

Chapter 6 Inner Product Spaces

6.1 Inner Products and Norms

Definition. Inner Product Let V be a vector space over F . An inner product on V is a function that assigns, to every ordered pair of vectors x and y in V , a scalar in F , denoted $\langle x, y \rangle$, such that for all $x, y, z \in V$ and all $c \in F$,

1. $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$
2. $\langle cx, y \rangle = c\langle x, y \rangle$
3. $\overline{\langle x, y \rangle} = \langle y, x \rangle$
4. $\langle x, x \rangle > 0$ if $x \neq 0$

First two condition requires inner product be linear in the first component. Also

$$\langle \sum_i a_i v_i, y \rangle = \sum_i a_i \langle v_i, y \rangle$$

Definition. Conjugate Transpose or Adjoint of a Matrix Let $A \in M_{m \times n}(F)$, the conjugate transpose or adjoint of A is an $n \times m$ matrix A^* such that $(A^*)_{ij} = \overline{A_{ji}}$ for all i, j . For $F = \mathbb{R}$, $A^* = A^T$

Definition. Inner Product Definition Example

1. **Standard Inner Product on F^n** For $x = (a_1, a_2, \dots, a_n)$ and $y = (b_1, b_2, \dots, b_n)$ in F^n , the standard inner product on F^n is given by

$$\langle x, y \rangle = \sum_{i=1}^n a_i \bar{b}_i$$

2. **Inner Product for Real-valued Continuous Functions on $[0, 1]$.** Let $V = C([0, 1])$, $f, g \in V$, define

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt$$

3. **Frobenius Inner Product for Matrices** Let $V = M_{n \times n}(F)$, $A, B \in V$, then

$$\langle A, B \rangle = \text{tr}(B^* A) = \sum_{i=1}^n (B^* A)_{ii}$$

Definition. Inner Product Space A vector space over F endowed with a specific inner product is called an inner product space. If $F = \mathbb{C}$, V is a complex inner product space; if $F = \mathbb{R}$, then V is a real inner product space

Theorem. 6.1 Properties From Inner Product Conditions Let V be an inner product space. Then for $x, y, z \in V$ and $c \in F$, the following statements are true

1. $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
2. $\langle x, cy \rangle = \bar{c}\langle x, y \rangle$
3. $\langle x, 0 \rangle = \langle 0, x \rangle = 0$
4. $\langle x, x \rangle = 0$ if and only if $x = 0$
5. If $\langle x, y \rangle = \langle x, z \rangle$ for all $x \in V$, then $y = z$

The inner product is conjugate linear in the second argument

Definition. Norm/Length Let V be an inner product space. For $x \in V$, define norm or length of x by

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Definition. 6.2 Properties of Norm Let V be an inner product space over F . Then for all $x, y \in V$ and $c \in F$, the following statements are true

1. $\|cx\| = |c| \cdot \|x\|$
2. $\|x\| = 0$ if and only if $x = 0$. In any case, $\|x\| \geq 0$
3. **Cauchy-Schwarz Inequality** $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$
4. **Triangular Inequality** $\|x + y\| \leq \|x\| + \|y\|$

Definition. Angle For $F = \mathbb{R}$, $x, y \neq 0$, and θ be angle between x and y

$$\cos \theta = \frac{\langle x, y \rangle}{\|x\| \|y\|} \quad \theta = \cos^{-1} \left(\frac{\langle x, y \rangle}{\|x\| \|y\|} \right)$$

Note

$$\left| \frac{\langle x, y \rangle}{\|x\| \|y\|} \right| \leq 1$$

So valid input to arccos function

Definition. Orthogonal Vectors Let V be an inner product space. Vectors x and y in V are orthogonal (perpendicular) if $\langle x, y \rangle = 0$.

Definition. Orthogonal Sets and Orthonormal Sets A subset S of V is orthogonal if any two distinct vectors in S are orthogonal. A vector x in V is a unit vector if $\|x\| = 1$. A subset S of V is orthonormal if S is orthogonal and consists entirely of unit vectors.

