# CSC446 A2

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## Problem 1 (chapter 9 q9.2)

Find explicitly the coefficients $a_{k,l}$ where $k, l = 1, 2, \cdots, m$ for

$$-y'' + y = f$$

assuming the space $\mathbb{H}_m^\circ$ is spanned by the chapeau function on an equidistant grid

*Proof.* This is a special case of the ODE we discussed in class. We get the expression for $a_{k,l}$ as

$$a_{k,l} = \left\langle \varphi_l'(x), \varphi_k'(x) \right\rangle + \left\langle \varphi_l(x), \varphi_k(x) \right\rangle = \int_0^1 \varphi_l'(x)\varphi_k'(x)dx + \int_0^1 \varphi_l(x)\varphi_k(x)dx$$

where $\varphi_k(x)$ is given by the chapeau function

$$\varphi_k(x) = \begin{cases} 1 - k + \frac{x}{h} & (k-1)h \le x \le kh \\ 1 + k - \frac{x}{h} & kh \le x \le (k+1)h \\ 0 & |x - kh| \ge h \end{cases}$$

$a_{k,l} \neq 0$ if and only if $|k - l| < 2$. Now we do integration and compute $a_{k,k-1}, a_{k,k}, a_{k,k+1}$

$$\begin{aligned} a_{k,k+1} &= \int_{kh}^{(k+1)h} \left\{ -\frac{1}{h}\frac{1}{h} + \left(1 + k - \frac{x}{h}\right)\left(-k + \frac{x}{h}\right) \right\} dx = \frac{h^2 - 6}{6h} \\ a_{k,k} &= \int_{(k-1)h}^{kh} \left\{ \frac{1}{h}\frac{1}{h} + \left(1 - k + \frac{x}{h}\right)^2 \right\} dx + \int_{kh}^{(k+1)h} \left\{ \frac{1}{h}\frac{1}{h} + \left(1 + k - \frac{x}{h}\right)^2 \right\} dx = \frac{2(h^2+3)}{3h} \\ a_{k,k-1} &= \int_{(k-1)h}^{kh} \left\{ -\frac{1}{h}\frac{1}{h} + \left(k - \frac{x}{h}\right)\left(1 - k + \frac{x}{h}\right) \right\} dx = \frac{h^2 - 6}{6h} \end{aligned}$$

with the help of online integral calculator. Therefore

$$a_{k,l} = \begin{cases} \frac{2(h^2+3)}{3h} & l = k \\ \frac{h^2-6}{6h} & l = k-1, k+1 \\ 0 & \text{otherwise} \end{cases}$$

□

## Problem 2 (chapter 9 q9.3)

Suppose equation (9.1)

$$-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right]+b(x)u=f$$

is solved by Galerkin method with chapeau functions on a non-equidistant grid. In other words, we are given $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$ such that each $\varphi_k$ is supported in $(t_{k-1},t_{k+1})$ for $k=1,2,\cdots,m$. Prove that the linear system (9.7)

$$\sum_{l=1}^{m} a_{k,l}\gamma_l=\int_0^1 f(\tau)\varphi_k(\tau)d\tau-a_{k,0} \qquad k=1,2,\cdots,m$$

where

$$a_{k,l}=\int_0^1\left[a(\tau)\varphi_l'(\tau)\varphi_k'(\tau)+b(\tau)\varphi_l(\tau)\varphi_k(\tau)\right]d\tau$$

is nonsingular (Hint: Use Gersgorin criterion). State any additional assumptions that you need to ensure that the system is nonsingular.

*Proof.* Idea is to show that 0 is not inside any Gersgorin disk. Consider expression for $\mathbb{S}_k$

$$\mathbb{S}_k=\{c\in\mathbb{C}\mid |z-a_{k,k}|\le|a_{k,k-1}+a_{k,k+1}|\}$$

It is trivial to show that $a_{k,k-1},a_{k,k+1}$ have the same sign; so it is OK to right 1 absolute sign on right hand side. Now we write formula for the hat function

$$\varphi_k(x)=\begin{cases}\frac{x-t_{k-1}}{t_k-t_{k-1}} & t\in[t_{k-1},t_k]\\ \frac{t_{k+1}-x}{t_{k+1}-t_k} & t\in[t_k,t_{k+1}]\\ 0 & \text{otherwise}\end{cases}$$

