This chapter and Chapter 20 present data structures known as ***mergeable heaps***, which support the following five operations.

MAKE-HEAP() creates and returns a new heap containing no elements.

INSERT(H, x) inserts node x, whose *key* field has already been filled in, into heap H.

MINIMUM(H) returns a pointer to the node in heap H whose key is minimum.

EXTRACT-MIN(H) deletes the node from heap H whose key is minimum, returning a pointer to the node.

UNION(H_1, H_2) creates and returns a new heap that contains all the nodes of heaps H_1 and H_2. Heaps H_1 and H_2 are "destroyed" by this operation.

In addition, the data structures in these chapters also support the following two operations.

DECREASE-KEY(H, x, k) assigns to node x within heap H the new key value k, which is assumed to be no greater than its current key value.[1]

DELETE(H, x) deletes node x from heap H.

As the table in Figure 19.1 shows, if we don't need the UNION operation, ordinary binary heaps, as used in heapsort (Chapter 6), work well. Operations other than UNION run in worst-case time $O(\lg n)$ (or better) on a binary heap. If the UNION operation must be supported, however, binary heaps perform poorly. By concatenating the two arrays that hold the binary heaps to be merged and then running MIN-HEAPIFY (see Exercise 6.2-2), the UNION operation takes $\Theta(n)$ time in the worst case.

[1]As mentioned in the introduction to Part V, our default mergeable heaps are mergeable min-heaps, and so the operations MINIMUM, EXTRACT-MIN, and DECREASE-KEY apply. Alternatively, we could define a ***mergeable max-heap*** with the operations MAXIMUM, EXTRACT-MAX, and INCREASE-KEY.

MINIMUM	$\Theta(1)$	$O(\lg n)$	$\Theta(1)$
EXTRACT-MIN	$\Theta(\lg n)$	$\Theta(\lg n)$	$O(\lg n)$
UNION	$\Theta(n)$	$O(\lg n)$	$\Theta(1)$
DECREASE-KEY	$\Theta(\lg n)$	$\Theta(\lg n)$	$\Theta(1)$
DELETE	$\Theta(\lg n)$	$\Theta(\lg n)$	$O(\lg n)$

Figure 19.1 Running times for operations on three implementations of mergeable heaps. The number of items in the heap(s) at the time of an operation is denoted by n.

In this chapter, we examine "binomial heaps," whose worst-case time bounds are also shown in Figure 19.1. In particular, the UNION operation takes only $O(\lg n)$ time to merge two binomial heaps with a total of n elements.

In Chapter 20, we shall explore Fibonacci heaps, which have even better time bounds for some operations. Note, however, that the running times for Fibonacci heaps in Figure 19.1 are amortized time bounds, not worst-case per-operation time bounds.

This chapter ignores issues of allocating nodes prior to insertion and freeing nodes following deletion. We assume that the code that calls the heap procedures deals with these details.

Binary heaps, binomial heaps, and Fibonacci heaps are all inefficient in their support of the operation SEARCH; it can take a while to find a node with a given key. For this reason, operations such as DECREASE-KEY and DELETE that refer to a given node require a pointer to that node as part of their input. As in our discussion of priority queues in Section 6.5, when we use a mergeable heap in an application, we often store a handle to the corresponding application object in each mergeable-heap element, as well as a handle to corresponding mergeable-heap element in each application object. The exact nature of these handles depends on the application and its implementation.

Section 19.1 defines binomial heaps after first defining their constituent binomial trees. It also introduces a particular representation of binomial heaps. Section 19.2 shows how we can implement operations on binomial heaps in the time bounds given in Figure 19.1.

binomial trees and proving some key properties. We then define binomial heaps and show how they can be represented.

19.1.1 Binomial trees

The ***binomial tree*** B_k is an ordered tree (see Section B.5.2) defined recursively. As shown in Figure 19.2(a), the binomial tree B_0 consists of a single node. The binomial tree B_k consists of two binomial trees B_{k-1} that are ***linked*** together: the root of one is the leftmost child of the root of the other. Figure 19.2(b) shows the binomial trees B_0 through B_4.

Some properties of binomial trees are given by the following lemma.

Lemma 19.1 (Properties of binomial trees)
For the binomial tree B_k,

1. there are 2^k nodes,
2. the height of the tree is k,
3. there are exactly $\binom{k}{i}$ nodes at depth i for $i = 0, 1, \ldots, k$, and
4. the root has degree k, which is greater than that of any other node; moreover if the children of the root are numbered from left to right by $k-1, k-2, \ldots, 0$, child i is the root of a subtree B_i.

Proof The proof is by induction on k. For each property, the basis is the binomial tree B_0. Verifying that each property holds for B_0 is trivial.

For the inductive step, we assume that the lemma holds for B_{k-1}.

1. Binomial tree B_k consists of two copies of B_{k-1}, and so B_k has $2^{k-1}+2^{k-1} = 2^k$ nodes.
2. Because of the way in which the two copies of B_{k-1} are linked to form B_k, the maximum depth of a node in B_k is one greater than the maximum depth in B_{k-1}. By the inductive hypothesis, this maximum depth is $(k-1)+1 = k$.
3. Let $D(k, i)$ be the number of nodes at depth i of binomial tree B_k. Since B_k is composed of two copies of B_{k-1} linked together, a node at depth i in B_{k-1} appears in B_k once at depth i and once at depth $i+1$. In other words, the number of nodes at depth i in B_k is the number of nodes at depth i in B_{k-1} plus

