Proof of the Independence of $\bar{X}$ and S^2 Under Normality and Related Results

Ofir Harari*

November 18, 2016

Let $X_1, \ldots, X_n \overset{\text{i.i.d}}{\sim} \mathcal{N}(\mu, \sigma^2)$ and recall the definition of the sample mean

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

as well as the sample variance

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2 .$$

Our primary goal here is to prove the (somewhat surprising) independence of $\bar{X}$ and S^2. To do that, we invoke the following two results from Probability theory.

Theorem 1. *Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be independent random variables (of any finite dimension) and let $f(\cdot)$ and $g(\cdot)$ be (measurable) real valued functions. Then $f(\boldsymbol{U})$ and $g(\boldsymbol{V})$ are also independent.*

Theorem 2. *Let $\boldsymbol{U} = (U_1, \ldots, U_n)$ be an absolutely continuous random vector with density*

*Department of Statistical Sciences, University of Toronto

$f_{\boldsymbol{U}}(u_1,\ldots,u_n)$ *and let* $\boldsymbol{V}=(V_1,\ldots,V_n)$ *be a transformation of* $\boldsymbol{U}$*, such that the mapping*

$$\begin{cases} V_1 = V_1(U_1,\ldots,U_n) \\ \vdots \\ V_n = V_n(U_1,\ldots,U_n) \end{cases}$$

is invertible. Then the density of $\boldsymbol{V}$ *is given by*

$$f_{\boldsymbol{V}}(v_1,\ldots,v_n) = f_{\boldsymbol{U}}\big(u_1(v_1,\ldots,v_n),\ldots,u_n(v_1,\ldots,v_n)\big)\left|\frac{\partial(u_1,\ldots,u_n)}{\partial(v_1,\ldots,v_n)}\right|,$$

where

$$\left|\frac{\partial(u_1,\ldots,u_n)}{\partial(v_1,\ldots,v_n)}\right| = \begin{vmatrix} \partial u_1/\partial v_1 & \partial u_2/\partial v_1 & \cdots & \partial u_n/\partial v_1 \\ \partial u_1/\partial v_2 & \partial u_2/\partial v_2 & \cdots & \partial u_n/\partial v_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial u_1/\partial v_n & \partial u_2/\partial v_n & \cdots & \partial u_n/\partial v_n \end{vmatrix} \tag{1}$$

is the determinant of the Jacobian matrix of the inverted transformation.

We will prove the result in several steps. First, let us prove the following proposition.

Proposition 1. *The sample variance* S^2 *and the random vector* $(X_1-\bar{X}, X_2-\bar{X},\ldots,X_n-\bar{X})$ *are independent.*

Proof. Without loss of generality, assume that $\mu=0$. First, recall that due to the independence of $X_1,\ldots,X_n$,

$$f_{\boldsymbol{X}}(x_1,\ldots,x_n) = \prod_{i=1}^n f_{X_i}(x_i) = \prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{x_i^2}{2\sigma^2}\right\}$$

$$= \frac{1}{\sigma^n(2\pi)^{n/2}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2\right\}.$$

Consider the following transformation of $(X_1, \ldots, X_n)$ –

$$\begin{cases} Y_1 = \bar{X}, \\ Y_2 = X_2 - \bar{X}, \\ \vdots \\ Y_n = X_n - \bar{X}. \end{cases}$$

whose inverse is given by

$$\begin{cases} X_2 = Y_2 + Y_1, \\ \vdots \\ X_n = Y_n + Y_1, \\ X_1 = n\bar{X} - \sum_{i=2}^n X_i = nY_1 - \sum_{i=2}^n (Y_i + Y_1) = Y_1 - Y_2 - \cdots - Y_n. \end{cases}$$

It follows that

$$\left|\frac{\partial(x_1, \ldots, x_n)}{\partial(y_1, \ldots, y_n)}\right| = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & 1 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix} = n,$$

Now, since $\left|\frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)}\right| = \left|\frac{\partial(x_1, \ldots, x_n)}{\partial(y_1, \ldots, y_n)}\right|^{-1} = \frac{1}{n}$, from Theorem 2 we learn that

$$f_{\boldsymbol{Y}}(y_1, \ldots, y_n) = f_{\boldsymbol{X}}\left(y_1 - \sum_{i=2}^n y_i, y_1 + y_2, \ldots, y_1 + y_n\right) \cdot \frac{1}{n}$$

$$= \frac{1}{n\sigma^n(2\pi)^{n/2}} \exp\left\{-\frac{1}{2\sigma^2}\left[\left(y_1 - \sum_{i=2}^n y_i\right)^2 + \sum_{i=2}^n (y_1 + y_i)^2\right]\right\}$$

(after some rearrangement of terms)

$$= \frac{1}{n\sigma^n(2\pi)^{n/2}} \exp\left\{-\frac{1}{2\sigma^2}\left[ny_1^2 + \sum_{i=2} y_i^2 + \left(\sum_{i=2} y_i\right)^2\right]\right\}$$

$$= \text{const} \times \exp\left\{-\frac{ny_1^2}{2\sigma^2}\right\} \times \exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{i=2} y_i^2 + \left(\sum_{i=2} y_i\right)^2\right]\right\}.$$

The factorization $f_{\mathbf{Y}}(y_1, \ldots, y_n) = f_1(y_1) f_2(y_2, \ldots, y_n)$ establishes the independence of $Y_1 = \bar{X}$ and $(Y_2, \ldots, Y_n) = \left(X_2 - \bar{X}, \ldots, X_n - \bar{X}\right)$ are independent. To complete the proof, note that

$$X_1 - \bar{X} = \sum_{i=1}^n X_i - \sum_{i=2}^n X_i - \bar{X} = (n-1)\bar{X} - \sum_{i=2}^n X_i = -\sum_{i=2}^n \left(X_i - \bar{X}\right),$$

hence $X_1 - \bar{X}$ is a function of $\left(X_2 - \bar{X}, \ldots, X_n - \bar{X}\right)$, and from Theorem 1 $X_1 - \bar{X}$ and $\bar{X}$ are independent, too. $\square$

All is set now to proof the main result.

Theorem 3. *Let* $X_1, \ldots, X_n \overset{\text{i.i.d}}{\sim} \mathcal{N}(\mu, \sigma^2)$. *Then the sample mean* $\bar{X}$ *and the sample variance* S^2 *are independent.*

Proof. Clearly S^2 is a function of $(X_1 - \bar{X}, X_2 - \bar{X}, \ldots, X_n - \bar{X})$, and thus, combining Theorem 1 with Proposition 1, we have the result. $\square$