University of Toronto

Faculty of Arts and Science
August 2016 Examination
MAT237Y1Y
Duration - 3 hours
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No Aids Permitted

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This exam contains 13 pages (including this cover page) and 9 problems. Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in. No aids are permitted on this examination. Examples of illegal aids include, but are not limited to textbooks, notes, calculators, or any electronic device. Unless otherwise indicated, you are required to show your work on each problem on this exam. The following rules apply:

- **For Question 1**, fill in your chosen answer $\{a, b, c, d, e\}$ in the box provided on this page. Failure to do so will result in you receiving zero on this question.

- **Organize your work** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work, will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

- **If you need more space**, use the back of the pages; clearly indicate when you have done this.

Question 1

Part	Answer
i	
ii	
iii	
iv	
v	
vi	
vii	
viii	
ix	
x	

Problem	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total:	100	

1. (i) (2 points) Let $f : [1,3] \to \mathbb{R}$ be a uniformly continuous function such that $\epsilon = 0.5$ and $\delta = 1$ satisfy the definition of uniform continuity. If $f(1) = 2$ estimate $f(3)$ as well as possible.

a) $1 \leq f(3) \leq 3$,
b) $-2 \leq f(3) \leq 6$,
c) $1.5 \leq f(3) \leq 2.5$,
d) $0 \leq f(3) \leq 4$,
e) $2 \leq f(3)$.

(ii) (2 points) Consider the function given by $F(x,y) = x^2 - y^2$. Which values of c make the locus given by $F(x,y) = c$ smooth?

a) All values of c,
b) $c > 0$,
c) $c \neq 0$,
d) $c = 0$,
e) No values of c.

(iii) (2 points) Consider the Taylor expansion of $f(x,y) = \cos(xy) - 1$ about $(x,y) = (0,0)$. Write this expansion as $f(x,y) = \sum_\alpha c_\alpha \mathbf{x}^\alpha$ where $c_\alpha \in \mathbb{R}$.
What is the least order $|\alpha|$ such that c_α, the coefficient of $\mathbf{x}^\alpha$, is non-zero in the expansion above?

a) $|\alpha| = 0$,
b) $|\alpha| = 1$,
c) $|\alpha| = 2$,
d) $|\alpha| = 3$,
e) $|\alpha| = 4$.

(iv) (2 points) Suppose that $S \subseteq \mathbb{R}^n$ is a path connected set. Which of the following statements is correct?

a) Every subset of S is connected,
b) Every open subset of S is connected,
c) Every compact subset of S is connected,
d) We cannot conclude anything about the connectedness of subsets.

(v) (2 points) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ and $\iint f dA$ is equal to
3, over the region $S_1 = [0,2] \times [0,2]$
4, over the region $S_2 = [1,3] \times [1,3]$
5, over the region $S_3 = [1,2] \times [1,2]$
Determine the integral of f over $S_1 \cup S_2 \cup S_3$.

a) 0,
b) 2,
c) 7,
d) 12,
e) The given data is not sufficient to calculate the integral.

(vi) (2 points) Let $\alpha = (1, 2, 3)$. Determine $\dfrac{\partial^\alpha \mathbf{x}^\alpha}{\alpha!}$ if $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$.

a) 1,
b) 0,
c) xy^2z^3,
d) $\dfrac{xy^2z^3}{24}$.

(vii) (2 points) Which of the following constrained optimization problems has a solution?

a) Minimize $f(x, y) = (x - 1)^2 + y^2$ subject to $x^2 + y^2 < 1$,
b) Maximize $f(x, y) = x^3$ subject to $xy = 0$,
c) Minimize $f(x, y) = e^{xy}\sin(x)$ subject to $0 \leq x \leq y$ and $x + y \leq 10$,
d) None of the above.

(viii) (2 points) Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 2x + 1$. Find all points x such that f is invertible in a small neighborhood of x.

a) $x \neq 1$,
b) $x > 1$,
c) $x > 0$,
d) $0 < x < 1$,
e) $x \neq 0$.

(ix) (2 points) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}^n$ are differentiable. What are the dimensions of $\mathbf{D}(g \circ f)(\mathbf{0})$?

a) $1 \times n$,
b) $n \times 1$,
c) 1×1,
d) $n \times n$.

(x) (2 points) If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and continuous at $\mathbf{x} = \mathbf{a}$ which of the following is not necessarily true?

a) $\partial_{\mathbf{u}} f$ exists for every $\mathbf{u}$,
b) ∇f exists,
c) f is continuous at $\mathbf{x} = \mathbf{a}$,
d) Each partial derivative is continuous at $\mathbf{x} = \mathbf{a}$.

2. In this question you will construct a curve $\gamma : [1, \infty) \to \mathbb{R}^2$ with dense image. Notice that it suffices to find a curve whose image contains every point of $\mathbb{Q}^2 \subseteq \mathbb{R}^2$.