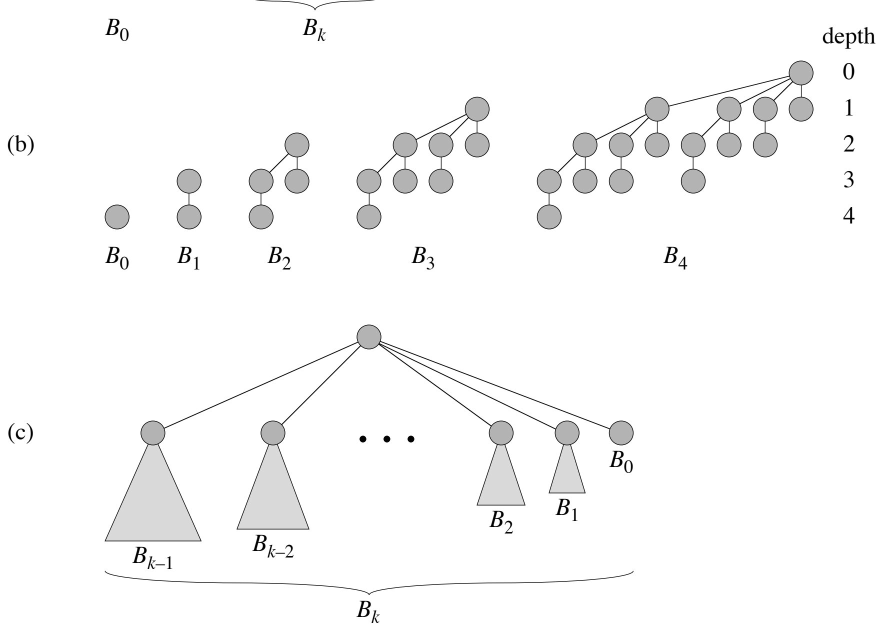


Figure 19.2 **(a)** The recursive definition of the binomial tree B_k. Triangles represent rooted subtrees. **(b)** The binomial trees B_0 through B_4. Node depths in B_4 are shown. **(c)** Another way of looking at the binomial tree B_k.

the number of nodes at depth $i - 1$ in B_{k-1}. Thus,

$$\begin{aligned} D(k,i) &= D(k-1,i) + D(k-1,i-1) && \text{(by the inductive hypothesis)} \\ &= \binom{k-1}{i} + \binom{k-1}{i-1} && \text{(by Exercise C.1-7)} \\ &= \binom{k}{i} . \end{aligned}$$

4. The only node with greater degree in B_k than in B_{k-1} is the root, which has one more child than in B_{k-1}. Since the root of B_{k-1} has degree $k-1$, the root of B_k has degree k. Now, by the inductive hypothesis, and as Figure 19.2(c) shows, from left to right, the children of the root of B_{k-1} are roots of $B_{k-2}, B_{k-3}, \ldots, B_0$. When B_{k-1} is linked to B_{k-1}, therefore, the children of the resulting root are roots of $B_{k-1}, B_{k-2}, \ldots, B_0$. ■

The term "binomial tree" comes from property 3 of Lemma 19.1, since the terms $\binom{k}{i}$ are the binomial coefficients. Exercise 19.1-3 gives further justification for the term.

19.1.2 Binomial heaps

A ***binomial heap*** H is a set of binomial trees that satisfies the following ***binomial-heap properties***.

1. Each binomial tree in H obeys the ***min-heap property***: the key of a node is greater than or equal to the key of its parent. We say that each such tree is ***min-heap-ordered***.
2. For any nonnegative integer k, there is at most one binomial tree in H whose root has degree k.

The first property tells us that the root of a min-heap-ordered tree contains the smallest key in the tree.

The second property implies that an n-node binomial heap H consists of at most $\lfloor \lg n \rfloor + 1$ binomial trees. To see why, observe that the binary representation of n has $\lfloor \lg n \rfloor + 1$ bits, say $\langle b_{\lfloor \lg n \rfloor}, b_{\lfloor \lg n \rfloor - 1}, \ldots, b_0 \rangle$, so that $n = \sum_{i=0}^{\lfloor \lg n \rfloor} b_i 2^i$. By property 1 of Lemma 19.1, therefore, binomial tree B_i appears in H if and only if bit $b_i = 1$. Thus, binomial heap H contains at most $\lfloor \lg n \rfloor + 1$ binomial trees.

Figure 19.3(a) shows a binomial heap H with 13 nodes. The binary representation of 13 is $\langle 1101 \rangle$, and H consists of min-heap-ordered binomial trees B_3, B_2, and B_0, having 8, 4, and 1 nodes respectively, for a total of 13 nodes.

Representing binomial heaps

As shown in Figure 19.3(b), each binomial tree within a binomial heap is stored in the left-child, right-sibling representation of Section 10.4. Each node has a *key* field and any other satellite information required by the application. In addition, each node x contains pointers $p[x]$ to its parent, $child[x]$ to its leftmost child, and $sibling[x]$ to the sibling of x immediately to its right. If node x is a root, then $p[x] = \text{NIL}$. If node x has no children, then $child[x] = \text{NIL}$, and if x is the rightmost child of its parent, then $sibling[x] = \text{NIL}$. Each node x also contains the field $degree[x]$, which is the number of children of x.

As Figure 19.3 also shows, the roots of the binomial trees within a binomial heap are organized in a linked list, which we refer to as the ***root list***. The degrees