1. $S = \{v_1, v_2, \dots\}$, then S is orthonormal if and only if $\langle v_i, v_j \rangle = \delta_{ij}$

2. We can **normalize** an orthogonal set S , by multiplying $1/\|x\|$ for each $x \in S$

3. No matter S , S^\perp is always a subspace.

$$\langle cx + y, z \rangle = c\langle x, z \rangle + \langle y, z \rangle = 0$$

for $c \in F$ and $x, y \in S^\perp$ and $z \in S$

4. $S \cap S^\perp \subset \{0\}$

$$x \perp x \rightarrow \langle x, x \rangle = 0 \rightarrow x = 0$$

for $x \in S \cap S^\perp$

Definition. Orthonormal Set Property Let V be inner product space and $S = \{s_1, s_2, \dots\} \subseteq V$ be an orthonormal set. Let $v \in \text{span}(S)$, then $v = a_1s_1 + \dots + a_ks_k$. Then

$$\langle v, s_j \rangle = a_j$$

by

$$\langle v, s_j \rangle = \langle \sum_i a_i s_i, s_j \rangle = \sum_i a_i \langle s_i, s_j \rangle = \sum_i a_i \delta_{ij} = a_j$$

Gram-Schmidt Orthogonalization Process and Orthogonal Complements

Definition. Orthonormal Basis Let V be an inner product space. A subset of V is an orthonormal basis for V if it is an ordered basis that is orthonormal

Definition. Every Inner Product Space has n Orthogonal Basis Let V be an inner product space and $S = \{v_1, v_2, \dots, v_k\}$ be an orthogonal subset of V consisting of nonzero vectors. If $y \in \text{span}(S)$, then

$$y = \sum_{i=1}^k \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i$$

Corollary. Special case for Orthonormal Set If, in addition to hypotheses of previous theorem, S is orthonormal and $y \in \text{span}(S)$, then

$$y = \sum_{i=1}^k \langle y, v_i \rangle v_i$$

Corollary. Nonzero Orthonormal Set is Linearly Independent Let V be an inner product space, and let S be an orthonormal subset of V consisting of nonzero vectors. Then S is linearly independent

Theorem. 6.4 Gram-Schmidt Process Let V be an inner product space and $S = \{w_1, w_2, \dots, w_n\}$ be a linearly independent subset of V . Define $S' = \{v_1, v_2, \dots, v_n\}$, where $v_1 = w_1$ and

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j \quad 2 \leq k \leq n$$

Then S' is an orthogonal set of nonzero vectors such that $\text{span}(S') = \text{span}(S)$

Theorem. 6.5 Every Finite Dimensional I.P.S has an Orthonormal Basis Let V be a nonzero finite-dimensional inner product space. Then V has an orthonormal basis β . Furthermore, if $\beta = \{v_1, v_2, \dots, v_n\}$ and $x \in V$, then

$$x = \sum_{i=1}^n \langle x, v_i \rangle v_i$$

Corollary. Expression for Matrix Representation of Transformation on Orthonormal Basis Let V be a finite-dimensional inner product space with an orthonormal basis $\beta = \{v_1, v_2, \dots, v_n\}$. Let T be a linear operator on V , and let $A = [T]_\beta$. Then for any i and j , $A_{ij} = \langle T(v_j), v_i \rangle$, i.e.

$$T(v_j) = \sum_{i=1}^n \langle T(v_j), v_i \rangle v_i$$

Definition. Fourier Coefficients Let β be an orthonormal subset (possibly infinite) of an inner product space V , and let $x \in V$. We define the Fourier coefficients of x relative to β to be the scalars $\langle x, y \rangle$, where $y \in \beta$

Definition. Example of infinite dimension vector space Define inner product on vector space H as

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t) \overline{g(t)} dt$$

Let $f_n(t) = e^{int}$ where $0 \leq t \leq 2\pi$ ($e^{int} = \cos nt + i \sin nt$). Let $S = \{f_n : n \text{ is an integer}\}$. $S \subseteq H$. For $m \neq n$, we have $\langle f_m, f_n \rangle = \delta_{mn}$, therefore an orthonormal subset of H

6.2 Example 7 showed that

Definition. Another example of infinite dimension vector space 1.2 example 5 Define a vector space of sequences $\sigma(n) = \{a_n\} = a_1, a_2, \dots$ where $\sigma(n) \neq 0$ for finitely many elements. Define inner product as

$$\langle \sigma, \mu \rangle = \sum_{n=1}^{\infty} \sigma(n) \overline{\mu(n)}$$

6.2.23 As an example, define $e_k(n) = \delta_{nk}$. We can prove that $\{e_1, e_2, \dots\}$ are orthonormal basis for V .