To show that 0 not in any $\mathbb{S}_k$, consider the region $[t_{k-1},t_k]$ which supports $\varphi_{k-1}(x)$ and $\varphi_k(x)$ but not $\varphi_{k+1}(x)$. So suffice to show $a_{k,k}>|a_{k,k-1}|$, i.e. $a_{k,k}+a_{k,k-1}>0$ and $a_{k,k}-a_{k,k-1}>0$

$$\begin{aligned}a_{k,k}+a_{k,k-1}&=\int_{t_{k-1}}^{t_k}a(x)\left(\frac{1}{t_k-t_{k-1}}\right)^2+b(x)\left(\frac{x-t_{k-1}}{t_k-t_{k-1}}\right)^2dx\\&+\int_{t_{k-1}}^{t_k}a(x)\left(-\frac{1}{t_k-t_{k-1}}\right)\left(\frac{1}{t_k-t_{k-1}}\right)+b(x)\left(\frac{t_k-x}{t_k-t_{k-1}}\right)\left(\frac{x-t_{k-1}}{t_k-t_{k-1}}\right)dx\\&=\int_{t_{k-1}}^{t_k}b(x)\frac{1}{(t_k-t_{k-1})^2}\left\{(x-t_{k-1})^2+(t_k-x)(x-t_{k-1})\right\}dx\\&=\int_{t_{k-1}}^{t_k}b(x)\frac{t_k-t_{k-1}}{(t_k-t_{k-1})^2}\left\{x-t_{k-1}\right\}dx\end{aligned}$$

which is greater than zero, since $b(x)\ge0$ and $x-t_{k-1}\ge0$ on $x\in[t_{k-1},t_k]$. Now consider

$$\begin{aligned}a_{k,k}-a_{k,k-1}&=\int_{t_{k-1}}^{t_k}a(x)\left(\frac{1}{t_k-t_{k-1}}\right)^2+b(x)\left(\frac{x-t_{k-1}}{t_k-t_{k-1}}\right)^2dx\\&-\int_{t_{k-1}}^{t_k}a(x)\left(-\frac{1}{t_k-t_{k-1}}\right)\left(\frac{1}{t_k-t_{k-1}}\right)+b(x)\left(\frac{t_k-x}{t_k-t_{k-1}}\right)\left(\frac{x-t_{k-1}}{t_k-t_{k-1}}\right)dx\\&=\int_{t_{k-1}}^{t_k}2a(x)\left(\frac{1}{t_k-t_{k-1}}\right)^2+b(x)\frac{1}{(t_k-t_{k-1})^2}\left\{(x-t_{k-1})^2-(t_k-x)(x-t_{k-1})\right\}dx\\&=\int_{t_{k-1}}^{t_k}2a(x)\left(\frac{1}{t_k-t_{k-1}}\right)^2dx+\frac{1}{(t_k-t_{k-1})^2}\int_{t_{k-1}}^{t_k}b(x)\left\{(x-t_{k-1})^2-(t_k-x)(x-t_{k-1})\right\}dx\end{aligned}$$

since $a(x) > 0$ on the support, the first term results to a value greater than zero. However, the second term is not always positive. We can show that the quadratic function of $x$ inside the curly bracket has roots inside $[t_{k-1}, t_k]$, i.e.

$$p(x) := (x - t_{k-1})^2 - (t_k - x)(x - t_{k-1}) = 2x^2 + (-3t_{k-1} - t_k)x + (t_{k-1}t_k + t_{k-1}^2)$$

is an upward facing parabolla with root at $t_{k-1}$ and $t_{mid} := \frac{1}{2}(t_{k-1} + t_k)$. We now try to derive a lower bound on the second term. Let $M$ be maximum value of $b(x)$ for $x \in [t_{k-1}, t_{mid}]$ and $m$ be the minimum value of $b(x)$ for $x \in [t_{mid}, t_k]$. This is possible by extreme value theorem with continuous $b(x)$. Then

$$\begin{aligned}
\int_{t_{k-1}}^{t_k} b(x)p(x)dx &= \int_{t_{k-1}}^{t_{mid}} b(x)p(x)dx + \int_{t_{mid}}^{t_k} b(x)p(x)dx \\
&\geq M \int_{t_{k-1}}^{t_{mid}} p(x)dx + m \int_{t_{mid}}^{t_k} p(x)dx \\
&= e(-M + 5m)
\end{aligned}$$

