STA302/STA1001, Weeks 9-10

Mark Ebden, 14 & 16 November 2017

With grateful acknowledgment to Alison Gibbs

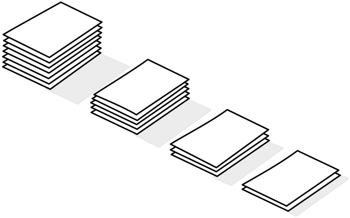
This week's lectures

- Poll regarding the Exam Jam on 8 December
- Chapter 5:
  - Matrix version of SLR
  - Multiple linear regression (MLR)

If you haven't retrieved your midterm

To pick up your test at SS 6027 CLTA:

1. Book an office-hours slot online for any Wednesday until 6 December, or
2. Drop in on Tuesday 14 November, 2-2:30 pm, or
3. Drop in on Thursday 16 November, 1-1:30 pm



To appeal/discuss a recent TA decision on regrading

Please express your appeal via your section's regrading email address.

If the TA who had examined your work doesn't reply within a week then please notify me.

Recap of our recent studies

The SLR model in matrix form is $\mathbf{Y} = \mathbf{X}\beta + \mathbf{e}$, in which:

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}, \qquad \mathbf{X} = \begin{pmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{pmatrix}, \qquad \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}, \qquad \mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$$

Setting the derivative of RSS(β) to zero yielded $\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ when rank($\mathbf{X}$) $= 2$. This plus the fact that $\mathrm{E}(\mathbf{e}) = \mathbf{0}$ gives

$$\widehat{\mathbf{Y}} = \mathbf{X}\hat{\beta} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} = \mathbf{HY}$$ hat matrix

We can write the residuals in terms of idempotent matrix $\mathbf{I} - \mathbf{H}$ as

$$\widehat{\mathbf{e}} = \mathbf{Y} - \widehat{\mathbf{Y}} = \mathbf{Y} - \mathbf{X}\hat{\beta} = \mathbf{Y} - \mathbf{HY} = (\mathbf{I} - \mathbf{H})\,\mathbf{Y}$$

We find $\mathrm{E}(\widehat{\mathbf{e}}) = \mathbf{0}$ and were about to try var($\widehat{\mathbf{e}}$), requiring the notion of a covariance matrix: $\mathrm{var}(\mathbf{X}) = \mathrm{E}\left[(\mathbf{X} - \mathrm{E}(\mathbf{X})\,(\mathbf{X} - \mathrm{E}(\mathbf{X}))'\right]$.

Five facts about idempotent matrices *(Weeks 8–9, slide 18)*

1. A square matrix $\mathbf{A}$ is idempotent iff $\mathbf{A}^2 = \mathbf{A}$
2. If $\mathbf{A}$ is idempotent then $\text{trace}(\mathbf{A}) = \text{rank}(\mathbf{A})$
3. $\mathbf{A}$ is idempotent iff $\text{rank}(\mathbf{A}) + \text{rank}(\mathbf{I} - \mathbf{A}) = n$ where the dimensions of $\mathbf{A}$ are $n \times n$ and $\mathbf{I}$ is the $n \times n$ identity matrix
4. For hat matrix $\mathbf{H}$ and matrix of all 1's $\mathbf{J}$, the following matrices are idempotent:

$$\mathbf{H} \qquad \mathbf{I} - \mathbf{H} \qquad \frac{1}{n}\mathbf{J} \qquad \mathbf{H} - \frac{1}{n}\mathbf{J}$$

5. If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are idempotent and $\mathbf{A} = \mathbf{B} + \mathbf{C}$, then $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) + \text{rank}(\mathbf{C})$

Variance of the residuals, in matrix form

note off-diagonal elements might be zero,
which is fine, since they are not necessarily independent
but their sum equates to 1

$$\begin{aligned}\mathsf{var}(\widehat{\mathbf{e}}) &= \mathrm{E}\left\{[\hat{\mathbf{e}} - \mathrm{E}(\hat{\mathbf{e}})]\,[\hat{\mathbf{e}} - \mathrm{E}(\hat{\mathbf{e}})]'\right\} \\ &= \mathrm{E}\left\{\underline{(\mathbf{I}-\mathbf{H})\,\mathbf{Y}}\underline{\mathbf{Y}'\,(\mathbf{I}-\mathbf{H})}\right\} \\ &= (\mathbf{I}-\mathbf{H})\,\mathrm{E}(\mathbf{Y}\mathbf{Y}')\,(\mathbf{I}-\mathbf{H})\end{aligned}$$

