Problem 1

A random variable X has moment-generating function $m(t) = e^{9(e^t-1)}$. Find $P(|X-\mu| \leq 2\sigma)$.

Solution.
Notice that because moment-generating function of a distribution is unique. $X \sim Poisson(\lambda = 9)$

$$P(|X-\mu| \leq 2\sigma) = P(|X-9| \leq 6) = P(3 \leq X \leq 15) = 0.9717321$$

The last step calculated using R with

$$ppois(15, 9, lower.tail = TRUE) - ppois(2, 9, lower.tail = TRUE)$$

Therefore the corresponding probability is 0.9717321 $\square$

Problem 2

The joint density of M and N is given by

$$f(m,n) = \begin{cases} me^{-(m+n)}, & m > 0, n > 0 \\ 0, & otherwise \end{cases}$$

1. Find the marginal distributions of M and N.

   Solution.

   $$f_M(m) = \int_0^\infty me^{-(m+n)}dn = -me^{-(m+n)}|_{n=0}^\infty = \begin{cases} me^{-m}, & m > 0 \\ 0, & otherwise \end{cases}$$

   $$f_N(n) = \int_0^\infty me^{-(m+n)}dm = -me^{-(m+n)} - e^{-(m+n)}|_{m=0}^\infty = \begin{cases} e^{-n}, & n > 0 \\ 0, & otherwise \end{cases}$$

   □

2. Find the conditional distribution of M given $N = c$.

   Solution.

   $$P(M|N=c) = \frac{f(m,c)}{f_N(c)} = \frac{me^{-(m+c)}}{e^{-c}} = \begin{cases} me^{-m}, m > 0 \\ 0, & otherwise \end{cases}$$

   □

3. Determine whether M and N are independent. How do you know?

   Solution.

   □

   We verified that $\forall m > 0, n > 0$,

   $$f(m,n) = me^{-(m+n)} = me^{-m} * e^{-n} = f_M(m)f_N(n)$$

   Then M and N are independent.

4. What is the distribution of M? N?

Solution.
According to marginal distribution of M, note with $\alpha = 2$ and $\beta = 1$, then

$$me^{-m} = \frac{1}{\Gamma(2)1^2} m^{2-1} e^{-\frac{x}{1}}$$

Therefore, since PDFs are unique, $M \sim Gamma(2, 1)$
According to marginal distribution of N, note with $\theta = 1$

$$e^{-n} = \frac{1}{1} e^{-\frac{n}{1}}$$

Therefore, $M \sim Exponential(1)$ $\square$

Problem 3

Let T_1 and T_2 denote the proportions of time (out of a school day) during which student A and student B spend working on this assignment. The joint density function of T_1 and $T2$ is modeled by:

$$f(t_1, t_2) = \begin{cases} t_1 + t_2, & 0 \le t_1 \le 1, 0 \le t_2 \le 1 \\ 0, & otherwise \end{cases}$$

1. Find $P(T_1 < \frac{1}{2}, T_2 > \frac{1}{4})$

   Solution.

$$\begin{aligned} P(T_1 < \frac{1}{2}, T_2 > \frac{1}{4}) &= \int_0^{\frac{1}{2}} \int_{\frac{1}{4}}^1 f(t_1, t_2) dt_2 dt_1 \\ &= \int_0^{\frac{1}{2}} \int_{\frac{1}{4}}^1 (t_1 + t_2) dt_2 dt_1 \\ &= \int_0^{\frac{1}{2}} (\frac{15}{32} + \frac{3}{4} t_1) dt_1 \\ &= \frac{15}{32} t_1 + \frac{3}{8} t_1^2 \Big|_{t_1=0}^{\frac{1}{2}} \\ &= \frac{21}{64} \end{aligned}$$

□

2. Find $P(T_1 + T_2 \le 1)$

   Solution.

$$\begin{aligned} P(T_1 + T_2 \le 1) &= \int_0^1 \int_0^{1-t_2} (t_1 + t_2) dt_1 dt_2 \\ &= \int_0^1 (\frac{1}{2} t_1^2 + t_2 t_1 \Big|_{t_1=0}^{1-t_2}) dt_2 \\ &= \int_0^1 (\frac{1}{2} - \frac{1}{2} t_2^2) dt_2 \\ &= \frac{1}{2} t_2 - \frac{1}{6} t_2^3 \Big|_{t_2=0}^1 \\ &= \frac{1}{3} \end{aligned}$$

□

3. Find the covariance of T_1 and T_2. Are T_1 and T_2 independent or dependent? How do you know?

Solution.

$$E(T_1T_2) = \int_0^1 \int_0^1 t_1t_2(t_1+t_2)dt_1dt_2 = \int_0^1 (\frac{1}{3}t_2 + \frac{1}{2}t_2^2)dt_2 = \frac{1}{3}$$

$$f_{T_1}(t_1) = \int_0^1 t_1 + t_2 dt_2 = t_1t_2 + \frac{1}{2}t_2^2|_{t_2=0}^1 = t_1 + \frac{1}{2}$$

$$f_{T_2}(t_2) = \int_0^1 t_1 + t_2 dt_1 = t_1t_2 + \frac{1}{2}t_1^2|_{t_1=0}^1 = t_2 + \frac{1}{2}$$

$$E(T_1) = \int_0^1 (t_1 f_{T_1}(t_1))dt_1 = \int_0^1 (t_1^2 + \frac{1}{2}t_1)dt_1 = \frac{7}{12}$$

$$E(T_2) = \int_0^1 (t_2 f_{T_2}(t_2))dt_2 = \int_0^1 \frac{3}{2}t_2^3 dt_2 = \frac{7}{12}$$

$$COV(T_1, T_2) = E(T_1T_2) - E(T_1)E(T_2) = \frac{1}{3} - \frac{7}{12}\frac{7}{12} = -0.006944$$

Since covariance of T_1 and T_2 is not 0, T_1 and T_2 are not independent. □

4. Find $P(0.22 < T_1 < 0.33|T_2 \geq 0.3)$

Solution.

$$\begin{aligned}
P(0.22 < T_1 < 0.33, T_2 \geq 0.3) &= \int_{0.22}^{0.33} \int_{0.3}^{1} (t_1 + t_2)dt_2dt_1 \\
&= \int_{0.22}^{0.33} (\frac{7}{10}t_1 + \frac{91}{200})dt_1 \\
&= \frac{7}{20}t_1^2 + \frac{91}{200}t_1|_{t_1=0.22}^{0.33} \\
&= 0.0712
\end{aligned}$$

$$P(T_2 \geq 0.3) = \int_{0.3}^1 f_{T_2}(t_2)dt_2 = \int_{0.3}^1 (t_2 + \frac{1}{2})dt_2 = 0.805$$

$$P(0.22 < T_1 < 0.33|T_2 \geq 0.3) = \frac{P(0.22 < T_1 < 0.33, T_2 \geq 0.3)}{P(T_2 \geq 0.3)} = 0.0884$$

□