Chapter 4 Random Walks and Markov Chains

Definition. ***Concepts***

1. ***Random Walk*** *A sequence of vertices generated from a start vertex by probabilitically selecting an incident edge, traversing the edge to a new vertex, and repeating the process.*

   (a) ***Strongly Connected*** *For any $x, y \in V$, the graph contains a path of directed edges starting at x and ending at y*

   (b) ***Probability distribution at time*** t

   $$\mathbf{p(t)}P = \mathbf{p(t+1)}$$

   where $\mathbf{p(t)}$ be a row vector with a component for each vertex specifiying the probability mass of the vertex at time t, i.e. the probability in state i at time t, and P be the ***transition matrix*** *with*

      i. *P_{ij} is the probability of the walk at vertex i selecting the edge to vertex j*

      ii. *$P_{ij} > 0$ with $\sum_j P_{ij} = 1$ for all i*

2. ***Markov Chain***

   (a) ***State (vertices)*** *A markov chain has finite set of states (vertices)*

   (b) ***Transition Probability (edge weights)*** *For each pair of state x and y, the transition probability p_{xy} is the probability of going from x to y, where for each x, $\sum_y p_{xy} = 1$*

   (c) ***Idea*** *Start at some state. At a given state, if it is in state x the next state y is selected randomly with probability p_{xy}*

   (d) ***Connected*** *A markov chain is connected if underlying graph is strongly connected*

   (e) ***Transition Probability Matrix*** *THer matrix P consisting of p_{xy}*

   (f) ***Persistent*** *Should a state be reached, the random process will return to it with probability one. Equivalent to say that state is in a strongly connected component with no out edges.*

   (g) ***Stationary Distribution*** *The long-term average probability, the average probability distribution of random walk over the first t steps, converges to a limiting distribution for connected chains*

4.1 Stationary Distribution

Definition. ***Long-term average probability distribution***

Let $\mathbf{p(t)}$ *be probability distribution after* t *steps of random walk, then*

$$\mathbf{a(t)} = \frac{1}{t}\left(\mathbf{p(0)} + \mathbf{p(1)} + \cdots + \mathbf{p(t-1)}\right)$$

Theorem. ***Fundamental theorem of Markov Chains*** *For a connected Markov chain there is a unique probability vector* $\boldsymbol{\pi}$ *satisfying* $\boldsymbol{\pi}P = \boldsymbol{\pi}$. *Moreover, for any starting distribution,* $\lim_{t\to\infty} \mathbf{a(t)}$ *exists and equals* $\boldsymbol{\pi}$

Lemma. *For a random walk on a strongly connected graph with probabilities on edges, if vector* $\boldsymbol{\pi}$ *satisfies* $\pi_x p_{xy} = \pi_y p_{yx}$ *for all* x *and* y *and* $\sum_x \pi_x = 1$, *then* $\boldsymbol{\pi}$ *is the stationary distribution of the walk*

4.2 Markov Chain Monte Carlo

Definition. ***Metropolis-Hasting Algorithm***

1. *For Markov Chain with a fixed stationary probability*
2. *Let* r *be maximum possible degree in the graph and* $\mathbf{p} = (p_1, \cdots)$ *be target distribution, then assign*

   $$p_{ij} = \frac{1}{r}\min(1, \frac{p_j}{p_i}) \qquad p_{ii} = 1 - \sum_{j\neq i} p_{ij}$$

   whereby the stationary probability is simply $\mathbf{p}$