Problem Set 1

Definitions

Definition 0.1. Let S be a set and choose 2 sets $A, B \subseteq S$. The **union** of A and B is

$$A \cup B = \{x \in S| \quad x \in A \vee x \in B\}$$

$$\bigcup_{i \in \mathbb{I}} A_i = \{x \in S| \quad \exists i \in I, x \in A_i\}$$

Definition 0.2. Let S be a set and choose 2 sets $A, B \subseteq S$. The **intersection** of A and B is

$$A \cap B = \{x \in S| \quad x \in A \wedge x \in B\}$$

$$\bigcap_{i \in \mathbb{I}} A_i = \{x \in S| \quad \forall i \in I, x \in A_i\}$$

Definition 0.3. If $A \subseteq S$, then the **complement** of A with respect to S is all elements which are not in A, that is

$$A^c = \{x \in S : x \notin A\}$$

Definition 0.4. Given 2 sets A, B, a **function** $f : A \to B$ is a map which assigns to every point in A a unique point of B, that is

$$f : a \mapsto f(a), \text{ where } a \in A, f(a) \in B$$

Definition 0.5. Let $f : A \to B$ be a function.

1. If $U \subseteq A$, then we define the **image** of U to be

$$f(U) = \{y \in B : \exists x \in U, f(x) = y\} = \{f(x) : x \in U\}$$

2. If $V \subseteq B$ we define the **pre-image** of V to be

$$f^{-1}(V) = \{x \in A : f(x) \in V\}$$

Remark. Note U, V are sets, not variable.

Definition 0.6. Let $f : A \to B$ be a function. We say that

1. f is **injective** if whenever $f(x) = f(y)$ then $x = y$
2. f is **surjective** if for every $y \in B$ there exists $x \in A$ such that $f(x) = y$
3. f is **bijective** if f is both injective and surjective

Remark. Testing injectivity by using the horizontal line test in $\mathbb{R}^2$: An injective function is one whose graph that never intersect any horizontal line twice. Test surjectivity by ensuring that every horizontal line in the domain is crossed at least once by the graph.

Problem 1

Let $A \subseteq S$ and $B \subseteq S$. Prove each of the following statements

1. $A \subseteq B$ if and only if $A \cup B = B$

   Proof.
   (if) Given $A \cup B = B$. Proof by contradiction (always assume the opposite of the conclusion is true). Assume $A \not\subseteq B$, which means $\exists x \in A, x \notin B$. Also $x \in A \cup B$. Because $A \cup B = B$, then $x \in B$. This contradicts with $x \notin B$. Therefore $A \subseteq B$ if $A \cup B = B$
   (only if) $A \cup B$ means $\exists x \in S, x \in A \lor x \in B$. Because given $A \subseteq B$, which means if $x \in A$ then $x \in B$, $\exists x \in S, x \in A \lor x \in B$ is equivalent to $\exists x \in S, x \in B$, which is the definition of B $\square$

2. $A^c \subseteq B$ if and only if $A \cup B = S$

   Proof.
   (if) Given $A \cup B = S$, then if $x \in S$, then $x \in A \lor x \in B$. Use proof by contradiction. Assume $A^c \not\subseteq B$, let $x \in S$ such that $x \in A^c$, then $x \notin B$. Therefore $\exists x \in S, x \notin A \land x \notin B$. This contradicts with the first sentence of the proof. Therefore $A \cup B = S \Rightarrow A^c \subseteq B$
   (only if) Given $A^c \subseteq B$. Proof by contradiction. Assume $\exists x \in S, x \notin A \cup B$, then $x \notin A \lor x \notin B$. $x \notin A$ is equivalent to $x \in A^c$, which implies $x \in B$ by $A^c \subseteq B$. Contradiction arises as x cannot be in both B and not in B. $\square$