6.5.16 shows that $W = \text{span}(e_2, e_4, \dots)$ and some unitary U such that W^\perp is not U -invariant

Orthogonal Complements

Definition. Orthogonal Complements Let S be a nonempty subset of an inner product space V . We define $S^\perp = \{x \in V : \langle x, y \rangle = 0 \text{ for all } y \in S\}$. The set S^\perp is called the orthogonal complement of S

1. $\{0\}^\perp = V$ and $V^\perp = \{0\}$

Theorem. 6.6 Finding Projection of a Vector onto a Subspace Let W be a finite-dimensional subspace of an inner product space V , and let $y \in V$. Then there exist unique vectors $u \in W$ and $z \in W^\perp$ such that $y = u + z$. Furthermore, if $\{v_1, v_2, \dots, v_k\}$ is an orthonormal basis for W , then

$$u = \sum_{i=1}^k \langle y, v_i \rangle v_i$$

where u is the orthogonal projection of y on W .

Corollary. Orthogonal Projection is Unique and Closest to Projected Vector In the notation of previous theorem, the vector u the unique vector in W that is closest to y ; that is, for any $x \in W$, $\|y - x\| \geq \|y - u\|$, and this inequality is an equality if and only if $x = u$

Theorem. 6.7 Orthonormal Basis and Subspaces Suppose that $S = \{v_1, v_2, \dots, v_k\}$ is an orthonormal set in an n -dimensional inner product space V . Then

1. S can be extended to an orthonormal basis $\{v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n\}$ for V .
2. If $W = \text{span}(S)$, then $S_1 = \{v_{k+1}, \dots, v_n\}$ is an orthonormal basis for W^\perp
3. If W is any subspace of V , then $\dim(V) = \dim(W) + \dim(W^\perp)$

If V finite dimensional, then

1. $W = (W^\perp)^\perp$
2. $V = W \oplus W^\perp$

6.3 The Adjoint of a Linear Operator

Definition. Dual Space is a space of all linear transformations from a vector space V to its field F .

Theorem. 6.8 Every Linear Transformation from V to F Can Be Written as a Inner Product Let V be a finite-dimensional inner product space over F , and let $g : V \rightarrow F$ be a linear transformation. Then there exists a unique vector $y \in V$ such that $g(x) = \langle x, y \rangle$ for all $x \in V$, where

$$y = \sum_i \overline{g(v_i)} v_i \quad \beta = \{v_1, \dots, v_n\} \text{ is orthonormal basis}$$

Definition. Adjoint Linear Operator Given inner product space V , let T be a linear operator on V . The adjoint of operator T , T^* , is the unique operator on V satisfying

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle \quad \text{for all } x, y \in V$$

Theorem. 6.9 Adjoint of an Linear Operator Exist for f.d. Inner Product Space Let V be a finite-dimensional inner product space, and let T be a linear operator on V . Then there exists a unique function, called the adjoint of T , $T^* : V \rightarrow V$ such that

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle$$

for all $x, y \in V$. Furthermore, T^* is linear. We can view the equation symbolically as adding an asterik $*$ to T when shifting position inside the inner product symbol

Theorem. 6.10 Adjoint of a Linear Operator in Matrix Form is the Adjoint of Matrix Form of that Linear Operator Let v be a finite-dimensional inner product space. Let β be an orthonormal basis for V . If T is a linear operator on V , then

$$[T^*]_{\beta} = [T]_{\beta}^*$$

Corollary. For Left-Matrix Transformation Let A be $n \times n$ matrix, then $L_{A^*} = (L_A)^*$. ([theorem 2.16](#))

Theorem. 6.11 Properties of Adjoint of Linear Operators

Let V be an inner product space, and let T, U be linear operators on V , then

1. $(T + U)^* = T^* + U^*$
2. $(cT)^* = \bar{c}T^*$ for any $c \in F$
3. $(TU)^* = U^*T^*$
4. $T^{**} = T$
5. $I^* = I$

assuming adjoints always exists.

Corollary. For Matrix

Let A and B be $n \times n$ matrix, then

1. $(A + B)^* = A^* + B^*$
2. $(cA)^* = \bar{c}A^*$ for all $c \in F$
3. $(AB)^* = B^*A^*$
4. $A^{**} = A$
5. $I^* = I$

Least Squares Approximation

Definition. Some notation For $x, y \in F^n$

1. $\langle x, y \rangle_n$ is the standard inner product of x and y in F^n
2. If x and y are column vectors, then $\langle x, y \rangle_n = y^*x$

Lemma. Let $A \in M_{m \times n}(F)$, $x \in F^n$ and $y \in F^m$, then

$$\langle Ax, y \rangle_m = \langle x, A^*y \rangle_n$$

Lemma. Let $A \in M_{m \times n}(F)$. Then $\text{rank}(A^*A) = \text{rank}(A)$

Corollary. If A is $m \times n$ matrix such that $\text{rank}(A) = n$, then A^*A is invertible

