Wavelet Tutorial

Learning Goals

- What are wavelet transforms for? Wavelet compression!
- Haar wavelets
    - Given a 1-D (or 2-D) image, find its Haar wavelet coefficients.
    - Given the Haar wavelet coefficients, recover the original image.
- Proof of minimum reconstruction error

Wavelet Compression

- An image can be thought of as a vector of dimension $w \times h$.
- This is really big! Makes for slow computation.
- GOAL: Reduce the dimensionality of the image without losing too much "important information".
- (Hint: We've already seen an example of a possible set of basis vectors: eigenfaces via PCA.)

Image transformation

Represent each image as a linear combination of basis vectors and coefficients.

(sorted from largest to smallest, ie. coarse-to-fine details)

(column vectors)

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m-1} \\ y_m \end{bmatrix} = \begin{bmatrix} B_1 & B_2 & \dots & B_{m-1} & B_m \end{bmatrix}^T \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \\ x_m \end{bmatrix}$$

wavelet coefficient vector
$(m \times 1)$

wavelet basis
$(m \times m)$

original image vector
$(m \times 1)$

Image reconstruction

Represent each image as a linear combination of basis vectors and coefficients.

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \\ x_m \end{bmatrix} = \begin{bmatrix} B_1 & B_2 & \dots & B_{m-1} & B_m \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m-1} \\ y_m \end{bmatrix}$$

original image vector	wavelet basis	wavelet coefficient vector
$(m \times 1)$	$(m \times m)$	$(m \times 1)$

Wavelet Compression

GOAL: Reduce the dimensionality of the image X without losing too much "important information".

ALGORITHM:

1. Compute the **wavelet transformation** $Y = B^T X$ of image X given a wavelet basis B.
2. Keep the k-largest coefficients in Y to obtain Y' with dimensionality k. (k is a fixed parameter.)
3. Keep the corresponding vectors in B to obtain B'.
4. Compute a new image by **wavelet reconstruction**, $X' = B'\, Y'$. This new image has dimensionality k.

Wavelet Compression

Alternatively, instead of fixing k beforehand, keep growing Y' until the difference between the reconstructed image X' and the original image X is small.

Haar Wavelet Transform

Let X be a 1-D image with pixel intensities $[\ 8\ \ 4\ \ 3\ \ 5\ ]$. Compute the Haar wavelet coefficients of X.

Haar Wavelet Transform

Let X be a 1-D image with pixel intensities [8 4 3 5]. Compute the Haar wavelet coefficients of X.

Method 1: Use the recursive algorithm.

1. For each pair of pixels $(2i, 2i + 1)$, compute the **average** and the **detail** coefficients (ie. $2i$ - average).
2. Recurse on the averages until there is only one left.
3. Return the vector containing: the last average and the detail coefficients.

8	4	3	5

Haar Wavelet Transform

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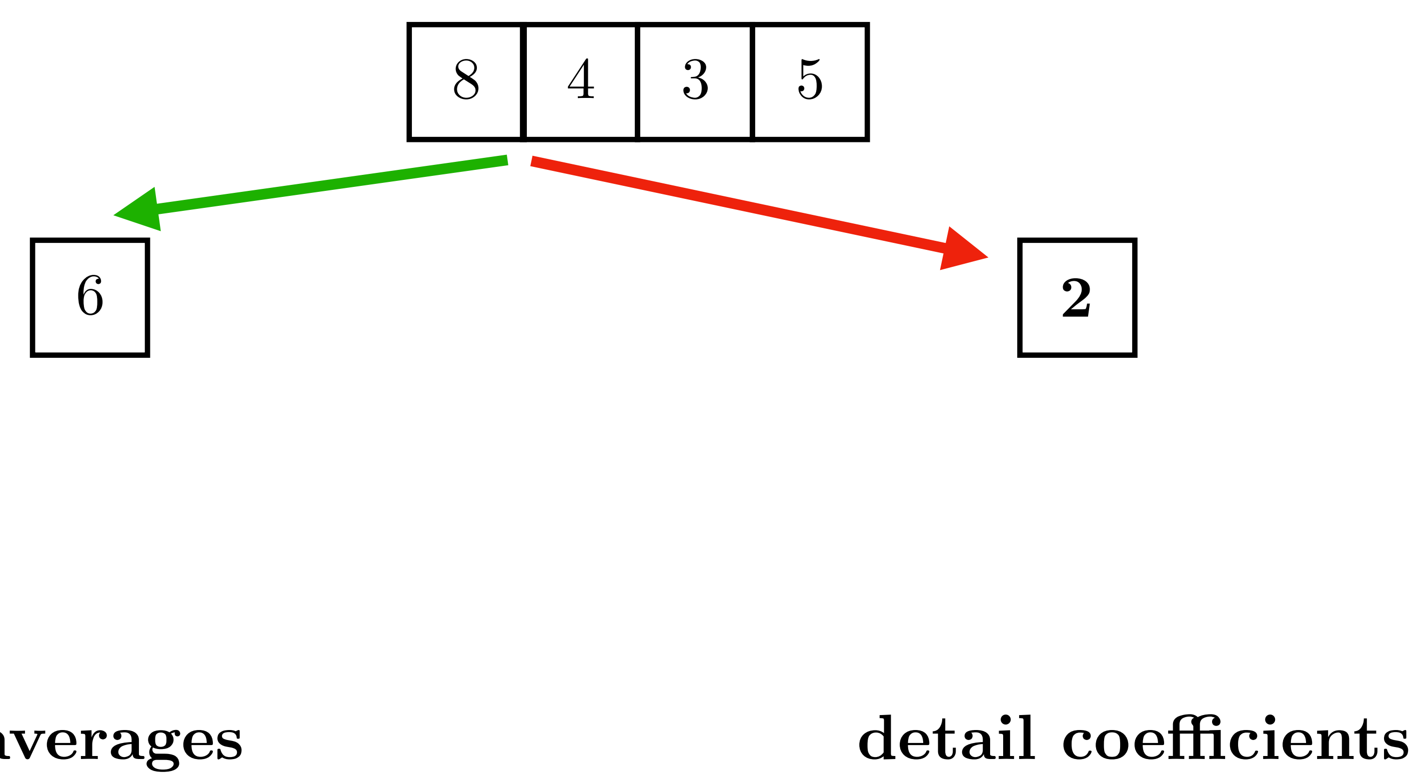