3. $A \subseteq B$ if and only if $B^c \subseteq A^c$

   Proof.
   (if) Given $B^c \subseteq A^c$. Proof by contradiction. Assume $A \not\subseteq B$. Then $\exists x \in A, x \notin B$. $x \notin B$ is equivalent to $x \in B^c$, then by $B^c \subseteq A^c$, $x \in A^c$. Contradiction happens when $x \in A$ and $x \in A^c$.
   (only if) Given $A \subseteq B$. Proof by contradiction. Assume $B^c \not\subseteq A^c$, then $\exists x \in S, x \in B^c \Rightarrow x \notin A^c$. Then $x \notin B \Rightarrow x \in A$, and by $a \subseteq B$, $x \notin B \Rightarrow x \in B$. Contradiction arises $\square$

4. $A \subseteq B^c$ if and only if $A \cap B = \emptyset$

   Proof.
   (if) Assume $A \cap B = \emptyset$. Proof by contradiction. Let $A \not\subseteq B^c$. So if arbitrary $a \in A, a \notin B^c$, which is the same as $a \in B$. Therefore $a \in A \Rightarrow a \in B$. Therefore $a \in A \cap B$. This contradict with the assumption that $A \cap B = \emptyset$
   (only if) Assume $A \subseteq B^c$, then for arbitrary $a \in A, a \notin B$. Therefore there can not be an a such that it is both in A and B. Therefore $A \cap B = \emptyset$ $\square$

Problem 2

Let A, B, and C be subsets of S. Show that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof.
Given $A \subseteq B$ and $B \subseteq C$. Let $x \in S, x \in A$, then $x \in B$ by $A \subseteq B$. Then $x \in C$ by $B \subseteq C$. Therefore, $x \in A \Rightarrow x \in C$, which means $A \subseteq C$. $\square$

Problem 3

Let $f : A \to B$ be a map of sets, and let $\{X_i\}_{i \in I}$ be an indexed collection of subsets of A.

1. Prove that $f\left(\bigcup_{i \in I} X_i\right) = \bigcup_{i \in I} f(X_i)$

   Proof.

$$
\begin{aligned}
LHS &= \{y \in B : \exists i \in \mathbb{I}, x \in X_i \text{ and } f(x) = y\} \\
&= \{y \in B : \exists i \in \mathbb{I}, y \in f(X_i)\} \\
&= RHS
\end{aligned}
$$

   Remark. Refer to definition (0.5) and definition (0.1) for proof. Think both forward and backward to find a common claim.

   The formal way of proving this should be proving that $f\left(\bigcup_{i \in I} X_i\right) \subseteq \bigcup_{i \in I} f(X_i)$ and $f\left(\bigcup_{i \in I} X_i\right) \supseteq \bigcup_{i \in I} f(X_i)$.
   Let $x \in f\left(\bigcup_{i \in I} X_i\right)$, then $x \in \{a \in A : \exists i, a \in X_i\}$

   $\square$

2. Prove that $f\left(\bigcap_{i \in I} X_i\right) \subset \bigcap_{i \in I} f(X_i)$

   Remark. Intuitively speaking, non-injective function will follow this claim.

   Proof.

$$
\begin{aligned}
LHS &= \{y \in B : \forall i \in \mathbb{I}, x \in X_i \text{ and } f(x) = y\} \\
&= \{y \in B : \forall i \in \mathbb{I}, y \in f(X_i)\}
\end{aligned}
$$

   dunno how to prove this... $\square$

3. When does equality of sets hold in the above part?

   Prove $f\left(\bigcap_{i\in\mathbb{I}} X_i\right) = \bigcap_{i\in I} f(X_i)$ if and only if f is injective