Compare to our previous work: $\mathsf{var}(\hat{e}_i) = \sigma^2(1 - h_{ii})$. Does the above match?

matrix form gives more information

NB: As before, the "$|\mathbf{X}$" is implicit — e.g. $\mathsf{var}(\hat{\mathbf{e}}|X)$ is abbreviated as $\mathsf{var}(\hat{\mathbf{e}})$.

Variance of the residuals, in matrix form

$$
\begin{aligned}
\text{The middle factor is } \mathrm{E}(\mathbf{Y}\mathbf{Y}') &= \mathrm{E}\left\{(\mathbf{X}\beta + \mathbf{e})(\mathbf{X}\beta + \mathbf{e})'\right\} \\
&= \mathrm{E}\left\{(\mathbf{X}\beta + \mathbf{e})(\beta'\mathbf{X}' + \mathbf{e}')\right\} \\
\text{missing X'?} &= \mathrm{E}\left\{\mathbf{X}\beta\beta'\mathbf{X} + \mathbf{X}\beta\mathbf{e}' + \mathbf{e}\beta'\mathbf{X}' + \mathbf{e}\mathbf{e}'\right\} \\
&= \mathrm{E}\left\{\mathbf{X}\beta\beta'\mathbf{X}\right\} + \mathbf{0} + \mathbf{0} + \mathrm{E}(\mathbf{e}\mathbf{e}') \\
\mathrm{E}(\mathbf{Y}\mathbf{Y}') &= \mathbf{X}\beta\beta'\mathbf{X} + \sigma^2\mathbf{I}
\end{aligned}
$$

Inserting the above into $\mathrm{var}(\widehat{\mathbf{e}})$ gives

$$
\begin{aligned}
\mathrm{var}(\widehat{\mathbf{e}}) &= (\mathbf{I} - \mathbf{H})(\mathbf{X}\beta\beta'\mathbf{X} + \sigma^2\mathbf{I})(\mathbf{I} - \mathbf{H}) \\
&= \left[\mathbf{I}\left(\mathbf{X}\beta\beta'\mathbf{X} + \sigma^2\mathbf{I}\right) - \mathbf{H}\left(\mathbf{X}\beta\beta'\mathbf{X} + \sigma^2\mathbf{I}\right)\right](\mathbf{I} - \mathbf{H}) \\
&= \left[\mathbf{X}\beta\beta'\mathbf{X} + \sigma^2\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\left(\mathbf{X}\beta\beta'\mathbf{X} - \sigma^2\mathbf{H}\right)\right](\mathbf{I} - \mathbf{H}) \\
&= \sigma^2(\mathbf{I} - \mathbf{H})(\mathbf{I} - \mathbf{H}) \\
\mathrm{var}(\widehat{\mathbf{e}}) &= \sigma^2(\mathbf{I} - \mathbf{H})
\end{aligned}
$$

indempotent

What's the rank of var($\widehat{\mathbf{e}}$)?

Recall the fifth of our *Five facts about idempotent matrices*:

$$\text{If } \mathbf{A} = \mathbf{B} + \mathbf{C}, \text{ then } \text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) + \text{rank}(\mathbf{C}).$$

Put another way, $\text{rank}(\mathbf{B}) = \text{rank}(\mathbf{A}) - \text{rank}(\mathbf{C})$.

Therefore, for example $\text{rank}(\mathbf{I} - \mathbf{H}) = \text{rank}(\mathbf{I}) - \text{rank}(\mathbf{H}) = n - 2$. We'll do other similar calculations when considering ANOVA in matrix terms.

