Homework 9 Solutions

1. Let V, W be finite dimensional inner product spaces with $\dim(V) \leq \dim(W)$. Prove there is a linear map $T : V \to W$ such that $\langle T(v), T(v')\rangle_W = \langle v, v'\rangle_V$ for all $v, v' \in V$.

Proof. Let $\{v_1, \ldots, v_m\}, \{w_1, \ldots, w_n\}$ be orthonormal bases for V, W respectively. These exist by the Gram-Schmidt process. By the dimension condition, $m \leq n$. Define T as the unique linear map with $T(v_k) = w_k$ for each $1 \leq k \leq m$. Then, for $v, v' \in V$, we write $v = \sum_{k=1}^m a_k v_k,\ v' = \sum_{k=1}^m b_k v_k$. We have

$$\langle T(v), T(v')\rangle_W = \langle \sum_{k=1}^m a_k T(v_k), \sum_{k=1}^m b_k T(v_k)\rangle = \langle \sum_{k=1}^m a_k w_k, \sum_{k=1}^m b_k w_k\rangle$$

As $\langle v_j, v_k\rangle = \langle w_j, w_k\rangle$ for each j, k, we have

$$\langle \sum_{k=1}^m a_k w_k, \sum_{k=1}^m b_k w_k\rangle = \langle \sum_{k=1}^m a_k v_k, \sum_{k=1}^m b_k v_k\rangle = \langle v, v'\rangle_V$$

2. Let $T : V \to V$ be an orthogonal projection on a subspace of an inner product space V. Prove that $\|T(v)\| \leq \|v\|$ for all $v \in V$.

Proof. For $v \in V$, write $v = w + z$ where $w \in R(T)$, $z \in R(T)^\perp$, so $T(v) = w$. Then,

$$\|T(v)\|^2 = \|w\|^2 \leq \|w\|^2 + \|z\|^2 = \|w + z\|^2 = \|v\|$$

as $\langle w, z\rangle = 0$.

3. Let V be a finite dimensional inner product space and $T : V \to V$ be a projection such that $\|T(v)\| \leq \|v\|$ for each $v \in V$. Prove T is an orthogonal projection.

Proof. Suppose T be projection along Z; that is, $N(T) = Z$. Suppose T is not an orthogonal projection. Thus, if $W = R(T)$, we have $Z \neq W^\perp$. Hence, $Z^\perp \neq W$ (if $Z^\perp = W$, then $(Z^\perp)^\perp = Z = W^\perp$). As $\dim Z^\perp = \dim W$, we have $Z^\perp \backslash W \neq \emptyset$. Let $x \in Z^\perp \backslash W$. We show $\|T(x)\| > \|x\|$. Write $T(x) = z + z'$ where $z \in Z$ and $z' \in Z^\perp$. This is possible as $V = Z \oplus Z^\perp$. Write $x = w_x + z_x$ where $w_x \in W$, $z_x \in Z$. Then, $T(x) = w_x = z + z'$, so $w_x - z' = z \in Z$. Thus, $x - z' = (w_x - z') + z_x \in Z$ and is also in $Z^\perp$, so $x = z'$. Hence, $\|T(x)\|^2 = \|z + z'\|^2 = \|z\|^2 + \|z'\|^2 = \|z\|^2 + \|x\|^2$. As $x \notin W$, $T(x) \neq x$, so $z \neq 0$. Thus, $\|z\|^2 > 0$, so $\|T(x)\|^2 > \|x\|^2$.

4. Let T be a normal operator on a finite dimensional inner product space. Suppose T is also a projection. Prove T is an orthogonal projection.

Proof. As T is normal, there is an orthonormal basis $\beta = \{v_1, \ldots, v_n\}$ for V such that $[T]_\beta$ is diagonal. As T is a projection, the eigenvalues of T all belong to $\{0, 1\}$. Thus, if $\{v_1, \ldots, v_m\}$ have eigenvalue 1 and the rest have eigenvalue 0, we have, for $v = \sum_{k=1}^n c_k v_k \in V$

$$\langle T(v), T(v)\rangle = \langle \sum_{k=1}^n c_k T(v_k), \sum_{k=1}^n c_k (v_k)\rangle = \langle \sum_{k=1}^m c_k v_k, \sum_{k=1}^m c_k v_k\rangle = \sum_{k=1}^m |c_k|^2 \leq \sum_{k=1}^n |c_k|^2 = \langle \sum_{k=1}^n c_k v_k, \sum_{k=1}^n c_k v_k\rangle = \langle v, v\rangle$$

so by problem 3, T is an orthogonal projection.

