Problem Set 2

You are strongly encouraged to solve the following exercises before next week's tutorial: Exercises 13, 23, 30 and 52 ((a), (b) and (c)) on Section 8.10, page 312 onwards (end of chapter 8).

Additional problem: We flip a coin until it comes up heads for the first time, and record the number of times it was flipped. Suppose that after repeating the experiment 10 times, we have collected the following numbers of tosses: $\underline{x} = \{2, 9, 6, 3, 1, 8, 1, 2, 1, 3\}$.

not asking for direct calculation here

(a) Find the Method of Moments estimator of p, and show that is is consistent.

(b) Find the maximum likelihood estimate of the probability to come up heads.

(i) by direct calculation, and

(ii) numerically, applying the Newton-Raphson algorithm. Start from $\widehat{p}_0 = 0.9$ and print out the value of $\widehat{p}$ at the end of each iteration.

(c) Find the Normal approximation to the sampling distribution of the MLE.

(d) Suppose that the probability of 'heads' is $p = 0.25$. Approximate the minimum sample size required to guarantee at a 95% probability that the MLE deviates from p by no more than 0.02.

(e) Denote by θ the probability of a coin coming up heads for the first time on the third attempt. Find the MLE of θ and its asymptotic sampling distribution.

invariance property of MLE

Solution:

(a) Clearly, here $X_1, \ldots, X_n \overset{\text{i.i.d}}{\sim} \text{Geom}(p)$. Solving

$$\mu_1 = \mathbb{E}[X] = \frac{1}{p} = m_1 = \overline{X}$$

one learns that the Method of Moments estimator is $\widehat{p}_{\text{MME}} = 1/\bar{X}$. Now, from the Law of Large Numbers

$$\overline{X} \xrightarrow{P} \mathbb{E}[X] = \frac{1}{p},$$

thus (from the Theorem on continuous mapping of consistent estimators)

$$\widehat{p}_{\text{MME}} = 1/\bar{X} \xrightarrow{P} \frac{1}{1/p} = p,$$

by property of convergence in probabilitry

and $\widehat{p}_{\text{MME}}$ is consistent.

(b) (i) The likelihood is given by

$$\mathcal{L}(p) = \prod_{i=1}^{n} (1-p)^{x_i - 1} p = (1-p)^{\sum_{i=1}^{n} x_i - n} p^n,$$

and the log-likelihood is

$$\ell(p) = \left(\sum_{i=1}^{n} x_i - n \right) \log(1-p) + n \log p = n(\bar{x} - 1) \log(1-p) + n \log p.$$

To find the MLE, we solve

so get the formula for mle first then plugin numbers

$$\ell'(p) = -\frac{n(\bar{x}-1)}{1-p} + \frac{n}{p} = 0 \Longrightarrow p = \frac{1}{\bar{x}}$$

hence $\widehat{p}_{\text{MLE}} = 1/\bar{X}$ (same as the method of moments estimator), and the estimate is $\dfrac{10}{2+9+\cdots+3} = 0.278.$

Now, to verify maximum

$$\ell''(p)\big|_{p=\widehat{p}_{\text{MLE}}} = -\frac{n(\bar{x}-1)}{(1-p)^2} - \frac{n}{p^2}\bigg|_{p=\widehat{p}_{\text{MLE}}} = \frac{n\bar{x}^2}{1-\bar{x}} - n\bar{x}^2$$

$$= \frac{n\bar{x}^3}{1-\bar{x}} = -179.45.$$

(ii) Differentiating

$$\ell'(p) = \frac{n(1-p\bar{x})}{p(1-p)} \text{ and } \ell''(p) = -\frac{n(\bar{x}p^2 - 2p + 1)}{p^2(1-p)^2},$$

we can write down the Newton-Raphson iterations (after rearrangement of terms)

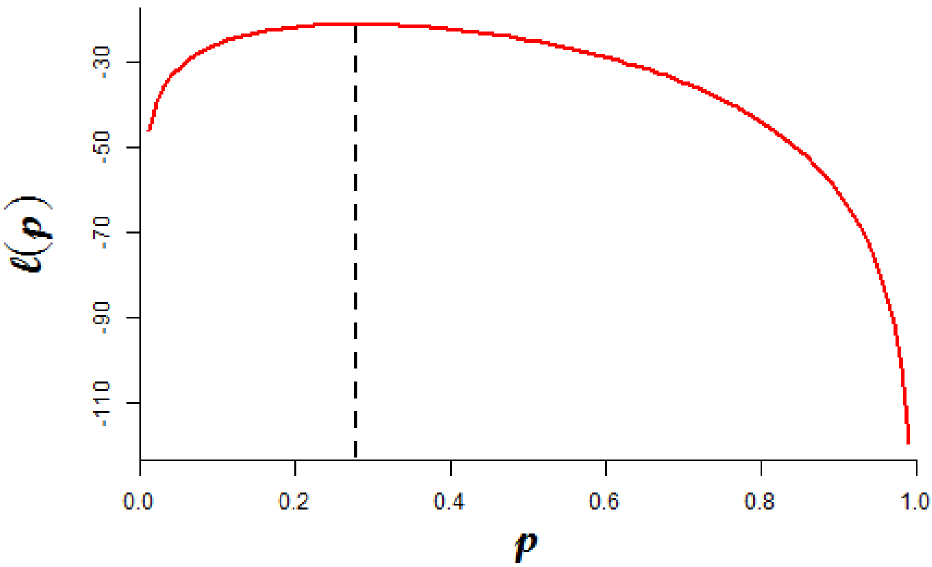
$$\widehat{p}_{\text{new}} = \widehat{p}_{\text{old}} - \frac{\ell'(\widehat{p}_{\text{old}})}{\ell''(\widehat{p}_{\text{old}})} = \widehat{p}_{\text{old}} + \frac{\widehat{p}_{\text{old}}(1-\widehat{p}_{\text{old}})(1-\widehat{p}_{\text{old}}\bar{x})}{\bar{x}\widehat{p}_{\text{old}}^2 - 2\widehat{p}_{\text{old}} + 1}$$