ANOVA in matrix terms

Recall from Week 3 that

$$\boxed{\text{SST} = \text{SSReg} + \text{RSS}}$$

where

$$\text{SST} = \sum_{i=1}^{n} (y_i - \bar{y})^2 = \sum_{i=1}^{n} y_i^2 - n\bar{y}^2$$

Exercise: Show that SST can be re-expressed as

$$\boxed{\text{SST} = \mathbf{Y}'\mathbf{Y} - \frac{1}{n}\mathbf{Y}'\mathbf{J}\mathbf{Y}}$$

where $\mathbf{J}$ is an $n \times n$ matrix of 1's. This means we can also write

$$\text{SST} = \mathbf{Y}'\left(\mathbf{I} - \frac{1}{n}\mathbf{J}\right)\mathbf{Y}$$

Some properties of $\mathbf{I} - \frac{1}{n}\mathbf{J}$

1. Note that $\mathbf{I} - \frac{1}{n}\mathbf{J}$ is symmetric. For a vector $\mathbf{Y}$ and symmetric matrix $\mathbf{A}$, you may recall from other courses that $\mathbf{Y}'\mathbf{A}\mathbf{Y}$ is a *quadratic form* (second-degree polynomial).

2. Since $\mathbf{I}$ is idempotent and $\frac{1}{n}\mathbf{J}$ is idempotent (from the five facts), $\mathbf{I} - \frac{1}{n}\mathbf{J}$ is also idempotent.

3. The rank of $\mathbf{I} - \frac{1}{n}\mathbf{J}$ is $\text{rank}(\mathbf{I}) - \text{rank}\left(\frac{1}{n}\mathbf{J}\right) = n - 1$.

   - This is the number of degrees of freedom for SST

rank = degree of freedom

Decomposing SST

Taking the first term of SST $= \mathbf{Y}'\mathbf{Y} - \frac{1}{n}\mathbf{Y}'\mathbf{J}\mathbf{Y}$,

$$
\begin{aligned}
\mathbf{Y}'\mathbf{Y} &= (\mathbf{Y} - \mathbf{Xb} + \mathbf{Xb})'(\mathbf{Y} - \mathbf{Xb} + \mathbf{Xb}) \\
&= (\mathbf{Y} - \mathbf{Xb})'(\mathbf{Y} - \mathbf{Xb}) + (\mathbf{Y} - \mathbf{Xb})'\mathbf{Xb} + (\mathbf{Xb})'(\mathbf{Y} - \mathbf{Xb}) + (\mathbf{Xb})'(\mathbf{Xb}) \\
&= \hat{\mathbf{e}}'\hat{\mathbf{e}} + \underbrace{\hat{\mathbf{e}}'\mathbf{Xb} + (\mathbf{Xb})'\hat{\mathbf{e}}}_{\text{equal scalars, both 0}} + \mathbf{b}'\mathbf{X}'\mathbf{Xb} \\
&= \hat{\mathbf{e}}'\hat{\mathbf{e}} + \mathbf{b}'\mathbf{X}'\mathbf{Xb}
\end{aligned}
$$

The middle terms were zero because

$$
\mathbf{X}'\hat{\mathbf{e}} = \mathbf{X}'(\mathbf{I} - \mathbf{H})\mathbf{Y} = \mathbf{X}'\mathbf{Y} - \mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} = \mathbf{0}
$$

So,

$$
\begin{aligned}
\text{SST} &= \mathbf{Y}'\mathbf{Y} - \frac{1}{n}\mathbf{Y}'\mathbf{J}\mathbf{Y} \\
\text{SST} &= \underbrace{\hat{\mathbf{e}}'\hat{\mathbf{e}}}_{\text{RSS}} + \underbrace{\mathbf{b}'\mathbf{X}'\mathbf{Xb} - \frac{1}{n}\mathbf{Y}'\mathbf{J}\mathbf{Y}}_{\text{SSReg}}
\end{aligned}
$$

A closer look at RSS

$$\boxed{\text{SST} = \text{SSReg} + \text{RSS}}$$

Making use of our expression $\hat{\mathbf{e}} = (\mathbf{I} - \mathbf{H})\mathbf{Y}$, we have

$$\begin{aligned} \text{RSS} &= \hat{\mathbf{e}}'\hat{\mathbf{e}} \\ &= \mathbf{Y}'(\mathbf{I} - \mathbf{H})'(\mathbf{I} - \mathbf{H})\mathbf{Y} \\ &= \mathbf{Y}'(\mathbf{I} - \mathbf{H})\mathbf{Y} \end{aligned}$$

This is another quadratic form in $\mathbf{Y}$. Also, $\text{rank}(\mathbf{I} - \mathbf{H}) = n - 2$ from earlier, the number of degrees of freedom for the error.

