a Since we are only interested in the sum, Let

$$\Omega = \{x \in \mathbb{Z} : 2 \le x \le 12\}$$

b Let X be a discrete random variable associated with the sum of two dices in one toss. When $X = 5$, they are 4 possible outcomes.

$$\{(1,4),(2,3),(3,2),(4,1)\}$$

The total number of outcome for rolling two dices is $6 \times 6 = 36$. Therefore

$$P(X = 5) = \frac{4}{36} = \frac{1}{9}$$

c When one of the dice is a 4, there are following possibilities of dice toss.

$$(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(1,4),(2,4),(3,4),(4,4),(5,4),(6,4)$$

We can see that only $(4,1)$ and $(1,4)$ has a sum of 5 and that one of the dice toss yield a 4. Therefore

$$P(X = 5|\text{one of the dice is a 4}) = \frac{P(X = 5 \cap \text{one of the dice is a 4})}{P(\text{one of the dice is a 4})} \quad = \frac{\frac{2}{36}}{\frac{12}{36}} = \frac{1}{6}$$

d

$$P(\text{one of the dice is a 4}|X = 5) = \frac{P(\text{one of the dice is a 4} \cap X = 5)}{P(X = 5)} \quad = \frac{\frac{2}{36}}{\frac{1}{9}} = \frac{1}{2}$$

e To find $P(X \ge 5)$. We find $P(X \le 4)$

$$\begin{aligned}
P(X \ge 5) &= 1 - P(X \le 4) \\
&= 1 - (P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)) \\
&= 1 - (0 + \frac{1}{36} + \frac{2}{36} + \frac{3}{36}) \\
&= \frac{5}{6}
\end{aligned}$$

This is because the dice pairs that satisfy $X = 1, 2, 3, 4$ are

$$\begin{aligned}
&\text{Not possible} && \text{(X=1)} \\
&(1,1) && \text{(X=2)} \\
&1,2,(2,1) && \text{(X=3)} \\
&(1,3),(2,2),(3,1) && \text{(X=4)}
\end{aligned}$$

a

$$
\begin{aligned}
P(C \cap A^c | B) &= \frac{P(C \cap A^c \cap B)}{P(B)} \\
&= \frac{P(B \cap C) - P(A \cap B \cap C)}{P(B)} \\
&= \frac{P(B \cap C)}{P(B)} - \frac{P(A \cap B \cap C)}{P(B)} \\
&= P(C|B) - P(A \cap C|B) \\
&= 0.5 - 0.1 \\
&= 0.4
\end{aligned}
$$

b

$$
\begin{aligned}
&P\Big((A \cap C^c) \cup (A^c \cap C) | B\Big) \\
&\qquad = \frac{P\Big(\big((A \cap C^c) \cup (A^c \cap C)\big) \cap B\Big)}{P(B)} \\
&\qquad = \frac{P\Big((A \cap B \cap C^c) \cup (A^c \cap B \cap C)\Big)}{P(B)} \\
&\qquad = \frac{P(A \cap B \cap C^c) + P(A^c \cap B \cap C)}{P(B)} && \text{(two sets are disjoint)} \\
&\qquad = \frac{P(A \cap B) - P(A \cap B \cap C)}{P(B)} + 0.4 && \Big(\frac{P(C \cap A^c \cap B)}{P(B)} = 0.4 \text{ from } a\Big) \\
&\qquad = P(A|B) - P(A \cap C|B) + 0.4 \\
&\qquad = 0.25 - 0.1 + 0.4 \\
&\qquad = 0.55
\end{aligned}
$$

c

$$
\begin{aligned}
P(A \cup C | B) &= \frac{P((A \cup C) \cap B)}{P(B)} \\
&= \frac{P((A \cap B) \cup (B \cap C))}{P(B)} \\
&= \frac{P(A \cap B) + P(B \cap C) - P(A \cap B \cap C)}{P(B)} \\
&= P(A|B) + P(C|B) - P(A \cap C|B) \\
&= 0.25 + 0.5 - 0.1 \\
&= 0.65
\end{aligned}
$$

Event associated with randomly selecting 4 shoes from 16 shoes without replacement and order does not matter has $\binom{16}{4}$ possible outcomes. Event associated with having exactly 1 complete pair has $\binom{8}{1}\binom{14}{2}$ outcomes. Therefore probability of having exactly 1 pair by selecting randomly 4 shoes from 8 pairs of shoes is

$$P = \frac{\binom{8}{1}\binom{14}{2}}{\binom{16}{4}} = 0.4$$