5. Let T be a normal operator on a finite dimensional complex inner product space such that $T^n = 0$ for some $n > 0$. Prove $T = 0$.

Proof. As T is normal, it has a basis of eigenvectors, say $\{v_1, \ldots, v_m\}$ with eigenvalues $\lambda_1, \ldots, \lambda_m$. Then, $T^n(v_k) = \lambda_k^n v_k = 0$ for all n, so $\lambda_k = 0$ for all k. Hence, $T = 0$ on a basis, so $T = 0$.

6. Let T be a normal operator on a finite dimensional complex inner product space. Prove for any integer $n > 1$ there is a linear operator S such that $T = S^n$.

Proof. As T is normal, there is an orthonormal basis $\beta = \{v_1, \ldots, v_m\}$ of eigenvectors, say with eigenvalues $\lambda_1, \ldots, \lambda_m$. For each λ_k, let μ_k be such that $\mu_k^n = \lambda_k$. These exist by the fundamental theorem of algebra. Define S by $S(v_k) = \mu_k v_k$. Then, $S^n(v_k) = \mu_k^n v_k = T(v_k)$ for each k, so $S^n = T$ as these transformations are equal on a basis.

7. Let T be a unitary operator on a finite dimensional inner product space and $W \subseteq V$ a finite dimensional T-invariant subspace. Prove that $W^\perp$ is also T-invariant.

Proof. Let $x \in W^\perp$ and $w \in W$. Then, $\langle w, T(x)\rangle = \langle T^*(w), x\rangle = \langle T^{-1}(w), x\rangle$ as $T^* = T^{-1}$. As T is unitary, it is an isomorphism, so T^{-1} is also an isomorphism. Hence, $\dim T^{-1}(X) = \dim(X)$ for any subspace $X \subseteq V$. In particular, $\dim T^{-1}(W) = \dim(W)$, but $W \subseteq T^{-1}(W)$ as W is T-invariant, so $W = T^{-1}(W)$. Thus, $T^{-1}(w) \in W$, so $\langle T^{-1}(w), x\rangle = 0$ as $x \in W^\perp$. As this holds for all $w \in W$, $T(x) \in W^\perp$.

8. Find an orthogonal matrix whose first row is $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$.

Solution. Take any basis where $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ is the first element and perform Gram-Schmidt. Then, use the results as the rows of the orthogonal matrix. For example, we could take $\{(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}), (-4, 1, 1), (0, 1, -1)\}$ (to start off orthogonally) and normalize to $\{(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}), (\frac{-4}{\sqrt{18}}, \frac{1}{\sqrt{18}}, \frac{1}{\sqrt{18}}), (0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\}$ giving the matrix

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-4}{\sqrt{18}} & \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{18}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

9. Let T be a unitary operator on a complex finite dimensional inner product space V. Prove there exists a unitary operator S such that $T = S^2$.

Proof. Let S be as constructed in problem 6. We show S is unitary. Let $\beta = \{v_1, \ldots, v_m\}$ be as in problem 6, so $S(v_k) = \mu_k v_k$, $T(v_k) = \lambda_k v_k$, $\mu_k^2 = \lambda_k$. Then, as β is an orthonormal basis, $S^*(v_k) = \overline{\mu_k} v_k$. Hence, $SS^*(v_k) = |\mu_k|^2 v_k = |\lambda_k| v_k$. Now, T is unitary, so $T^*(v_k) = \overline{\lambda_k} v_k$ and, as $T^* = T^{-1}$, we have $T^*(v_k) = T^{-1}(v_k) = \frac{1}{\lambda_k} v_k$. Thus, $\lambda_k \overline{\lambda_k} v_k = v_k$, which is $|\lambda_k|^2 = 1$. Hence, $SS^*(v_k) = v_k$ for all k, so $SS^* = Id$.

10. Let T be a self-adjoint positive definite operator on a finite dimensional inner product space V. Prove there is an operator S such that $S^*S = T$.

Proof. Let $\beta = \{v_1, \ldots, v_n\}$ be an orthonormal basis for V such that $T(v_k) = \lambda_k v_k$ for each $v_k \in \beta$. This exists as T is self-adjoint. As T is positive definite, $\lambda_k > 0$ for each k. Let $S : V \to V$ by $S(v_k) = \sqrt{\lambda_k} v_k$ (the positive square root). Then, as β is an orthonormal basis, $S^*(v_k) = \overline{\sqrt{\lambda_k}} v_k = \sqrt{\lambda_k} v_k$ for each v_k. Thus, $S^*S(v_k) = \lambda_k v_k = T(v_k)$ for all k, so $S^*S = T$.