Local Invertibility

Implicit Function Theorem

Definition. *A **level set** of a real-valued function f of n variables is a set of the form*

$$L_c(f) = \{(x_1, \cdots, x_n) | f(x_1, \cdots, x_n) = c\}$$

That is, a set where the functino takes on a given constant value c.

Definition. *The **level set** of a real-valued function f of n real variables is a set of the form*

$$L_c(f) = \{(x_1, \cdots, x_n) \mid f(x_1, \cdots, x_n) = c\}$$

that is, a set where the function takes on a given constant value c.

Theorem. *If the function f is differentiable, the gradient of f at a point is either zero, or perpendicular to the level set of f at that point.*

Definition. *the **zero set** of a real valued function $f : X \to \mathbb{R}$ is the subset $f^{-1}(0)$ of X (the inverse image of $\{0\}$)*

Definition. *A **hyperplane** is a subspace of one dimension less than its ambient space.*

Definition. *Let f be a function whose domain is the set X with image as Y. Then f is **invertible** if there exists a function g with domain Y and image X with property*

$$f(x) = y \iff g(y) = x$$

Remark. $f = x^2$ is not injective, so the function is not invertible.

Definition. ***Horizontal Line Test*** *Let $f : \mathbb{R} \to \mathbb{R}$. If any horizontal line $y = c$ intersects the graph in more than one point, the function is not injective. The function f is surjective (i.e., onto) if and only if its graph intersects any horizontal line at least once.*

Definition. *The **inverse** of a square matrix A is a matrix A^{-1} such that $AA^{-1} = I$*

Motivation If one is given the zero locus of a C^1 function $F : \mathbb{R}^{n+k} \to \mathbb{R}^k$, that is,

$$F(x_1, \cdots, x_n, y_1, \cdots, y_k) = 0$$

and is asked to determine y as a function of x. More precisely if there exists $f_i : \mathbb{R}^n \to \mathbb{R}$ such that $y_i = f(x_1, \cdots, x_n)$. Note here y_i are placed after x_i for convenience sake. The implicit functoin thoerem is a tool that allows relation to be converted to functions of several variables by representing the relation $f(x_1, \cdots, x_n) = (y_1, \cdots, y_k)$ as the graph of a function $F(x_1, \cdots, x_n, y_1, \cdots, y_k) = 0$

Theorem. ***Implicit Function Theorem for scalar valued function (k=1)*** *If $F(x, y)$ is C^1 on some neighbourhood of $U \subseteq \mathbb{R}^{n+1}$ of the point $(a, b) \in \mathbb{R}^{n+1}$, $F(a, b) = 0$, and $\dfrac{\partial F}{\partial y}(a, b) \neq 0$, then there exists an $r \in \mathbb{R}_{\geq 0}$ together with a unique C^1 function $f : B_a(r) \to \mathbb{R}$ such that $F(x, f(x)) = 0$ for all $x \in B_a(r)$*

Proof. We prove the theorem using the graph of F, specifically,

$$G : \mathbb{R}^n \to \mathbb{R}^2 (x, y) \mapsto (x, F(x, y))$$

So at point (a, b) we have $G(a, b) = (a, 0)$. Note since $\dfrac{\partial F}{\partial y}(a, b) \neq 0$ we have

$$DG(a, b) = \begin{bmatrix} 1 & 0 \\ \dfrac{\partial F}{\partial x}(a, b) & \dfrac{\partial F}{\partial y}(a, b) \end{bmatrix}$$

invertible. Therefore, G is a bijection and hence $G^{-1}(x, 0)$ has unique point $(x, f(x))$

$\square$

Remark. The function is not only existent but also differentiable in the neighbourhood

$$\frac{\partial f}{\partial x_i} = -\frac{\frac{\partial F}{\partial x_i}(x_0, y_0)}{\frac{\partial F}{\partial y}(x_0, y_0)}$$

Corollary. *If $F \in C^1(\mathbb{R}^{n+1}, \mathbb{R})$ satisfies $\nabla F \neq 0$, then for every $x_0 \in S = \{x : F(x) = 0\}$ there is a neighbourhood N containing x_0 such that $S \cap N$ is the graph of a C^1 function.*

Remark. Follows from the theorem because $\nabla F \neq 0$ means that at every point, one of the coponent $\partial_j F \neq 0$ We can apply the theorem to solve for x_j in terms of the remaining variable.