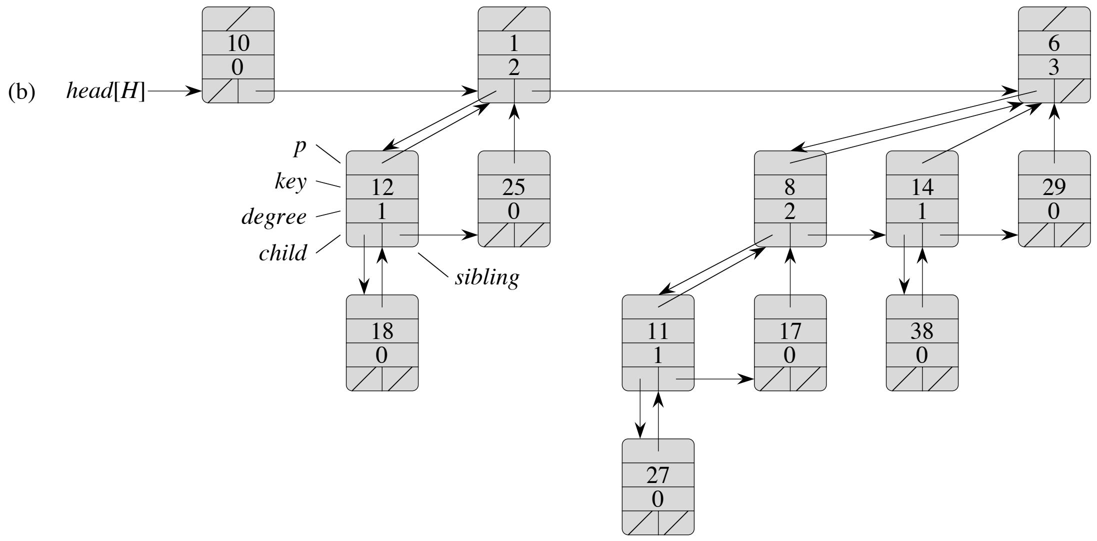


Figure 19.3 A binomial heap H with $n = 13$ nodes. **(a)** The heap consists of binomial trees B_0, B_2, and B_3, which have 1, 4, and 8 nodes respectively, totaling $n = 13$ nodes. Since each binomial tree is min-heap-ordered, the key of any node is no less than the key of its parent. Also shown is the root list, which is a linked list of roots in order of increasing degree. **(b)** A more detailed representation of binomial heap H. Each binomial tree is stored in the left-child, right-sibling representation, and each node stores its degree.

of the roots strictly increase as we traverse the root list. By the second binomial-heap property, in an n-node binomial heap the degrees of the roots are a subset of $\{0, 1, \ldots, \lfloor \lg n \rfloor\}$. The *sibling* field has a different meaning for roots than for nonroots. If x is a root, then $sibling[x]$ points to the next root in the root list. (As usual, $sibling[x] = \text{NIL}$ if x is the last root in the root list.)

A given binomial heap H is accessed by the field $head[H]$, which is simply a pointer to the first root in the root list of H. If binomial heap H has no elements, then $head[H] = \text{NIL}$.

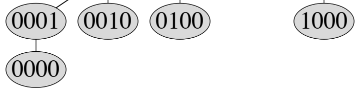


Figure 19.4 The binomial tree B_4 with nodes labeled in binary by a postorder walk.

Exercises

19.1-1

Suppose that x is a node in a binomial tree within a binomial heap, and assume that $sibling[x] \neq \text{NIL}$. If x is not a root, how does $degree[sibling[x]]$ compare to $degree[x]$? How about if x is a root?

19.1-2

If x is a nonroot node in a binomial tree within a binomial heap, how does $degree[x]$ compare to $degree[p[x]]$?

19.1-3

Suppose we label the nodes of binomial tree B_k in binary by a postorder walk, as in Figure 19.4. Consider a node x labeled l at depth i, and let $j = k - i$. Show that x has j 1's in its binary representation. How many binary k-strings are there that contain exactly j 1's? Show that the degree of x is equal to the number of 1's to the right of the rightmost 0 in the binary representation of l.

19.2 Operations on binomial heaps

In this section, we show how to perform operations on binomial heaps in the time bounds shown in Figure 19.1. We shall only show the upper bounds; the lower bounds are left as Exercise 19.2-10.

Creating a new binomial heap

To make an empty binomial heap, the MAKE-BINOMIAL-HEAP procedure simply allocates and returns an object H, where $head[H] = \text{NIL}$. The running time is $\Theta(1)$.

there are no keys with value ∞. (See Exercise 19.2-5.)

```
BINOMIAL-HEAP-MINIMUM($H$)
y ← NIL
x ← head[H]
min ← ∞
while x ≠ NIL
      do if key[x] < min
            then min ← key[x]
                 y ← x
         x ← sibling[x]
return y
```

Since a binomial heap is min-heap-ordered, the minimum key must reside in a root node. The BINOMIAL-HEAP-MINIMUM procedure checks all roots, which number at most $\lfloor \lg n \rfloor + 1$, saving the current minimum in *min* and a pointer to the current minimum in y. When called on the binomial heap of Figure 19.3, BINOMIAL-HEAP-MINIMUM returns a pointer to the node with key 1.

Because there are at most $\lfloor \lg n \rfloor + 1$ roots to check, the running time of BINOMIAL-HEAP-MINIMUM is $O(\lg n)$.

Uniting two binomial heaps

The operation of uniting two binomial heaps is used as a subroutine by most of the remaining operations. The BINOMIAL-HEAP-UNION procedure repeatedly links binomial trees whose roots have the same degree. The following procedure links the B_{k-1} tree rooted at node y to the B_{k-1} tree rooted at node z; that is, it makes z the parent of y. Node z thus becomes the root of a B_k tree.

```
BINOMIAL-LINK(y, z)
p[y] ← z
sibling[y] ← child[z]
child[z] ← y
degree[z] ← degree[z] + 1
```

The BINOMIAL-LINK procedure makes node y the new head of the linked list of node z's children in $O(1)$ time. It works because the left-child, right-sibling representation of each binomial tree matches the ordering property of the tree: in a B_k tree, the leftmost child of the root is the root of a B_{k-1} tree.

is sorted by degree into monotonically increasing order. The BINOMIAL-HEAP-MERGE procedure, whose pseudocode we leave as Exercise 19.2-1, is similar to the MERGE procedure in Section 2.3.1.

```
BINOMIAL-HEAP-UNION($H_1$, $H_2$)
 1  $H \leftarrow$ MAKE-BINOMIAL-HEAP()
 2  $head[H] \leftarrow$ BINOMIAL-HEAP-MERGE($H_1$, $H_2$)
 3  free the objects $H_1$ and $H_2$ but not the lists they point to
 4  if $head[H] =$ NIL
 5     then return $H$
 6  $prev\text{-}x \leftarrow$ NIL
 7  $x \leftarrow head[H]$
 8  $next\text{-}x \leftarrow sibling[x]$
 9  while $next\text{-}x \neq$ NIL
10      do if ($degree[x] \neq degree[next\text{-}x]$) or
                ($sibling[next\text{-}x] \neq$ NIL and $degree[sibling[next\text{-}x]] = degree[x]$)
11            then $prev\text{-}x \leftarrow x$                           ▷ Cases 1 and 2
12                 $x \leftarrow next\text{-}x$                            ▷ Cases 1 and 2
13            else if $key[x] \leq key[next\text{-}x]$
14                    then $sibling[x] \leftarrow sibling[next\text{-}x]$      ▷ Case 3
15                         BINOMIAL-LINK($next\text{-}x$, $x$)            ▷ Case 3
16                    else if $prev\text{-}x =$ NIL                   ▷ Case 4
17                            then $head[H] \leftarrow next\text{-}x$        ▷ Case 4
18                            else $sibling[prev\text{-}x] \leftarrow next\text{-}x$  ▷ Case 4
19                         BINOMIAL-LINK($x$, $next\text{-}x$)            ▷ Case 4
20                         $x \leftarrow next\text{-}x$                     ▷ Case 4
21         $next\text{-}x \leftarrow sibling[x]$
22  return $H$
```

Figure 19.5 shows an example of BINOMIAL-HEAP-UNION in which all four cases given in the pseudocode occur.

