(chain rule)} \\
&= \frac{\partial^2 u}{\partial x^2}\left[\frac{\partial x}{\partial s}\right]^2 + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial y}{\partial s}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial x}\frac{\partial^2 x}{\partial s^2}
\end{aligned}$$

Similar computation can be applied to the latter term. Then

$$\frac{\partial^2 u}{\partial s^2} = \frac{\partial^2 u}{\partial x^2}\left[\frac{\partial x}{\partial s}\right]^2 + \frac{\partial^2 u}{\partial y^2}\left[\frac{\partial y}{\partial s}\right]^2 + 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial y}{\partial s}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial x}\frac{\partial^2 x}{\partial s^2} + \frac{\partial u}{\partial y}\frac{\partial^2 y}{\partial s^2}$$

$\square$

Definition 1.2. ***Higher Order Partials*** *If $U \subseteq \mathbb{R}^n$ is an open set, then for $k \in \mathbb{N}$ we define $C^k(U, \mathbb{R})$ to be the collection of functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the k-th order partial derivatives of f all exist and are continuous on U. If the partials exist and are continuous for all k, we say that f is of type $C^\infty(U, \mathbb{R})$*

Theorem 1.2. ***Generalized Clairuit's Theorem*** *If $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is of type C^k, then*

$$\partial_{i_1 \dots i_k} f = \partial_{j_1 \dots j_k} f$$

whenever $(i_1, \dots, i_k)$ and $(j_1, \dots, j_k)$ are re-orderings of each other.

Definition 1.3. ***Multi-index notation*** *A multi-index α is a tuple of non-negative integers*

$$\alpha = (\alpha_1, \dots, \alpha_n)$$

*The **order** of α is the sum of its components*

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$$

*We define the multi-index **factorial** to be*

$$\alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$$

*If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ then the multi-index **exponential** is*

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$$

and if $f : \mathbb{R}^n \to \mathbb{R}$ we write

$$\partial^\alpha = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$$

2 Taylor Series

2.1 review

Derivatives can be a tool for linearly approximating a function

$$f(x) \approx f(a) + f'(a)(x - a)$$

We can go beyong just linear approximation and introduce quadratic, cubic, quartic approximations.

$$p_{n,a}(x) = \sum_{k=0}^{n} c_k (x - a)^k \text{ , where } c_k = \frac{f^{(k)}(a)}{k!}$$

Note here p is an expansion of f

$$T_n(x) = p_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k$$

is a taylor polynomial of f of degree n with center a

Theorem. *Let a function f be C^∞, let T be the Taylor polynomial of f of degree n with center a. Then for all $k \in [0, n]$,*

$$T^{(k)}(a) = f^{(k)}(a)$$

Definition. ***Single variable Taylor's Theorem*** *Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$. Let $n > 0$ $n \in \mathbb{Z}$. Suppose $f^{(n)}$ is continuous on $[a, b]$ and $f^{(n+1)}(x)$ exists on (a, b). Let $\alpha, \beta \in [a, b]$. Then Taylor polynomial of degree n of function f at point x, is denoted as*

$$p(x) = p_{n,\alpha} = \sum_{k=0}^{n} C_k(t - \alpha)^k, \ , \text{ where } C_k = \frac{f^{(k)}(\alpha)}{k!} \in \mathbb{R}$$

Remark. Here $p(x)$ and $f(x)$ have derivatives at α that agree up to order n; that is

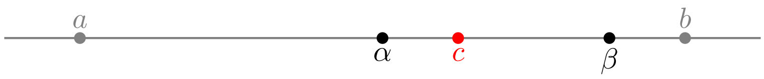
$$\forall k \in \{1, \ldots, n\} : p^{(k)}(\alpha) = f^{(k)}(\alpha)$$

Also note that

$$f(x) = p_{n,\alpha}(x) + r_{n,\alpha}(x)$$

If f is defined above, then for each β there eists a point c between α, β such that

$$f(\beta) = p_{n,\alpha}(\beta) + \frac{f^{(n+1)}(c)}{(n+1)!}(\beta - \alpha)^{n+1}$$



