1 Structures in $\mathbb{R}^n$

Definition 1.1. *A* ***Vector Space*** *is a collection of objects called vectors, which may be added together and multiplied ("scaled") by numbers, called scalars in this context. The set V and the operations of addition and multiplication must adhere to a number of requirements called axioms. let u, v and w be arbitrary vectors in V, and a and b scalars in F. First of all $u + v \in V$ and $au \in V$ and*

$$
\begin{align}
u + (v + w) &= (u + v) + w \tag{1}\\
u + v &= v + u \tag{2}\\
\exists 0 \in V, v + 0 &= v \forall v \in V \tag{3}\\
\forall v, \exists - v, v + (-v) &= 0 \tag{4}\\
a(bv) &= (ab)v \tag{5}\\
1v &= v \tag{6}\\
a(u + v) &= au + av \tag{7}\\
(a + b)v &= av + bv \tag{8}
\end{align}
$$

Definition 1.2. *The* ***Euclidean inner product****, or dot product given two vectors $x = (x_1, \dots, x_n)$, and $y = (y_1, \dots, y_n)$ in $\mathbb{R}^n$, which are two equal-length sequences of numbers, and returns a single number.*
Algebraically, it is the sum of the products of the corresponding entries of the two sequences of numbers.

$$\langle x, y\rangle = x \cdot y := \sum_{i=1}^{n} x_i y_i = x_1 y_2 + x_2 y_2 + \cdots + x_n y_n$$

Geometrically, it is the product of the Euclidean magnitudes of the two vectors and the cosine of the angle between them.

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \; \|\mathbf{b}\| \cos(\theta)$$

where $\|x\|$ is the length of x, $\|y\|$ is the length of y, θ is the angle between x, y.

Note.
When $a \cdot b = 0$, a and b are **orthogonal**.
When $a \cdot b = \|a\|\|b\|$, a and b are **co-directional**.

Here is a list of properties of the ***inner product space****. Given $a, b, c \in V$ and $r \in \mathbb{R}$*

$$a \cdot b = b \cdot a \tag{Commutative}$$
$$a \cdot (b + c) = a \cdot b + a \cdot c \tag{Distributive over vector addition}$$
$$a \cdot (rb + c) = r(a \cdot b) + (a \cdot c) \tag{Bilinear}$$
$$(c_1 a) \cdot (c_2 b) = c_1 c_2 (a \cdot b) \tag{Scalar multiplication}$$
$$\textit{two non-zero vectors are orthogonal} \iff a \cdot b = 0 \tag{Orthogonality}$$
$$a \cdot b = a \cdot c \textit{ does not imply } b = c \tag{No cancellation}$$
$$a \cdot b \geq 0 \textit{ and is euqal to zero if and only if } x = 0 \tag{Non-negative}$$

Remark. Mostly proved by using algebraic definition of inner dot product

Definition 1.3. *The scalar projection is*

$$comp_b a = \frac{a \cdot b}{\|b\|}$$

Proof. $b \cdot (a - b) = 0$ because they are orthogonal to each other. Arrange and with Biliear property for inner dot product we arrive at $comp_b a = \dfrac{a \cdot b}{\|b\|}$ □

Then the projection of a into b is the scalar projection multiply by the unit vector $\dfrac{b}{\|b\|}$

$$proj_b a = \frac{\langle a, b \rangle}{\|b\|^2} b$$

Definition 1.4. ***Norm*** *is a way of measuring the length of a vector. Here we define $\| \cdot \| : \mathbb{R}^n \to \mathbb{R}$ as the function*

$$\|x\| := \sqrt{\langle x, x \rangle} = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} = \sqrt{(x_1^2 + x_2^2 + \cdots + x_n^2)}$$

Proof. Use the algebraic definition of inner product $\mathbf{x} \cdot x = \|x\| \, \|x\| \cos(0) = \|x\|^2$ □

The ***normed space*** *has the following properties. Let $x, y \in \mathbb{R}^n$ and $c \in R$,*

$$\|x\| \geq 0 \textit{ with equality if and only if } x = 0 \tag{Non-degeneracy}$$
$$\|cx\| = |c| \|x\| \tag{Normality}$$
$$\|x + y\| \leq \|x\| + \|y\| \tag{Triangle Inequality}$$
$$|\langle x, y \rangle| \leq \|x\| \|y\| \tag{Couchy Schwarz Inequality}$$

Remark. proofs for Couchy Schwarz Inequality can be derived from geometric definition of inner dot product on the condition that $cos(x) \leq 1$. Proofs for Triangle Inequality requires Couchy Schwarz Inequality.

Definition 1.5. ***Metric*** *is a method for determining the distance between the two vectors.*

$$d(x,y) = \|x-y\| = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2} = \sqrt{(x_1-x_2)^2 + \cdots + (x_n - y_n)^2}$$

The metric space satisfies the following properties. Let $x, y, z \in \mathbb{R}^n$

$$d(x,y) = d(y,x) \tag{Symmetry}$$

$$d(x,y) \geq 0 \text{ with equality if and only if } x = y \tag{Non-degeneracy}$$

$$d(x,z) \leq d(x,y) + d(y,z) \tag{Triangle Inequality}$$

Definition 1.6. *In* $\mathbb{R}^3$ *the cross product of two vectors is a way of determining a third vector which is orthogonal ot the original two. If* $v = (v_1, v_2, v_3)$ *and* $w = (w_1, w_2, w_3)$ *then*

$$v \times w = (v_2w_3 - w_2v_3, w_1v_3 - v_1w_3, v_1w_2 - w_1v_2))$$

or we can use determinants to solve $v \times w$*. Here* i, j, k *represent standard unit vectors in* R^3

$$v \times w = det\begin{pmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = i\begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix} - j\begin{pmatrix} v_1 & v_3 \\ w_1 & w_3 \end{pmatrix} + k\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}$$

Note. Determinants of $2X2$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad - bc$