The BINOMIAL-HEAP-UNION procedure has two phases. The first phase, performed by the call of BINOMIAL-HEAP-MERGE, merges the root lists of binomial heaps H_1 and H_2 into a single linked list H that is sorted by degree into monotonically increasing order. There might be as many as two roots (but no more) of each degree, however, so the second phase links roots of equal degree until at most one root remains of each degree. Because the linked list H is sorted by degree, we can perform all the link operations quickly.

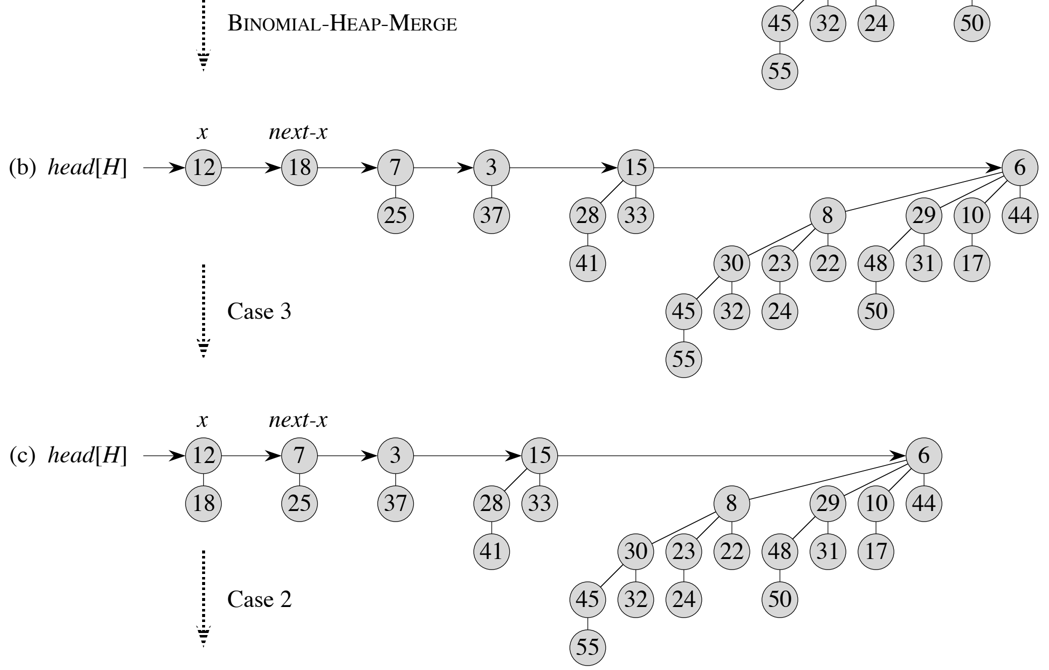
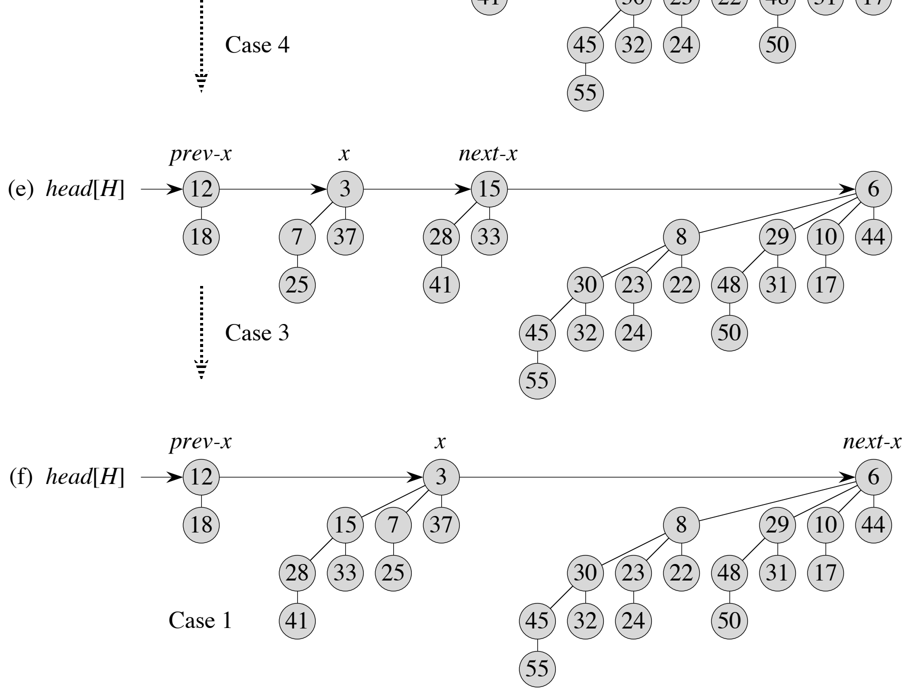


Figure 19.5 The execution of BINOMIAL-HEAP-UNION. **(a)** Binomial heaps H_1 and H_2. **(b)** Binomial heap H is the output of BINOMIAL-HEAP-MERGE(H_1, H_2). Initially, x is the first root on the root list of H. Because both x and *next-x* have degree 0 and $key[x] < key[next\text{-}x]$, case 3 applies. **(c)** After the link occurs, x is the first of three roots with the same degree, so case 2 applies. **(d)** After all the pointers move down one position in the root list, case 4 applies, since x is the first of two roots of equal degree. **(e)** After the link occurs, case 3 applies. **(f)** After another link, case 1 applies, because x has degree 3 and *next-x* has degree 4. This iteration of the **while** loop is the last, because after the pointers move down one position in the root list, $next\text{-}x = \text{NIL}$.