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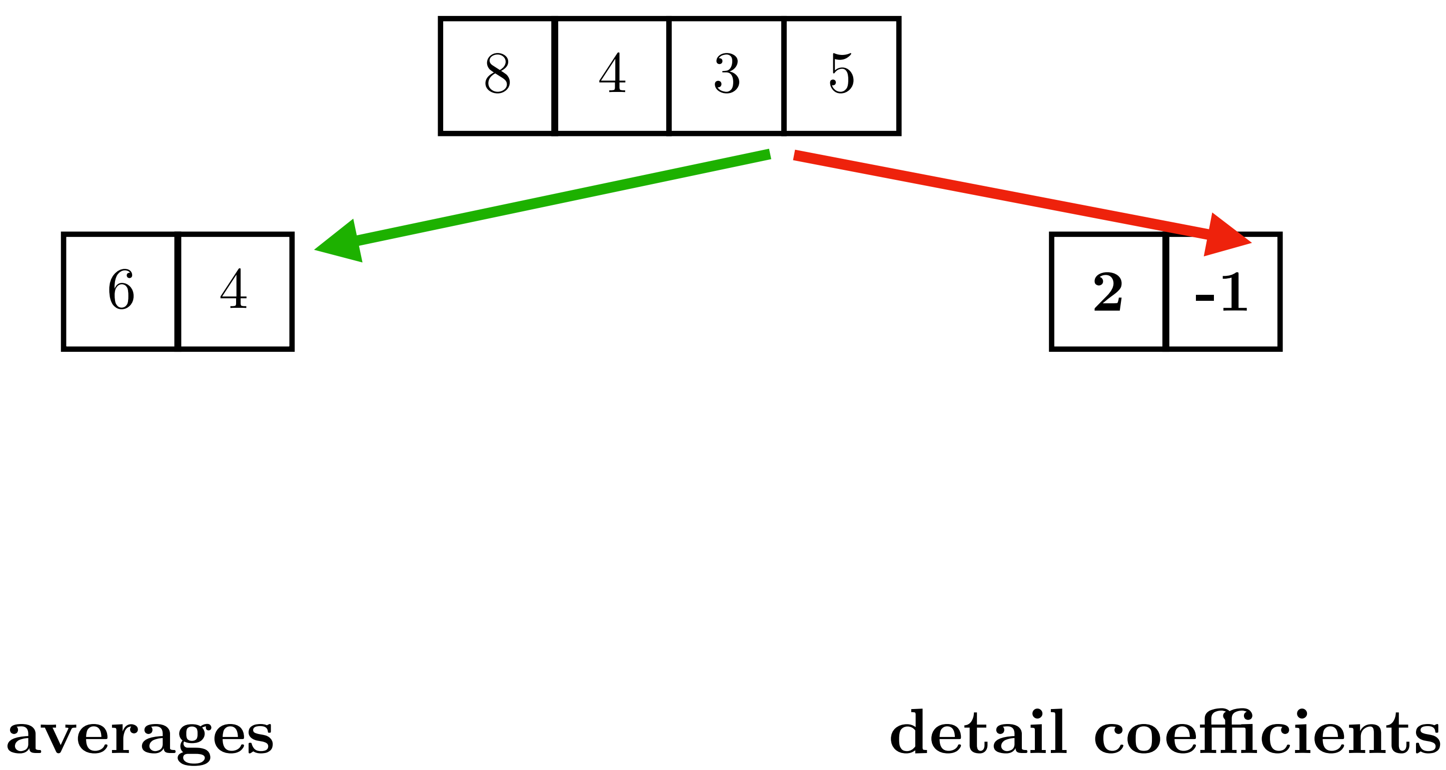

