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1.1 Ordinary differential equations and Lipschitz condition

1. **(Goal)** Approximate solution to

$$\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y}) \quad \text{with initial condition} \quad \boldsymbol{y}(t_0) = \boldsymbol{y}_0$$

   where $t > t_0$ and $\boldsymbol{f} : [t_0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is a sufficiently well behaved function

2. **(Lipschitz condition)** Given $\boldsymbol{f}$ and norm $\|\cdot\|$, the Lipschitz condition is defined by

$$\|\boldsymbol{f}(t, \boldsymbol{x}) - \boldsymbol{f}(t, \boldsymbol{y})\| \leq \lambda \|\boldsymbol{x} - \boldsymbol{y}\| \qquad \text{for all} \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d \ , \ t > t_0$$

   where $\lambda \in \mathbb{R}$ is called Lipschitz constant.

3. **(Picard Lindelof theorem)** Consider initial value problem

$$y'(t) = f(t, y(t)) \quad y(t_0) = y_0$$

   If f is uniformly Lipschitz continous in y and continous in t, then for some $\epsilon > 0$, there exists unique solution $y(t)$ to the initial value problem on the interval $[t_0 - \epsilon, t_0 + \epsilon]$

4. **(Analytic function)** A function $\boldsymbol{f}$ is an analytic function if it is a function that is locally given by a convergent power series, i.e. an infinitely differentiable function such that at any point $(t, \boldsymbol{y}_0) \in [0, \infty) \times \mathbb{R}^d$ in its domain, the Taylor series converges to $\boldsymbol{f}(\boldsymbol{x})$ for $\boldsymbol{x}$ in a neighborhood of $(t, \boldsymbol{y}_0)$.

   (a) (example) polynomial, exponential, trigonometric, logarithm, power function

   (b) (note) if $\boldsymbol{f}$ is analytic, solution $\boldsymbol{y}$ to the initial value problem is also analytic

5. **(Taylor Expansion)** Given $f \in C^\infty(\mathbb{R})$, the Taylor expansion of f at a is given by

$$\begin{aligned} T(a) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \end{aligned}$$

6. **(Taylor's Theorem)** Let $k \geq 1$ and function $f : \mathbb{R} \to \mathbb{R}$ be k times differentiable at a point $a \in \mathbb{R}$, then exists $h_k : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n + h_k(x)(x-a)^k$$

   and $\lim_{x \to a} h_k(x) = 0$, i.e. the reminder term $R_k(x) = f(x) - P_k(x)$ is asymptotically trivial. If f is $k+1$ times differentiable on the open interval and f^k continous on the closed interval $[a, x]$, then the Lagrange remind is given by

$$R_k(x) = \frac{f^{k+1}(\zeta)}{(k+1)!}(x-a)^{k+1}$$

   for soem $\zeta \in [a, x]$ by the mean value theorem

7. **($\mathcal{O}$ notation)** $f(x) = \mathcal{O}(g(x))$ describes asymptotic behavior of function f

   (a) (as $x \to \infty$) if there exists $M \geq 0$ and $x_0 \in \mathbb{R}$ such that $|f(x)| \leq Mg(x)$ for all $x > x_0$.

   (b) (as $x \to a$) if there exists $M \geq 0$ and $\delta \in \mathbb{R}$ such that $|f(x)| \leq Mg(x)$ when $0 < |x - a| < \delta$. Alternatively we can say

$$\limsup_{x \to a} \left| \frac{f(x)}{g(x)} \right| < \infty$$

1.2 Euler's method

Definition. ***(Euler's Method)*** *Given initial value problem $\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y})$ for $t \geq t_0$ and initial value $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$. If we assume $\boldsymbol{y}'(t) = \boldsymbol{f}(t, \boldsymbol{y}(t)) \approx \boldsymbol{f}(t_0, \boldsymbol{y}(t_0))$ for $t \in [t_0, t_0 + h)$ (i.e. deriviative in $[t_n, t_{n+1}]$ is approximated by value of derivative at t_n) for some sufficiently small time step $h > 0$, we can approximate the value of $\boldsymbol{y}(t)$ by*

$$\begin{aligned} \boldsymbol{y}(\mathbf{t}) &= \boldsymbol{y}(t_0) + \int_{t_0}^{t} \boldsymbol{f}(\tau, \boldsymbol{y}(\tau)) d\tau \\ &\approx \boldsymbol{y}_0 + (t - t_0)\boldsymbol{f}(t_0, \boldsymbol{y}_0) \end{aligned}$$

