Lemma. ***5: Farkas's Lemma*** *For any $m \times n$ matrix A and $m \times 1$ vector b, exactly one of following is true*

1. *There exists $x \in \mathbb{R}^n$ such that $x \geq 0$ and $Ax = b$*
2. *There exists $y \in \mathbb{R}^m$ such that $A^T y \leq 0$ and $b^T y > 0$*

Lemma. ***7: Farkas's Lemma variant*** *For any $m \times n$ matrix A and $m \times 1$ vector b, exactly one of following is true*

1. *There exists $x \in \mathbb{R}^n$ such that $x \geq 0$ and $Ax \leq b$*
2. *There exists $y \in \mathbb{R}^m$ such that $y \geq 0$, $A^T y \geq 0$ and $b^T y < 0$*

Question 1: LP Theory

1. use lemma 5 prove lemma 7

   Proof. Equivalent to proving (**7.1**) $\iff \neg$(**7.2**)

   (a) ($\Rightarrow$) Assume (**7.1**) true, 2 cases

      i. Exists $x \in \mathbb{R}^m$ such that $x \geq 0$ and $Ax = b$, then (**5.1**) is true and (**5.2**) is false. By contradiction, assume (**7.2**) is true, let $y' \in \mathbb{R}^m$ such that $y' \geq 0$, $A^T y' \geq 0$ and $b^T y' < 0$. Let $y = -y'$, therefore, there exists $y \in \mathbb{R}^m$ such that $A^T y \leq 0$ and $b^T y > 0$, which is saying (**5.2**) is true, hence contradiction. Therefore (**7.2**) is false

      ii. Otherwise, exists $x \in \mathbb{R}^m$ such that $x \geq 0$ and $Ax < b$, then (**5.1**) is false and (**5.2**) is true. Let $y \in \mathbb{R}^m$ be any vector satisfying (**5.2**), we claim $y > 0$. By contradiction assume $y \leq 0$, then $b < 0$ since $b^T y > 0$ (**5.2**). Use $b^T y > 0$ again, $y \neq 0$, so $y < 0$. Therefore $Ax < b < 0$. Multiply $y < 0$ to both sides, we have

      $$y^t(Ax) > 0 \quad \rightarrow \quad (A^T y)^T x > 0$$

      Contradiction since $x \geq 0$ and $A^T y \leq 0$ (**5.2**). Therefore $y > 0$ for all y satisfying (**5.2**). Therefore, no $y \in \mathbb{R}^m$ exists satisfying (**5.2**) for which $y \leq 0$, i.e.

      $$\nexists y \in \mathbb{R}^m \,,\; y \leq 0 \quad A^T y \leq 0 \quad b^T y > 0$$

      therefore

      $$\nexists y \in \mathbb{R}^m \,,\; y \geq 0 \quad A^T y \geq 0 \quad b^T y < 0$$

      which is equivalent to (**7.2**) false

   (b) ($\Leftarrow$) Idea is that given (**7.2**) false, we want to construct matrices $\tilde{A}$ and $\tilde{b}$ satisfying constraints for (**7.2**) as well as (**5.2**),

   $$\underset{m\times(m+n)}{\tilde{A}} = \begin{pmatrix} -A & -I \end{pmatrix} \qquad \underset{m\times 1}{\tilde{b}} = \begin{pmatrix} -b \end{pmatrix}$$

   Therefore, (**7.2**) can be rewritten as,

   $$\nexists y \in \mathbb{R}^m \,,\; y \geq 0 \quad A^T y \geq 0 \quad b^T y < 0 \qquad \rightarrow \qquad \nexists y \in \mathbb{R}^m \,,\; \tilde{A}^T y \leq 0 \quad \tilde{b}^T y > 0$$

   where the latter formulation is simply saying (**5.2**) is false, therefore (**5.1**) is true, i.e.

