HW5 Solutions

49. Gamma function

a) Show that $\Gamma(1)=1$

$$\Gamma(1)=\int_0^\infty u^{1-1}e^{-u}du=\int_0^\infty e^{-u}du=\left[-e^{-u}\right]_0^\infty=1-e^{-\infty}=1$$

b) Show that $\Gamma(x+1)=x\,\Gamma(x)$

$$\Gamma(x+1)=\int_0^\infty u^{x+1-1}e^{-u}du=\int_0^\infty u^x e^{-u}du$$

Let $s=u^x, dt=e^{-u}du$ then $ds=xu^{x-1}du, t=-e^{-u}$ and

$$\Gamma(x+1)=\int_0^\infty u^x e^{-u}du=\int_0^\infty sdt=[st]_0^\infty-\int_0^\infty tds=\left[u^x(-e^{-u})\right]_0^\infty-\int_0^\infty xu^{x-1}(-e^{-u})du$$

Hence $\Gamma(x+1)=0+x\int_0^\infty u^{x-1}e^{-u}du=x\,\Gamma(x)$ since $0^x=0$ and $\infty^x e^{-\infty}=0$

c) Show that $\Gamma(n)=(n-1)!$

Using a) and b) we have:

$$\Gamma(n)=(n-1)\Gamma(n-1)=(n-1)(n-2)\Gamma(n-2)=\ldots=(n-1)(n-2)..1\,\Gamma(1)=(n-1)!$$

d) Use $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ to show that if n is an odd integer $\Gamma\left(\frac{n}{2}\right)=\dfrac{\sqrt{\pi}(n-1)!}{2^{n-1}(\frac{n-1}{2})!}$

Applying b) again:

$$\Gamma\left(\frac{n}{2}\right)=(\frac{n}{2}-1)\Gamma\left(\frac{n}{2}-1\right)=(\frac{n}{2}-1)(\frac{n}{2}-2)\Gamma\left(\frac{n}{2}-2\right)=\ldots=(\frac{n}{2}-1)(\frac{n}{2}-2)..(\frac{1}{2})\Gamma\left(\frac{1}{2}\right)$$

Using $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ we have:

$$\Gamma\left(\frac{n}{2}\right)=(\frac{n}{2}-1)(\frac{n}{2}-2)..(\frac{1}{2})\sqrt{\pi}=(\frac{n-2}{2})(\frac{n-4}{2})..(\frac{1}{2})\sqrt{\pi}$$

We multiply and divide by $(\frac{n-1}{2})(\frac{n-3}{2})(\frac{n-5}{2})..(\frac{2}{2})$ to get

$$\Gamma\left(\frac{n}{2}\right)=\frac{(\frac{n-1}{2})(\frac{n-2}{2})(\frac{n-3}{2})(\frac{n-4}{2})..(\frac{2}{2})(\frac{1}{2})\sqrt{\pi}}{(\frac{n-1}{2})(\frac{n-3}{2})(\frac{n-5}{2})..(\frac{2}{2})}=\frac{\frac{(n-1)!}{2^{n-1}}\sqrt{\pi}}{(\frac{n-1}{2})!}=\frac{(n-1)!\sqrt{\pi}}{2^{n-1}(\frac{n-1}{2})!}$$

53. Let $X\sim N(5,10^2)$ Find:

a) $P(X>10)$

$$P(X>10)=P(\frac{X-5}{10}>\frac{10-5}{10})=P(Z>\frac{1}{2})=1-P(Z\leqslant\frac{1}{2})=1-0.6915=0.3085$$

When $Z=\dfrac{X-5}{10}\sim N(0,1)$

b) $P(-20<X<15)$

$$P(-20<X<15)=P(\frac{-20-5}{10}<\frac{X-5}{10}<\frac{15-5}{10})=P(-2.5<Z<1)=P(Z\leqslant 1)-P(Z\leqslant -2.5)$$

From Table 2 in Appendix B, we have $P(Z\leqslant 1)=0.8413$,
$P(Z\leqslant -2.5)=P(Z\geqslant 2.5)=1-P(Z\leqslant -2.5)=1-0.9938=0.0062$

Hence $P(-20<X<15)=P(Z\leqslant 1)-P(Z\leqslant -2.5)=0.8413-0.0062=0.8351$

c) The value of x such that $P(X>x)=0.05$

We know that $P(X>x)=P(\frac{X-5}{10}>\frac{x-5}{10})=P(Z>\frac{x-5}{10})=1-P(Z\leqslant \frac{x-5}{10})$.