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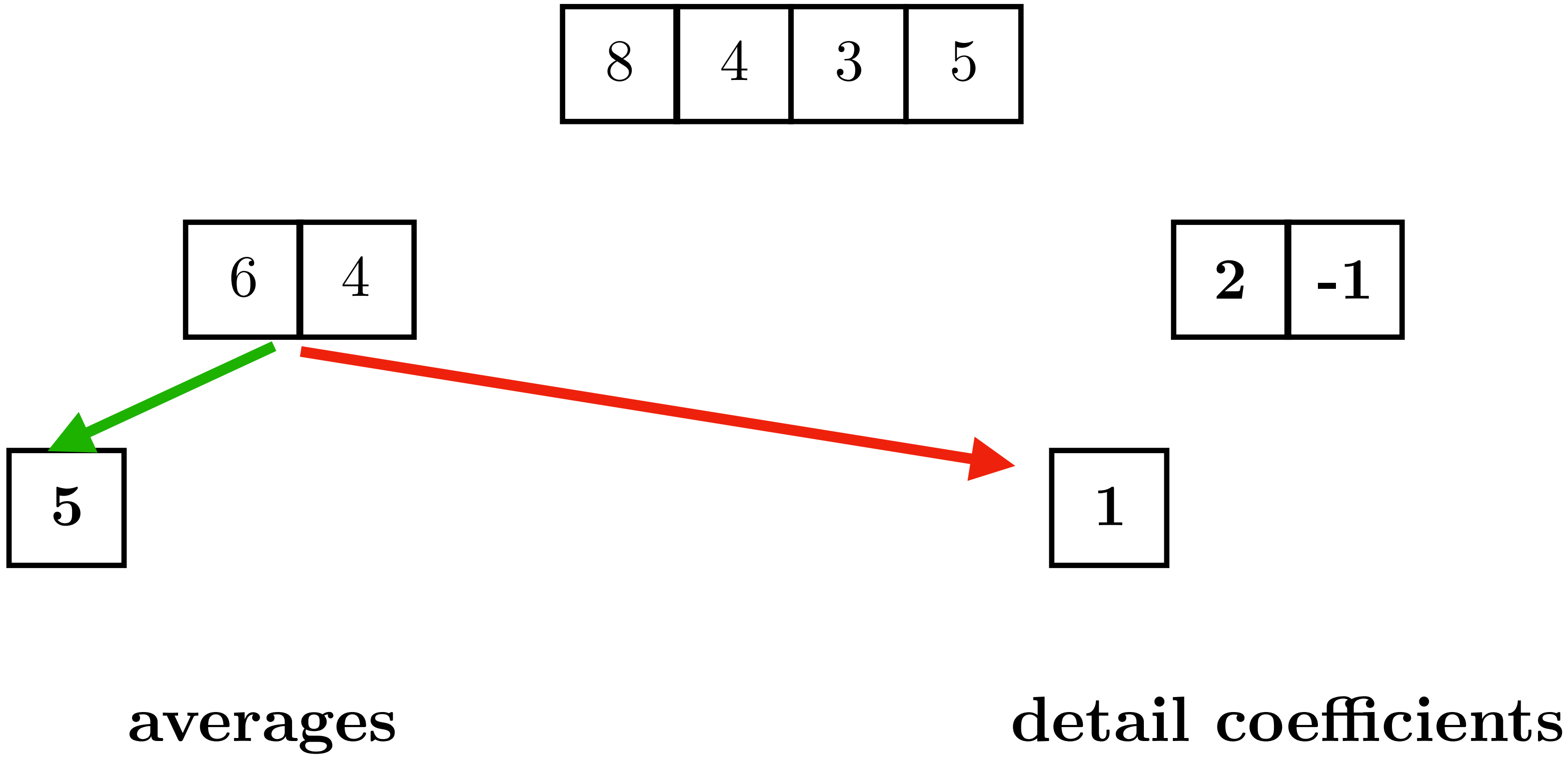