   $$\exists \tilde{x} \in \mathbb{R}^{m+n} \,,\; \tilde{x} \geq 0 \quad \tilde{A}\tilde{x} = \tilde{b}$$

let $\tilde{x} = (x, x')$ where $x \in \mathbb{R}^n, x' \in \mathbb{R}^m$, note $x, x' \geq 0$, also

$$\begin{pmatrix} -A & -I \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} -b \end{pmatrix} \qquad \iff \qquad -Ax - x' = -b$$

therefore $Ax = b - x' \leq b$ since $x' \geq 0$, therefore,

$$\exists x \in \mathbb{R}^n \,,\, x \geq 0 \quad Ax \leq b$$

hence (**7.1**) is true.

$\square$

2. Use lemma 7, prove for any $m \times n$ matrix A and $m \times 1$ matrix b such that $\{x : Ax \leq b \,,\, x \geq 0\} \neq \emptyset$, and any $n \times 1$ vector c, the system of inequalities

$$c^T x \geq 1 \qquad Ax \leq b \qquad x \geq 0$$

is infeasible if and only if there exists a $y \in \mathbb{R}^m$, $y \geq 0$, such that $A^T y \geq c$ and $b^T y < 1$

Proof. Define

$$\tilde{A} = \begin{pmatrix} A \\ -c^T \end{pmatrix} \qquad \tilde{b} = \begin{pmatrix} b \\ -1 \end{pmatrix}$$

Then proving the following suffices

$$\nexists x \in \mathbb{R}^n \,,\, \tilde{A}x \leq \tilde{b} \quad x \geq 0 \qquad \iff \qquad \exists y \in \mathbb{R}^m \,,\, y \geq 0 \quad A^T y \geq c \quad b^T y < 1$$

(a) ($\Leftarrow$) let $\tilde{y} = \begin{pmatrix} y & 1 \end{pmatrix}$, then,

$$\exists \tilde{y} \in \mathbb{R}^{m+1} \,,\, \tilde{y} \geq 0 \quad \tilde{A}^T \tilde{y} \geq 0 \quad \tilde{b}^T \tilde{y} < 0$$

lhs follows by Farkas's lemma.

(b) ($\Rightarrow$) Assume lhs true, by Farkas's lemma,

$$\exists \tilde{y} \in \mathbb{R}^{m+1} \,,\, \tilde{y} \geq 0 \quad \tilde{A}^T \tilde{y} \geq 0 \quad \tilde{b}^T \tilde{y} < 0$$

Let $\tilde{y} = \begin{pmatrix} y & a \end{pmatrix}$ satisfying the condition above. If $a \neq 0$, let $\tilde{y}' = {}^1/_a \tilde{y} = \begin{pmatrix} {}^1/_a y & 1 \end{pmatrix}$. Since $a > 0$,

$$\tilde{A}\tilde{y}' = \frac{1}{a}\tilde{A}\tilde{y} \geq 0 \qquad \tilde{b}^T \tilde{y}' = \frac{1}{a}\tilde{b}^T \tilde{y} < 0$$

Since $\tilde{A}\tilde{y}' = A^T y - c$ and $\tilde{b}^T \tilde{y}' = b^T y - 1$, then $A^T y \geq c$ and $b^T y < 1$. This proves rhs true. Now we consider the case where $a = 0$. We claim $a = 0$ is not possible. Since if $a = 0$, then we note y in $\tilde{y}$ statifies $A^T y \geq 0$ and $b^T y \leq 0$

$$\exists y \in \mathbb{R}^m \,,\, y \geq 0 \quad A^T y \geq 0 \quad b^T y \leq 0 \qquad \overset{Farkas}{\rightarrow} \qquad \nexists x \in \mathbb{R}^n \,,\, x \geq 0 \quad Ax \leq b$$

which contradicts the assumption that $\{x : Ax \leq 0 \quad x \geq 0\} \neq \emptyset$