A closer look at SSReg

$$\boxed{\text{SST} = \text{SSReg} + \text{RSS}}$$

Making use of $\hat{\beta} = \mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$,

$$\begin{aligned}
\text{SSReg} &= \mathbf{b}'\mathbf{X}'\mathbf{X}\mathbf{b} - \mathbf{Y}'\frac{1}{n}\mathbf{J}\mathbf{Y} \quad \text{from slide 12} \\
&= \mathbf{Y}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} - \mathbf{Y}'\frac{1}{n}\mathbf{J}\mathbf{Y} \\
&= \mathbf{Y}'\underbrace{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'}_{\mathbf{H}}\mathbf{Y} - \mathbf{Y}'\frac{1}{n}\mathbf{J}\mathbf{Y} \\
&= \mathbf{Y}'\left(\mathbf{H} - \frac{1}{n}\mathbf{J}\right)\mathbf{Y}
\end{aligned}$$

This is again a quadratic form in $\mathbf{Y}$, since the middle matrix is symmetric. Also, $\text{rank}(\mathbf{H} - \frac{1}{n}\mathbf{J}) = \text{rank}(\mathbf{H}) - \text{rank}\left(\frac{1}{n}\mathbf{J}\right) = 2 - 1 = 1$, the number of degrees of freedom for SSReg.

Using RSS to estimate σ^2

In $S^2 = \text{RSS}/(n-2)$, we have an unbiased estimator for σ^2. We can show it's unbiased using matrices by showing that $\mathrm{E}(\text{RSS}) = (n-2)\sigma^2$ as we did without matrices — i.e. when we considered $\mathrm{E}(\text{RSS}) = \mathrm{E}\left(\sum_{i=1}^n \hat{e}_i^2\right)$. = 0 (unbiased)

a scalar so trace equal to itself

$$
\begin{aligned}
\mathrm{E}(\text{RSS}) &= \mathrm{E}\left(\hat{\mathbf{e}}'\hat{\mathbf{e}}\right) \\
&= \mathrm{E}\left\{\mathbf{Y}'(\mathbf{I}-\mathbf{H})\mathbf{Y}\right\} && \text{from slide 13} \\
&= \mathrm{E}\left\{\text{trace}\left[\mathbf{Y}'(\mathbf{I}-\mathbf{H})\mathbf{Y}\right]\right\} && \text{since } \mathbf{Y}'(\mathbf{I}-\mathbf{H})\mathbf{Y} \text{ is a scalar} \\
&= \mathrm{E}\left\{\text{trace}\left[(\mathbf{I}-\mathbf{H})\mathbf{Y}\mathbf{Y}'\right]\right\} && \text{since trace}(\mathbf{AB}) = \text{trace}(\mathbf{BA}) \\
&= \text{trace}\left[(\mathbf{I}-\mathbf{H})\,\mathrm{E}(\mathbf{Y}\mathbf{Y}')\right] \\
&= \text{trace}\left[(\mathbf{I}-\mathbf{H})\left(\sigma^2\mathbf{I} + \mathbf{X}\beta\beta'\mathbf{X}'\right)\right] && \text{from slide 8} \\
&= \text{trace}\left[(\mathbf{I}-\mathbf{H})\,\sigma^2 + \mathbf{X}\beta\beta'\mathbf{X}' - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\beta\beta'\mathbf{X}'\right] \\
&= \text{trace}(\mathbf{I}-\mathbf{H})\,\sigma^2 \\
\mathrm{E}(\text{RSS}) &= (n-2)\sigma^2
\end{aligned}
$$

so unbiased estimator

where we have used $\text{trace}(\mathbf{A}+\mathbf{B}) = \text{trace}(\mathbf{A}) + \text{trace}(\mathbf{B})$, and thus $\text{trace}(\mathbf{I}-\mathbf{H}) = \text{trace}(\mathbf{I}) - \text{trace}(\mathbf{H}) = n - \sum_{i=1}^n h_{ii} = n-2$.