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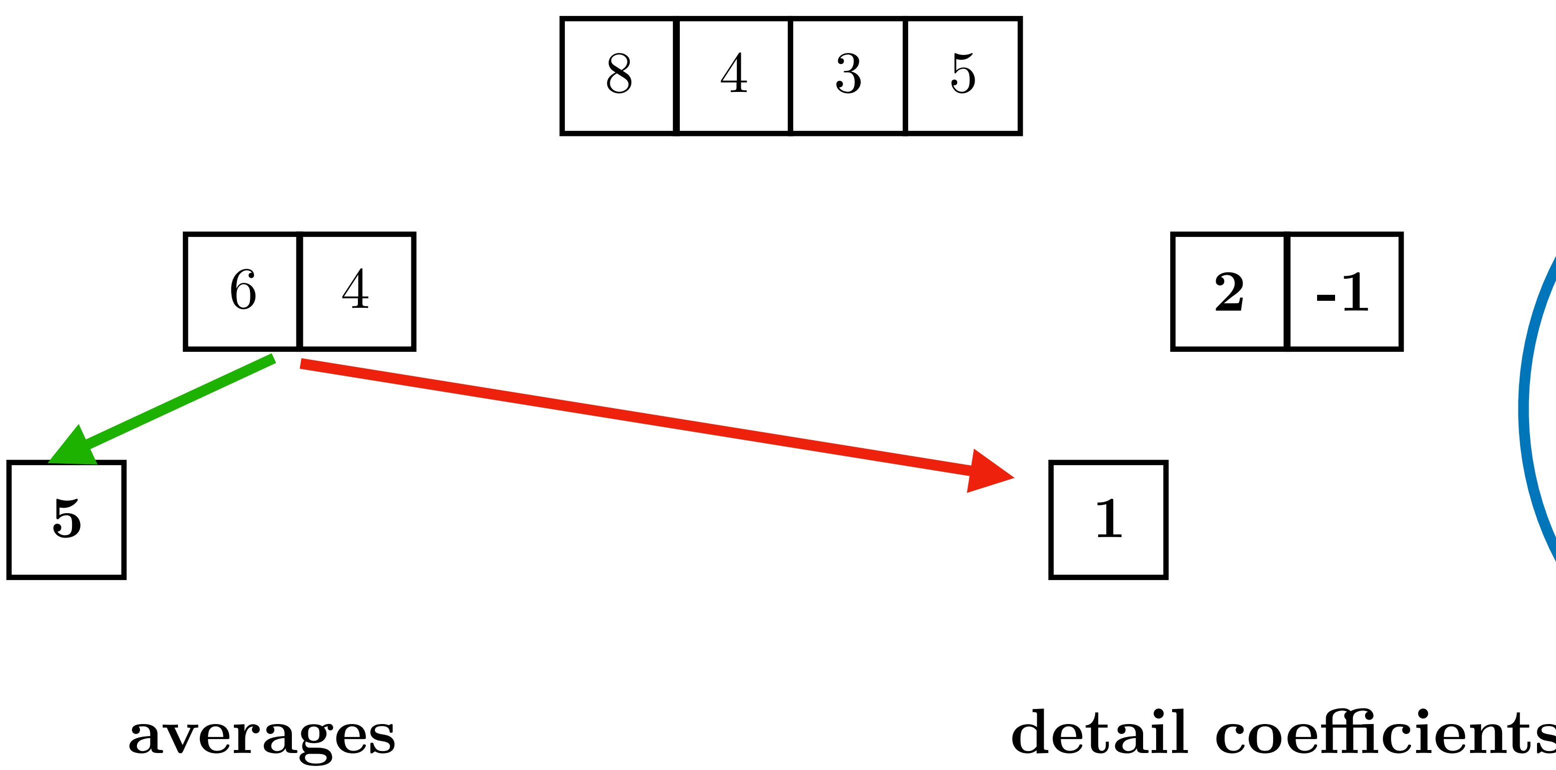

$$Y = \begin{bmatrix} 5 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

Haar Wavelet Reconstruction

Let $Y = [\ 6\ \ 1\ \ 2\ \ \text{-}2\ ]^{\mathrm{T}}$ be the Haar wavelet coefficients of some image X. Recover the pixel intensities of the original image.

Method 1: Reverse the algorithm from class.

1. The first coefficient is the average. Use it and the first detail coefficient to determine **two averages** from "the level above".
2. Recurse on the new averages and given detail coefficients no detail coefficients remain.
3. Return the resulting vector.