In detail, the procedure works as follows. Lines 1–3 start by merging the root lists of binomial heaps H_1 and H_2 into a single root list H. The root lists of H_1 and H_2 are sorted by strictly increasing degree, and BINOMIAL-HEAP-MERGE returns a root list H that is sorted by monotonically increasing degree. If the root lists of H_1 and H_2 have m roots altogether, BINOMIAL-HEAP-MERGE runs in $O(m)$ time by repeatedly examining the roots at the heads of the two root lists and appending the root with the lower degree to the output root list, removing it from its input root list in the process.



The BINOMIAL-HEAP-UNION procedure next initializes some pointers into the root list of H. First, it simply returns in lines 4–5 if it happens to be uniting two empty binomial heaps. From line 6 on, therefore, we know that H has at least one root. Throughout the procedure, we maintain three pointers into the root list:

- x points to the root currently being examined,
- *prev-x* points to the root preceding x on the root list: $sibling[prev\text{-}x] = x$ (since initially x has no predecessor, we start with *prev-x* set to NIL), and
- *next-x* points to the root following x on the root list: $sibling[x] = next\text{-}x$.

Initially, there are at most two roots on the root list H of a given degree: because H_1 and H_2 were binomial heaps, they each had at most one root of a given degree. Moreover, BINOMIAL-HEAP-MERGE guarantees us that if two roots in H have the same degree, they are adjacent in the root list.

In fact, during the execution of BINOMIAL-HEAP-UNION, there may be three roots of a given degree appearing on the root list H at some time. We shall see for a precise loop invariant.)

Case 1, shown in Figure 19.6(a), occurs when $degree[x] \neq degree[next\text{-}x]$, that is, when x is the root of a B_k-tree and $next\text{-}x$ is the root of a B_l-tree for some $l > k$. Lines 11–12 handle this case. We don't link x and $next\text{-}x$, so we simply march the pointers one position farther down the list. Updating $next\text{-}x$ to point to the node following the new node x is handled in line 21, which is common to every case.

Case 2, shown in Figure 19.6(b), occurs when x is the first of three roots of equal degree, that is, when

$$degree[x] = degree[next\text{-}x] = degree[sibling[next\text{-}x]] .$$

We handle this case in the same manner as case 1: we just march the pointers one position farther down the list. The next iteration will execute either case 3 or case 4 to combine the second and third of the three equal-degree roots. Line 10 tests for both cases 1 and 2, and lines 11–12 handle both cases.

Cases 3 and 4 occur when x is the first of two roots of equal degree, that is, when

$$degree[x] = degree[next\text{-}x] \neq degree[sibling[next\text{-}x]] .$$

These cases may occur in any iteration, but one of them always occurs immediately following case 2. In cases 3 and 4, we link x and $next\text{-}x$. The two cases are distinguished by whether x or $next\text{-}x$ has the smaller key, which determines the node that will be the root after the two are linked.

In case 3, shown in Figure 19.6(c), $key[x] \leq key[next\text{-}x]$, so $next\text{-}x$ is linked to x. Line 14 removes $next\text{-}x$ from the root list, and line 15 makes $next\text{-}x$ the leftmost child of x.

In case 4, shown in Figure 19.6(d), $next\text{-}x$ has the smaller key, so x is linked to $next\text{-}x$. Lines 16–18 remove x from the root list; there are two cases depending on whether x is the first root on the list (line 17) or is not (line 18). Line 19 then makes x the leftmost child of $next\text{-}x$, and line 20 updates x for the next iteration.

Following either case 3 or case 4, the setup for the next iteration of the **while** loop is the same. We have just linked two B_k-trees to form a B_{k+1}-tree, which x now points to. There were already zero, one, or two other B_{k+1}-trees on the root list resulting from BINOMIAL-HEAP-MERGE, so x is now the first of either one, two, or three B_{k+1}-trees on the root list. If x is the only one, then we enter case 1 in the next iteration: $degree[x] \neq degree[next\text{-}x]$. If x is the first of two, then we enter either case 3 or case 4 in the next iteration. It is when x is the first of three that we enter case 2 in the next iteration.

The running time of BINOMIAL-HEAP-UNION is $O(\lg n)$, where n is the total number of nodes in binomial heaps H_1 and H_2. We can see this as follows. Let H_1

