Lemma. ***5: Farkas's Lemma*** *For any $m \times n$ matrix A and $m \times 1$ vector b, exactly one of following is true*

1. *There exists $x \in \mathbb{R}^n$ such that $x \geq 0$ and $Ax = b$*
2. *There exists $y \in \mathbb{R}^m$ such that $A^T y \leq 0$ and $b^T y > 0$*

Lemma. ***7: Farkas's Lemma variant*** *For any $m \times n$ matrix A and $m \times 1$ vector b, exactly one of following is true*

1. *There exists $x \in \mathbb{R}^n$ such that $x \geq 0$ and $Ax \leq b$*
2. *There exists $y \in \mathbb{R}^m$ such that $y \geq 0$, $A^T y \geq 0$ and $b^T y < 0$*

Question 1: LP Theory

1. use lemma 5 prove lemma 7

   Proof. Equivalent to proving (**7.1**) $\iff \neg$(**7.2**)

   (a) ($\Rightarrow$) Assume (**7.1**) true, 2 cases

   i. Exists $x \in \mathbb{R}^m$ such that $x \geq 0$ and $Ax = b$, then (**5.1**) is true and (**5.2**) is false. By contradiction, assume (**7.2**) is true, let $y' \in \mathbb{R}^m$ such that $y' \geq 0$, $A^T y' \geq 0$ and $b^T y' < 0$. Let $y = -y'$, therefore, there exists $y \in \mathbb{R}^m$ such that $A^T y \leq 0$ and $b^T y > 0$, which is saying (**5.2**) is true, hence contradiction. Therefore (**7.2**) is false

   ii. Otherwise, exists $x \in \mathbb{R}^m$ such that $x \geq 0$ and $Ax < b$, then (**5.1**) is false and (**5.2**) is true. Let $y \in \mathbb{R}^m$ be any vector satisfying (**5.2**), we claim $y > 0$. By contradiction assume $y \leq 0$, then $b < 0$ since $b^T y > 0$ (**5.2**). Use $b^T y > 0$ again, $y \neq 0$, so $y < 0$. Therefore $Ax < b < 0$. Multiply $y < 0$ to both sides, we have

   $$y^t(Ax) > 0 \quad \rightarrow \quad (A^T y)^T x > 0$$

   Contradiction since $x \geq 0$ and $A^T y \leq 0$ (**5.2**). Therefore $y > 0$ for all y satisfying (**5.2**). Therefore, no $y \in \mathbb{R}^m$ exists satisfying (**5.2**) for which $y \leq 0$, i.e.

   $$\nexists y \in \mathbb{R}^m \,,\ y \leq 0 \quad A^T y \leq 0 \quad b^T y > 0$$

   therefore

   $$\nexists y \in \mathbb{R}^m \,,\ y \geq 0 \quad A^T y \geq 0 \quad b^T y < 0$$

   which is equivalent to (**7.2**) false

   (b) ($\Leftarrow$) Idea is that given (**7.2**) false, we want to construct matrices $\tilde{A}$ and $\tilde{b}$ satisfying constraints for (**7.2**) as well as (**5.2**),

   $$\underset{m \times (m+n)}{\tilde{A}} = \begin{pmatrix} -A & -I \end{pmatrix} \qquad \underset{m \times 1}{\tilde{b}} = \begin{pmatrix} -b \end{pmatrix}$$

   Therefore, (**7.2**) can be rewritten as,

   $$\nexists y \in \mathbb{R}^m \,,\ y \geq 0 \quad A^T y \geq 0 \quad b^T y < 0 \qquad \rightarrow \qquad \nexists y \in \mathbb{R}^m \,,\ \tilde{A}^T y \leq 0 \quad \tilde{b}^T y > 0$$

   where the latter formulation is simply saying (**5.2**) is false, therefore (**5.1**) is true, i.e.