$\boxed{\mathbf{2}}\boxed{\mathbf{-2}}$

$\boxed{\mathbf{6}}$ $\boxed{\mathbf{1}}$

averages **detail coefficients**

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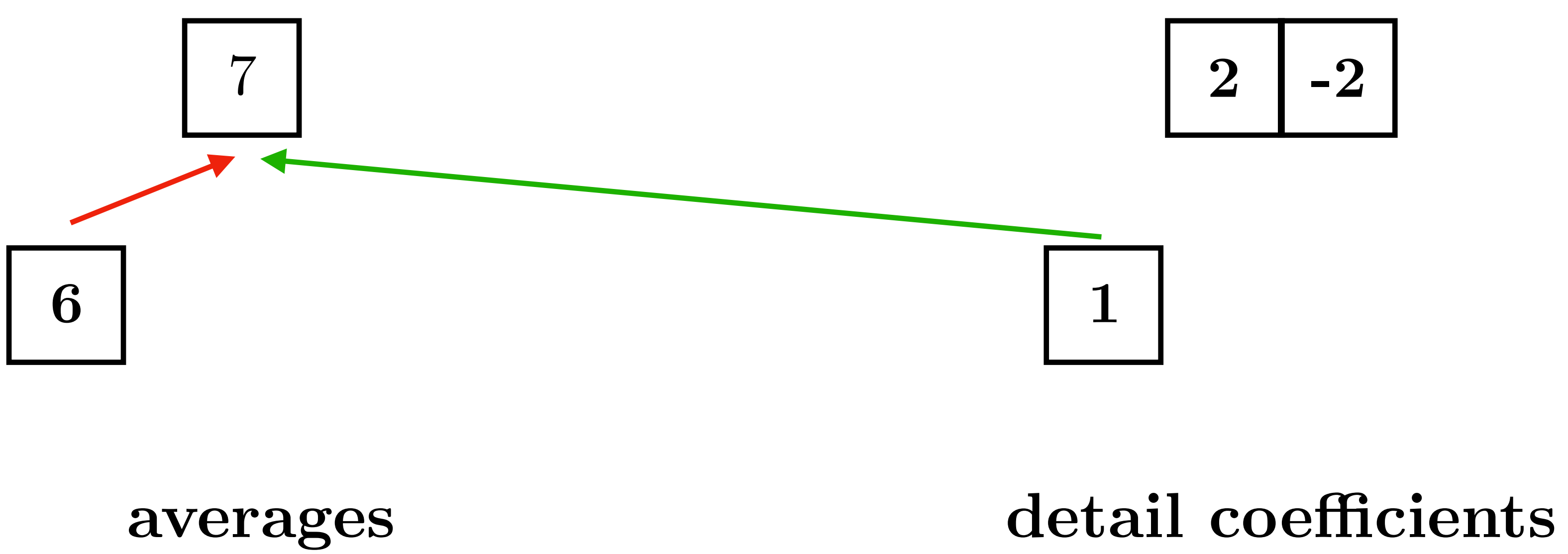

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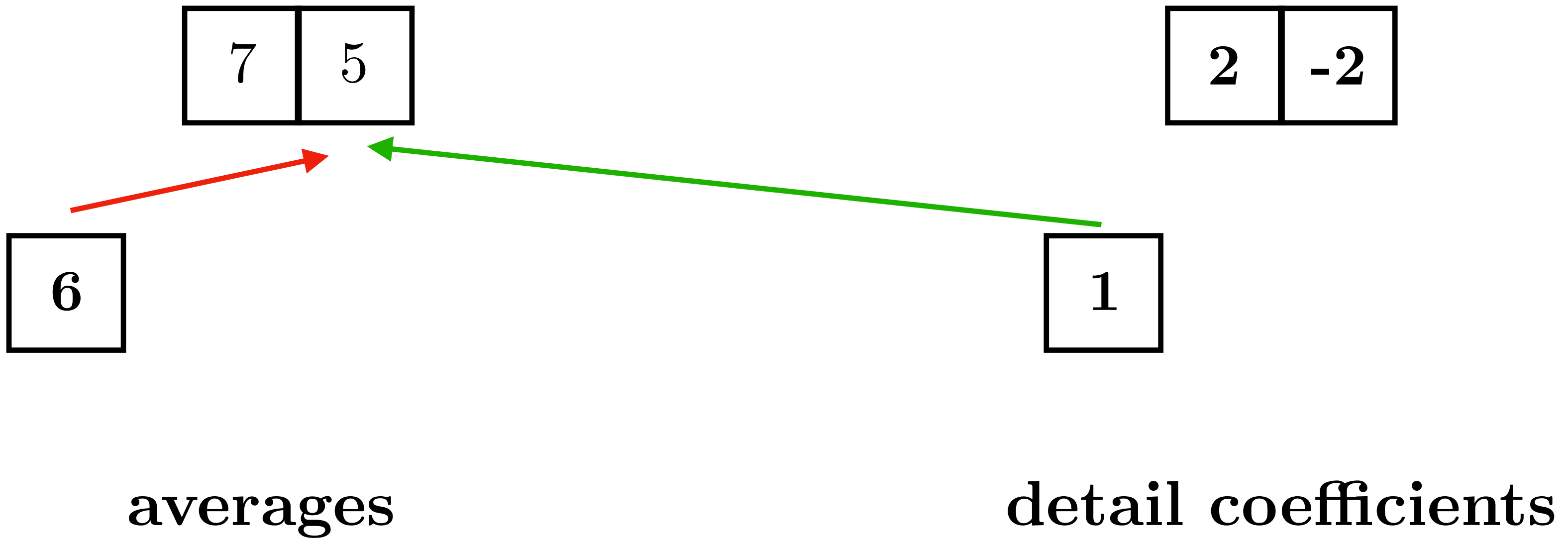

