Direct Sum Definition

Definition. ***Summation of Sets*** *If S_1 and S_2 are nonemtpy subsets of a vector space V, then the **sum** of S_1 and S_2, denoted $S_1 + S_2$, is the set*

$$\{x + y : x \in S_1 \text{ and } y \in S_2\}$$

1. *$W_1 + W_2$ is a subspace of V containing both W_1 and W_2*

2. *If for a subset $S \subseteq V$, $W_1 \subseteq S$ and $W_2 \subseteq S$, then $W_1 + W_2 \subseteq S$*

Definition. ***Direct Sum*** *A vector space V is called the direct sum of W_1 and W_2 if W_1 and W_2 are subspaces of V such that $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. We denote that V is the direct sum of W_1 and W_2 by writing $V = W_1 + W_2$. We denote V is the direct sum of W_1 and W_2 by writing $V = W_1 \oplus W_2$*

1. *Direct sum of the set of upper triangular like matrices and lower triangular matrices is $M_{m\times n}(F)$*

2. *The trick of decomposing vector space into direct sums is that the intersection of subsets yield the zero vector*

5.4 Invariant Subspaces and Direct Sum

Chapter 7 Canonical Forms

7.1 The Jordan Canonical Form

Definition. ***Jordan Block and Jordan Canonical Form*** *Select ordered basis whose union is an ordered basis β, the Jordan caconical basis for T, for V such that*

$$[T]_\beta = \begin{pmatrix} A_1 & O & \cdots & O \\ O & A_2 & \cdots & O \\ \vdots & \vdots & & \vdots \\ O & O & \cdots & A_k \end{pmatrix}$$

where A_i are jordan block corresponding to λ

$$A_i = \begin{pmatrix} \lambda \end{pmatrix} \qquad or \qquad A_i = \begin{pmatrix} \lambda & 1 & O & \cdots & O & O \\ O & \lambda & 1 & \cdots & O & O \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ O & O & O & \cdots & \lambda & 1 \\ O & O & O & \cdots & O & \lambda \end{pmatrix}$$

Definition. ***Generalized Eigenvector*** *Let T be a linear operator on a vector space V, and let λ be a scalar. A nonzero vector x in V is called a generalized eigenvector of T corresponding to λ if $(T - \lambda I)^p(x) = 0$ for some positive integer p*

1. *For v in a Jordan canonical basis for T, $(T-\lambda I)^p(v) = 0$ for sufficiently large p. Eigenvectors satisfy this condition for $p = 1$*

2. *If x is a generalized eigenvector of T corresponding to λ, and p is smallest positive integer for which $(T-\lambda I)^p(x) = 0$, then $(T-\lambda I)^{p-1}(x) = 0$ is an eigenvector of T corresponding to λ*

$$(T-\lambda I)(v) = 0 \qquad \text{where eigenvector } v = (T-\lambda I)^{p-1}(x) \neq 0$$

Definition. ***Generalized Eigenspace*** *Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. The generalized eigenspace of T corresponds to λ, denoted K_λ, is the subset of V defined by*

$$K_\lambda = \{x \in V : (T-\lambda I)^p(x) = 0 \text{ for some positive integer } p\} = \bigcup_{p\geq 1} N((T-\lambda I)^p)$$

1. *Note*
$$N(U) \subseteq N(U^2) \subseteq \cdots \subseteq N(U^k) \subseteq N(U^{k+1}) \subseteq \cdots$$

Theorem. ***7.1 Properties of Generalized Eigenspace*** *Let T be linear operator on a vector space V, and let λ be an eigenvalue of T. Then*

1. *K_λ is a T-invariant subspace of V containing E_λ (the eigenspace of T corresponding to λ)*

2. *For any scalar $\mu \neq \lambda$, the restiction $T-\mu I$ to K_λ is one-to-one.*

Note

1. *Second property implies $E_\mu = N(T-\mu I) = 0$ for all $\mu \neq \lambda$, so K_λ contains only one eigenspace E_λ and λ is the only eigenvalue of T_{λ_k}*

2. *For $\mu \neq \lambda$, $K_\lambda = K_\mu = 0$*

Theorem. ***7.2 Property of Generalized Eigenspace When Characteristic Polynomial Splits*** *Let T be a linear operator on a finite-dimensional vector space V such that the characteristic polynomial of T splits. Suppose that λ is an eigenvalue of T with multiplicity m. Then*

1. *$dim(K_\lambda) \leq m$*

2. *$K_\lambda = N((T-\lambda I)^m)$*

For proofs

1. *Use theorem 5.21 T-invariant $W \subseteq V$ have $P_{T_W}(t) \mid P_T(t)$, we have $[T_W]_\beta$ in the form of a Jordan Block, therefore*

$$h(t) = P_{T_W}(t) = (-1)^d(t-\lambda)^d$$

2. *Prove forward direction* $(\Rightarrow)$ *Use* theorem 5.23 Cayley-Hamilton $f(T) = T_0$, *i.e. linear operator satisfies its characteristic equation, on* T_W

$$h(T_W) = (-1)^d (T - \lambda I)^d = T_0$$

So $(T - \lambda I)^d(x) = 0$ *for all* $x \in W$ *where* $d \leq m$, *so* $K_\lambda \subseteq N((T - \lambda I)^m)$