So we need to find x such that $P(Z\leqslant \frac{x-5}{10})=1-0.05=0.95$

From Table 2 (inside out), we have $\frac{x-5}{10}=1.645$. Hence $x=16.45+5=21.45$

60. Find pdf (density function) of $Y=e^Z$ where $Z\sim N(\mu,\sigma^2)$

Here we find pdf of Y when $Y=g(Z)$ with $g(z)=e^z$. In order to apply Prop. B, we have to find:

- the inverse function of g: do that by solving this equation: $y=e^z$ we get $z=\ln(y)$. So $g^{-1}(y)=\ln(y)$ and $\frac{d}{dy}g^{-1}(y)=\frac{1}{y}$
- pdf of Z: $f_Z(z)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(z-\mu)^2/2\sigma^2},\ -\infty<z<\infty$

From Prop. B, we have:

$$f_Y(y)=f_Z(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)|=f_Z(\ln(y))|\frac{1}{y}|=\frac{1}{\sigma\sqrt{2\pi}}e^{-(\ln(y)-\mu)^2/2\sigma^2}\frac{1}{y},\ 0<y<\infty$$

67. Weibull cdf is

$$F(x)=1-e^{-(\frac{x}{\alpha})^{\beta}},\quad x\geqslant 0,\ \alpha>0,\ \beta>0$$

a) Find the density function (pdf):

Take the derivative of cdf F(x) we get pdf f(x):

$$f(x)=F'(x)=-e^{-(\frac{x}{\alpha})^{\beta}}\beta(-(\frac{x}{\alpha})^{\beta-1})\frac{1}{\alpha}=\frac{\beta}{\alpha^{\beta}}x^{\beta-1}e^{-(\frac{x}{\alpha})^{\beta}},\quad x\geqslant 0,\ \alpha>0,\ \beta>0$$

b) Show that if W follows a Weibull distribution, then $X=(\frac{W}{\alpha})^{\beta}$ follows an exponential distribution.

First, we find pdf of X when $X=g(W)$ with $g(w)=(\frac{w}{\alpha})^{\beta}$. In order to apply Prop. B, we have to find:

- the inverse function of g: do that by solving this equation: $x=(\frac{w}{\alpha})^{\beta}$ we get $w=\alpha x^{\frac{1}{\beta}}$. So $g^{-1}(x)=\alpha x^{\frac{1}{\beta}}$ and $\frac{d}{dy}g^{-1}(x)=\frac{\alpha}{\beta}x^{\frac{1}{\beta}-1}$
- pdf of W: $f_W(w)=\frac{\beta}{\alpha^{\beta}}w^{\beta-1}e^{-(\frac{w}{\alpha})^{\beta}},\quad w\geqslant 0,\ \alpha>0,\ \beta>0$

From Prop. B, we have:

$$f_X(x)=f_W(g^{-1}(x))|\frac{d}{dx}g^{-1}(x)|=f_W(\alpha x^{\frac{1}{\beta}})|\frac{\alpha}{\beta}x^{\frac{1}{\beta}-1}|=\frac{\beta}{\alpha^\beta}(\alpha x^{\frac{1}{\beta}})^{\beta-1}e^{-(\frac{(\alpha x^{\frac{1}{\beta}})}{\alpha})^\beta}|\frac{\alpha}{\beta}x^{\frac{1}{\beta}-1}|$$

So $f_X(x)=\frac{\beta}{\alpha^\beta}\alpha^{\beta-1}(x^{\frac{1}{\beta}})^{\beta-1}e^{-(x^{\frac{1}{\beta}})^\beta}\frac{\alpha}{\beta}x^{\frac{1}{\beta}-1}=\frac{\beta}{\alpha}\frac{\alpha}{\beta}x^{1-\frac{1}{\beta}}x^{\frac{1}{\beta}-1}e^{-x}=e^{-x}, \quad x\geqslant 0$

Which means X follows an exponential distribution.

c) How could Weibull distribution random variables be generated from a uniform random generator number generator?

Let W is a Weibull distribution, then cdf of W is

$F_W(w)=1-e^{-(\frac{w}{\alpha})^\beta}, \quad w\geqslant 0,\ \alpha>0,\ \beta>0$

Using Prop. C we know that if we set $U=F_W(W)$ then U has a uniform distribution on [0,1]. Hence $W=F_W^{-1}(U)$ is generated from uniform random variable U. We have to find F_W^{-1} to complete our job. Doing that by solving this equation for w:

$u=1-e^{-(\frac{w}{\alpha})^\beta}, \quad w\geqslant 0,\ \alpha>0,\ \beta>0$. From that, we have $-(\frac{w}{\alpha})^\beta=\ln(1-u)$ or $w=\alpha(-\ln(1-u))^{\frac{1}{\beta}}$. So $F_W^{-1}(u)=\alpha(-\ln(1-u))^{\frac{1}{\beta}}$ and $W=\alpha(-\ln(1-U))^{\frac{1}{\beta}}$. We generate U first and then use above formula to get W.