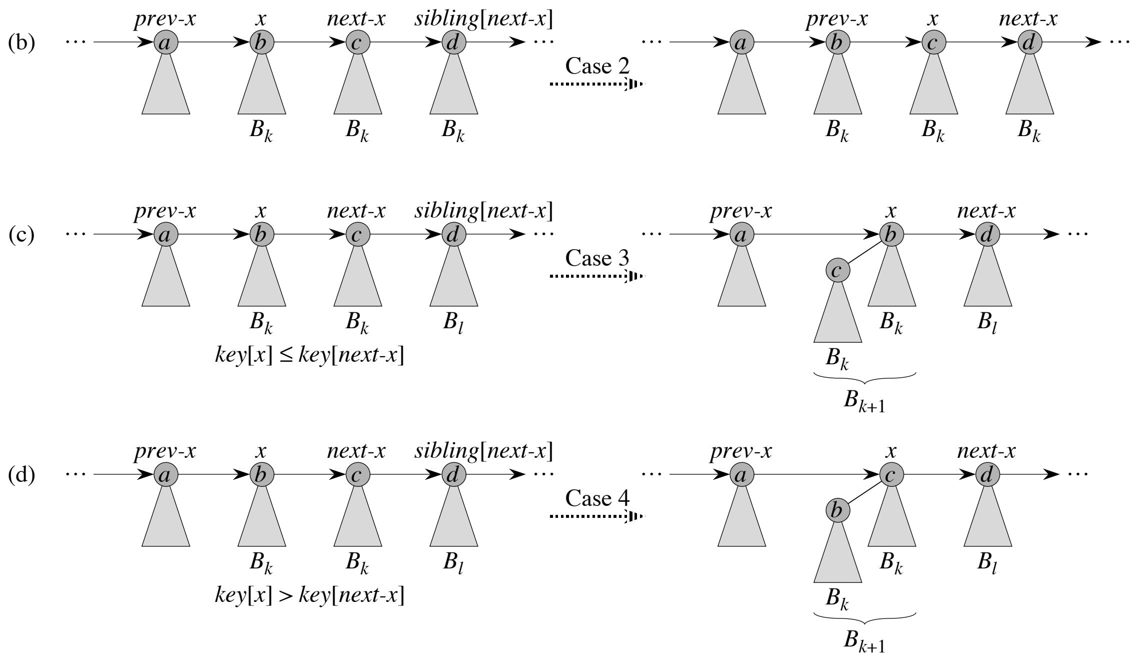


Figure 19.6 The four cases that occur in BINOMIAL-HEAP-UNION. Labels a, b, c, and d serve only to identify the roots involved; they do not indicate the degrees or keys of these roots. In each case, x is the root of a B_k-tree and $l > k$. **(a)** Case 1: $degree[x] \neq degree[next\text{-}x]$. The pointers move one position farther down the root list. **(b)** Case 2: $degree[x] = degree[next\text{-}x] = degree[sibling[next\text{-}x]]$. Again, the pointers move one position farther down the list, and the next iteration executes either case 3 or case 4. **(c)** Case 3: $degree[x] = degree[next\text{-}x] \neq degree[sibling[next\text{-}x]]$ and $key[x] \le key[next\text{-}x]$. We remove *next-x* from the root list and link it to x, creating a B_{k+1}-tree. **(d)** Case 4: $degree[x] = degree[next\text{-}x] \neq degree[sibling[next\text{-}x]]$ and $key[next\text{-}x] \le key[x]$. We remove x from the root list and link it to *next-x*, again creating a B_{k+1}-tree.

contain n_1 nodes and H_2 contain n_2 nodes, so that $n = n_1 + n_2$. Then H_1 contains at most $\lfloor \lg n_1 \rfloor + 1$ roots and H_2 contains at most $\lfloor \lg n_2 \rfloor + 1$ roots, and so H contains at most $\lfloor \lg n_1 \rfloor + \lfloor \lg n_2 \rfloor + 2 \le 2 \lfloor \lg n \rfloor + 2 = O(\lg n)$ roots immediately after the call of BINOMIAL-HEAP-MERGE. The time to perform BINOMIAL-HEAP-MERGE is thus $O(\lg n)$. Each iteration of the **while** loop takes $O(1)$ time, and there are at most $\lfloor \lg n_1 \rfloor + \lfloor \lg n_2 \rfloor + 2$ iterations because each iteration either advances the

The following procedure inserts node x into binomial heap H, assuming that x has already been allocated and *key*[x] has already been filled in.

```
BINOMIAL-HEAP-INSERT(H, x)
1  H' ← MAKE-BINOMIAL-HEAP()
2  p[x] ← NIL
3  child[x] ← NIL
4  sibling[x] ← NIL
5  degree[x] ← 0
6  head[H'] ← x
7  H ← BINOMIAL-HEAP-UNION(H, H')
```

The procedure simply makes a one-node binomial heap H' in $O(1)$ time and unites it with the n-node binomial heap H in $O(\lg n)$ time. The call to BINOMIAL-HEAP-UNION takes care of freeing the temporary binomial heap H'. (A direct implementation that does not call BINOMIAL-HEAP-UNION is given as Exercise 19.2-8.)

Extracting the node with minimum key

The following procedure extracts the node with the minimum key from binomial heap H and returns a pointer to the extracted node.

```
BINOMIAL-HEAP-EXTRACT-MIN(H)
1  find the root x with the minimum key in the root list of H,
           and remove x from the root list of H
2  H' ← MAKE-BINOMIAL-HEAP()
3  reverse the order of the linked list of x's children,
           and set head[H'] to point to the head of the resulting list
4  H ← BINOMIAL-HEAP-UNION(H, H')
5  return x
```

This procedure works as shown in Figure 19.7. The input binomial heap H is shown in Figure 19.7(a). Figure 19.7(b) shows the situation after line 1: the root x with the minimum key has been removed from the root list of H. If x is the root of a B_k-tree, then by property 4 of Lemma 19.1, x's children, from left to right, are roots of B_{k-1}-, B_{k-2}-, $\ldots$, B_0-trees. Figure 19.7(c) shows that by reversing the list of x's children in line 3, we have a binomial heap H' that contains every node