where $e = (d - c)^3/24 > 0$. Therefore the second term is greater than zero if and only if $M < 5m$. Therefore one additional assumption is to make sure the grid $t_k$s are adapted according to $b(x)$ such that $M < 5m$ holds. This is always possible since for sufficiently dense grid values, $M \approx m$. With a similar argument, we consider region $[t_k, t_{k+1}]$ and show $a_{k,k} > |a_{k,k+1}|$. It is clear that this would hold by symmetry of the hat function. Now we have shown that

$$a_{k,k} > |a_{k,k-1} + a_{k,k+1}|$$

Note we have covered the boundary cases, i.e. $\mathbb{S}_1, \mathbb{S}_m$, by virtue of how we structured our proof. Hence $0 \notin \mathbb{S}_k$ for all $k = 1, \cdots, m$ implying $0 \notin \cup_k \mathbb{S}_k$. Since all eigenvalues are inside union of Gersgorin disks, 0 cannot be an eigenvalue of $A$. Therefore the linear system (9.7) is nonsingular. $\square$

## Problem 3 (chapter 9 q9.4)

Let $a$ be a given positive univariate function and

$$\mathcal{L} = \frac{\partial^2}{\partial x^2}\left[a(x)\frac{\partial^2}{\partial x^2}\right]$$

Assuming zero Dirichlet boundary conditions, i.e. on $[c, d]$, $v(x)$ satisfies Dirichlet boundary if

$$v(c) = v'(c) = v(d) = v'(d) = 0$$

prove that $\mathcal{L}$ is positive definite in the Euclidean norm

*Proof.* To show $\mathcal{L}$ is positive definite, we show that $\mathcal{L}$ is

1. self-adjoint ($\langle \mathcal{L}v, w\rangle = \langle v, \mathcal{L}w\rangle$ for all $v, w \in \mathring{\mathbb{H}}$) and
2. elliptic ($\langle \mathcal{L}v, v\rangle > 0$ for all $v \in \mathring{\mathbb{H}}$)

Consider

$$\begin{aligned}
\langle \mathcal{L}v, w\rangle &= \int_0^1 \frac{\partial^2}{\partial x^2}\left[a(x)\frac{\partial^2 v(x)}{\partial x^2}\right] w(x)dx \\
&= \left[\frac{\partial}{\partial x}\left[a(x)\frac{\partial^2 v(x)}{\partial x^2}\right] w(x)\right]_0^1 - \int_0^1 \frac{\partial}{\partial x}\left[a(x)\frac{\partial^2 v(x)}{\partial x^2}\right]\frac{\partial w(x)}{\partial x}dx \\
&= \left[\frac{\partial}{\partial x}\left[a(x)\frac{\partial^2 v(x)}{\partial x^2}\right] w(x)\right]_0^1 - \left(\left[a(x)\frac{\partial^2 v(x)}{\partial x^2}\frac{\partial w(x)}{\partial x}\right]_0^1 - \int_0^1 a(x)\frac{\partial^2 v(x)}{\partial x^2}\frac{\partial^2 w(x)}{\partial x^2}dx\right) \\
&= \int_0^1 a(x)\frac{\partial^2 v(x)}{\partial x^2}\frac{\partial^2 w(x)}{\partial x^2}dx
\end{aligned}$$

where last equality holds by $w \in \mathring{\mathbb{H}}$ satisfying Dirichlet boundary condition. Note the resulting expression is symmetric in $v$ and $w$, therefore $\langle \mathcal{L}v, w\rangle = \langle v, \mathcal{L}w\rangle$ for arbitrary $v, w \in \mathring{\mathbb{H}}$. If $w = v$, we have

$$\langle \mathcal{L}v, v\rangle = \int_0^1 a(x)\left(\frac{\partial^2 v(x)}{\partial x^2}\right)^2 dx > 0$$

by $a(x) > 0$ in its domain and $v$ cannot be zero almost everywhere. So, $\mathcal{L}$ is a positive definite operator □