Theorem. 6.12 Close Form Solution for Least Squared Problem Let $A \in M_{m \times n}(F)$ and $y \in F^m$. Then there exists $x_0 \in F^n$ such that $(A^*A)x_0 = A^*y$ and $\|Ax_0 - y\| \leq \|Ax - y\|$ for all $x \in F^n$. Furthermore, if $\text{rank}(A) = n$, then $x_0 = (A^*A)^{-1}A^*y$

6.4 Normal and Self-Adjoint Operators

Definition. Motivation Condition for orthonormal basis of eigenvectors in

1. $F = \mathbb{C}$, T normal
2. $F = \mathbb{R}$, T self-adjoint

Lemma. Condition on Existence of Eigenvector for Adjoint Linear Operators Let T be a linear operator on a finite-dimensional inner product space V . If T has an eigenvector, then so does T^* . If λ is an eigenvalue of T , then $\bar{\lambda}$ is an eigenvalue of T^*

Proof. Let v be eigenvector of T with corresponding eigenvalue λ , then for any $x \in V$,

$$0 = \langle 0, x \rangle = \langle (T - \lambda I)(v), x \rangle = \langle v, (T - \lambda I)^*(x) \rangle = \langle v, (T^* - \bar{\lambda}I)(x) \rangle$$

Let $W = \text{span}(\{v\})$, so $R(T^* - \bar{\lambda}I) \subseteq W^\perp$. Note $\text{rank}(T^* - \bar{\lambda}I) \leq \dim(W^\perp) = n - 1$, then $N(T^* - \bar{\lambda}I) \neq \{0\}$. So exists $u \in N(T^* - \bar{\lambda}I)$ such that $T^*(u) = \bar{\lambda}u$ \square

Theorem. 6.14 (Schur's Theorem)

$P_T(t)$ Splits Implies Exists O.N. Basis st. $[T]_\beta$ is Upper Triangular

Let T be a linear operator on a finite-dimensional inner product space V . Suppose that the characteristic polybnonomial of T splits. Then there exists an orthonormal basis β for V such that the matrix $[T]_\beta$ is upper triangular

Proof. With induction, idea is to construct an orthonormal basis $\beta = \gamma \cup \{z\}$, where γ is an orthonormal basis for W^\perp and $z \in W = \text{span}(z)$, where z is unit eigenvector for T^* whose existence ensured by previous lemma. The induction hypothesis mandates

1. W^\perp is a T -invariant subspace as an assumption, i.e. if $y \in W^\perp, x \in W$, then $\langle T(y), x \rangle = 0$
2. $P_{T_{W^\perp}}(t) | P_T(t)$, so characteristic polynomial of T_{W^\perp} splits

to get the orthonormal basis γ , for which $[T_{W^\perp}]_\gamma$ is upper triangular. \square

Definition. Normal Linear Operator Let V be an inner product space, and let T be a linear operator on V . We say that T is normal if $TT^* = T^*T$. An $n \times n$ real or complex matrix A is normal if $AA^* = A^*A$ (Commutativity).

1. Motivation is that if $[T]_\beta$ is diagonal, then T^* also diagonal, hence T and T^* commutes
2. T is normal if and only if $[T]_\beta$ is normal, where β is an orthonormal basis
3. Skew-symmetric matrix ($A^t = -A$) is normal by $A^t A = -A^2 = AA^t$
4. Normality not sufficient to guarantee an orthonormal basis of eigenvectors. However, normality suffices if V is a complex inner product space

Theorem. 6.15 Properties of Normal Operator

Let V be an inner product space, and let T be a normal operator on V . Then the following are true

1. $\|T(x)\| = \|T^*(x)\|$ for all $x \in V$
2. $T - cI$ is normal for every $c \in F$
3. If x is an eigenvector of T , then x is also an eigenvector of T^* . In fact, if $T(x) = \lambda x$, then $T^*(x) = \bar{\lambda}x$
4. If λ_1 and λ_2 are distinct eigenvalues of T with corresponding eigenvectors x_1 and x_2 , then x_1 and x_2 are orthogonal, i.e. $\langle x_1, x_2 \rangle = 0$

Theorem. 6.16 Normal Operator iff Diagonalizable ($F = \mathbb{C}$)

Let T be a linear operator on a finite-dimensional **complex** inner product space V . Then T is normal if and only if there exists an orthonormal basis for V consisting of eigenvectors of T .

