Problem Set 3

You are strongly encouraged to solve the following exercises before next week's tutorial:

On page 317 onwards (end of section 8), exercises 19 (a)-(c), 53 (a)-(d), 54, 55 (a)-(b), 57 (c), 60 (a)-(e) (you might consider trying the additional exercise first).

Additional problem:

(a) Let $X_1, \ldots, X_n \overset{\text{i.i.d}}{\sim} \text{Exp}(\lambda)$. Show that $\sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \lambda)$. (Hint: use the moment generating function)

(b) Show that if $U \sim \text{Gamma}(n, \lambda)$ then $2\lambda U \sim \chi^2_{2n}$.

know pdf of U find pdf of a function of U by transformation method; see if it matches that of chi squared

(c) Use part (b) to find a $(1-\alpha)100\%$ confidence interval for λ.

(d) Suppose that we collected 30 observations from the Exp(λ) distribution, and the sample mean turned out to be 0.9. Compare the resultant 95% confidence interval of part (c) with the approximate (large sample) 95% confidence interval for λ derived in class.

Solution:

(a) The moment generating function of $X \sim \text{Exp}(\lambda)$ is $M_X(t) = \dfrac{\lambda}{\lambda - t}$, and since the X_i's are independent

$$M_{\sum_{i=1}^{n}}(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} \frac{\lambda}{\lambda - t} = \left(\frac{\lambda}{\lambda - t}\right)^n.$$

This is the moment generating function of a Gamma(n, λ), and the uniqueness of the mgf seals the deal.

(b) The pdf of $U \sim \text{Gamma}(n, \lambda)$ is $f_U(u) = \dfrac{\lambda^n}{(n-1)!} \mathrm{e}^{-\lambda u} u^{n-1}$, $u \geqslant 0$. If we now define

$V = 2\lambda U$, then its cdf is given by

$$F_V(v) = \mathbb{P}(V \leqslant v) = \mathbb{P}\left(U \leqslant \frac{v}{2\lambda}\right) = F_U\left(\frac{v}{2\lambda}\right),$$

and thus the pdf is

by transformation method

$$f_V(v) = \frac{\mathrm{d}F_V(v)}{\mathrm{d}v} = f_U\left(\frac{v}{2\lambda}\right) \cdot \frac{1}{2\lambda} = \frac{\lambda^n}{(n-1)!}\mathrm{e}^{-\lambda\left(\frac{v}{2\lambda}\right)}\left(\frac{v}{2\lambda}\right)^{n-1} \cdot \frac{1}{2\lambda}$$

$$= \frac{2^{-n}}{(n-1)!}e^{-v/2}v^{n-1}.$$

Incidentally, this is the pdf of a Gamma$(n, 1/2) = \chi^2_{2n}$ random variable.

(c)

$$1 - \alpha = \mathbb{P}\left(\chi^2_{2n,\alpha/2} \leqslant 2\lambda \sum_{i=1}^{n} X_i \leqslant \chi^2_{2n,1-\alpha/2}\right) = \mathbb{P}\left(\frac{\chi^2_{2n,\alpha/2}}{2\sum_{i=1}^{n} X_i} \leqslant \lambda \leqslant \frac{\chi^2_{2n,1-\alpha/2}}{2\sum_{i=1}^{n} X_i}\right),$$

hence $\left[\frac{\chi^2_{2n,\alpha/2}}{2\sum_{i=1}^{n} X_i}, \frac{\chi^2_{2n,1-\alpha/2}}{2\sum_{i=1}^{n} X_i}\right]$ is a $(1-\alpha)100\%$ confidence interval for λ.

(d) Substituting $\alpha = 0.05$, $n = 30$, $\overline{X} = 0.9$, $\sum_{i=1}^{30} X_i = 27$, $\chi^2_{60,0.025} = 40.48$ and $\chi^2_{60,0.975} = 83.3$ in the expression of part (c) yields an exact 95% confidence interval for λ –

$$\left[\frac{40.48}{2 \times 27}, \frac{83.3}{2 \times 27}\right] = [0.75, 1.54].$$

In class we found an approximate $(1-\alpha)100\%$ confidence interval for λ of the form –

$$\left[1\big/\overline{X} - \frac{z_{1-\alpha/2}}{\overline{X}\sqrt{n}}, 1\big/\overline{X} + \frac{z_{1-\alpha/2}}{\overline{X}\sqrt{n}}\right].$$

Recalling that $z_{0.975} = 1.96$, we learn that

$$\left[\frac{1}{0.9} - \frac{1.96}{0.9 \times \sqrt{30}}, \frac{1}{0.9} + \frac{1.96}{0.9 \times \sqrt{30}}\right] = [0.71, 1.51].$$

Overall the two CIs in this case are very similar and of an almost identical length.