Theorem. ***Rolle's Theorem*** *If a real-valued function f is continuous on a proper closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$, then there exists at least one c in the open interval (a, b) such that*

$$f'(c) = 0$$

Proof. Since $[a, b]$ closed and bounded, intermediate value theorem applies here; that is, $f(x)$ achieves its maximum and minimum over $[a, b]$. Let $c \in [a, b]$. If $c \in (a, b)$, since f is differentiable on (a, b), $f'(c) = 0$ because it is an extremum. If $c \in \{a, b\}$ or maximum and minimum occurs at endpoints. Because $f(a) = f(b)$, then it means that $f(x)$ cannot be greater or smaller than $f(a) = f(b)$, then $f(x)$ is a constant function and $f'(x)$ is therefore 0 over (a, b) $\square$

Remark. Rolle's Theorem is used to prove the Mean Value Theorem. We will prove this here

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Define

$$g(x) = f(x) - rx$$

for some $r \in \mathbb{R}$. Now we want to choose r so that $g(x)$ satisfies Rolle's Theorem, specifically,

$$g(a) = g(b) \iff f(a) - ra = f(b) - rb \iff r = \frac{f(b) - f(a)}{b - a}$$

By Rolle's theorem since $g(x)$ is differentiable and $g(a) = g(b)$, then there exists $c \in (a, b)$ for which $g'(c) = 0$, i.e.,

$$g'(c) = f'(c) - r \iff f'(c) = g'(c) + r = \frac{f(b) - f(a)}{b - a}$$

□

Theorem. ***Higher Order Rolle's Theorem*** *Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ and $n + 1$ times differentiable on $[a, b]$. If $f(a) = f(b)$ and $f^{(k)}(a) = 0$ for all $k \in \{1, \ldots, n\}$ then there exists a $c \in (a, b)$ such that $f^{(n+1)}(c) = 0$*

Proof. Note that $f^{(k)}(a) = 0$ for all $k \in \{1, \cdots, n\}$ while no such constraint on the other endpoint b. All condition of Rolle's Theorem apply here with $f(a) = f(b)$, so there exists $\theta_1 \in (a, b)$ such that $f'(\theta_1) = 0$. Again with $f'(a) = f'(\theta_1) = 0$, there exists $\theta_2 \in (a, \theta_1)$ such that $f''(\theta_2) = 0$. We can continue inductively in this fashion until $f^{(n)}(a) = f^{(n)}(\theta_n)$, so that there exists $c := \theta_{n+1} \in (a, \theta_n) \subseteq (a, b)$ such that $f^{(n+1)}(c) = 0$ as required □

Theorem. ***Taylor's Theorem with Lagrange Reminder*** *Suppose that f is $n+1$ times differentiable on an interval I with $a \in I$. For each $x \in I$ there is a point c between a and x such that*

$$r_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}$$

so if f is k times differentiable at the point a, then

$$f(x) = p_{n,a}(x) + r_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}$$

Corollary. ***Taylor reminder is a good approximation*** *If f is of type C^{n+1} on an open interval I with $a \in I$, then*

$$\lim_{x \to a} \frac{r_{n,a}(x)}{|x - a|^n} = 0$$

Proof. Since f is of type C^{n+1} we know that $f^{(n+1)}$ is continuous on I. Since I is open and $a \in I$. We can find a closed inteval J such that $a \in J \subseteq I$. Therefore J is bounded. By Extreme Value Theorem, there exists $M > 0$ such that $|f^{(n+1)}(x)| \leq M$ for all $x \in J$. Since

f is $n+1$ times differentiable in a. We can construct Taylor polynomial with Lagrange reminder

$$\begin{aligned}\lim_{x\to a}\frac{|r_{n,a}(x)|}{|x-a|^n} &= \lim_{x\to a}\frac{|f^{(n+1)}(c)|}{(n+1)!}\frac{|x-a|^{n+1}}{|x-a|^n}\\ &= \lim_{x\to a}\frac{M}{(n+1)!}|x-a|\\ &= 0\end{aligned}$$

Then by Squeeze theorem, $\lim_{x\to a}\frac{r_{n,a}(x)}{|x-a|^n}=0$ Moreover, we could bound $r_{n,a}(x)$ as

$$|r_{n,a}(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \text{ , for some } M>0$$

□

Remark. This corollary just shows that the Taylor reminder is a good approximation, since error vanishes faster than order n. Also we can determine error bounds on Taylor series with the last formula since f attains its maximum by Extreme Value Theorem.

