$$
\begin{aligned}
&Max \quad \sum_{(s,u)\in E} f_{sv} - \sum_{(v,s)\in \mathbb{E}} f_{vs} && Max \quad d^T f \\
&st. \sum_{(v,u)\in E} f_{vu} - \sum_{(u,v)\in E} f_{uv} = 0 \qquad \forall u \in V \setminus \{s,t\} && s.t. \quad Af \geq b \quad Af \leq b \\
&0 \leq f_e \leq c_e \qquad \forall e \in E && f_e \geq 0
\end{aligned}
$$

where

$$
\underset{m\times 1}{d} \qquad \underset{(n-2+m)\times m}{A} = \begin{pmatrix} \underset{(n-2)\times m}{B} \\ I_m \end{pmatrix} \qquad \underset{(n-2+m)\times 1}{b} = \begin{pmatrix} \underset{(n-2)\times 1}{0} \\ C \end{pmatrix}
$$

$$
d_{uv} = \begin{cases} 1 & u = s \quad (s,v) \in E \\ -1 & v = s \quad (u,s) \in E \\ 0 & otherwise \end{cases} \qquad B_{ve} = \begin{cases} 1 & (\cdot, v) = e \\ -1 & (v, \cdot) = e \\ 0 & otherwise \end{cases}
$$

Let $|E| = m$ and $|v| = n$. PLP has m variables and $2(n-2)+m$ varialbes. Supposedly, the dual has $2(n-2)+m$ variables and m constraints. However, we can reduce number of dual variables to $n-2+m$ by noticing that

$$
Ax = b \iff Ax \leq b \quad (-A)x \leq (-b) \iff \tilde{A}x = \begin{pmatrix} A \\ -A \end{pmatrix} x \leq \begin{pmatrix} b \\ -b \end{pmatrix} = \tilde{b}
$$

Given dual varaibles $y = \begin{pmatrix} y_1 & y_2 \end{pmatrix} \in \mathbb{R}^{2m}$ where $y_1, y_2 \in \mathbb{R}^{m+}$ have the following dual constraints

$$
\tilde{A}^T y = \begin{pmatrix} A^T & -A^T \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A^T(y_1 - y_2) = A^T y' \geq c
$$

for some $y' = y_1 - y_2 \in \mathbb{R}^m$. Note we reduce number of dual variables by half, i.e. non-negative $y_1, y_2 \in \mathbb{R}^{m+}$ to $y' \in \mathbb{R}^m$, which can be negative

$$
\begin{aligned}
&Min \sum_{e\in E} c_e y_e && Min \quad b^T \tilde{y} = b^T \begin{pmatrix} x \\ y \end{pmatrix} \\
&s.t. \quad x_v - x_u + y_{uv} \geq 0 \qquad \forall (u,v) \in E && s.t. A^T \tilde{y} \geq d \\
&x_s = 1 \quad x_t = 0 && y \geq 0 \\
&y_e \geq 0 \qquad \forall e \in E &&
\end{aligned}
$$

where $\underset{(n-2)\times 1}{x}$ has one value for each $V \setminus \{s,t\}$, and $\underset{m\times 1}{y}$ has one value for each $e \in E$

$$
b^T \tilde{y} = \begin{pmatrix} \underset{(n-2)\times 1}{0} & C \end{pmatrix} \begin{pmatrix} \underset{(n-2)\times 1}{x} \\ \underset{m\times 1}{y} \end{pmatrix} = \sum_{e\in E} c_e y_e \qquad \text{and} \qquad y \geq 0 \quad \to \quad y_e \geq 0 \forall e \in E
$$

$$A^T\tilde{y} = \begin{pmatrix} \underset{m\times(n-2)}{B^T} & I_m \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = B^T x + I_m y \qquad (I_m y)_e = y_e$$

$$(B^T)_{ev} = B_{ve} = \begin{cases} 1 & (\cdot, v) = e \\ -1 & (v, \cdot) = e \\ 0 & otherwise \end{cases} \quad \rightarrow \quad (B^T x)_{e=(v_1,v_2)} = \sum_{v\in V\setminus\{s,t\}} (B^T)_{ev} x_v = B_{v_1 e} x_{v_1} + B_{v_2 e} x_{v_2} = x_{v_1} - x_{v_2}$$

therefore for edges whose vertices consists of neither s nor t

$$A^T\tilde{y} \geq d \quad \rightarrow \quad x_v - x_u + y_{uv} \geq 0 \quad \forall (u,v) \in E : u, v \neq s, t$$

For edges that connects s, we have

$$x_u + y_{su} \geq 1 \quad x_s + y_{us} \geq -1$$

if we set variables $x_s = 1$, then we can write above as

$$x_u - x_s + y_{su} \geq 0 \quad x_s - x_u + y_{us} \geq 0$$

to fit with how the above formulation. Setting $x_t = 0$ does the trick for edges that connects t