STA247 - Probability with Computer Applications
Midterm 1 Fall 2016
Oct. 14, 2016
50 Minutes

Name (Print): Answer Key Student Number: ____________

This test contains 5 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and write your name on the top of every page, in case the pages become separated.

This is a closed book test. You are allowed a **non-programmable** calculator on this test

- **Show all your work**. Answers, correct or not, unsupported by calculations, algebraic work, or explanation will not earn any marks.
- **Clearly state and define any variables and/or distributions**. These are part of the problem solving process and are worth marks.
- **Organize your work** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page that cannot be understood will not earn full marks.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- Do your best! If you get stuck, there's no need to panic. Take a few slow, deep breaths, look at another question before returning to your problem. You got this!

Problem	Points	Score
1	8	
2	14	
3	10	
4	3	
Total:	35	

Do not write in the table to the right.

1. (8 points) A basket has 4 blue balls, 5 yellow balls, and 3 red balls. You take two turns drawing one ball at a time. On your first draw, if the ball is either yellow or blue you put it back in the basket. If the ball is red, you remove it from the basket.

   (a) (2 points) How many ways, in general, are there to arrange all 12 balls?

$$\frac{12!}{4!5!3!} = 27{,}720$$

∴ 27,720 diff unique arrangements of all 12 balls

   (b) (2 points) What is the probability that your second draw will be a red ball, if your first draw was also red?

If first ball was red, it was removed leaving 11 balls, 2 red

$P(\text{2nd red} \mid \text{1st red}) = 2/11$

$P(\text{2nd red} \cap \text{1st red}) = \frac{3\times 2}{12\times 11}$

$P(\text{1st red}) = 3/12 = 1/4$ 0.5

$P(\text{2nd red} \mid \text{1st red}) = \frac{3\times2}{12\times11} \div \frac{1}{4}$

$= 2/11$ 0.5

   (c) (4 points) What is the probability that your first draw was a red ball, if you drew a blue ball on your second turn?

$P(\text{2nd Blue}) = P(\text{1st red} \cap \text{2nd blue}) + P(\text{1st not red} \cap \text{2nd blue})$

$= 4/11 + 4/12 = 23/33$

$$P(\text{1st red} \mid \text{2nd blue}) = \frac{P(\text{1st red} \cap \text{2nd blue})}{P(\text{2nd blue})}$$

$$= \frac{4/11}{23/33} = \frac{12}{23}$$

2. (14 points) A basketball team called *The Statisticians* tends to have a 46.4% chance of winning a game. Assuming that each game's outcome is independent of others,

   (a) (4 points) Find the probability that *The Statisticians* will play 3 games *before* their first win.

   Let G = # of losses before first win ✓, $G \sim$ geometric(0.464) ✓ → If they only named it, it's still acceptable

   $P(G=3) = (1-0.464)^3(0.464) = 0.0715$ ✓

   ∴ 7.15% chance they will play 3 games before their first win. ✓

   (b) (4 points) Find the probability that they will win 3 games *by their fifth game.*

   BOTH ACCEPTED

   Let X = # of games won out of 5 ✓

   $X \sim Bin(5, 0.464)$ ✓

   $P(X=3) = \binom{5}{3}(0.464)^3(1-0.464)^2$

   $= 0.2870$ ✓

   ∴ 28.70% chance will win 3 by 5th game. ✓

   Let Y = # of losses before third win ✓

   $Y \sim NB(r=3, p=0.464)$ ✓ → If they named it (Negative binomial...) it's acceptable

   $P(Y=2) = \binom{4}{2}(0.464)^3(1-0.464)^2$ ✓

   $= 0.1722$

   ∴ 17.22% chance they win the 3rd game on the fifth game. ✓

   (c) (6 points) In a series of 5 games, the first team to win 3 games will in the series. What is the probability that *The Statisticians* will win the series? *Hint: You may want to brainstorm some cases before you start computing.*