Theorem 2.1. ***Multi-variable Taylor's Theorem*** *Let $f:\mathbb{R}^n\to\mathbb{R}$ where $f\in C^{k+1}(S,\mathbb{R})$ where $S\subseteq\mathbb{R}^n$ be an open and convex set. Let $a=(a^1,\dots,a^n)\in S$ and $x=(x^1,\dots,x^n)\in S$. Then multivariate Taylor polynomial is given by*

$$f(x)=\sum_{|\alpha|\le n}\frac{(\partial^\alpha f)(a)}{\alpha!}(x-a)^\alpha + r_{n,a}(x) \text{ where } r_{n,a}(x)=\sum_{|\alpha|=n+1}\frac{\partial^\alpha f(c)}{\alpha!}(x-a)^\alpha$$

Or consider $h=x-a$, then

$$f(a+h)=\sum_{|\alpha|\le n}\frac{(\partial^\alpha f)(a)}{\alpha!}h^\alpha+\sum_{|\alpha|=n+1}\frac{\partial^\alpha f(c)}{\alpha!}h^\alpha$$

for some c on the line joining a to x, i.e. there exists $t\in[0,1]$ such that

$$c=a(1-t)+tx=a+t(x-a)$$

Remark. Taylor polynomials are *unique*; that is if we ahve an order k polynomial approximation to a function whose error vanishes in order $k+1$, then that polynomial is necessarily Taylor polynomial. This implies that the Taylor series of any polynomial is that polynomial itself.

Definition. *Here is a list of common taylor series*

1. $$\frac{1}{1-x}=1+x+x^2+x^3+\cdots=\sum_{k=0}^{\infty}x^k$$

2.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

3.

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{k=0}^{\infty} = \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

4.

$$cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

5.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$$

2.2 The Hessian Matrix

Definition 2.1. ***Hessian Matrix*** *If $f : \mathbb{R}^n \to \mathbb{R}$ is of class C^2 then the Hessian matrix of f at $a \in \mathbb{R}^n$ is the symmetric (i.e. $H = H^T$) $n \times n$ matrix of second order partial derivatives*

$$H(a) = \begin{bmatrix} \partial_{11} f(a) & \partial_{12} f(a) & \dots & \partial_{1n} f(a) \\ \partial_{21} f(a) & \partial_{22} f(a) & \dots & \partial_{2n} f(a) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{n1} f(a) & \partial_{n2} f(a) & \dots & \partial_{nn} f(a) \end{bmatrix}$$

Remark. We can use notion of Hessian matrix to simplify Taylor series formula. For first term of polynomial expansion

$$\sum_{|\alpha|=1}^{\infty} \frac{1}{\alpha!} (\partial^\alpha f)(a)(x_0 - a)^\alpha = \nabla f(a) \dot{(} x_0 - a)$$

For second order polynomial expansion

$$\sum_{|\alpha|=2}^{\infty} \frac{2}{\alpha!} (\partial^\alpha f)(a)(x_0 - a)^\alpha = (x_0 - a)^T H(a)(x_0 - a)$$

So second-order Taylor polynomial is just

$$f(x_0) = f(a) + \nabla f(a)(x_0 - a) + \frac{1}{2}(x_0 - a)^T H(a)(x_0 - a) + r_{2,a}(x_0)$$

Now we can compute simple Taylor polynomial not only from formula given but also from gradient and Hessian matrix.

Theorem 2.2. ***Spectral Theorem*** *If $A : \mathbb{R}^n \to \mathbb{R}^n$ is a symmetric matrix then there exists an orthonormal basis consisting of eigenvectors of A. And all of its eigenvalues are real numbers. That is, there exists real numbers $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, and vectors $v_1, \cdots, v_n$, such that*

$$Av_i = \lambda_i v_i$$

and

$$v_i \cdot v_j = \begin{cases} 1 & \textit{if } i = j \\ 0 & \textit{if } i \neq j \end{cases}$$