UNIVERSITY OF TORONTO
Faculty of Arts and Science

APRIL 2017 EXAMINATIONS

CSC473H1S

Duration – 3 hours

Examination Aids: Both sides of an A4 or US Letter format handwritten aid sheet.

- Check that your exam booklet has 13 pages (including this cover page).
- This exam has 6 questions.
- Put all answers in this booklet, in the spaces provided. If you run out of space, you can use the back of the page, provided that you ***clearly indicate which question you are answering***.
- For rough work, use pages 10 to 13; ***these will not be marked***.
- Your answers will be marked based on the correctness and completeness of your answers ***and*** the clarity of your explanations.
- ***A mark of at least 40% on the final exam is necessary to pass the course.***
- Good luck!

PLEASE COMPLETE THIS SECTION

First name ____________________

Family name ____________________

Student ID ____________________

Problem	Marks Received	Marks Worth
1.		25
2.		10
3.		25
4.		30
5.		15
6.		20
TOTAL		125

Question 1. (25 marks) Let $G = (V, E)$ be an undirected connected graph, which we treat as an electric network: a battery is hooked to two vertices s and t, and each edge $e \in E$ has resistance r_e. For this question we allow the network to have multiple parallel edges between the same pair of vertices, possibly with different resistances.

a. Suppose that G has a vertex x of degree 2 (assume $x \notin \{s, t\}$), and let y and z be its neighbors. Create a new network G' by removing x along with its two incident edges, and adding a new edge $e = (y, z)$ with resistance r_e. (See Figure 1.) The rest of the network is unmodified. What should r_e be, so that, if we keep the same voltage at every vertex in $V \setminus \{x\}$, then the current flowing across e from y to z in G' is equal to the current flowing across (y, x) from y to x in G. You do not need to justify your answer.

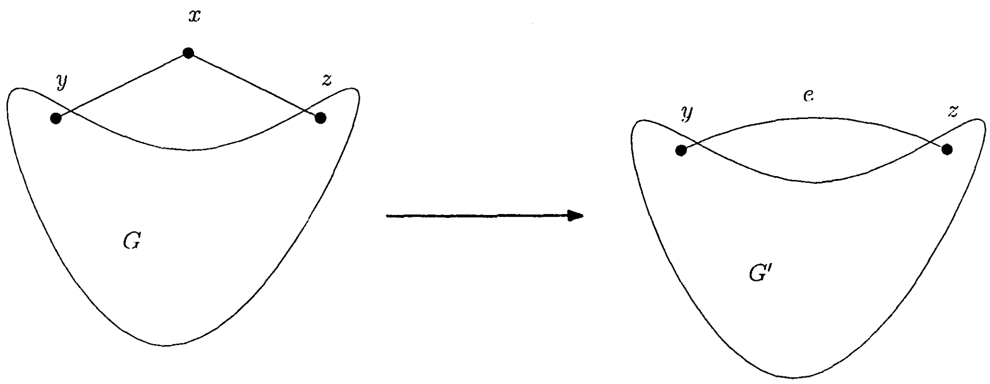

Figure 1: The path from y to z through x is replaced by the edge e.

b. Suppose that G has two parallel edges e_1 and e_2, both connecting vertices y and x, with resistances respectively r_{e_1} and r_{e_2}. Create a new network G' by removing both e_1 and e_2 and replacing them with a new edge e with resistance r_e. (See Figure 2.) The rest of the network is unmodified. What should r_e be, so that, if we keep the same voltage at every vertex in V, then the current flowing across e from x to y in G' is equal to the sum of the currents flowing across e_1 and e_2 from x to y in G. You do not need to justify your answer.

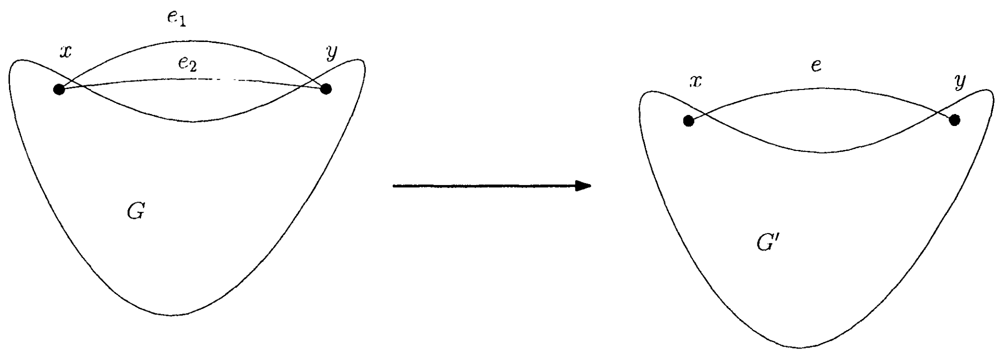

Figure 2: The two parallel edges e_1 and e_2 are replaced by the edge e.

c. Use your solution to the two subproblems above to answer the following question. Let C_n be a cycle with n vertices, numbered consecutively from 1 to n. I.e. $C_n = (V, E)$ is a graph on the vertices $V = \{1, \ldots, n\}$, with edge set $E = \{(1,2), (2,3), \ldots, (n,1)\}$. For each $i \in \{2, \ldots, n\}$, give the *exact* expected number of times a simple random walk started at 1 returns to 1 before being absorbed at i. Justify your answer. (Note: the number of *returns* to 1 is equal to the number of *visits* to 1 minus one.)