The following code –

```
> x <- c(2, 9, 6, 3, 1, 8, 1, 2, 1, 3)
> xBar <- mean(x)
> pHat <- 0.9
> epsilon <- 5e-16
> delta <- 1
>
> while(delta > epsilon){
+ cat(paste(pHat, "\n"))
+ pHatNew <- pHat + pHat*(1-pHat)*(1-pHat*xBar)/(xBar*pHat^2-2*pHat+1)
+ delta <- abs(pHat-pHatNew)
+ pHat <- pHatNew
+ }
```

produces the output

```
0.9
0.804725897920605
0.631597731804025
0.378908747887343
0.266030200624667
0.277456101219601
0.27777754820735
0.277777777777661
0.277777777777778
>
```



(c) Since we have already calculated

$$\ell''(p) = -\frac{n(\bar{x}-1)}{(1-p)^2} - \frac{n}{p^2} = -\frac{\sum x_i - n}{(1-p)^2} - \frac{n}{p^2},$$

The Fisher Information is

E[X] = 1/p by geometric distribution

$$\mathcal{I}(p) = -\mathbb{E}\left\{\ell''(p)\right\} = \frac{\sum \mathbb{E}[X_i] - n}{(1-p)^2} + \frac{n}{p^2} = \frac{n/p - n}{(1-p)^2} + \frac{n}{p^2} = \frac{n}{p^2(1-p)},$$

therefore

$$1\Big/\overline{X} \sim AN\left(p, \frac{p^2(1-p)}{n}\right).$$

(d) We need to find n such that

$$\mathbb{P}\left(|\widehat{p}_{\text{MLE}} - p| \leqslant 0.02\right) = \mathbb{P}\left(|\widehat{p}_{\text{MLE}} - 0.25| \leqslant 0.02\right) \geqslant 0.95.$$

Now, using the Normal approximation,

$$\frac{\widehat{p}_{\text{MLE}} - p}{\sqrt{\frac{p^2(1-p)}{n}}} = \frac{\widehat{p}_{\text{MLE}} - 0.25}{\sqrt{\frac{0.25^2 \times 0.75}{n}}} \sim \mathcal{N}(0,1).$$

Based on this,

$$\mathbb{P}\left(|\widehat{p}_{\text{MLE}} - 0.25| \leqslant 0.02\right) = \mathbb{P}\left(\left|\frac{\widehat{p}_{\text{MLE}} - 0.25}{\sqrt{\frac{0.25^2 \times 0.75}{n}}}\right| \leqslant \frac{0.02}{\sqrt{\frac{0.25^2 \times 0.75}{n}}}\right) \approx 2\Phi\left(\frac{0.02}{\sqrt{\frac{0.25^2 \times 0.75}{n}}}\right) - 1$$

$$= 2\Phi\left(0.092\sqrt{n}\right) - 1,$$

and

$$2\Phi\left(0.092\sqrt{n}\right) - 1 \geqslant 0.95 \iff \Phi\left(0.092\sqrt{n}\right) \geqslant 0.975$$

$$\iff 0.092\sqrt{n} \geqslant z_{0.975} = 1.96 \iff n \geqslant 453.7,$$

therefore we will need a sample of size $n \geqslant 454$.

(e) Clearly $\theta(p) = (1-p)^2 p$, and from the Theorem on transformations of MLEs,

$$\widehat{\theta}_{\text{MLE}} = \left(1 - \widehat{p}_{\text{MLE}}\right)^2 \widehat{p}_{\text{MLE}} = \frac{\left(\overline{X} - 1\right)^2}{\overline{X}^3}.$$

Moreover, $\theta'(p) = 2p(1-p) + (1-p)^2 = 1 - p^2$, hence the asymptotic variance of $\widehat{\theta}_{\text{MLE}}$ is given by

calulation wrong here ..

$$\left[\theta'(p)\right]^2 \mathcal{I}^{-1}(p) = \frac{p^2(1-p)^2(1+p)}{n}.$$

Finally,

$$\widehat{\theta}_{\text{MLE}} = \frac{\left(\overline{X} - 1\right)^2}{\overline{X}^3} \sim AN\left((1-p)^2 p\,,\ \frac{p^2(1-p)^2(1+p)}{n}\right).$$