The big picture

We have expressed our ANOVA identity in matrix form:

$$\mathrm{SST} = \mathrm{SSReg} + \mathrm{RSS}$$

$$\underbrace{\mathbf{Y}'\left(\mathbf{I} - \frac{1}{n}\mathbf{J}\right)\mathbf{Y}}_{\mathrm{SST}} = \underbrace{\mathbf{Y}'\left(\mathbf{H} - \frac{1}{n}\mathbf{J}\right)\mathbf{Y}}_{\mathrm{SSReg}} + \underbrace{\mathbf{Y}'\left(\mathbf{I} - \mathbf{H}\right)\mathbf{Y}}_{\mathrm{RSS}}$$

What does R have to say about this?

ANOVA in R

The anova command is one way in R to produce an ANOVA table (see Week 3, slide 42), in addition to analysing it. For example, for the 654-point SLR problem in Assignment 2, question 1:

```
a2 = read.table("DataA2.txt",sep=" ",header=T) # Load the data set
fev <- a2$fev; age <- a2$age
mod1 = lm(fev~age)
anova(mod1)
```

```
## Analysis of Variance Table
##
## Response: fev
##            Df Sum Sq Mean Sq F value    Pr(>F)
## age         1 280.92 280.919  872.18 < 2.2e-16 ***
## Residuals 652 210.00   0.322
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

ANOVA in R

The p-value will match that obtained from the `summary(lm...)` command:

```
summary(mod1)
```

```
##
## Call:
## lm(formula = fev ~ age)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.57539 -0.34567 -0.04989  0.32124  2.12786
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 0.431648   0.077895   5.541 4.36e-08 ***
## age         0.222041   0.007518  29.533  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.5675 on 652 degrees of freedom
## Multiple R-squared:  0.5722, Adjusted R-squared:  0.5716
## F-statistic: 872.2 on 1 and 652 DF,  p-value: < 2.2e-16
```

same as anova p value

This week's lectures

- Poll regarding the Exam Jam on 8 December
- Chapter 5:
  - Matrix version of SLR
  - **Multiple linear regression (MLR)**

Multiple regression

1. handles more than 1 explanatory variable
2. fitting curlinear model
3. compare regression line with multiple groups

Multiple regression is used when we have more than one explanatory variable. Multiple x's can arise naturally. In addition, sometimes we want to:

- Control for some x's to consider the effect on y of other x's over and above the control variables
- Fit a polynomial
- Compare the regression line for two or more groups

In multiple linear regression (MLR), generally we let p represent the number of explanatory variables in the model, i.e.

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_p x_{ip} + e_i$$

for $i \in \{1, \ldots, n\}$. How many parameters do we need to estimate? p+2

And therefore, how many observations do we need at a minimum?

p+2

Matrix version of MLR

each column is a explanatory variable
each row is an observation

Our main equation is unchanged: $\mathbf{Y} = \mathbf{X}\beta + \mathbf{e}$

However, the **design matrix X** and β are bigger:

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} 1 & X_{11} & X_{12} & \cdots & X_{1p} \\ 1 & X_{21} & X_{22} & & X_{2p} \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & X_{n1} & X_{n2} & \cdots & X_{np} \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$$

A design matrix gives the explanatory variables (often without the column of 1's). Each row is an observation and each column corresponds to a different kind of variable.

Gauss-Markov assumptions for MLR

The key equations are unchanged:

$$\mathrm{E}(\mathbf{e}) = \mathbf{0} \quad \text{and} \quad \text{var}(\mathbf{e}) = \sigma^2 \mathbf{I}$$

For our inference methods (CIs etc), we need $\mathbf{e}$ to have a multivariate normal distribution as before.

The expression for residuals is still $\widehat{\mathbf{e}} = \mathbf{Y} - \widehat{\mathbf{Y}} = \mathbf{Y} - \mathbf{X}\mathbf{b} = (\mathbf{I} - \mathbf{H})\mathbf{Y}$, where now we have

$$\mathbf{b} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_p \end{pmatrix}$$

Estimating σ^2 in MLR

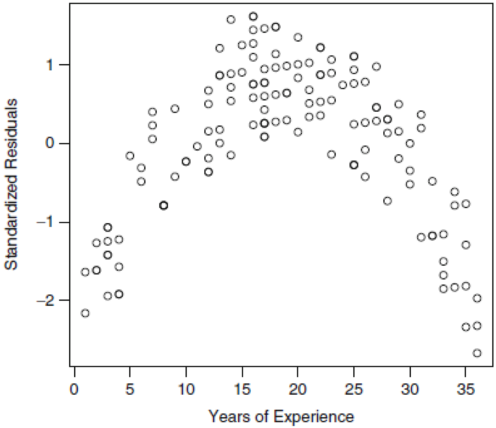
Recall that $S^2 = \text{MSE} = \sum_{i=1}^{n} \hat{e}_i^2 / \text{df} = \widehat{\mathbf{e}}'\widehat{\mathbf{e}} / \text{df}$. The degrees of freedom was $n-2$ in SLR, and is $n-p-1$ in MLR. To see this, recall that $\text{RSS} = \widehat{\mathbf{e}}'\widehat{\mathbf{e}} = \mathbf{Y}'(\mathbf{I}-\mathbf{H})\mathbf{Y}$. Using our five key properties of idempotent matrices again, $\text{rank}(\mathbf{I}-\mathbf{H}) = \text{rank}(\mathbf{I}) - \text{rank}(\mathbf{H}) = n-(p+1)$ assuming that the columns of X are linearly independent.

To show that S^2 is unbiased in MLR, similar to before we can show $\mathrm{E}(\text{RSS}) = (n-p-1)\sigma^2$. The proof is akin to the SLR proof except that:

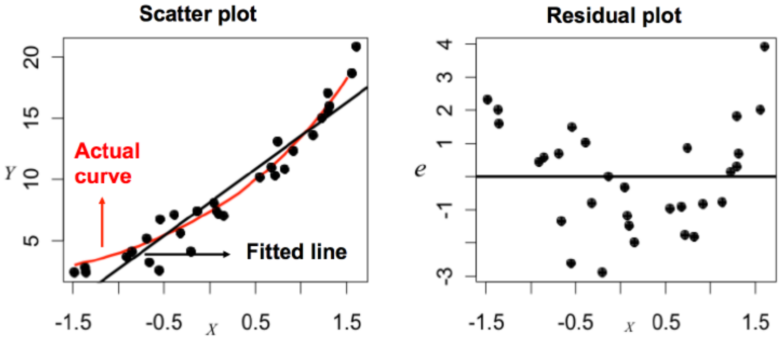
$$\begin{aligned}
\text{trace}(\mathbf{I}-\mathbf{H}) &= \text{trace}(\mathbf{I}) - \text{trace}(\mathbf{H}) \\
&= n - \text{trace}\left[\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\right] \\
&= n - \text{trace}\left[\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\right] \\
&= n - \text{trace}(\mathbf{I}_2) \\
&= n - (p+1)
\end{aligned}$$

Example of MLR: Fitting a polynomial

A professional-salary database contains 143 ordered pairs: (years of experience, salary). Generally, but not monotonically, salary increases with years of experience. Using SLR, our model is $Y_i = \beta_0 + \beta_1 x_i + e_i$. After fitting a straight line, this is the plot of standardized residuals:



Example of a nonlinear relationship *(Weeks 4–5, Slide 41)*

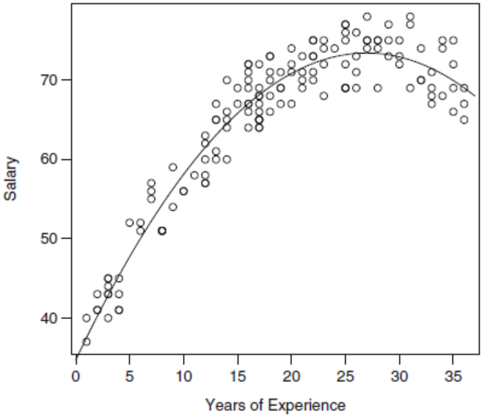


Remedial measure: If the regression function isn't linear,

- In some cases, a variable transformation can make the data "more linear"
- Otherwise, a different (e.g. nonlinear) model might be better

Back to our salary database

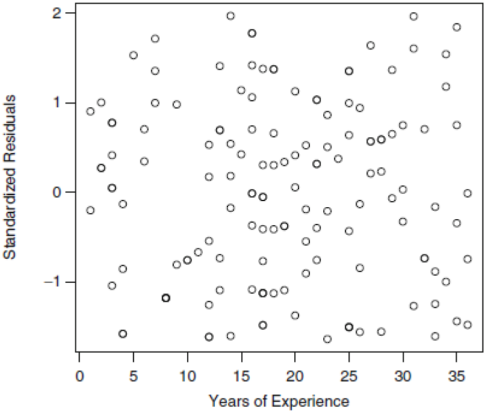
A simple nonlinear model is MLR in which we fit a parabola, i.e. incorporate x and x^2. The model is $Y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + e_i$ and the plot is:



Here, $\beta_0 \approx 35$, $\beta_1 \approx 2.87$, and $\beta_2 \approx -0.053$, each with $p < 2 \times 10^{-16}$.

MLR example: fitting a polynomial

The residuals no longer have a pattern:



R code for MLR

person#, salary, experience