Proof. Idea is the orthonormal basis that makes T an upper triangular matrix (using Schur's Theorem) happens to be a set of eigenvectors \square

1. [example showing theorem does not work on infinite dimension vector spaces](#) with problem definition [here](#). Specifically an example where T is normal and that T has no eigenvectors
2. Normality not sufficient for existence of orthonormal basis of eigenvectors for real inner product spaces

Definition. Self-Adjoint (Hermitian) Let T be a linear operator on an inner product space V . We say that T is self-adjoint (Hermitian) if $T = T^*$. An $n \times n$ real or complex matrix A is self-adjoint (Hermitian) if $A = A^*$

1. If β is orthonormal basis, then T is self-adjoint if and only if $[T]_\beta$ is self-adjoint (symmetric matrix for $F = \mathbb{R}$)
2. If T is self-adjoint, then T is normal

Lemma. Properties of Self-Adjoins

Let T be a self-adjoint operator on a finite-dimensional inner product space V . Then

1. Every eigenvalue of T is real
2. Suppose that V is a real inner product space ($F = \mathbb{R}$). Then the characteristic polynomial of T splits

Theorem. 6.17 Self-Adjoins iff Diagonalizable ($F = \mathbb{R}$)

Let T be a linear operator on a finite-dimensional real inner product space V . Then T is self-adjoint if and only if there exists an orthonormal basis β for v consisting of eigenvectors of T .

Definition. Computing Squared Root of Imaginary Number

Relies on Euler's formula

$$e^{ix} = \cos x + i \sin x$$

Therefore we have

$$e^{i\pi} = -1 \quad i = \sqrt{-1} = e^{i\pi/2} \quad \sqrt{i} = e^{i\pi/4} = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}$$

Definition. Positive Definite/Semidefinite

A linear operator T on a finite-dimensional inner product space is called positive definite (positive semidefinite) if T is

1. self-adjoint, and
2. $\langle T(x), x \rangle > 0$ ($\langle T(x), x \rangle \geq 0$) for all $x \neq 0$

An $n \times n$ matrix A with entries from \mathbb{R} or \mathbb{C} is positive definite (positive semidefinite) if L_A is positive definite (positive semidefinite)

6.4 Unitary and Orthogonal Operators

Definition. Unitary/Orthogonal Operator Let T be a linear operator on a finite-dimensional inner product space V (over F). If $\|T(x)\| = \|x\|$ for all $x \in V$, we call T a unitary operator if $F = \mathbb{C}$ and an orthogonal operator if $F = \mathbb{R}$

1. **Isomtry** for infinite-dimensional case
2. **Motivation** Linear operator that preserves length
3. T is onto for finite dimensional V

Theorem. 6.18 Property of Unitary/Orthogonal Operator Let T be a linear operator on a finite-dimensional inner product space V . Then the following statements are equivalent

1. $TT^* = T^*T = I$ (unitary/orthogonal operators are **normal**)
2. $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all $x, y \in V$
3. If β is an orthonormal basis for V , then $T(\beta)$ is an orthonormal basis for V
4. There exists an orthonormal basis β for V such that $T(\beta)$ is an orthonormal basis for V
5. $\|T(x)\| = \|x\|$ for all $x \in V$ (T is a **unitary/orthogonal**)

Proof. (2 to 3)

$$\langle T(\beta_i), T(\beta_j) \rangle = \langle \beta_i, \beta_j \rangle = \delta_{ij}$$

(4 to 5) Given $x = a_1\beta_1 + \cdots + a_n\beta_n$ and $T(x) = a_1T(\beta_1) + \cdots + a_nT(\beta_n)$, we have

$$\langle x, x \rangle = |a_1|^2 + \cdots + |a_n|^2 = \langle T(x), T(x) \rangle$$

□

Lemma. Let U be a self-adjoint operator on a finite-dimensional inner product space V . If $\langle x, U(x) \rangle = 0$ for all $x \in V$, then $U = T_0$

1. **Interpretation** All vectors are orthogonal to $\{0\}$, the range of T_0

Corollary. $F = \mathbb{R}$ Orthogonal and Self-Adjoint Operator has All Eigenvalues ± 1 Let T be a linear operator on a finite-dimensional real inner product space V . Then V has an orthonormal basis of eigenvectors of T with corresponding eigenvalues of **absolute value 1** if and only if T is both **self-adjoint** and **orthogonal**

1. **Interpretation** Length-preserving operator has eigenvalue of ± 1

Corollary. $F = \mathbb{C}$ Unitary Operator has All Eigenvalues ± 1 Let T be a linear operator on a finite-dimensional complex inner product space V . Then V has an orthonormal basis of eigenvectors of T with corresponding eigenvalues of absolute value 1 if and only if T is unitary

