Problem Set 7

You are strongly encouraged to solve the following exercises before next week's tutorial:

- Starting on page 362 (end of Chapter 9): 12 (note that the test is a GLRT and not an LRT as stated in the question), 13 (a-c), 23, 26 (a-e, use "greater than or equal to" instead of "less than or equal to" in (a)) and 28.
- Starting on page 459 (end of Chapter 11): 1 (d-g), 11 and 21 (a, also test the assumption that the variances of the two groups are equal at the 5% level).

Additional Exercise:

Let $X_1, \dots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$, and consider the problem of testing

$$\begin{cases} \mathcal{H}_0 : \sigma^2 = \sigma_0^2 \\ \mathcal{H}_1 : \sigma^2 \neq \sigma_0^2 \end{cases}$$

at level α.

(a) Calculate the generalized likelihood ratio statistic $\Lambda(\underline{X})$.

(b) Denote $\mathcal{X}^2 = \dfrac{1}{\sigma_0^2}\sum_{i=1}^{n}(X_i - \overline{X})^2$. Express $\Lambda(\underline{X})$ in terms of $\mathcal{X}^2$.

(c) Show that Λ, as a function of $\mathcal{X}^2$, has a single minimum. Conclude that

$\Lambda \geqslant c \iff \mathcal{X}^2 \leqslant c_1$ or $\mathcal{X}^2 \geqslant c_2$ for some constants c_1 and c_2.

(d) What is the distribution of $\mathcal{X}^2$ under $\mathcal{H}_0$? Find a test at level α.

Solution:

(a) The likelihood is of course

$$\mathcal{L}(\mu, \sigma^2) = \left(2\pi\sigma^2\right)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2\right\},$$

and under no constrains, the MLEs are $\widehat{\mu}=\overline{X}$ and $\widehat{\sigma}^2=\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^2$, hence

$$\mathcal{L}(\widehat{\mu},\widehat{\sigma}^2)=\left(2\pi\mathrm{e}\widehat{\sigma}^2\right)^{-n/2}.$$

Under $\mathcal{H}_0$, the MLE for μ remains unchanged, and obviously $\widehat{\sigma}_0^2=\sigma_0^2$, hence

$$\mathcal{L}(\widehat{\mu}_0,\widehat{\sigma}_0^2)=\left(2\pi\sigma_0^2\right)^{-n/2}\exp\left\{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(X_i-\overline{X})^2\right\}.$$

Finally,

$$\Lambda(\underline{X})=\frac{\mathcal{L}(\widehat{\mu},\widehat{\sigma}^2)}{\mathcal{L}(\widehat{\mu}_0,\widehat{\sigma}_0^2)}=\left(\frac{\sigma_0^2}{\mathrm{e}\widehat{\sigma}^2}\right)^{n/2}\exp\left\{\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(X_i-\overline{X})^2\right\}.$$

(b) Since $\widehat{\sigma}^2=\frac{\sigma_0^2\mathcal{X}^2}{n}$, we can rewrite

$$\Lambda=\left(\frac{n}{\mathcal{X}^2}\right)^{n/2}\exp\left\{\frac{\mathcal{X}^2-n}{2}\right\}.$$

(c) Write $\Lambda(t)=n^{n/2}t^{-n/2}\mathrm{e}^{\frac{t-n}{2}}$, then it is easy to verify that

$$\lim_{t\to 0}\Lambda(t)=\lim_{t\to\infty}\Lambda(t)=\infty \tag{1}$$

why is this: find the shape of distribution of Lambda

and that

$$\boxed{\Lambda'(t)=\frac{n^{n/2}}{2}t^{-\frac{n+1}{2}}(t-n)\mathrm{e}^{\frac{t-n}{2}}.}$$

The function then has a single extremum at $t=n$, and using (1) we know that its shape is as in Figure 1. From the figure it becomes clear that the rejection region of the GLRT

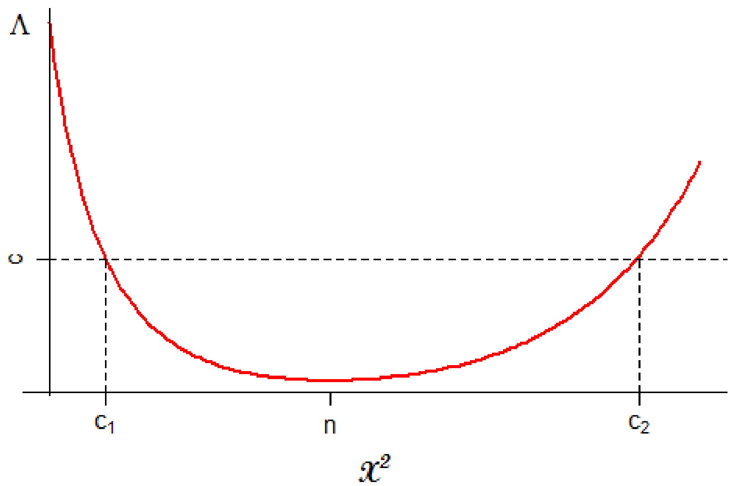

Figure 1: Λ as a function of $\mathcal{X}^2$.

is

$$\mathcal{C} = \left\{\Lambda(\underline{X}) \geqslant c\right\} = \left\{\mathcal{X}^2 \leqslant c_1\right\} \bigcup \left\{\mathcal{X}^2 \geqslant c_2\right\}.$$

(d) Clearly $\mathcal{X}^2 \overset{\mathcal{H}_0}{\sim} \chi^2_{n-1}$, hence a rejection region of a test at level α would be –

n-1 not n

$$\mathcal{C} = \left\{\mathcal{X}^2 \leqslant \chi^2_{n-1,\alpha/2}\right\} \bigcup \left\{\mathcal{X}^2 \geqslant \chi^2_{n-1,1-\alpha/2}\right\}.$$