$\square$

Question 2: Primal-Dual Algorithm Given directed $G = (V, E)$ where $w_e > 0$. Given $s, t \in V$ and a flow from s to t. weight of flow $w(f) = \sum_{e \in E} w_e f_e$. Objective to send one unit of flow from s to t so that weight of flow minimized

1. Primal

$$\begin{aligned}
\min \quad & \sum_{e \in E} w_e f_e \\
st & \\
\forall u \in V \setminus \{s,t\} \quad & \sum_{(v,u)\in E} f_{vu} - \sum_{(u,v)\in E} f_{uv} = 0 \\
& \sum_{(v,s)\in E} f_{vs} - \sum_{(s,v)\in E} f_{sv} = -1 \\
& \sum_{(v,t)\in E} f_{vt} - \sum_{(t,v)\in E} f_{tv} = 1 \\
\forall e \in E \quad & f_e \geq 0
\end{aligned}$$

Dual is given by

$$\begin{aligned}
\max \quad & y_t - y_s \\
st & \\
\forall (u,v) \in E \quad & y_v - y_u \leq w_{uv}
\end{aligned}$$

2. Given feasible y, characterize when there exists feasible flow f such that y, f satisfies complementary slackness conditions

Feasible f, y are optimal if following complementary slackness condition holds

$$\forall (u,v) \in E \quad (w_{uv} + y_u - y_v) f_{uv} = 0$$

Therefore, *f, y optimal if and only if there exists a flow f such that $f_{uv} \neq 0$ for any $w_{uv} + y_u - y_v = 0$, i.e. flow only operates on tight edges*

3. Give a primal-dual algorithm which finds a pair of feasible solutions primal and dual linear programs

> $y \leftarrow 0$
> $S \leftarrow \{s\}$
> **while** $t \notin S$ **do**
> $\delta \leftarrow \min\{w_{uv} + y_u - y_v : u \in S \wedge v \notin S \wedge (u,v) \in E\}$
> $Q = \{v \notin S : (u,v) \in E \wedge u \in S \wedge w_{uv} + (y_u - \delta) - y_v = 0\}$
>   **for** $u \in S$ **do**
> $y_u \leftarrow y_u - \delta$
> $S \leftarrow S \cup Q$
> $p \leftarrow$ a path from s to t consists of edges in S
> $f_e \leftarrow \begin{cases} 1 & e \in p \\ 0 & otherwise \end{cases}$
> **return** (f, y)

We will prove the following points

(a) y remain feasible in each iteration

Proof. For all $u \in V$, $y_u = 0$ before loop starts and hence is a feasible solution to the dual. At any iteration, assume y is feasible we prove that $y'_u = y_u - \delta$ $(u \in S)$ $y'_u = 0$ $(u \notin S)$ remains feasible. For any $(u, v) \in E$ such that $u, v \in S$, we have

$$w_{uv} + y'_u - y'_v = w_{uv} + y_u - \delta - y_v + \delta = w_{uv} + y_u - y_v \geq 0$$

For any $(u, v) \in E$ such that $u \in S$ and $v \notin S$, then

$$\begin{aligned} w_{uv} + y'_u - y'_v = w_{uv} + y_u - \delta - y_v = &(w_{uv} + y_u - y_v) \\ &- \min\{w_{uv} + y_u - y_v : u \in S \wedge v \notin S \wedge (u, v) \in E\} \geq 0 \end{aligned}$$

For any $(u, v) \in E$ such that $u \notin S$, $v \in S$, then

$$w_{uv} + y'_u - y'_v = w_{uv} + y_u - y_v + \delta \geq \delta \geq 0$$

For any $(u, v) \in E$ such that $u, v \notin S$, $y'_u = y'_v = 0$ satisfies dual constraints. □

(b) complementary slackness is maintained, i.e. *for all* $e \in S$, $w_{uv} + y_u - y_v = 0$