Theorem. ***General Impicit Function Theorem*** *Let $F : \mathbb{R}^{n+k} \to \mathbb{R}^k$ be C^1 function, and write $(x_1, \cdots, x_n, y_1, \cdots, y_k)$ for the coordinates in $\mathbb{R}^{n+k}$. If (a, b) satisfies $F(a, b) = 0$ and the Jacobian matrix $J_{F,y}(a, b) = [\dfrac{\partial F_i}{\partial y_j}(a, b)]$ is invertible, there exists $r > 0$ and a unique C^1 function $f : U = B_r(a) \to \mathbb{R}^k$ such that for all $x \in B_r(a)$, $F(x, f(x)) = 0$*

Remark. The partial derivative of arbitrary $f_k(x)$ can be obtained by diffeentiation the equation $F(x, f(x)) = 0$ with respect to x_j to determine the result, but this does not clear things up as much as a simple example

Definition. *A bijection from the set X to the set Y has an inverse function from Y to X*

Theorem. ***Inverse Function Theorem*** *Let $U, V \subseteq \mathbb{R}^n$ and fix some point $a \in U$. If $f : U \to V$ is of class C^1 and $Df(a)$ is invertible, then there exists some neighbourhoods $U' \subseteq U$ of a and $V' \subseteq V$ of $f(a)$ such that $f|_{U'} : U' \to V'$ is bijective with C^1 inverse $(f|_U)^{-1} : V' \to U'$. Moreover, if $b = f(a)$ then the derivative of the inverse map is given by*

$$[Df^{-1}](b) = [Df(a)]^{-1}$$

Remark. The inverse function theorem gives conditino for a function to be invertible in a neighborhood of a point. Informally, if F is C^1 with invertible Jacobian matrix at a point a, then F is invertible in a neighborhood of a. That is, an inverse function to F exists in some neighborhood of $F(a)$. Moreover, the inverse function F^{-1} is also C^1

Remark. Given $U, V \in \mathbb{R}$. Think of a point $a \in U$ such that $f'(a) = 0$. Here the curve in the neighborhood of a is not injective, so inverse does not exist. Whereas if $f'(a) \neq 0$, f is a bijection and inverse exists.

0.1 Parameterization

Definition. *The **graph** of a function f is the collection of all ordered pairs $(x, f(x))$.*

Remark. If the function input is scalar, the graph is 2-dimensinoal and for a continuous function is a **curve**. if the function is an ordered pair (x_1, x_2), the graph is the collection of all ordered triples $(x_1, x_2, f(x_1, f_2))$ and for a continuous function is a **surface**. note that only injective function can be represented as a graph. And scalar-valued function are easily represented using graph

Definition. *A **curve** is a map $\gamma : (a, b) \in \mathbb{I} \to \mathbb{R}^n$ such that $t \mapsto (f_1(t), \cdots, f_n(t))$. A curve is **simple** if γ is injective; Geometrically, a simple curve is one that does not intersect with itself. A curve is **smooth** if*

1. $\gamma \in C^1$
2. $\gamma'(t) \neq 0$ *for all* $t \in (a, b)$

Remark. Intuitively, a smooth curve have no cornes. There are two ways it could have a corner

1. parameterization fail to be differentiable
2. parameterization could slow to a stop, then start up again in a different direction.

Theorem. *Let $\gamma : (a, b) \in \mathbb{R} \to \mathbb{R}^n : t \mapsto (f_i(t))$ be a parameterization such that $\gamma'(t_0) \neq 0$ where $t_0 \in (a, b)$. Then in the neighborhood of $\gamma(t_0)$, $\gamma(a, b)$ can be written as the graph of a C^1 function f*

$$\gamma(t) = (\gamma(t), (f \circ \gamma)(t))$$

Definition. *A **Spherical Coordinate** is a coordinate system for 3-dimensional space where position of a point is specified by 3 numbers: the **radial distance** τ from a fixed origin, its **polar angle** ϕ measured from a fixed zenith direction, and the **azimuth angle** θ of its orthogonal projection on the reference plane that passes through the origin and is orthogonal to the zenith, measured from a fixed reference direction on that plane. Here we define $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ that maps spherical coordinates to cartesian coordinates. Specifically*

$$(\theta, \phi) \to (x, y, z) \text{ where } x = \tau \sin\phi cos\theta, y = \tau \sin\phi sin\theta, z = \tau\cos\phi$$