## Problem 4 (chapter 9 q9.7)

Let $\mathcal{L}$ be elliptic differential operator and $f$ a given bounded function. Let $\mathbb{H}$ be such that all $v \in \mathbb{H}$ satisfies boundary conditions $v(0) = \alpha$ and $v(1) = \beta$ while $\mathring{\mathbb{H}}$ be function space satisfying the zero boundary condition. Note

$$\langle v, v \rangle = \int_0^1 [v(x)]^2 \, dx < \infty \qquad \text{and} \qquad \langle v', v' \rangle = \int_0^1 \left[v'(x)\right]^2 dx < \infty$$

Moreover

$$\tilde{a}(v, v) = \int_0^1 \left\{ a(x) \left[v'(x)\right]^2 + b(x) \left[v(x)\right]^2 \right\} dx$$

where $a(x) > 0$ and $b(x) \geq 0$ for all $x \in [0, 1]$.

1. Prove that

$$c_1 = \min_{v \in \mathring{\mathbb{H}} \, \|v\|=1} \tilde{a}(v, v) \qquad \text{and} \qquad c_2 = \max_{v \in \mathring{\mathbb{H}} \, \|v\|=1} \langle f, v \rangle$$

   are bounded and that $c_1 > 0$

2. Given $w \in \mathring{\mathbb{H}}$, prove

$$\tilde{a}(w, w) - 2 \langle f, w \rangle \geq c_1 \|w\|^2 - 2c_2 \|w\|$$

   (Hint: write $w = \kappa v$, where $\|v\| = 1$ and $|\kappa| = \|w\|$)

3. Deduce that

$$\tilde{a}(w, w) - 2 \langle f, w \rangle \geq -\frac{c_2^2}{c_1} \quad w \in \mathring{\mathbb{H}}$$

   thereby proving that the functional $\mathcal{J}$ from (9.21) has a bounded minimum

*Proof.* 1. $c_1$ is bounded since the operator $\mathcal{L}$ is elliptic. Now we need to show that the minimum is in fact greater than zero. Since $a(x)$ is a differentiable (hence continuous) function on a compact set $[0, 1]$, by extreme value theorem, there exists $d \in \mathbb{R}$ such that $a(x) \geq a(d)$ for all $x \in [0, 1]$. Note since $a(x) > 0$, $d > 0$. Since $v \in \mathring{\mathbb{H}}$, $v$ satisfies zero boundary condition $v(0) = v(1) = 0$. Since $\|v\| = 1$, $v'(x)$ must not be zero almost everywhere, i.e. $v'(x) > 0$. Therefore,

$$\begin{aligned}
c_1 &= \min_{v \in \mathring{\mathbb{H}} \, \|v\|=1} \tilde{a}(v, v) \\
&= \min_{v \in \mathring{\mathbb{H}} \, \|v\|=1} \int_0^1 \left\{ a(x) \left(v'(x)\right)^2 + b(x) \left(v(x)\right)^2 \right\} dx \\
&\geq d \min_{v \in \mathring{\mathbb{H}} \, \|v\|=1} \int_0^1 \left(v'(x)\right)^2 dx \\
&> 0
\end{aligned}$$

Now we show $c_2$ is bounded above

$$\begin{aligned}
c_2 &= \max_{v \in \mathring{\mathbb{H}} \, \|v\|=1} \langle f, v \rangle \\
&\leq \max_{v \in \mathring{\mathbb{H}} \, \|v\|=1} |\langle f, v \rangle| \\
&\leq \max_{v \in \mathring{\mathbb{H}} \, \|v\|=1} \|f\| \, \|v\| && \text{(CauchySchwarz inequality)} \\
&= \max_{v \in \mathring{\mathbb{H}} \, \|v\|=1} \|f\|
\end{aligned}$$

which is bounded above since $f$ is a bounded function

2. Let $w = \kappa v$ where $w \in \mathring{\mathbb{H}}$ and $\|v\| = 1$, then

$$\|w\| = \sqrt{\int_0^1 w(x)^2 dx} = \sqrt{\int_0^1 \kappa^2 v(x)^2 dx} = |\kappa| \sqrt{\int_0^1 v(x)^2 dx} = |\kappa| \, \|v\| = |\kappa|$$