Given a sequence of times $(t_n)_{n \in \mathbb{N}} = (t_0, t_0 + h, \cdots)$ we have numerical approximation $(\boldsymbol{y}_n)_{n\in\mathbb{N}}$ by

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{f}(t_n, \boldsymbol{y}_n)$$

1. *(intuition) euler's method is a time-stepping numerical method that covers interval by an equidistant grid and produe numerical solution at the grid points. we can show that euler's method is convergent, i.e. as $h \to 0$, grid is refined, the numerical solution tends to exact solution*

Definition. ***(convergent method)*** *Given a time-stepping numerical method on a compact inteval $[t_0, t_0 + t^*]$, we can compute numerical solutions dependent upon h*

$$\boldsymbol{y}_n = \boldsymbol{y}_{n,h} \quad \textit{for} \quad n = 0, 1, \cdots, \lfloor t * /h \rfloor$$

A method is said to be convergent if for every ODE with Lipschitz function $\boldsymbol{f}$, the numerical solution tends to the true solution as the grid becomes increasingly fine. More rigorously, if every ODE with Lipschitz function $\boldsymbol{y}$ and for every $t^* > 0$, then following holds

$$\lim_{h \to 0^+} \max_{n=0,1,\cdots,\lfloor t*/h \rfloor} \|\boldsymbol{y}_{n,h} - \boldsymbol{y}(t_n)\| = 0$$

Theorem. ***(Euler's method is convergent)***

Proof. Assume $\boldsymbol{f}$ and therefore also $\boldsymbol{y}$ is analytic, i.e. convergent Taylor expansion. Let $\boldsymbol{e}_{n,h} = \boldsymbol{y}_{n,h} - \boldsymbol{y}(t_n)$ be the numerical error. Show $\lim_{h\to 0^+} \max_n \|\boldsymbol{e}_{n,h}\| = 0$. By Taylor's theorem

$$\boldsymbol{y}(t_{n+1}) = \boldsymbol{y}(t_n) + h\boldsymbol{y}'(t_n) + \mathcal{O}(h^2) = \boldsymbol{y}(t_n) + h\boldsymbol{f}(t_n, \boldsymbol{y}(t_n)) + \mathcal{O}(h^2)$$

given $\boldsymbol{y}$ continously differentiable, $\mathcal{O}(h^2)$ can be bounded uniformly for all $h > 0$ by a term ch^2 for some $c > 0$. Subtract previous from iterative formula of euler's method

$$\boldsymbol{e}_{n+1,h} = \boldsymbol{e}_{n,h} + h\left(\boldsymbol{f}(t_n, \boldsymbol{y}(t_n) + \boldsymbol{e}_{n,h}) - \boldsymbol{f}(t_m, \boldsymbol{y}(t_n))\right) + \mathcal{O}(h^2)$$

By triangle inequality and Lipschitz condition

$$\begin{aligned} \|\boldsymbol{e}_{n+1,h}\| &\leq \|\boldsymbol{e}_{n,h}\| + h\,\|\boldsymbol{f}(t_n, \boldsymbol{y}(t_n) + \boldsymbol{e}_{n,h}) - \boldsymbol{f}(t_n, \boldsymbol{y}(t_n))\| + ch^2 \\ &\leq (1 + h\lambda)\,\|\boldsymbol{e}_{n,h}\| + ch^2 \end{aligned}$$