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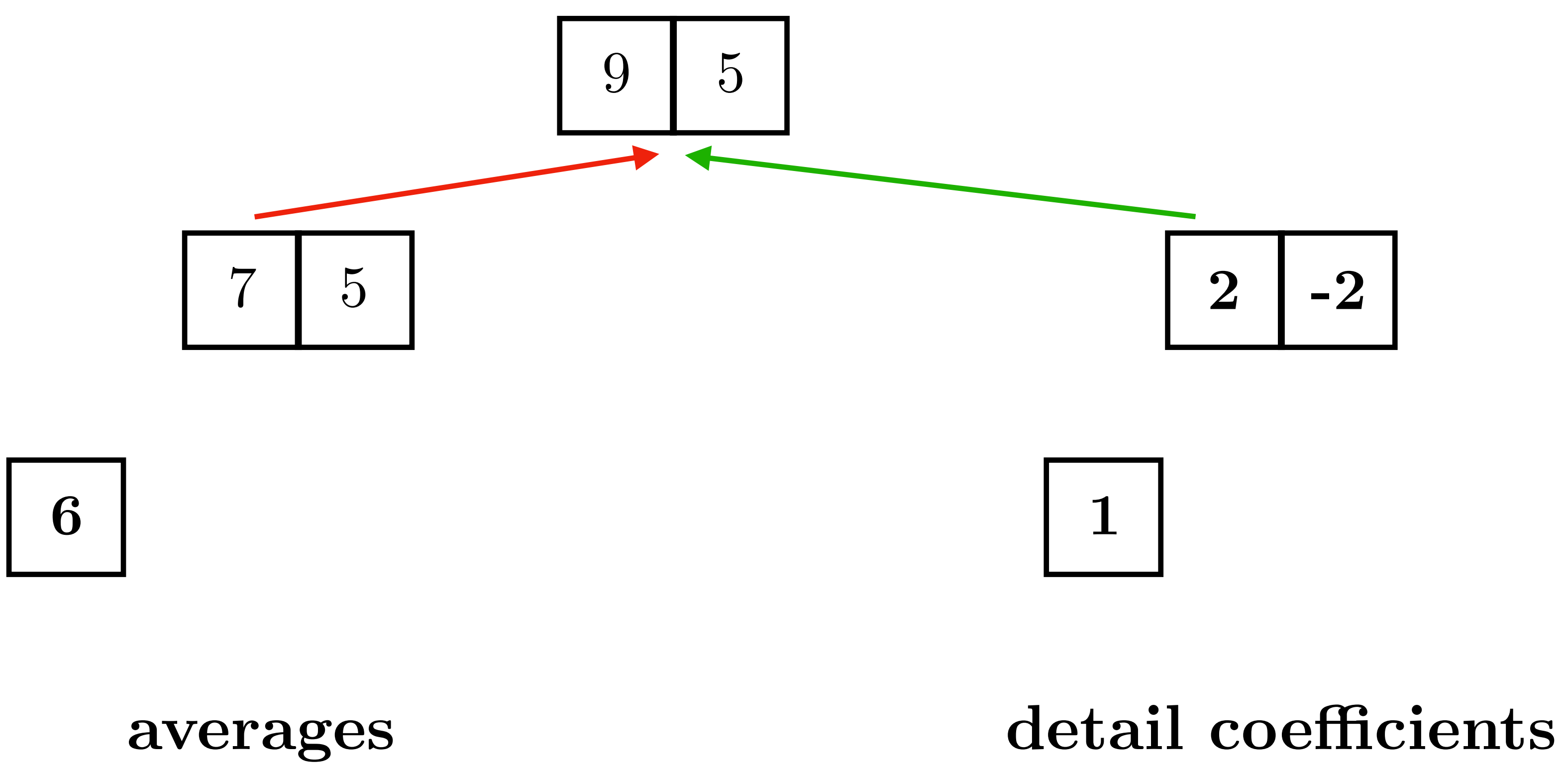