Therefore

$$\begin{aligned}
\tilde{a}(w,w) - 2\langle f, w\rangle &= \int_0^1 a(x)\left(w'(x)\right)^2 + b(x)\left(w(x)\right)^2 dx - 2\int_0^1 f(x)w(x)dx \\
&= \int_0^1 a(x)\kappa^2\left(v'(x)\right)^2 + b(x)\kappa^2\left(v(x)\right)^2 dx - 2\int_0^1 f(x)\kappa v(x)dx \\
&\geq (|\kappa|)^2 \int_0^1 a(x)\left(v'(x)\right)^2 + b(x)\left(v(x)\right)^2 dx - |\kappa| \, 2\int_0^1 f(x)v(x)dx \\
&= \|w\|^2 \tilde{a}(v,v) - \|w\| \, 2\langle f, v\rangle \\
&\geq c_1 \|w\|^2 - 2c_2 \|w\|
\end{aligned}$$

where the last inequality follows from definition of $c_1, c_2$

3. Note

$$\begin{aligned}
\tilde{a}(w,w) - 2\langle f, w\rangle &\geq c_1 \|w\|^2 - 2c_2 \|w\| \\
&= \frac{1}{c_1}\left(c_1^2 \|w\|^2 - 2c_1c_2 \|w\| + c_2^2\right) - \frac{c_2^2}{c_1} \\
&= \frac{1}{c_1}\left(c_1 \|w\| - c_2\right)^2 - \frac{c_2^2}{c_1} \\
&\geq -\frac{c_2^2}{c_1}
\end{aligned}$$

Hence functional $\mathcal{J}$ has a bounded minimum

$\square$

## Problem 5

*Proof.* Similar in Q1, we can compute the integral for the expression with an online calculator

$$a_{k,l} = \begin{cases} \frac{2(10h^2+3)}{3h} & l = k \\ \frac{5h^2-3}{3h} & l = k-1, k+1 \\ 0 & \text{otherwise} \end{cases}$$

Let $\varphi_0(x) = 1$ and so $\varphi_0'(x) = 0$ for $x \in (0, 1)$. Now we compute the right hand side of the linear system

$$\begin{aligned} b_k &= \langle f, \varphi_k \rangle - a_{k,0} \\ &= \int_0^1 x\varphi_k(x)dx - \int_0^1 \varphi_0'(x)\varphi_k'(x) + 10\varphi_0(x)\varphi_k(x)dx \\ &= \int_0^1 x\varphi_k(x)dx - 10\int_0^1 \varphi_k(x)dx \\ &= kh^2 - 10h \end{aligned}$$

Implementation for setting up and solving the 2d bvp problem is in appendix. Maximum error is given as follows

| m | error | error_ratio |
|---|---|---|
| 9 | 0.0022901713 | 1.0000000000 |
| 19 | 0.0005687657 | 0.2483507075 |
| 39 | 0.0001421082 | 0.2498536936 |
| 79 | 0.0000355124 | 0.2498969312 |
| 159 | 0.0000088772 | 0.2499742346 |
| 319 | 0.0000022192 | 0.2499935590 |
| 639 | 0.0000005548 | 0.2499992486 |

As $h$ halves, maximum error decreases by a factor of 4. □

# Appendix

## Problem 5 code

```
global m;
ms = [9 19 39 79 159 319 639];
fprintf('m\terror\t\terror_ratio\n')
for n_hat = ms
    m = n_hat;
    h = 1/(m+1);
    [A b] = make_Ab();
    c = A \ b;
    e = zeros(m);
    for i = 1:m
        xi = i*h;
        e(i) = abs(y(xi) - (c(i)+1));
    end
    max_e = max(e, [], 'all');
    if m == 9
        e_norm = max_e;
    end
    fprintf('%d\t%.10f\t%.10f\n', m, max_e, max_e/e_norm);
    e_norm = max_e;
end

function [A b] = make_Ab()
    global m;
    h = 1/(m+1);
    A = sparse(m,m);
    b = zeros(m,1);
    for k = 1:m
        for l = 1:m
            if l == k
                A(k,l) = 2*(10*h^2+3) / (3*h);
            elseif l == k-1 || l == k+1
                A(k,l) = (5*h^2-3) / (3*h);
            end
        end
        b(k) = k*h^2 - 10*h;
    end
end

function yx = y(x)
    l = sqrt(10);
    c_2 = (exp(l) + 1/l^2 - 1) / (exp(l) - exp(-l));
    c_1 = 1 - c_2;
    yx = c_1*exp(l*x) + c_2*exp(-l*x) + x/l^2;
end
```