   Proof.
   ($\Leftarrow$)
   Assume f is injective. Let $y \in \bigcap_{i\in\mathbb{I}} f(X_i)$. This means $y \in f(X_i)$ for all $i \in \mathbb{I}$, which means $\forall i \in \mathbb{I}, \exists x_i \in X : f(x_i) = y$. Since f is injective, $f(x_i) = y$ for all $i \in \mathbb{I}$ (the preimage is just one x) means $x_1 = \cdots = x_i = \cdots = x$. So $x \in \bigcap_{i\in\mathbb{I}} X_i$. So $y = f(x) \in f(\bigcap_{i\in\mathbb{I}} X_i)$
   ($\Rightarrow$)
   Assume $f\left(\bigcap_{i\in\mathbb{I}} X_i\right) = \bigcap_{i\in I} f(X_i)$. Think about what it means for f to not be injective — when $x \neq y, f(x) = f(y)$

   $$\emptyset = f(\emptyset) = f(\{x\} \cap \{y\}) = f(\{x\}) \cap f(\{y\}) = f(\{x\})$$

   Thus there is no such x, y. This is a contradiction. □

   Two ways of proving. Proof by contradiction really works here!

Problem 4

Consider the map $f : \mathbb{R} \to \mathbb{R}, x \mapsto \sin(x)$

1. Find the image of the set $[0, \pi]$; that is, determine $I = f([0, \pi])$.

   $$I = f([0, \pi]) = \{f(x) : x \in [0, \pi]\} = [0, 1]$$

2. Let I be as in the previous question. Determine the preimage $f^{-1}(I)$.

   $$f^{-1}(I) = \{x \in R : f(x) \in [0, \pi]\} = \bigcup_{i\in\mathbb{I}} [2\pi i, 2\pi(i + 1)]$$

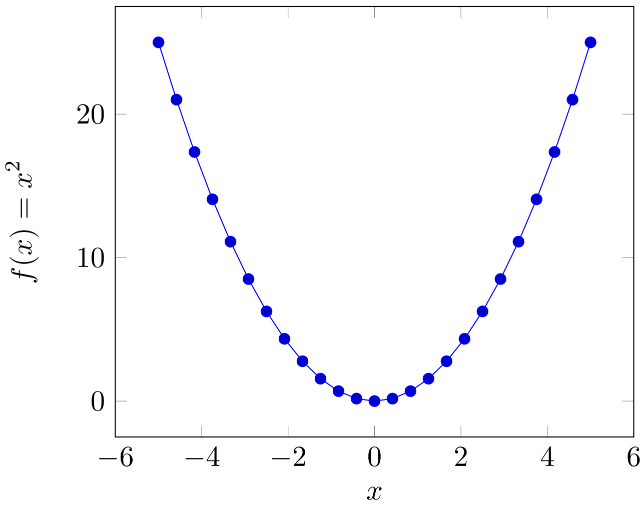
   Remark. Use [definition (0.5) to solve the problem

Problem 5

Consider the function $F : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto y - x^2$

1. Graph the set $F^{-1}(0)$ in $\mathbb{R}^2$

$$\begin{aligned} F^{-1}(0) &= \{(x,y) \in \mathbb{R}^2 : F((x,y)) = 0\} \\ &= \{(x,y) \in \mathbb{R}^2 : y - x^2 = 0\} \end{aligned}$$



2. More generally, if $c \in \mathbb{R}$ is any real number, what does the set $F^{-1}(c)$ look like?

$$\begin{aligned} F^{-1}(c) &= \{(x,y) \in \mathbb{R}^2 : F((x,y)) = c\} \\ &= \{(x,y) \in \mathbb{R}^2 : y - x^2 = c\} \end{aligned}$$

The plot would be $y = x^2$ that translate vertically c units.