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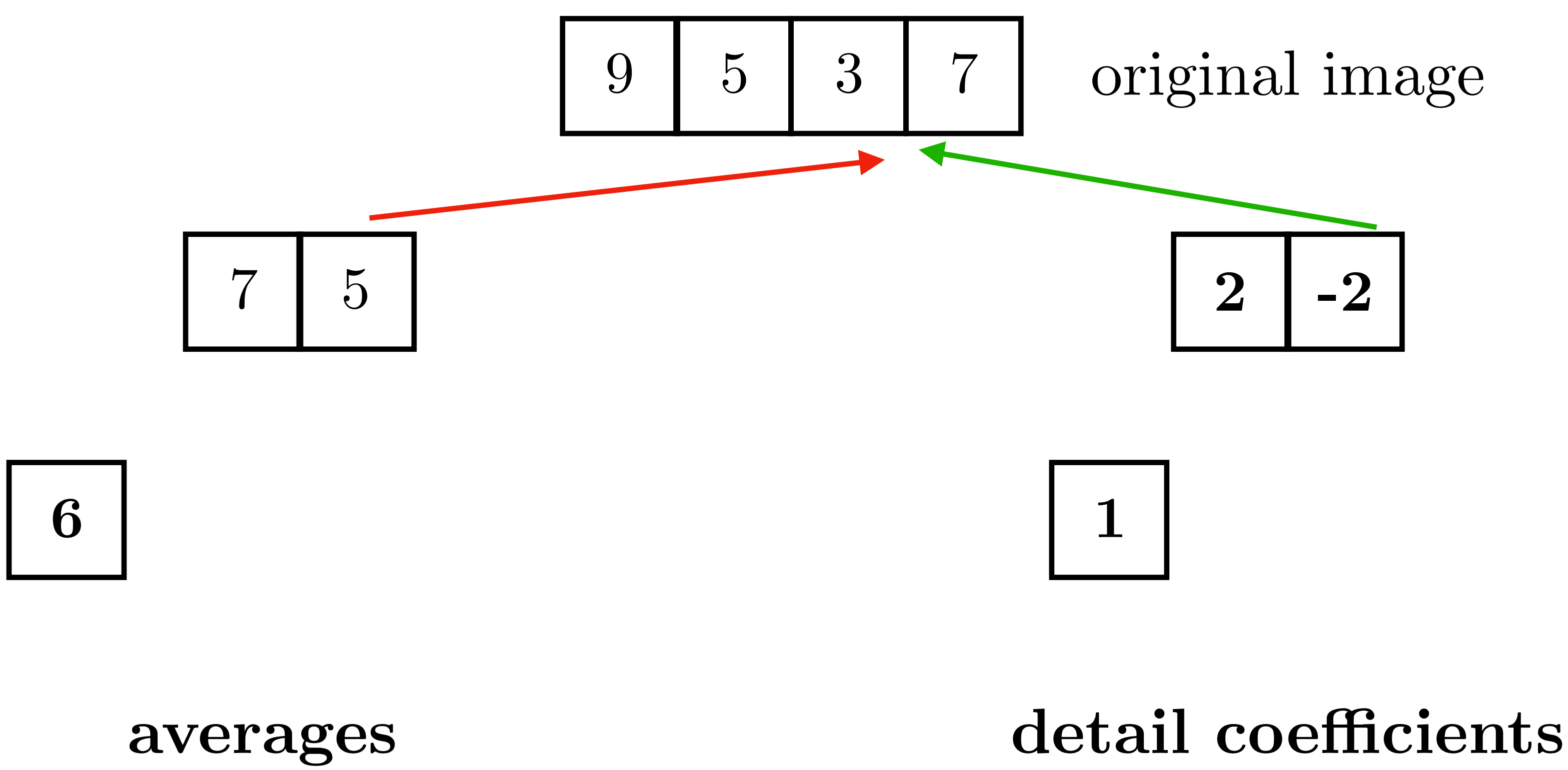

Haar Wavelet Transform

Let X be a 1-D image with pixel intensities $[\ 8\ \ 4\ \ 3\ \ 5\ ]^{\mathrm{T}}$. Compute the Haar wavelet coefficients of X.

Method 2: Use the Haar matrix product. Compute $Y = WX$ where W is:

$$W = \begin{bmatrix} \phi_0^0 \\ \varphi_0^0 \\ \varphi_0^1 \\ \varphi_1^1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Haar Wavelet Reconstruction

Let $Y = [\ 6\ \ 1\ \ 2\ \ \text{-}2\ ]^{\mathrm{T}}$ be the Haar wavelet coefficients of some image X. Recover the pixel intensities of the original image.

Exercise: Do this using matrix products!

2-D Haar Wavelet Transform

Let X be a 2-D image with pixel intensities as indicated. Compute the Haar wavelet coefficients of X.

1	3	8	4
5	7	6	2
4	2	7	3
8	4	3	7

original image

2-D Haar Wavelet Transform

Let X be a 2-D image with pixel intensities as indicated. Compute the Haar wavelet coefficients of X.

Method 1: Use the algorithm from class:

1. For every discrete 2-by-2 patch of the image, compute the average and detail coefficients.
2. Recurse on the averages until only one remains.

1	3	8	4
5	7	6	2
4	2	7	3
8	4	3	7

original image

2-D Haar Wavelet Transform

Let X be a 2-D image with pixel intensities as indicated. Compute the Haar wavelet coefficients of X.

p_1	p_2
p_3	p_4

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1	**3**	8	4
5	**7**	6	2
4	2	7	3
8	4	3	7

original image

4		**-1**	
-2		**0**	

intermediate image

$$A = \frac{1}{4}(p_1 + p_2 + p_3 + p_4)$$

$$d_2 = \frac{1}{4}(p_1 - p_2 + p_3 - p_4)$$

$$d_3 = \frac{1}{4}(p_1 + p_2 - p_3 - p_4)$$

$$d_4 = \frac{1}{4}(p_1 - p_2 - p_3 + p_4)$$

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1	3	**8**	**4**
5	7	**6**	**2**
4	2	7	3
8	4	3	7

original image

4	**5**	-1	**2**
-2	**1**	0	**0**

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5	7	6	2
4	2	7	3
8	4	3	7

original image

recurse on these intermediate averages

4	5	-1	2
4.5	5	1.5	0
-2	1	0	0
-1.5	0	-0.5	2

intermediate image

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1	3	8	4
5	7	6	2
4	2	7	3
8	4	3	7

original image

4	5	-1	2
4.5	5	1.5	0
-2	1	0	0
-1.5	0	-0.5	2

intermediate image

4.63	0.38	-1	2
-0.13	-0.13	1.5	0
-2	1	0	0
-1.5	0	-0.5	2

final image

2-D Haar Wavelet Re-construction

Let Y be a 2-D matrix of Haar wavelet coefficients of some image X. Recover the original pixel intensities.