   (i) (2 points) Write a formula for the straight line segment from $\mathbf{a}$ to $\mathbf{b}$ where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$. Your formula should be of the form $\gamma_{\mathbf{a},\mathbf{b}}(t)$. Your answer should satisfy $\gamma_{\mathbf{a},\mathbf{b}}(0) = \mathbf{a}$ and $\gamma_{\mathbf{a},\mathbf{b}}(1) = \mathbf{b}$.

   (ii) (2 points) Let $\mathbf{f} : \mathbb{N} \to \mathbb{Q}^2 \subset \mathbb{R}^2$ be a bijective function. You may assume, without proof, that such a function exists. Write a path from $\mathbf{f}(n)$ to $\mathbf{f}(n+1)$. *Hint*: Use part (i).

   (iii) (4 points) Construct $\gamma : [1, \infty) \to \mathbb{R}^2$ with dense image by using a piecewise function defined in terms of $\mathbf{f}$ and your functions defined in part (i) and (ii).

   (iv) (2 points) In at most 30 words, describe the function $\gamma : [1, \infty) \to \mathbb{R}^2$ is doing.

3. (i) (5 points) Suppose that

$$M = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{pmatrix}$$

is a symmetric matrix ($M^T = M$) with real entries ($m_{ij} \in \mathbb{R}$). Write down an explicit polynomial $p : \mathbb{R}^n \to \mathbb{R}$ such that Hessian matrix $H(p)(\mathbf{x})$ satisfies $H(p)(\mathbf{x}) = M$ for all $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

(ii) (5 points) Compute the Taylor series of

$$f(x, y, z) = x^2 y + z$$

about the point $\mathbf{a} = (x, y, z) = (1, 2, 1)$. Your final answer must be of the form $f(\mathbf{x}) = \sum_\alpha c_\alpha (\mathbf{x} - \mathbf{a})^\alpha$ where $c_\alpha \in \mathbb{R}$. *Hint:* $c_\alpha = 0$ for infinitely many α.

4. (10 points) Evaluate $\iint_S \operatorname{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}}\, \mathrm{d}\mathbf{A}$ where $\mathbf{F}(x, y, z) = xyz\mathbf{i} + xy\mathbf{j} + x^2yz\mathbf{k}$ and S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outward.

5. Let C be a smooth curve in $\mathbb{R}^3$ parameterized by a vector function $\mathbf{r} : [a, b] \to \mathbb{R}^3$.

   (i) (3 points) Prove that $\dfrac{\mathrm{d}}{\mathrm{d}t}|\mathbf{r}(t)|^2 = 2\mathbf{r}(t) \cdot \mathbf{r}'(t)$.

   (ii) (3 points) Show that $\displaystyle\int_C \mathbf{r} \cdot \mathrm{d}\mathbf{r} = \frac{1}{2}\left(|\mathbf{r}(b)|^2 - |\mathbf{r}(a)|^2\right)$.

   (iii) (2 points) What can you conclude about $\int_C \mathbf{r} \cdot \mathrm{d}\mathbf{r}$ when C is a closed curve?

   (iv) (2 points) Now let $\mathbf{r} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field. **Briefly** (i.e. in at most 5 sentences) explain whether or not part (iii) shows that $\mathbf{r}$ is a conservative vector field.

6. Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ be C^1 functions. Let $\mathbf{F}, \mathbf{G}$ be smooth vector fields in $\mathbb{R}^3$. Exactly one of the following statements is always true.

   A. $\nabla(fg) = \nabla f \cdot \nabla g$.

   B. $\mathrm{curl}(f\mathbf{G}) = f\mathrm{curl}\mathbf{G} + (\mathrm{curl} f) \times \mathbf{G}$.

   C. $\mathrm{div}(f\mathbf{G}) = f\mathrm{div}\mathbf{G} + (\nabla f) \cdot \mathbf{G}$.

   The correct choice is:

   (i) (6 points) Indicate which identity is correct in the box above. Prove the identity.

   (ii) (4 points) Using **one sentence each**, describe why the other two identities cannot be correct.

7. Consider the transformation $\mathbf{F}:\mathbb{R}^2 \to \mathbb{R}^2$ given by $\mathbf{F}(x,y) = (x^2 - y^2, 2xy)$.

   (i) (5 points) Determine all $(x,y) \in \mathbb{R}^2$, such that $\mathbf{F}$ is invertible in a neighbourhood of (x,y).

   (ii) (5 points) For those points (x,y) where the Inverse Function Theorem does not apply, show that $\mathbf{F}$ is **not** invertible in a neighbourhood of (x,y). *Hint:* Try to replace (x,y) with $(-x,-y)$.

8. (10 points) Let $F(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, and take S to be the surface of the cube $0 \leq x, y, z \leq 10$, oriented so that the positive normal points out of the region bounded by S. Using the Divergence Theorem, or by any other means, evaluate the surface integral $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}A$.

9. Let $R \subseteq \mathbb{R}^2$ be the region in the 1^{st} quadrant bounded by the curves $y = x^2$, $y = x^2/5$, $xy = 2$, and $xy = 4$.

   (i) (2 points) Draw the region R.

   (ii) (8 points) Using the change of variables $u = x^2/y$ and $v = xy$, compute the area of R.

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