UNIVERSITY OF TORONTO
Faculty of Arts and Science

DECEMBER 2015 EXAMINATIONS

STA247H1F

Duration- 2 hours

Student Number: ______________

Family Name:________________ **First Name:**________________

Instructions:

1. **Aids:** a non-programmable calculator.
2. There are 14 multiple choice questions and 4 long answer questions.
3. Submit your answers for the multiple choice questions to the answer sheet given in page 2.
4. Points for each question are indicated in parentheses. Total points: 45.
5. There are 18 pages including this page. The last 3 pages contain an empty page as scrap paper, the normal distribution table, and the formula sheet. You can rip them off if you want.

Answer Sheet

1.	a	b	c	d	e
2.	a	b	c	d	e
3.	a	b	c	d	e
4.	a	b	c	d	e
5.	a	b	c	d	e
6.	a	b	c	d	e
7.	a	b	c	d	e
8.	a	b	c	d	e
9.	a	b	c	d	e
10.	a	b	c	d	e
11.	a	b	c	d	e
12.	a	b	c	d	e
13.	a	b	c	d	e
14.	a	b	c	d	e

Professor's use only:

Part I		
Part II	1	
	2	
	3	
	4	
Total		

PART I (Multiple Choice Questions): Circle the correct answer. **Submit your answers to the answer sheet given in page 2.** Each correct answer will receive 2 points.

1. In how many ways can 10 people be seated in a row so that a certain pair of them are not next to each other?

   (a) 2903040 (b) 3628800 (c) 3265920 (d) 725760 (e) $P_{10,2}$

2. A person is known to speak truth 4 out of 5 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

   (a) 0.30 (b) 0.80 (c) 0.22 (d) 0.44 (e) 0.62

3. If a certain machine makes electrical resistors having a mean resistance of 40 ohms and a standard deviation of 2 ohms, what is the probability that a random sample of 36 of these resistors will have an average resistance of more than 40.5 ohms?

(a) 0.0228 (b) 0.4332 (c) 0.0343 (d) 0.4657 (e) 0.0668

4. If $f(x) = kx^2,\ \ 0 < x < 2$ is the probability density function (pdf) of a random variable X, then the 25^{th} percentile of X is

(a) 1.63 (b) 1.26 (c) 1.00 (d) 1.43 (e) 0.83

5. If X and Y are two random variables such that $E(X) = E(Y) = 0$, $Var(X) = 1$, $Var(Y) = 4$ and $Corr(X, Y) = \rho = -0.5$. Find $Var(2X - 3Y + 1)$.

(a) 46 (b) 26 (c) 52 (d) 37 (e) 28

6. A computer shop sells an electronic game powered by a single battery that has an exponential lifetime with a mean of 5 hours. I purchased 4 batteries for my game. Let T be the total playing time (the total life of my batteries). Find $E(T(T-1))$.

(a) 120 (b) 480 (c) 100 (d) 80 (e) 420

7. Consider the situation in problem #6. What is the probability that exactly two of my four batteries last longer than 5 hours?

(a) 0.3679 (b) 0.4656 (c) 0.3245 (d) 0.4323 (e) 0.2218

8. Suppose $X \sim Poisson(\lambda)$. If $E(X^2) = 6$, find $P(X \geq 2)$.

(a) 0.3528 (b) 0.3233 (c) 0.8009 (d) 0.5769 (e) 0.5940

9. It is known that 20% of the computer chips produced by a manufacturer are defective. The Faculty of Arts & Science has just purchased 36 new computers, each containing a chip produced by this manufacturer. Approximately what is the probability that at most 10 of the computers contain a defective chip?

(a) 0.9162 (b) 0.3737 (c) 0.7054 (d) 0.02946 (e) 0

10. Suppose that X is a normal random variable with mean 5. If $P(X > 9) = 0.025$, find $P(X < 4)$.

(a) 0.6711 (b) 0.3342 (c) 0.3121 (d) .3598 (e) 0.2268

11. The bus I take to work each morning leaves my stop at precisely 7:10. If the time that I arrive at the bus stop is uniformly distributed between 7:00 and 7:10 and my arrival times are independent from one day to the next, what is the probability that Friday is the first day this week that I have to wait more than 2 minutes? (The work week begins on Monday.)

(a) 0.00128 (b) 0.08192 (c) 0.0064 (d) 0.4096 (e) 0.8000

12. There are two roads from A to B and two roads from B to C. Each of the four roads has probability 0.25 of being blocked by snow, independently of all the others. What is the probability that there is an open road from A to C?

(a) 0.3475 (b) 0.1914 (c) 0.6545 (d) 0.5625 (e) 0.8789

13. The moment generating function of a random variable X is given by

$$M(t) = \frac{1}{1-t^2}, \quad \text{for } |t| < 1.$$

Find $Var(-0.5X + 3)$.

(a) 0 (b) 2 (c) 4 (d) 1 (e) 0.5

14. Suppose that X and Y are continuous random variables with joint probability density distribution function

$$f(x,y) = \begin{cases} e^{-x-y} & x \geq 0, y \geq 0 \\ 0 & \text{Otherwise} \end{cases}.$$

Find $P(X < 2Y)$.

(a) 0.50 (b) 0.33 (c) 0.60 (d) 0.67 (e) 0.40

PART II (Long Answer Questions): Solve the following questions.