A	d_2
d_3	d_4

Method 1: Reverse the algorithm from class:

1. Begin with the upper-right 2-by-2 patch. Use the detail coefficients to recover the averages from "one level up".
2. Recurse on the intermediate averages until no detail coefficients remain.

4.63	0.38	-1	2
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original coefficients

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original coefficients

4	**5**	-1	2
4.5	**5**	1.5	0
-2	1	0	0
-1.5	0	-0.5	2

intermediate coefficients

$$p_1 = A + d_2 + d_3 + d_4$$

$$p_2 = A - d_2 + d_3 - d_4$$

$$p_3 = A + d_2 - d_3 - d_4$$

$$p_4 = A - d_2 - d_3 + d_4$$

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2. Recurse.

4	5	**-1**	2
4.5	5	1.5	0
-2	1	**0**	0
-1.5	0	-0.5	2

intermediate coefficients

1	**3**		
5	**7**		

original image

$p_1 = A + d_2 + d_3 + d_4$

$p_2 = A - d_2 + d_3 - d_4$

$p_3 = A + d_2 - d_3 - d_4$

$p_4 = A - d_2 - d_3 + d_4$

2-D Haar Wavelet Re-construction

Let Y be a 2-D matrix of Haar wavelet coefficients of some image X.
Recover the original pixel intensities.

A	d_2
d_3	d_4

Method 1: Reverse the algorithm from class:

1. Begin with the upper-right 2-by-2 patch. Use the detail coefficients to recover the averages from "one level up".
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4	5	-1	2
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intermediate coefficients

1	3	8	4
5	7	6	2
4	2	7	3
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original image

$$p_1 = A + d_2 + d_3 + d_4$$

$$p_2 = A - d_2 + d_3 - d_4$$

$$p_3 = A + d_2 - d_3 - d_4$$

$$p_4 = A - d_2 - d_3 + d_4$$

2-D Haar Wavelet Transform

Let X be a 2-D image with pixel intensities as indicated. Compute the Haar wavelet coefficients of X.

The 2-D Haar wavelet transformation is a **separable convolution**! Hence, it can be represented as a series of 1-D transformations.

1	3	8	4
5	7	6	2
4	2	7	3
8	4	3	7

original image

2-D Haar Wavelet Transform

Let X be a 2-D image with pixel intensities as indicated. Compute the Haar wavelet coefficients of X.

Method 2: Compute a bunch of 1-D Haar wavelet transforms:

1. Compute the 1-D Haar wavelet transform for each column to form an intermediate image, I'.
2. Compute the 1-D Haar wavelet transform for each row of I'.

1	3	8	4
5	7	6	2
4	2	7	3
8	4	3	7

original image

2-D Haar Wavelet Transform

Let X be a 2-D image with pixel intensities as indicated. Compute the Haar wavelet coefficients of X.

Method 2: Compute a bunch of 1-D Haar wavelet transforms:

1. Compute the 1-D Haar wavelet transform for each column to form an intermediate image, I'.
2. Compute the 1-D Haar wavelet transform for the **first half of the rows** of I'.

1	3	8	4
5	7	6	2
4	2	7	3
8	4	3	7

original image

4.5	4	6	4
-1.5	1	1	-1
-2	-2	1	1
-2	-1	2	-2

I'

4.63	0.38	-1	2
-0.13	-0.13	1.5	0
-2	1	0	0
-1.5	0	-0.5	2

final image

2-D Haar Wavelet Reconstruction

Let Y be a 2-D matrix of Haar wavelet coefficients of some image X. Recover the original pixel intensities.

Method 2: Compute a bunch of 1-D Haar wavelet reconstructions:

1. Compute the 1-D Haar wavelet reconstruction for each row of I to form an intermediate image, I'.
2. Compute the 1-D Haar wavelet reconstruction for each column of I'.

4.63	0.38	-1	2
-0.13	-0.13	1.5	0
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original image

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original image

4.5	4	6	4
-1.5	1	1	-1
-2	-2	1	1
-2	-1	2	-2

I'

1	3	8	4
5	7	6	2
4	2	7	3
8	4	3	7

final image

2-D Haar Wavelet Transform

Let X be a 2-D image with pixel intensities as indicated. Compute the Haar wavelet coefficients of X.

(Bonus!) Method 3: Derive the 2-D Haar wavelet basis.

- Each component B_i of the basis will be a matrix with the same dimensions as the image X.
- Each Haar wavelet coefficient Y_i is the dot product $X \bullet B_i$. Ask your prof how the dot product between two matrices is defined...