Proof. Assume S consists of only 'tight' edges, we now prove that edges in S remains tight after updating y_u. Let $(u, v) \in S$, then $w_{uv} + y_u - y_v = 0$, updating $y'_u = y_u - \delta$ and $y'_v = y_v = \delta$, we have

$$w_{uv} + y'_v - y'_v = w_{uv} - \delta - (y_v - \delta) = w_{uv} + y_u - y_v = 0$$

Also note Q consists of edges that becomes 'tight' after updating $u \in S$ by $-\delta$. Therefore $S \cup A$ contains 'tight' edges only □

(c) dual objective strictly increases in each iteration

Proof. Inside the loop $y'_t = y_t = 0$, and $y'_s = y_s - \delta$. Therefore

$$y'_t - y'_s = y_t - y_s + \delta$$

the dual increases by δ during each iteration □

(d) algorithm terminates

Proof. The loop terminates because in each iteration $|S|$ increases by $|Q| \geq 1$ since we always pick a δ that makes at least one edge $(u, v) \in E$, $u \in S$, $v \notin S$ 'tight'. The loop body runs in polynomial time, therefore the algorithm terminates in polynomial time □

(e) the algorithm is correct

Proof. The returned primal solution is a feasible solution in that it represent a flow that pushes one unit from s to t while satisfying conservation constraint. The corresponding dual solution y is also feasible and f, y satisfies the complementary slackness constraint and hence f, y are optimal solution. Note f is an integral solution □

(f) the algorithm has runtime of $O(nm)$

Proof. The loop runs at most n iterations. At each iteration of the for loop, finding δ and Q requires $O(m)$ and updating y requires $O(n)$, hence $O(m+n)$ for each iteration. In total, the algorithm termiantes in $O(n(m+n))$ $\square$

Question 3: Simple Randomized Approximation Algorithm Given directed $G = (V, E)$ with $w_e \geq 0$ for all $e \in E$. Goal is to partition vertices into k disjoint sets $V_1, \cdots, V_k$ such that the objective is maximized

$$\sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \sum_{u,v:(u,v)\in E, u\in V_i, v\in V_j} w_{uv}$$

i.e. total weights of edges that go from V_i from V_j such that $i < j$. Give a randomized polynomial approximation algorithm that achieves approximation ratio of $\frac{k-1}{2k}$ in expectation. Justify approx ratio and running time

Proof. The algorithm works by assigning each $v \in V$ to $V_1, \cdots, V_k$ with uniform probability, i.e. $\frac{1}{k}$. We prove that such assignment guarantees a $\frac{k-1}{2k}$-approximation algorithm. Let $X : V \to \{1, \cdots, k\}$ be a random variable representing the random assignment of v to $V_1, \cdots, V_k$. Note

$$\mathsf{P}\left(X_u < X_v\right) = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \mathsf{P}\left(X_u = i, X_v = j\right) = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \frac{1}{k^2} = \frac{k-1}{2k}$$

Let $Y_{uv} = \mathbb{1}_{u\in V_i, v\in V_j, i<j}$. Note

$$\mathsf{E}\left\{Y_{uv}\right\} = \mathsf{P}\left(X_u < X_v\right) = \frac{k-1}{2k}$$

We then derive expectation of objective as follows

$$\mathsf{E}\left\{\sum_{(u,v)\in E} Y_{uv} w_{uv}\right\} = \sum_{(u,v)\in E} w_{uv} \mathsf{E}\left\{Y_{uv}\right\} = \frac{k-1}{2k} \sum_{(u,v)\in E} w_{uv} \geq \frac{k-1}{2k}\, opt$$

where last inequality follow from the fact that optimal assignment is upper bounded by the sum of weight of all edges. The upper bound on expectation of objective is *opt* by definition. Therefore, the algorithm is $\frac{k-1}{2k}$-approximation algorithm. The algorithm has polynomial complexity $O(m)$, where $m = |E|$ since the algorithm simply samples uniformly from $\{1, \cdots, k\}$. $\square$