1. The following table is the joint probability distribution of X and Y:

		x = 0	1	2
	0	1/9	2/9	1/9
y	1	2/9	2/9	0
	2	1/9	0	0

(a) Determine whether X and Y are independent. Justify your answer. **[1 points]**

(b) Find $E(X(Y-1))$. **[2 points]**

(c) Find $P(Y \leq 1 | X = 0)$. **[1 points]**

2. Suppose that X and Y are continuous random variables with joint probability density function (pdf)

$$f(x,y) = \begin{cases} \frac{1}{3}(x+y) & 0 \le x \le 1, 0 \le y \le 2 \\ 0 & \text{Otherwise} \end{cases}.$$

You are also given that: $E(X) = 5/9$, $E(Y) = 11/9$, $E(X^2) = 7/18$ and $E(Y^2) = 16/9$.

(a) Find the marginal pdf of Y. **[1 points]**

(b) Find $E(XY)$. **[1 points]**

(c) Find the correlation between X and Y. **Hint:** use the additional information given in the question. **[1 points]**

(d) Find $P(X \leq 0.75|Y = 0.5)$. **[2 points]**

3. Let X and Y be independent random variables, both with $Poisson(\lambda)$ distribution, for some $\lambda > 0$. Define $Z = X + Y$.

   (a) Find the distribution of Z. **Hint:** $M_X(t) = E(e^{tX}) = e^{\lambda(e^t - 1)}$. **[1 points]**

   (b) For any non-negative integer n, find the conditional probability mass function of X given $Z = n$. **[2 points]**

   (c) State the name of the conditional distribution of X given $Z = n$. **[1 points]**

4. A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Note: the transition probability matrix is (T: catches rain; C: drives car)

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cc} T & C \end{array} \\ \begin{array}{c} T \\ C \end{array} & \begin{pmatrix} 0 & 1 \\ 0.5 & 0.5 \end{pmatrix} \end{array}.$$

(a) Suppose that on the first day of the week (i.e. Monday), the man tossed a fair die and drove to work if and only if a 6 appeared. Find the probability that he takes a train on the third day (i.e. Wednesday). **[2 points]**

(b) Find the probability that he drives to work in the long run. **[2 points]**

Scratch Page

Formula Sheet

Distribution	Notation	pmf/pdf	Mean	Variance
Bernoulli	$X \sim Bernoulli(p)$	$p(x) = p^x(1-p)^{1-x},\ x = 0, 1$	p	$p(1-p)$
Binomial	$X \sim Binomial(n, p)$	$p(x) = \binom{n}{x}p^x(1-p)^{n-x},\ x = 0, 1, \ldots, n$	np	$np(1-p)$
Geometric	$X \sim Geometric(p)$	$p(x) = p(1-p)^{x-1},\ x = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	$X \sim Poisson(\lambda)$	$p(x) = \frac{\lambda^x e^{-\lambda}}{x!},\ x = 0, 1, \ldots$	λ	λ
Uniform	$X \sim U(a, b)$	$f(x) = \frac{1}{b-a},\ a \le x \le b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential	$X \sim Exponential(\lambda)$	$f(x) = \lambda e^{-\lambda x},\ x \ge 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma	$X \sim Gamma(\alpha, \lambda)$	$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x},\ x > 0$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$
Normal	$X \sim \mathcal{N}(\mu, \sigma^2)$	$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}e^{-(x-\mu)^2/2\sigma^2},\ -\infty < x < \infty$	μ	σ^2

$P_{n,r} = \frac{n!}{(n-r)!} = n \times (n-1) \cdots \times (n-r+1)$
$\binom{n}{r} = C_{n,r} = \frac{n!}{r!(n-r)!}$
$P(A') = 1 - P(A)$
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$
$P(A) = P(A \cap B) + P(A \cap B')$
$P(A|B) = \frac{P(A \cap B)}{P(B)}$
Bayes Theorem: $P(A_k|B) = \frac{P(B|A_k)P(A_k)}{P(B)} = \frac{P(B|A_k)P(A_k)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$
$Var(X) = E(X^2) - \mu^2$
$F(x) = P(X \le x)$
$(100p)^{th}$ percentile$=F^{-1}(p)$
If $X \sim Exponential(\lambda)$, then $F(x) = 1 - e^{-\lambda x}$
For $\alpha > 1$, $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$
$\Gamma(1/2) = \sqrt{\pi}$
If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$
Normal Approximation to the Binomial: if $X \sim Binomial(n, p)$, and if $np \ge 10$, $n(1-p) \ge 10$, then $X \sim \mathcal{N}(\mu = np, \sigma^2 = np(1-p))$ approximately.
$E(\overline{X}) = \mu_{\overline{X}} = \mu$
$Var(\overline{X}) = \sigma^2_{\overline{X}} = \frac{\sigma^2}{n}$
$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}$
Moment generating function: $M(t) = E(e^{tX})$
$Var(aX + b) = a^2Var(X)$
$Var(aX + bY + c) = a^2Var(X) + b^2Var(Y) + 2abCov(XY)$.
$Cov(X, Y) = E(XY) - E(X)E(Y)$
$Corr(X, Y) = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y}$
$Corr(aX + b, cY + d) = Corr(X, Y)$ if $ac > 0$
$Corr(aX + b, cY + d) = -Corr(X, Y)$ if $ac < 0$