   $Y \sim NB(r=3, p=0.464)$ ✓, Y = # of losses before 3rd win ✓

   → win by 3rd : $P(Y=0) = (0.464)^3 = 0.0999$ ✓

   → win by 4th game : $P(Y=1) = \binom{3}{1}(0.464)^3(1-0.464) = 0.1606$ ✓

   → win by 5th game : $P(Y=2) = \binom{4}{2}(0.464)^3(1-0.464)^2 = 0.1722$ ✓

   0.4327

   ∴ 43.27% chance they will win the series. ✓

3. (10 points) Consider the following probability distribution function (PDF) for the amount in winnings in a card game:

X	-2	1	3	6
$f(x) = P(X = x)$	$0.11k$	$0.01k^2 - 0.01k$	$0.03k$	0.1
	$= 0.55$	$= 0.2$	$= 0.15$	

(a) (4 points) Compute k such that $f(x)$ is a valid PDF.

$0.11k + 0.01k^2 - 0.01k + 0.03k + 0.1 = 1$ ✓

$0.01k^2 + 0.13k - 0.9 = 0$) ×100

$k^2 + 13k - 90 = 0$

$(k+18)(k-5) = 0$ ✓

~~k=-18~~ or $k=5$ ✓ since $k=-18$ would make some probabilities <0. ✓

one mark for solving another for identifying the acceptable k.

(b) (2 points) What is the expected winnings of this game?

$E(X) = \sum_{x \in X} x \cdot f(x) = -2(0.55) + 1(0.2) + 3(0.15) + 6(0.1)$ ✓

$= 0.15$ ✓

∴ expected winnings of \$0.15

(c) (3 points) What is the variance in the expected winnings of this game?

$E(X^2) = \sum_{x \in X} x^2 \cdot f(x) = (-2)^2(0.55) + (1)^2(0.2) + (3)^2(0.15) + (6)^2(0.1)$ ✓

$= 7.35$

$V(X) = E(X^2) - E(X)^2$ ✓ $= 7.35 - (0.15)^2 = 7.3275$ ✓

(d) (1 point) Find $P(X > 0 | X \leq 3)$.

$P(X > 0 \cap X \leq 3) = P(X=1) + P(X=3)$

$= 0.2 + 0.15 = 0.35$

$P(X \leq 3) = P(X=-2) + P(X=1) + P(X=3) = 0.9$ ✓

$P(X > 0 | X \leq 3) = \frac{0.35}{0.9} = 0.3\overline{8} = 7/18$

4. (3 points) 6 numbers are selected randomly without replacement from the set {1, 4, 7, 8, 9, 13, 24, 25, 28, 11, 6}. What is the probability that the largest third largest number is 11?

3rd largest ⇒ 2 are larger

⇒ 3 are smaller

(Brainstorm — some evidence of thinking/reasoning)

Set of larger than 11 : {13, 24, 25, 28} 0.5

Set of smaller than 11 : {1, 4, 7, 8, 9, 6}

Set of exactly 11 : {11}

determining appropriate choices

Total choices of 6 #s : $_{11}C_6 = 462$ 0.5

Choices where 11 is 3rd largest : $_4C_2 \times {_1C_1} \times {_6C_3} = 120$ ✓

$\therefore P(\text{11 is third largest}) = \frac{120}{462} = \frac{20}{77} = 0.2597$ ✓

∴ 25.97% chance that 11 is 3rd largest

Aid Sheet

Binomial	$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$	$E(X) = np$	$V(X) = np(1-p)$
Geometric	$P(X=x) = (1-p)^x p$	$E(X) = \frac{1-p}{p}$	$V(X) = \frac{1-p}{p^2}$
Negative Binomial	$P(X=x) = \binom{x+r-1}{r-1} p^r (1-p)^x$	$E(X) = \frac{r(1-p)}{p}$	$V(X) = \frac{r(1-p)}{p^2}$
Poisson	$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$	$E(X) = \lambda$	$V(X) = \lambda$

Variance: $V(X) = E[(x-\mu)^2]$

Chebyshev's Inequality: $P(|x-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$