Problem 6

Let I be an index for a collection of subsets $A_i \subseteq S, i \in I$. Show that for every $k \in I$, $\bigcap\limits_{i \in I} A_i \subseteq A_k$

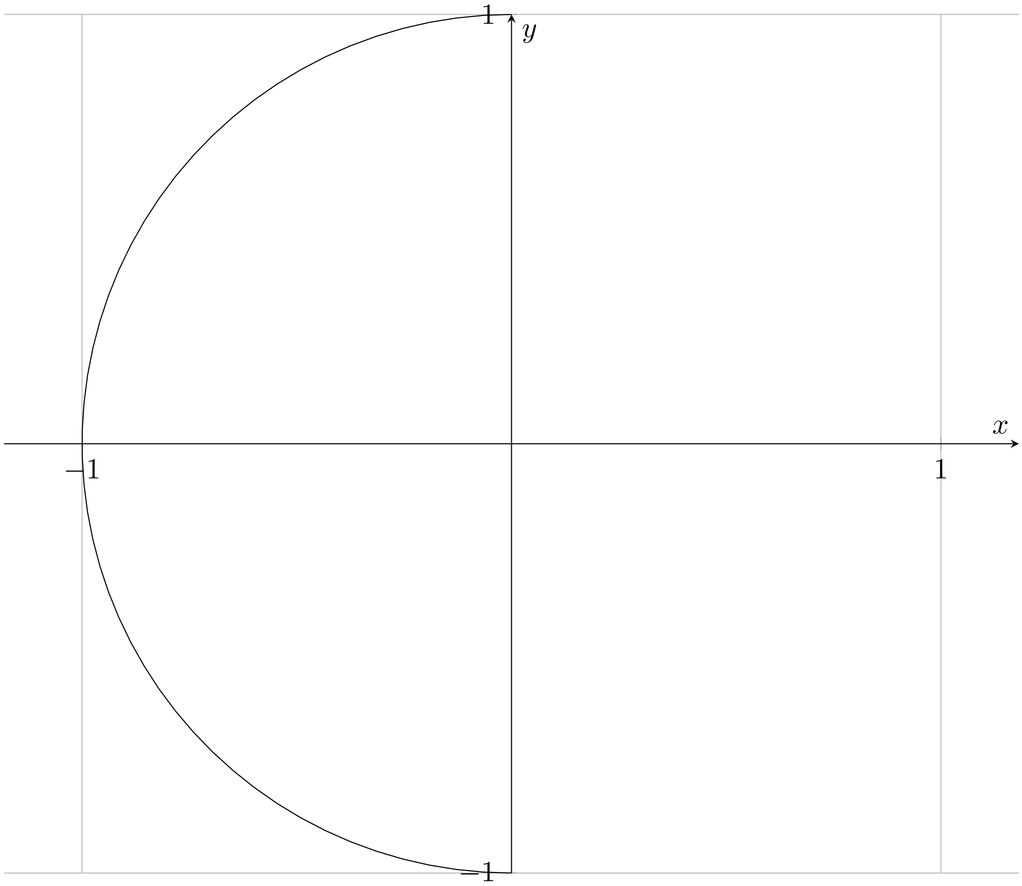
Solution.
Assume $\bigcap\limits_{i \in I} A_i \subseteq A_k$. Let $a \in \bigcap\limits_{i \in I} A_i$, then $a \in \{x \in S : \forall i \in I, x \in A_i\}$. Therefore for every $k \in I, a \in A_k$ by previous definition. This proves $\bigcap\limits_{i \in I} A_i \subseteq A_k$. $\square$

Problem 7

Give an example of a function $f(x) = (f_1(x), f_2(x))$, such that neither of the component functions f_1, f_2 are injective, but f is injective. Be sure to specify the domain of f.

Solution.
Let $f_1 : \mathbb{R} \to \mathbb{R}^+, x \mapsto x^2$, and $f_2 : \mathbb{R} \to \mathbb{R}^+, x \mapsto (x^2 - 1)^2$. Both f_1, f_2 are not injective. To prove f is injective, we show that $a = b$ when $f(a) = f(b)$, where $a, b \in \mathbb{R}$. Given $f(a^2, (a-1)^2) = f(b^2, (b-1)^2)$, this is only true when $a^2 = b^2 \wedge (a-1)^2 = (b-1)^2$. $(a-1)^2 = (b-1)^2 \Rightarrow a^2 - 2a + 1 = b^2 - 2b + 1$, and since $a^2 = b^2$, $-2a + 1 = -2b + 1 \Rightarrow a = b$. Therefore f is injective.

