Chapter 4 Determinants

4.1 Determinants of Order 2

Definition. ***Determinant*** *If A is $2 \times n$ matrix with entries from a field F, then we define the determinant of A, denoted $det\left(A\right)$ or $|A|$, to be the scalar $ad - bc$*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Determinant $M_{2\times n}(F) \to F$ is not a linear transformation since $det\left(A+B\right) \neq det\left(A\right) + det\left(B\right)$

Theorem. ***4.1 Determinant is Linear in Each Row*** *The function $\det : M_{2\times n}(F) \to F$ is a linear function of each row of a $2 \times n$ matrix when the other row is held fixed. That is, if u, v and w are in F^2 and k is a scalar, then*

$$det\begin{pmatrix} u+kv \\ w \end{pmatrix} = det\begin{pmatrix} u \\ w \end{pmatrix} + k det\begin{pmatrix} v \\ w \end{pmatrix} \qquad det\begin{pmatrix} w \\ u+kv \end{pmatrix} = det\begin{pmatrix} w \\ u \end{pmatrix} + k det\begin{pmatrix} w \\ v \end{pmatrix}$$

Theorem. ***4.2 Nonzero Determinant Implies Invertibility*** *Let $A \in M_{n\times n}(F)$. Then the determinant of A is nonzero if and only if A is invertible. Moreover, if A is invertible, then*

$$A^{-1} = \frac{1}{det\left(A\right)} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

4.2 Determinants of Order n

Definition. ***Matrix for Computing Minors*** *Let $A \in M_{n\times n}(F)$, for $n \geq 2$, denote $(n-1) \times (n-1)$ matrix obtained from A by deleting row i and column j by $\tilde{A}_{ij}$*

Definition. ***Determinants*** *Let $A \in M_{n\times n}(F)$. If $n = 1$, $A = (A_{11})$, define $det\left(A\right) = A_{11}$. For $n \geq 2$, we define $det\left(A\right)$ recursively as*

$$det\left(A\right) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot det\left(\tilde{A}_{1j}\right)$$

The scalar $det\left(A\right)$ is called the determinant of A and is also denoted by $|A|$. The scalar

$$c_{ij} = (-1)^{i+j} det\left(\tilde{A}_{ij}\right)$$

is called the ***cofactor*** *of the entry of A in row i, and column j. So determinant can be expressed as a* ***cofactor expansion along the first row*** *of A*

$$det\left(A\right) = A_{11}c_{11} + A_{12}c_{12} + \cdots + A_{1n}c_{1n}$$

Definition. ***Determinant for Identity*** *Determinant of $n \times n$ identity matrix is 1*

Theorem. ***4.3 Determinant is Linear in Each Row*** *The determinant of an $n \times n$ matrix is a linear function of each row when the remaining rows are held fixed.That is, for $1 \leq r \leq n$, we have*

$$det \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u + kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = det \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k det \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

where $k \in F$, and $u, v, a_i \in F^n$ are row vectors

Corollary. *If $A \in M_{n\times n}(F)$ has a row consisting entirely of zeros, then $det\left(A\right) = 0$*

Lemma. *Let $B \in M_{n\times n}(F)$, where $n \geq 2$. If row i of B equals e_k for some k $(1 \leq k \leq n)$, then*

$$det\left(B\right) = (-1)^{i+k} det\left(\tilde{B}_{ik}\right)$$

Theorem. ***4.4 Cofactor Expansion Along Any Row Yields Determinant***
The determinants of a square matrix can be evaluated by cofactor expansion along any row. That is, if $A \in M_{n\times n}(F)$, then for any integer i $(1 \leq i \leq n)$,

$$det\left(A\right) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot det\left(\tilde{A}_{ij}\right)$$

Corollary. ***Same Rows Implies Zero Determinant*** *If $A \in M_{n\times n}(F)$ has two identical rows, then $det\left(A\right) = 0$*

Theorem. ***4.5 Switching Rows Yields Negative Determinant***
If $A \in M_{n\times n}(F)$ and B is a matrix obtained from AE by interchanging any two rows of A, then $det\left(B\right) = -det\left(A\right)$

Theorem. ***4.6 Adding a Multiple of One Row to Another Do Not Change Determinant***
Let $A \in M_{n\times n}(F)$, and let B be a matrix obtained by adding a multiple of one row of A to another row of A. Then $det\left(B\right) = det\left(A\right)$

Corollary. *If $A \in M_{n\times n}(F)$ has rank less than n, then $det\left(A\right) = 0$*

Definition. ***Simplifying Operations for Computing Determinants***
Applying elementary operation to A yields B

1. *Exchanging two rows of A, then $det\left(B\right) = -det\left(A\right)$. $(det\left(E\right) = -1)$*

2. *Multiply a row of A by a scalar, then* $det\left(B\right) = kdet\left(A\right)$. *(* $det\left(E\right) = k$ *)*

3. *Add a multiple of one row of A to another row of A, then* $det\left(B\right) = det\left(A\right)$. *(* $det\left(E\right) = 1$ *)*

4. $det\left(E^t\right) = det\left(E\right)$

Definition. ***Upper Triangular Matrix*** *The determinant of an upper triangular matrix is the product of its diagonal entries.*

Can use type 1,3 elementary operation to convert a matrix to upper triangular form, then compute determinants

4.3 Properties of Determinants

Theorem. ***4.7 Multiplication Rule for Determinants*** *For any* $A, B \in M_{n\times n}(F)$,

$$det\left(AB\right) = det\left(A\right) \cdot det\left(B\right)$$

Corollary. ***Determinant of Inverse Matrix*** *A matrix* $A \in M_{n\times n}(F)$ *is invertible if and only if* $det\left(A\right) \neq 0$. *Furthermore, if A is invertible, then* $det\left(A^{-1}\right) = \dfrac{1}{det\left(A\right)}$

Theorem. ***4.8 Matrix Transpose Have Equal Determinant*** *For any* $A \in M_{n\times n}(F)$,

$$det\left(A^t\right) = det\left(A\right)$$

Implies argument to rows can be applied to columns equally well

Theorem. ***4.9 Cramer's Rule***