Definition. Reflection Let L be a one-dimensional subspace of \mathbb{R}^2 . We may view L as a line in the plane through the origin. A linear operator T on \mathbb{R}^2 is called a reflection of \mathbb{R}^2 about L if $T(x) = x$ for all $x \in L$ and $T(x) = -x$ for all $x \in L^\perp$

1. Reflection transformation is an orthogonal operator

Definition. Orthogonal/Unitary Matrix A square matrix A is called an orthogonal matrix if $A^T A = AA^T = I$ and unitary if $A^* A = AA^* = I$

1. **Inverse** of unitary matrix $A^{-1} = A^*$
2. $AA^* = I$ if and only if rows of A form orthonormal basis for F^n , similarly for columns of A (prove $\langle r_i, r_j \rangle = \delta_{ij}$)
3. For $F = \mathbb{R}$, if columns of A consists of orthonormal basis of eigenvectors β , then rows of A^T are β , so then A is orthogonal
4. T unitary/orthogonal if and only if $[T]_\beta$ is unitary/orthogonal for some orthonormal basis β for V

Definition. Unitarily Equivalent Matrices A and B are unitarily equivalent (orthogonally equivalent) if and only if there exists a unitary (orthogonal) matrix P such that $A = P^* B P$

Theorem. 6.19 Condition for Normal ($F = \mathbb{C}$) Matrix Let $A \in M_n(\mathbb{C})$. Then A is normal if and only if A is unitarily equivalent to a diagonal matrix

Proof. (\Rightarrow) Since A complex normal, exists orthonormal basis β consisting of eigenvectors for V . So $D = Q^{-1} A Q$ where D is diagonal and Q has columns consisting of vectors in β , therefore Q unitary/orthogonal. Therefore A and D is unitarily equivalent
(\Leftarrow) Let $A = P^* D P$ for some unitary P and diagonal D , therefore,

$$AA^* = (P^* D P)(P^* D P)^* = P^* D P P^* D^* P = P^* D D^* P = P^* D^* D P = (P^* D^* P)(P^* D P) = A^* A$$

□

Theorem. 6.20 Condition for Symmetric ($F = \mathbb{R}$) Matrix Let $A \in M_n(\mathbb{R})$. Then A is symmetric if and only if A is orthogonally equivalent to a real diagonal matrix.

1. Idea is we can diagonalize a symmetric matrix by finding orthogonal matrix P whose column consists of orthonormalized (Gram-Schmit) set of eigenvectors (find basis for eigenspaces)

Theorem. 6.21 Schur Let $A \in M_{n \times n}(F)$ be a matrix whose characteristic polynomial splits over F

1. If $F = \mathbb{C}$, then A is unitarily equivalent to a complex upper triangular matrix
2. If $F = \mathbb{R}$, then A is orthogonally equivalent to a real upper triangular matrix

As an example, unitary P with columns consists of orthonormal basis of eigenvectors satisfies $P^* A P = U$ for some upper triangular matrix U . Note U happens to be diagonal if A is normal in \mathbb{C} or self-adjoint in \mathbb{R}

5.2 Direct Sums

Definition. Sum Let W_1, W_2, \dots, W_k be subspaces of a vector space V . The sum of these subspaces is the set

$$\{v_1 + \dots + v_k : v_i \in W_i \text{ for } 1 \leq i \leq k\}$$

which we denote by $W_1 + \dots + W_k$ or $\sum_{i=1}^k W_i$

Definition. Direct Sum Let W_1, W_2, \dots, W_k be subspaces of a vector space V . We call V the direct sum of the subspaces W_1, W_2, \dots, W_k and write $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$ if

$$V = \sum_{i=1}^k W_i \quad \text{and} \quad W_j \cap \sum_{i \neq j} W_i = \{0\} \text{ for each } 1 \leq j \leq k$$

1. dimension of direct sum is sum of dimension of the subspaces in the sum

$$\dim(V) = \dim(W_1) + \dots + \dim(W_k)$$

Theorem. 5.10 Equivalence Condition for Direct Sum

Let W_1, \dots, W_k be subspaces of finite-dimensional vector space V . The following results are equivalent