Remark. Also think of f as a curve in a cartesian plane with coordinates the output of f_1 and f_2. Such as the unit circle $f = (cos(x), sin(x))$. It is obvious that $cos(x)$ and $sin(x)$ are not injective. To make f injective, we just restrict the domain to $[{}^1/_2\pi, {}^3/_2\pi]$ so that the curve passes the horizontal line test.



□

Problem 8

Let $f : A \to B$ be a function.

1. For every $X \subseteq A$, $X \subseteq f^{-1}(f(X))$

   Remark. Recall definition (0.5). This question is implying that the preimage of an image contains the domain. It may exceed the size of domain.

   Proof.
   Assume $X \subseteq A$, so $f(X) \in B$.

   $$\begin{aligned} f^{-1}(f(X)) &= \{a \in A : f(a) \in f(X)\} \\ &= \{a \in A : \exists x \in X, f(a) = f(x)\} \end{aligned}$$

   Note. Combination of definition for image and preimage

   Let $x \in X$, try to get to $x \in f^{-1}(f(X)) = \{a \in A : \exists x \in X, f(x) = f(a)\}$. x satisfies $\exists x \in X, f(x) = f(x)$. Also $x \in X \subseteq A$, hence $x \in f^{-1}(f(X))$ by definition

   □

2. For every $Y \subseteq B$, $Y \supseteq f(f^{-1}(Y))$

   Proof. Assume $Y \subseteq B$, so $f^{-1}(Y) \in B$

   $$\begin{aligned} f(f^{-1}(Y)) &= \{b \in B : \exists x \in f^{-1}(Y), f(x) = b\} \\ &= \{b \in B : \exists a \in A, f(a) \in Y \wedge f(a) = b\} \\ &= \{y \in Y : \exists a \in A, y = f(a)\} \end{aligned}$$

   Let $y \in f(f^{-1}(Y))$, then by definition $y \in Y$. Therefore $Y \supseteq f(f^{-1}(Y))$ □

3. If $f : A \to B$ is injective, then for every $X \subseteq A$ we have $X = f^{-1}(f(X))$

   Proof. Refer to question 1, we already proved that for every $X \subseteq A$, $X \subseteq f^{-1}(f(X))$. Just have to prove $X \supseteq f^{-1}(f(X))$ to show equality, on the condition that $f : A \to B$ is injective. Recall that $f^{-1}(f(X)) = \{a \in A : \exists x \in X, f(a) = f(x)\} = \{a \in A : \exists x \in X, a = x\}$ using injectivity of f. Let $x \in f^{-1}(f(X))$, then $x \in X$ by definition of $f^{-1}(f(X))$. □

4. If $f : A \to B$ is surjective, then for every $Y \subseteq B$ we have $Y = f(f^{-1}(Y))$

   Proof. Refer to question 2, we already proved that for every $Y \subseteq B$, $Y \supseteq f(f^{-1}(Y))$. Just have to prove $Y \subseteq f(f^{-1}(Y))$ to show equality, on the condition that $f : A \to B$ is surjective. Recall that $f(f^{-1}(Y)) = \{y \in Y : \exists a \in A, y = f(a)\}$. Let $y \in Y$, then $\exists a \in A, y = f(a)$ by definition of surjectivity. This is equivalent to the set notation of $f(f^{-1}(Y))$, hence the arbitrary $y \in f(f^{-1}(Y))$. Therefore $Y \subseteq f(f^{-1}(Y))$. Also $Y \supseteq f(f^{-1}(Y))$, $Y = f(f^{-1}(Y))$ for every $X \subseteq A$. □