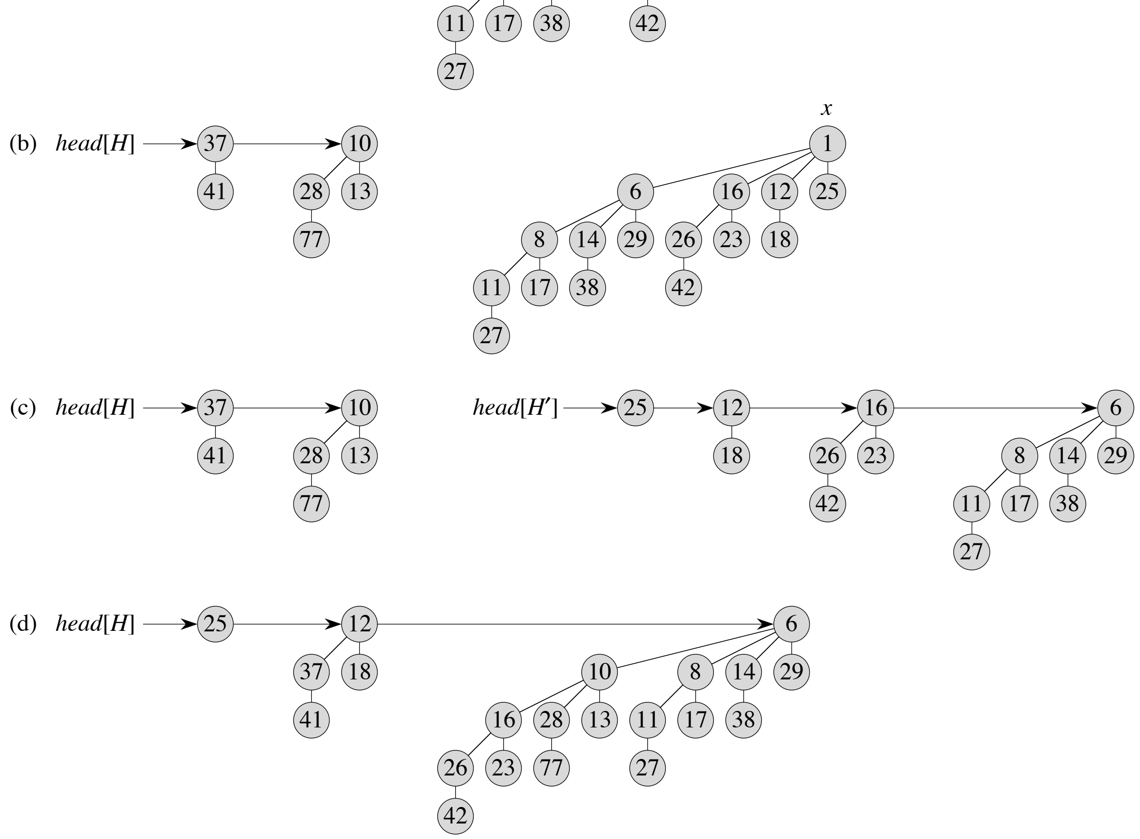


Figure 19.7 The action of BINOMIAL-HEAP-EXTRACT-MIN. **(a)** A binomial heap H. **(b)** The root x with minimum key is removed from the root list of H. **(c)** The linked list of x's children is reversed, giving another binomial heap H'. **(d)** The result of uniting H and H'.

in x's tree except for x itself. Because x's tree was removed from H in line 1, the binomial heap that results from uniting H and H' in line 4, shown in Figure 19.7(d), contains all the nodes originally in H except for x. Finally, line 5 returns x.

Since each of lines 1–4 takes $O(\lg n)$ time if H has n nodes, BINOMIAL-HEAP-EXTRACT-MIN runs in $O(\lg n)$ time.

```
BINOMIAL-HEAP-DECREASE-KEY(H, x, k)
 1  if $k > key[x]$
 2     then error "new key is greater than current key"
 3  $key[x] \leftarrow k$
 4  $y \leftarrow x$
 5  $z \leftarrow p[y]$
 6  while $z \neq$ NIL and $key[y] < key[z]$
 7       do exchange $key[y] \leftrightarrow key[z]$
 8          ▷ If y and z have satellite fields, exchange them, too.
 9          $y \leftarrow z$
10          $z \leftarrow p[y]$
```

As shown in Figure 19.8, this procedure decreases a key in the same manner as in a binary min-heap: by "bubbling up" the key in the heap. After ensuring that the new key is in fact no greater than the current key and then assigning the new key to x, the procedure goes up the tree, with y initially pointing to node x. In each iteration of the **while** loop of lines 6–10, $key[y]$ is checked against the key of y's parent z. If y is the root or $key[y] \geq key[z]$, the binomial tree is now min-heap-ordered. Otherwise, node y violates min-heap ordering, and so its key is exchanged with the key of its parent z, along with any other satellite information. The procedure then sets y to z, going up one level in the tree, and continues with the next iteration.

The BINOMIAL-HEAP-DECREASE-KEY procedure takes $O(\lg n)$ time. By property 2 of Lemma 19.1, the maximum depth of x is $\lfloor \lg n \rfloor$, so the **while** loop of lines 6–10 iterates at most $\lfloor \lg n \rfloor$ times.

Deleting a key

It is easy to delete a node x's key and satellite information from binomial heap H in $O(\lg n)$ time. The following implementation assumes that no node currently in the binomial heap has a key of $-\infty$.

```
BINOMIAL-HEAP-DELETE(H, x)
1  BINOMIAL-HEAP-DECREASE-KEY(H, x, $-\infty$)
2  BINOMIAL-HEAP-EXTRACT-MIN(H)
```

The BINOMIAL-HEAP-DELETE procedure makes node x have the unique minimum key in the entire binomial heap by giving it a key of $-\infty$. (Exercise 19.2-6

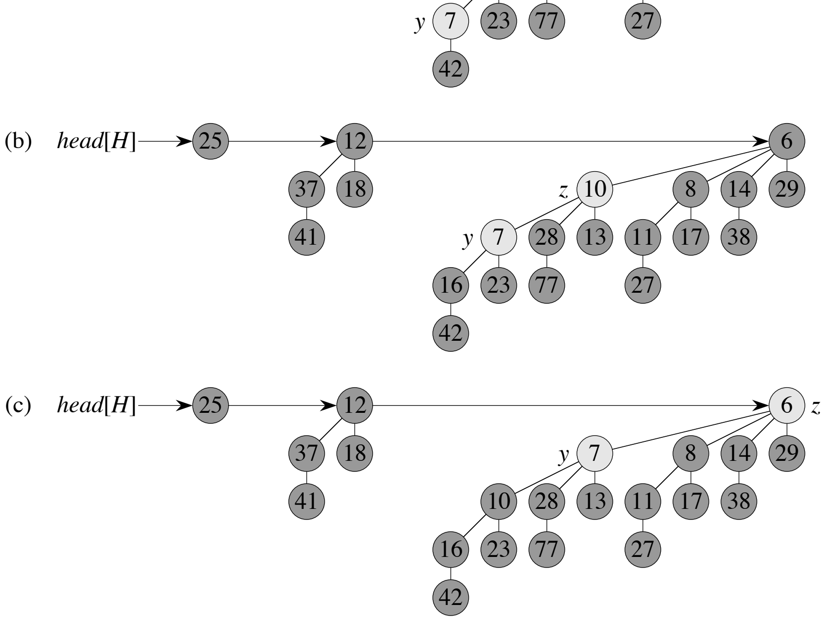


Figure 19.8 The action of BINOMIAL-HEAP-DECREASE-KEY. **(a)** The situation just before line 6 of the first iteration of the **while** loop. Node y has had its key decreased to 7, which is less than the key of y's parent z. **(b)** The keys of the two nodes are exchanged, and the situation just before line 6 of the second iteration is shown. Pointers y and z have moved up one level in the tree, but min-heap order is still violated. **(c)** After another exchange and moving pointers y and z up one more level, we find that min-heap order is satisfied, so the **while** loop terminates.

deals with the situation in which $-\infty$ cannot appear as a key, even temporarily.) It then bubbles this key and the associated satellite information up to a root by calling BINOMIAL-HEAP-DECREASE-KEY. This root is then removed from H by a call of BINOMIAL-HEAP-EXTRACT-MIN.