1. $V = W_1 \oplus \dots \oplus W_k$
2. $V = \sum_{i=1}^k W_i$ and, for any vector v_1, \dots, v_k such that $v_i \in W_i$ ($1 \leq i \leq k$), if $v_1 + \dots + v_k = 0$, then $v_i = 0$ for all i .
3. Each vector $v \in V$ can be uniquely written as $v = v_1 + v_2 + \dots + v_k$ where $v_i \in W_i$
4. If γ_i is an ordered basis for W_i ($1 \leq i \leq k$), then $\gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_k$ is an ordered basis for V
5. For each $i = 1, 2, \dots, k$, there exists an ordered basis γ_i for W_i such that $\gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_k$ is an ordered basis for V

Theorem. 5.11 A linear operator T on a finite-dimensional vector space V is diagonalizable if and only if V is the direct sum of the eigenspaces of T

6.6 Orthogonal Projection and the Spectral Theorem

Definition. Projection on a Subspace

Let V be a vector space and W_1 and W_2 be subspaces of V such that $V = W_1 \oplus W_2$. A function $T : V \rightarrow V$ is called the projection on W_1 along W_2 if, for $x = x_1 + x_2$ with $x_1 \in W_1$ and $x_2 \in W_2$, we have $T(x) = x_1$

1. $R(T) = W_1 = \{x \in V : T(x) = x\}$ and $N(T) = W_2$

2. $V = R(T) \oplus N(T)$, i.e.

- (a) every projection uniquely determined by its range and nullspace
- (b) W_1 does not uniquely determine T
- (c) If T orthogonal projection, then T uniquely determined by its range

3. **T is a projection if and only if $T = T^2$**

Proof. Proving T is a projection iff $T^2 = T$. Forward direction,

$$T^2(x) = T(T(x)) = T(x_1) = x_1 = T(x)$$

For reverse direction, assume $T^2 = T$, then $T(I-T) = 0_V$. Let $W_1 = R(T)$ and $W_2 = N(T)$, now claim $V = W_1 \oplus W_2$.

1. To prove $x = x_1 + x_2$ where $x \in W_1$ and $x_2 \in W_2$. Observe for any $x \in V$,

$$I(x) = T(x) + (I-T)(x) \quad \longrightarrow \quad V = R(T) + R(I-T)$$

Therefore, we want to prove $N(T) = R(I-T)$. Since $T(I-T) = 0$, then $R(I-T) \subseteq N(T)$. Conversely, if $x \in N(T)$, then $(I-T)(x) = x - T(x) = x$, i.e. $x \in R(I-T)$.

2. To prove uniqueness, i.e. $\{0\} = R(T) \cap N(T)$. Let $T(x) \in R(T) \cap N(T)$ as $T(x) \in R(T)$ by default and let $T(x) \in N(T)$. Then $T(x) = 0$.

Therefore $V = W_1 \oplus W_2$. Then we can write $x = x_1 + x_2$ where $x_1 \in R(T)$ and $x_2 \in N(T)$, hence

$$T(x) = T(x_1 + x_2) = T(x_1) + T(x_2) = T(x_1) = x_1$$

where the last equality given by letting $x_1 = T(y)$, then $T(x_1) = T^2(y) = T(y) = x_1$. So T is a projection on $R(T)$ along $N(T)$ \square

Definition. Orthogonal Projection Let V be an inner product space, and let $T : V \rightarrow V$ be a linear operator. We say that T is an orthogonal projection if

- 1. T is a projection, and
- 2. $R(T)^\perp = N(T)$ and $N(T)^\perp = R(T)$

Note

- 1. If V finite-dimensional, need to assume either condition in 2. hold.
- 2. Orthogonal projection T is uniquely determined by its range W , so instead call T the orthogonal projection of V on W

Proposition. Projection of Vector to a Subspace is an Orthogonal Projection Let W be subspace of V . there exists $u \in W$ and $z \in W^\perp$ and $y \in V$ such that $y = u + z$. If we define linear operator $T : V \rightarrow V$ by

$$T(y) = u = \sum_{i=1}^k \langle y, v_i \rangle v_i$$

where $\{v_1, \dots, v_n\}$ is an orthonormal basis for W . Then T is an orthogonal projection.