Question 2. (10 marks) Let $A[1 \mathinner{..} n]$ be an array of distinct integers, where $n \geq 2$ is even. Let the array $B[1 \mathinner{..} k]$ be constructed by the following procedure:

for $i = 1$ **to** k
    Sample j uniformly at random from $\{1, \ldots, n\}$
    Set $B[i] = A[j]$

Let X be the *fraction* of entries in B that are among the $n/2$ smallest integers in A. I.e. X is defined as $\frac{|\{i : B[i] \leq m\}|}{k}$, where m is the element of rank $\frac{n}{2}$ of A.

a. What is the expected value of X? What is the variance of X? Justify your answers.

b. Give an *upper bound* on the probability that $X \notin [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$ for any $\varepsilon \in (0, \frac{1}{2})$. Be as precise as possible, and justify your answer.

Question 3. (25 marks) Consider the following optimization problem. We are given as input an undirected bipartite graph $G = (A \cup B, E)$, with A and B the two sides of the bipartition, and an edge cost c_e for each $e \in E$. The goal is to find a set of edges $F \subseteq E$ such that:

- for any $a \in A$, there is *at least* one edge in F incident on a;
- for any $b \in B$ there is *at most* one edge in F incident on b;
- the cost $c(F) = \sum_{e \in F} c(e)$ is minimized.

a. Write this problem as an equivalent integer program. The program should have variables $x_e \in \{0, 1\}$ for each $e \in E$. Each feasible solution x to the program should be such that the set of edges F defined by $F = \{e : x_e = 1\}$ is a feasible solution to the problem above. The optimal solution of the program should have value equal to the cost of the optimal solution of the problem above. You do not need to justify your answer.

b. Relax the integer program from the previous subproblem to a linear program by relaxing the integrality constraints, and derive the dual of the relaxation. You do not need to justify your answer.

c. Suppose the relaxed linear program has an optimal integral solution for any graph G and costs c. Using complementary slackness, characterize pairs (F, y), where F is an optimal solution to the original problem, and y is an optimal solution to the dual of the linear programming relaxation from the previous subproblem. You do not need to justify your answer

Question 4. (30 marks) Let $G = (V, E)$ be an undirected graph, and let s, t be two vertices in V. Suppose we are given a nonnegative real value c_e for every edge $e \in E$. Consider the following optimization problem:

$$\min \sum_{(u,v) \in E} c_e |x_u - x_v| \tag{1}$$

$$\text{s.t.}$$

$$x_s = 1 \tag{2}$$

$$x_t = 0 \tag{3}$$

$$x_u \in \mathbb{R} \quad \forall u \in V \tag{4}$$

Above $|x_u - x_v|$ is the absolute value of the difference $x_u - x_v$.

a. Prove that (1)–(4) is a relaxation of the minimum $s - t$ cut problem in G, where c is the vector of costs. I.e. show that for each partition of V into two disjoint sets $S, \bar{S}$ such that $s \in S$, $t \in \bar{S}$, there exists a solution x satisfying (2)–(4) such that

$$\sum_{(u,v) \in E} c_e |x_u - x_v| = c(S, \bar{S}),$$

where $c(S, \bar{S}) = \sum_{(u,v) \in E: u \in S, v \in \bar{S}} c_{uv}$.

b. Give a linear program which is equivalent to (1)–(4). Your linear program should have the same optimal value as (1)–(4), and it should be possible to take a feasible solution of your linear program and compute (in polynomial time) a feasible solution of (1)–(4) with the same or smaller value. Describe your linear program and justify why it has the required properties.

c. Consider a randomized algorithm which takes a feasible solution x to (1)–(4), picks a uniformly random value θ in $[0, 1]$ and outputs the sets $S = \{u \in V : x_u \geq \theta\}$, $\bar{S} = \{u \in V : x_u < \theta\}$. Prove the tightest upper bound you can on the ratio

$$\frac{\mathbb{E}[c(S, \bar{S})]}{\sum_{(u,v)\in E} c_e |x_u - x_v|},$$

where $c(S, \bar{S})$ is as defined in the first subproblem, and $\mathbb{E}[c(S, \bar{S})]$ is the expected value of $c(S, \bar{S})$ when S and $\bar{S}$ are the random sets defined above. Justify your answer.

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Question 5. (15 marks) In the *maximum directed cut* problem you are given as input a *directed* graph $G = (V, E)$, and a nonnegative weight w_e for each directed edge $e \in E$. The goal is to partition V into two disjoint subsets S and $\bar{S}$ so that the weight of edges going from S to $\bar{S}$ is maximized. I.e. you want to maximize $w(S, \bar{S}) = \sum_{(u,v) \in E: u \in S, v \in \bar{S}} w_{uv}$. Consider the simple randomized approximation algorithm that puts each $u \in V$ in S with probability $1/2$, and puts it in $\bar{S}$ with probability $1/2$. The choice for each vertex u is independent of the choices for all other vertices. What is the approximation ratio achieved by this algorithm? Justify your answer.

Question 6. (20 marks) Design an algorithm in the streaming model that, given a value ϕ such that $0 < \phi < 1$, processes a stream σ of m updates from $\{1, \ldots, n\}$, and outputs a set $S \subseteq \{1, \ldots, n\}$ such that:

1. If i is such that $f_i > \phi m$, then $i \in S$;
2. If i is such that $i < \frac{\phi}{2} m$, then $i \notin S$.

Your algorithm should use $O(\lceil \frac{1}{\phi} \rceil)$ words of memory. You can use streaming algorithms that we have studied in class as a black box. Describe your algorithm carefully and justify why it satisfies the required properties.

THE END

Page for rough work: not graded.

Page for rough work: not graded.

Page for rough work: not graded.

Page for rough work: not graded.

Total Pages = 13 Total Marks = 125