for $n = 0, 1, \cdots, \lfloor t^*/h \rfloor - 1$. By induction on n, we can show $\|\boldsymbol{e}_{n,h}\| \leq \frac{c}{\lambda} h\left((1 + h\lambda)^n - 1\right)$. Since $1 + h\lambda < e^{h\lambda}$ we have $(1 + h\lambda)^n < e^{nh\lambda} < e^{\lfloor t*/h \rfloor h\lambda} \leq e^{t^*\lambda}$. Therefore

$$\|\boldsymbol{e}_{n,h}\| \leq \frac{c}{\lambda}(e^{t^*\lambda} - 1)h$$

for $n = 0, 1, \cdots, \lfloor t * /h \rfloor$. This is an upper bound on the error that is independent of h, hence $\lim_{h\to 0} \|\boldsymbol{e}_{n,h}\| = 0$. from which we can infer that error decays globally as $\mathcal{O}(h)$ □

Definition. ***(order p method)*** *Given arbitrary time-stepping method*

$$\boldsymbol{y}_{n+1} = \boldsymbol{\mathcal{Y}}_n(\boldsymbol{f}, h.\boldsymbol{y}_0, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_n) \qquad n = 0, 1, \cdots$$

for initial value problem, it is of order p if

$$\boldsymbol{y}(t_{n+1}) - \boldsymbol{\mathcal{Y}}_n(\boldsymbol{f}, h, \boldsymbol{y}(t_0), \boldsymbol{y}(t_1), \cdots, \boldsymbol{y}(t_n)) = \mathcal{O}(h^{p+1})$$

for every analytic $\boldsymbol{f}$ and $n = 0, 1, \cdots$. Intuitively, a method is of order p if it recovers exactly every polynomial oslution of degrees p or less.

1. *(intuition) order of a method gives information about local behavior, i.e. advancing from t_n to t_{n+1} where $h > 0$ is sufficiently small, we are incurring an error of $\mathcal{O}(h^{p+1})$. Generally want the the global (convergence) behavior of the method instead.*

2. ***(fact)** euler's method is order 1*

   Proof. Euler's method can be written as $\boldsymbol{y}_{n+1} - (\boldsymbol{y}_n + h\boldsymbol{f}(t_n, \boldsymbol{y}_n)) = 0$. Replace $\boldsymbol{y}_k$ by $\boldsymbol{y}(t_k)$ and expand terms of Taylor series about t_n we have

$$\boldsymbol{y}(t_{n+1}) - (\boldsymbol{y}(t_n) + h\boldsymbol{f}(t_n, \boldsymbol{y}(t_n))) = \big(\boldsymbol{y}(t_n) + h\boldsymbol{y}'(t_n) + \mathcal{O}(h^2)\big) - \big(\boldsymbol{y}(t_n) + h\boldsymbol{y}'(t_n)\big) = \mathcal{O}(h^2)$$

   □

1.3 The trapezoidal rule

Definition. ***(Trapezoidal Rule)*** *Instead of approximating derivative by a constant in $[t_n, t_{n+1}]$, namely by its value at t_n, the trapezoidal rule approximates the value of the derivate by average of values at the endpoints. We can approximate solution $\boldsymbol{y}(t)$ by*

$$\begin{aligned}\boldsymbol{y}(t) &= \boldsymbol{y}(t_n) + \int_{t_n}^{t} \boldsymbol{f}(\tau, \boldsymbol{f}(\tau))d\tau \\ &\approx \boldsymbol{y}(t_n) + \frac{1}{2}(t - t_n)\left(\boldsymbol{f}(t_n, \boldsymbol{y}(t_n)) + \boldsymbol{f}(t, \boldsymbol{y}(t))\right)\end{aligned}$$

Given a sequence of times $(t_n)_{n\in\mathbb{N}} = (t_0, t_0 + h, \cdots)$ we have numerical approximation $(\boldsymbol{y}_n)_{n\in\mathbb{N}}$ by