```
X <- read.csv("profsalary.txt",sep="\t")
mod1 <- lm(Salary ~ Experience + I(Experience^2), data=X)
summary(mod1)
```

Typing `I(.)` is a way to express formulae within a call to `lm`.

The + sign indicates that more than one explanatory variable is being used. To have four variables, use e.g. `y ~ x1 + x2 + x3 + x4`

R output for MLR

```
##
## Call:
## lm(formula = Salary ~ Experience + I(Experience^2), data = X)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -4.5786 -2.3573  0.0957  2.0171  5.5176
##
## Coefficients:
##                  Estimate Std. Error t value Pr(>|t|)
## (Intercept)     34.720498   0.828724   41.90   <2e-16 ***
## Experience       2.872275   0.095697   30.01   <2e-16 ***
## I(Experience^2) -0.053316   0.002477  -21.53   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.817 on 140 degrees of freedom
## Multiple R-squared:  0.9247, Adjusted R-squared:  0.9236
## F-statistic: 859.3 on 2 and 140 DF,  p-value: < 2.2e-16
```

Using the R model

Interpolate at 5 years of experience:

```
e <- 5; mod1$coefficients%*%c(1,e,e^2)
```

matrix multiplication

```
##          [,1]
## [1,] 47.74897
```

Alternatively, use the `predict` command:

```
predict(mod1,data.frame(Experience=5))
```

```
##        1
## 47.74897
```

The data frame passed to `predict` names and initializes all of the information used towards making the predictor variables. Another example would be:

```
predict(lm(y~x1+x2),data.frame(x1=5,x2=3))
```

Interpreting MLR coefficients

How should we interpret β_j, or similarly their estimates b_j — i.e. what's the meaning of the coefficients of MLR predictor variables?

In general, β_j is the change in the mean value of Y associated with a one-unit change in the predictor variable x_j, with *all other variables held constant*.

For our salary database example, this is impossible. The closest interpretations we can make are of this sort:

x cant change when x^2 constant

- If Experience increases from 5 years to 6 years, the estimated change in mean Salary is $2.87 - 0.053(36 - 25) \approx 2.3$
- If Experience increases from 35 years to 36 years, the estimated change in mean Salary is $2.87 - 0.053(36^2 - 35^2) \approx -0.9$

Do we need a polynomial fit?

We can quantify whether the quadratic term is 0 or not using familiar hypothesis testing:

$$H_0 : \beta_2 = 0 \quad \text{vs} \quad H_a : \beta_2 \neq 0$$

Exercise: Try this on the salary database. What do you find?

quantify if MLR is valid

Do we need the *j*th predictor?

dropping some x variables, the betas generally go from significant to not significant

In general, a test of $H_0 : \beta_j = 0$ gives an indication of whether or not the *j*th predictor variable statistically significantly contributes to the estimation/prediction of Y *over and above* the other predictor variables.

That is, the test assumes that the other variables are in the model.

Next steps

- Solutions to **HW2** were posted during the Study Break
- Complete Chapter 5's **question 1**
- The lecture on Tuesday **21 November** will start at 11:10 am (not 10:10 am) and finish at the usual time. Sections 1 and 2 will then be resynchronized
- **Further ahead:**
  - In Chapter 6, we won't cover Marginal Model Plots, Inverse Response Plots, or Box-Cox transformations
  - In Chapter 7, mainly we'll cover §7.2.3 and p. 252