The BINOMIAL-HEAP-DELETE procedure takes $O(\lg n)$ time.

Exercises

19.2-1

Write pseudocode for BINOMIAL-HEAP-MERGE.

19.2-3
Show the binomial heap that results when the node with key 28 is deleted from the binomial heap shown in Figure 19.8(c).

19.2-4
Argue the correctness of BINOMIAL-HEAP-UNION using the following loop invariant:

> At the start of each iteration of the **while** loop of lines 9–21, x points to a root that is one of the following:
>
> - the only root of its degree,
> - the first of the only two roots of its degree, or
> - the first or second of the only three roots of its degree.
>
> Moreover, all roots preceding x's predecessor on the root list have unique degrees on the root list, and if x's predecessor has a degree different from that of x, its degree on the root list is unique, too. Finally, node degrees monotonically increase as we traverse the root list.

19.2-5
Explain why the BINOMIAL-HEAP-MINIMUM procedure might not work correctly if keys can have the value ∞. Rewrite the pseudocode to make it work correctly in such cases.

19.2-6
Suppose there is no way to represent the key $-\infty$. Rewrite the BINOMIAL-HEAP-DELETE procedure to work correctly in this situation. It should still take $O(\lg n)$ time.

19.2-7
Discuss the relationship between inserting into a binomial heap and incrementing a binary number and the relationship between uniting two binomial heaps and adding two binary numbers.

19.2-8
In light of Exercise 19.2-7, rewrite BINOMIAL-HEAP-INSERT to insert a node directly into a binomial heap without calling BINOMIAL-HEAP-UNION.

19.2-10

Find inputs that cause BINOMIAL-HEAP-EXTRACT-MIN, BINOMIAL-HEAP-DECREASE-KEY, and BINOMIAL-HEAP-DELETE to run in $\Omega(\lg n)$ time. Explain why the worst-case running times of BINOMIAL-HEAP-INSERT, BINOMIAL-HEAP-MINIMUM, and BINOMIAL-HEAP-UNION are $\overset{\infty}{\Omega}(\lg n)$ but not $\Omega(\lg n)$. (See Problem 3-5.)

Problems

19-1 2-3-4 heaps

Chapter 18 introduced the 2-3-4 tree, in which every internal node (other than possibly the root) has two, three, or four children and all leaves have the same depth. In this problem, we shall implement ***2-3-4 heaps***, which support the mergeable-heap operations.

The 2-3-4 heaps differ from 2-3-4 trees in the following ways. In 2-3-4 heaps, only leaves store keys, and each leaf x stores exactly one key in the field $key[x]$. There is no particular ordering of the keys in the leaves; that is, from left to right, the keys may be in any order. Each internal node x contains a value $small[x]$ that is equal to the smallest key stored in any leaf in the subtree rooted at x. The root r contains a field $height[r]$ that is the height of the tree. Finally, 2-3-4 heaps are intended to be kept in main memory, so that disk reads and writes are not needed.

Implement the following 2-3-4 heap operations. Each of the operations in parts (a)–(e) should run in $O(\lg n)$ time on a 2-3-4 heap with n elements. The UNION operation in part (f) should run in $O(\lg n)$ time, where n is the number of elements in the two input heaps.

a. MINIMUM, which returns a pointer to the leaf with the smallest key.

b. DECREASE-KEY, which decreases the key of a given leaf x to a given value $k \le key[x]$.

c. INSERT, which inserts leaf x with key k.

d. DELETE, which deletes a given leaf x.

e. EXTRACT-MIN, which extracts the leaf with the smallest key.

Chapter 23 presents two algorithms to solve the problem of finding a minimum spanning tree of an undirected graph. Here, we shall see how binomial heaps can be used to devise a different minimum-spanning-tree algorithm.

We are given a connected, undirected graph $G = (V, E)$ with a weight function $w : E \to \mathbf{R}$. We call $w(u, v)$ the weight of edge (u, v). We wish to find a minimum spanning tree for G: an acyclic subset $T \subseteq E$ that connects all the vertices in V and whose total weight

$$w(T) = \sum_{(u,v)\in T} w(u, v)$$

is minimized.

The following pseudocode, which can be proven correct using techniques from Section 23.1, constructs a minimum spanning tree T. It maintains a partition $\{V_i\}$ of the vertices of V and, with each set V_i, a set

$$E_i \subseteq \{(u, v) : u \in V_i \text{ or } v \in V_i\}$$

of edges incident on vertices in V_i.

```
MST(G)
 1  $T \leftarrow \emptyset$
 2  for each vertex $v_i \in V[G]$
 3       do $V_i \leftarrow \{v_i\}$
 4          $E_i \leftarrow \{(v_i, v) \in E[G]\}$
 5  while there is more than one set $V_i$
 6       do choose any set $V_i$
 7          extract the minimum-weight edge $(u, v)$ from $E_i$
 8          assume without loss of generality that $u \in V_i$ and $v \in V_j$
 9          if $i \neq j$
10            then $T \leftarrow T \cup \{(u, v)\}$
11                 $V_i \leftarrow V_i \cup V_j$, destroying $V_j$
12                 $E_i \leftarrow E_i \cup E_j$
```

Describe how to implement this algorithm using binomial heaps to manage the vertex and edge sets. Do you need to change the representation of a binomial heap? Do you need to add operations beyond the mergeable-heap operations given in Figure 19.1? Give the running time of your implementation.

ied their properties in detail.