Proof. Prove that $T^2 = T = T^*$. For any $v_j \in \beta$, we have

$$T(T(v_j)) = T\left(\sum_{i=1}^k \langle v_j, v_i \rangle v_i\right) = T(v_j)$$

therefore $T^2 = T$ since linear operator characterized by basis entirely. Let $x, y \in V$,

$$\langle T(x), y \rangle = \left\langle \sum_i \langle x, v_i \rangle v_i, y \right\rangle = \sum_i \langle x, v_i \rangle \langle v_i, y \rangle = \sum_i \overline{\langle y, v_i \rangle} \langle x, v_i \rangle = \langle x, \sum_i \langle y, v_i \rangle v_i \rangle = \langle x, T(y) \rangle$$

therefore $T = T^*$ by theorem 6.24, T is orthogonal projection. \square

Proof. Alternatively we can prove that $N(T)$ and $R(T)$ are reciprocally orthogonal sets. Since T is a projection, we have $V = R(T) \oplus N(T)$, where $N(T) = \{x \in V : \langle x, v_i \rangle = 0 \text{ for all } i\} = W^\perp$ and $R(T) = W$. Therefore $N(T) = R(T)^\perp$ and $N(T)^\perp = (R(T)^\perp)^\perp \supseteq R(T)$. Now we show if $x \in N(T)^\perp$, then $x \in R(T) = W$. Now by direct sum, we can write $x = y + w$ where $y \in N(T)$ and $w \in R(T) = W$, then

$$0 = \langle x, y \rangle = \langle y, y \rangle + \langle w, y \rangle \rightarrow \langle y, y \rangle = 0 \rightarrow y = 0$$

therefore $x = w \in W$. \square

Theorem. 6.24 $T^2 = T = T^*$ **iff Orthogonal Projection**

Let V be an inner product space, and let T be a linear operator on V . Then T is an orthogonal projection if and only if T has an adjoint T^* and $T^2 = T = T^*$

1. Let T be orthogonal projection of V on W , and $\beta = \{v_1, \dots, v_n\}$ is an orthonormal basis for V , and $\{v_1, \dots, v_k\}$ is a basis for W , then

$$[T]_\beta = \begin{pmatrix} I_k & O \\ O & O \end{pmatrix}$$

2. Let U be any projection on W , then exists γ such that $[U]_\gamma$ has same form as above, but γ need not be orthonormal

Proof. \Rightarrow . $T^2 = T$ iff projection. So left with proving $T = T^*$, i.e. for all $x, y \in V$, $\langle x, T(y) \rangle = \langle T(x), y \rangle$

\Leftarrow . $T^2 = T$ iff projection. So left with proving $R(T) = N(T)^\perp$ and $R(T)^\perp = N(T)$ (not equivalent in infinite dimensional V). Note that $x \in R(T)$ iff $T(x) = x$. A way to prove that $T(x) = x$ is to prove $\|x - T(x)\| = 0$ \square

Theorem. 6.25 The Spectral Theorem

Suppose T a linear operator on a finite-dimensional inner product space V over F with distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Assume T is normal if $F = \mathbb{C}$ and that T is self-adjoint if $F = \mathbb{R}$. (i.e. guarantees orthonormal basis of eigenvectors). For each i ($1 \leq i \leq k$), let W_i be the eigenspace of T corresponding to the eigenvalue λ_i , and let T_i be the orthogonal projection of V on W_i . Then the following statements are true

1. $V = W_1 \oplus \dots \oplus W_k$
2. If W'_i denotes the direct sum of subspaces W_j for $j \neq i$, then $W_i^\perp = W'_i$
3. $T_i T_j = \delta_{ij} T_i$ for $1 \leq i, j \leq k$
4. $I = T_1 + T_2 + \dots + T_k$ (**resolution of identity operator induced by T**)
5. $T = \lambda_1 T_1 + \lambda_2 T_2 + \dots + \lambda_k T_k$ (**spectral decomposition**)

note

1. Note since T_i orthogonal projection, we have $N(T_i) = R(T_i)^\perp = W_i^\perp = W'_i$
2. **Spectrum** The set $\{\lambda_1, \dots, \lambda_k\}$ is called spectrum of T
3. Let β be union of orthonormal basis of W_i 's and let $m_i = \dim(W_i)$, then

$$[T]_\beta = \begin{pmatrix} \lambda_1 I_{m_1} & O & \dots & O \\ O & \lambda_2 I_{m_2} & \dots & O \\ \vdots & \vdots & & \vdots \\ O & O & \dots & \lambda_k I_{m_k} \end{pmatrix}$$

4. Let $g(t)$ be any polynomial in F , then

$$g(T) = g(\lambda_1)T_1 + \dots + g(\lambda_k)T_k$$

Corollary. Condition for Normal If $F = \mathbb{C}$, then T is normal if and only if $T^* = g(T)$ for some polynomial

Corollary. Condition for Unitary If $F = \mathbb{C}$, then T is unitary if and only if T is normal and $|\lambda| = 1$ for every eigenvalue λ of T

Corollary. Condition for Self-Adjoint If $F = \mathbb{C}$ and T is normal, then T is self-adjoint if and only if every eigenvalue of T is real