Theorem. ***7.3 Lemma for Proving That Basis for E_λs' Spans Entire Space*** *Let T be a linear operator on a finite-dimensional vector space V such that the characteristic polynomial of T splits, and let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be the distinct eigenvalues of T. Then for every $x \in V$, there exists vectors $v_1 \in K_{\lambda_i}$, $1 \leq i \leq k$, such that*

$$x = v_1 + v_2 + \cdots + v_k$$

Proof. Cayley-Hamilton theorem works on some special case of characteristic polynomial of the form $(t - \lambda)^d$ yields the zero transformation, which makes some subset of the vector space satisfy condition for generalized eigenspace, i.e. $(T - \lambda I)(x) = 0$ □

Theorem. ***7.4 Basis for Generalized Eigenspace***
Let T be a linear operator on a finite-dimensional vector space V sucht hat the characteristic polynomial of T splits, and let $\lambda_1, \cdots, \lambda_k$ be the distinct eigenvalues of T with corresponding multiplicity $m_1, \cdots, m_k$. For $1 \leq i \leq k$, let β_i be an ordered basis for K_{λ_i}. Then the following statements are true

1. $\beta_i \cap \beta_j = \emptyset$ *for* $i \neq j$
2. $\beta = \beta_1 \cap \beta_2 \cap \cdots \cap \beta_k$ *is an ordered basis for* V
3. $dim(K_{\lambda_i}) = m_i$ *for all* i

Corollary. ***Assumption for Diagonalizability*** *Let T be a linear operator on a finite-dimensional vector space V sucht hat the characteristic polynomial of T splits. Then T is diagonalizable if and only if $E_\lambda = K_\lambda$ for every eigenvalue λ of T*

Definition. ***Cycle of Generalized Eigenvectors*** *Let T be a linear operator on a vector space V, and let x be a generalized eigenvector of T corresponding to the eigenvalue λ. Suppose that p is the smallest positive integer for which $(T - \lambda I)^p(x) = 0$. Then the ordered set*

$$\{(T - \lambda I)^{p-1}(x), (T - \lambda I)^{p-2}(x), \cdots, (T - \lambda I)(x), x\}$$

is called a cycle of generaliezd eigenvectors of T corresponding to λ. The vectors $(T - \lambda I)^{p-1}(x)$ and x are called the initial vector and the end vector of the cycle, respectively. We say that the length of the cycle is p (number of vectors).

Theorem. ***7.5 Jordan Canonical Basis as Disjoint Union of Cycles of Generalized Eigenvectors***
Let T be a linear operator on a finite-dimensional vector space V whose characteristic polynomial splits, and suppose that β is a basis for V such that β is a disjoint union of cycles of generalized eigenvectors of T. Then the following are true

1. *For each cycle γ of generalized eigenvectors contained in β, $W = span(\gamma)$ is T-invariant and $[T_W]_\gamma$ is a Jordan block*

2. *β is a Jordan canonical basis for V*

Theorem. ***7.6 Existence of Disjoint Union of Cycles of Generalized Eigenvectors*** *Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. Suppose that $\gamma_1, \gamma_2, \cdots, \gamma_q$ are cycles of generalized eigenvectors of T corresponding to λ such that the initial vectors of the γ_i's are distinct and form a linearly independent set. Then the γ_i's are disjoint, and their union $\gamma = \bigcup_{i=1}^{q} \gamma_i$ is linearly independent*

Corollary. *Every cycle of generalized eigenvectors of a linear operator is linearly independent*

Theorem. ***7.7 Existence of Disjoint Union in Generalized Eigenspace*** *Let T be a linear operator on a finite-dimensional vector space V, and let λ be an eigenvalue of T. Then K_λ has an ordered basis consisting of a union of disjoint cycles of generalized eigenvectors corresponding to λ*

Corollary. ***$P_T(t)$ Splits Ensures Existence of Jordan Canonical Form*** *Let T be a linear opertor on a finite-dimensional vector space V whose characteristic polynomial splits. Then T has a Jordan Canonical Form*

Definition. ***Jordan Canonical Form for Matrices*** *Let $A \in M_{n\times n}(F)$ be such that the characteristic polynomial of A (and hence of L_A) splits. Then the Jordan canonical form of A is defined to be the Jordan canonical form of the linear operator L_A on F^n*

Corollary. *Let A be $n \times n$ matrix whose characteristic polynomial splits. Then A has a Jordan canonical form J, and A is similar to J*

7.2 The Jordan Canonical Form II