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \frac{1}{2}h\left(\boldsymbol{f}(t_n, \boldsymbol{y}_n) + \boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})\right)$$

1. ***(theorem)** order of trapezoidal rule is 2*

   Proof. Compute by performing Taylor expansion on $\boldsymbol{y}(t_{n+1})$ and $\boldsymbol{y}'(t_{n+1})$ about t_n

$$\boldsymbol{y}(t_{n+1}) - \left\{\boldsymbol{y}(t_n) + \frac{1}{2}h\left\{\boldsymbol{f}(t_n, \boldsymbol{y}(t_n)) + \boldsymbol{f}(t_{n+1}, \boldsymbol{y}(t_{n+1}))\right\}\right\} = \mathcal{O}(h^3)$$

   □

2. ***(theorem)** trapezoidal rule is convergent*

   Proof. Detail of proof here . We can show error is bounded by

$$\|\boldsymbol{e}_{n,h}\| \le \frac{ch^2}{\lambda} exp\left(\frac{t^*\lambda}{1 - \frac{1}{2}h\lambda}\right)$$

   from which we can infer that error decays globally as $\mathcal{O}(h^2)$ □

3. *(note) euler's method is explicit, since we can compute $\boldsymbol{y}_{n+1}$ with a few arithmetic operations by computing $\boldsymbol{f}$, a function of a known $\boldsymbol{y}_n$. Trapezoidal rule is implicit, i.e. finding $\boldsymbol{y}_{n+1}$ is not trivial and $\boldsymbol{f}$ is a function of both $\boldsymbol{y}_n$ and $\boldsymbol{y}_{n+1}$. We might need to solve a nonlinear equation of $\boldsymbol{y}_{n+1}$*

$$\boldsymbol{y}_{n+1} - \frac{1}{2}h\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1}) = \boldsymbol{v}$$

   where $\boldsymbol{v} = \boldsymbol{y}_n + \frac{1}{2}h\boldsymbol{f}(t_n, \boldsymbol{y}_n)$ can be evaluated easily from assumptions.

1.4 The theta method

Definition. ***(theta method)*** *is a generalization of Euler's method ($\theta = 1$) and the trapezoidal rule ($\theta = 1/2$), whereby the derivates are assumed to be piecewise constant and provided by a linear combination of derivatives at the endpoints of each interval. The numerical approximates are,*

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\left(\theta \boldsymbol{f}(t_n, \boldsymbol{y}_n) + (1-\theta)\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})\right) \qquad n = 0, 1, \cdots$$

for some fixed $\theta \in [0, 1]$

1. *(fact) theta method is explicit for $\theta = 1$ and implicit otherwise*
2. ***(theorem)*** *theta method is of order 2 for $\theta = 1/2$ and order 1 otherwise.*
3. ***(theorem)*** *theta method is convergent for every $\theta \in [0, 1]$*

2 Multistep Method

3 8 Finite Differences Schemes

3.1 8.1 Finite differences

Definition. ***(finite difference operators)*** *Given real sequences $\boldsymbol{z} = \{z_k\}_{k\in\mathbb{Z}}$, finite difference operators map the space $\mathbb{R}^{\mathbb{Z}}$ of all such sequences to itself.*

1. ***(shift)*** $(\boldsymbol{\mathcal{E}z})_k = z_{k+1}$
2. ***(forward difference)*** $(\triangle_+\boldsymbol{z})_k = z_{k+1} - z_k$
3. ***(backward difference)*** $(\triangle_-\boldsymbol{z})_k = z_k - z_{k-1}$
4. ***(central difference)*** $(\triangle_0\boldsymbol{z})_k = z_{k+\frac{1}{2}} - z_{k-\frac{1}{2}}$
5. ***(averaging)*** $(\boldsymbol{\Gamma}_0\boldsymbol{z})_k = \frac{1}{2}(z_{k-\frac{1}{2}} + z_{k+\frac{1}{2}})$

1. *(note) central difference and averaging operator are not maps of $\boldsymbol{z}$ onto itself.*