   $$\exists \tilde{x} \in \mathbb{R}^{m+n} \,,\ \tilde{x} \geq 0 \quad \tilde{A}\tilde{x} = \tilde{b}$$

let $\tilde{x} = (x, x')$ where $x \in \mathbb{R}^n, x' \in \mathbb{R}^m$, note $x, x' \geq 0$, also

$$\begin{pmatrix} -A & -I \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} -b \end{pmatrix} \qquad \Longleftrightarrow \qquad -Ax - x' = -b$$

therefore $Ax = b - x' \leq b$ since $x' \geq 0$, therefore,

$$\exists x \in \mathbb{R}^n \, , \, x \geq 0 \quad Ax \leq b$$

hence (**7.1**) is true.

$\square$

2. Use lemma 7, prove for any $m \times n$ matrix A and $m \times 1$ matrix b such that $\{x : Ax \leq b \, , \, x \geq 0\} \neq \emptyset$, and any $n \times 1$ vector c, the system of inequalities

$$c^T x \geq 1 \qquad Ax \leq b \qquad x \geq 0$$

is infeasible if and only if there exists a $y \in \mathbb{R}^m$, $y \geq 0$, such that $A^T y \geq c$ and $b^T y < 1$

Proof. Define

$$\tilde{A} = \begin{pmatrix} A \\ -c^T \end{pmatrix} \qquad \tilde{b} = \begin{pmatrix} b \\ -1 \end{pmatrix}$$

Then proving the following suffices

$$\nexists x \in \mathbb{R}^n \, , \, \tilde{A}x \leq \tilde{b} \quad x \geq 0 \qquad \Longleftrightarrow \qquad \exists y \in \mathbb{R}^m \, , \, y \geq 0 \quad A^T y \geq c \quad b^T y < 1$$

(a) ($\Leftarrow$) let $\tilde{y} = \begin{pmatrix} y & 1 \end{pmatrix}$, then,

$$\exists \tilde{y} \in \mathbb{R}^{m+1} \, , \, \tilde{y} \geq 0 \quad \tilde{A}^T \tilde{y} \geq 0 \quad \tilde{b}^T \tilde{y} < 0$$

lhs follows by Farkas's lemma.

(b) ($\Rightarrow$) Assume lhs true, by Farkas's lemma,

$$\exists \tilde{y} \in \mathbb{R}^{m+1} \, , \, \tilde{y} \geq 0 \quad \tilde{A}^T \tilde{y} \geq 0 \quad \tilde{b}^T \tilde{y} < 0$$

Let $\tilde{y} = \begin{pmatrix} y & a \end{pmatrix}$ satisfying the condition above. If $a \neq 0$, let $\tilde{y}' = {}^1\!/_a \tilde{y} = \begin{pmatrix} {}^1\!/_a y & 1 \end{pmatrix}$. Since $a > 0$,

$$\tilde{A}\tilde{y}' = \frac{1}{a}\tilde{A}\tilde{y} \geq 0 \qquad \tilde{b}^T \tilde{y}' = \frac{1}{a}\tilde{b}^T \tilde{y} < 0$$

Since $\tilde{A}\tilde{y}' = A^T y - c$ and $\tilde{b}^T \tilde{y}' = b^T y - 1$, then $A^T y \geq c$ and $b^T y < 1$. This proves rhs true. Now we consider the case where $a = 0$. We claim $a = 0$ is not possible. Since if $a = 0$, then we note y in $\tilde{y}$ statifies $A^T y \geq 0$ and $b^T y \leq 0$

$$\exists y \in \mathbb{R}^m \, , \, y \geq 0 \quad A^T y \geq 0 \quad b^T y \leq 0 \qquad \overset{Farkas}{\rightarrow} \qquad \nexists x \in \mathbb{R}^n \, , \, x \geq 0 \quad Ax \leq b$$

which contradicts the assumption that $\{x : Ax \leq 0 \quad x \geq 0\} \neq \emptyset$

$\square$