Direct Sum Definition

Definition. ***Summation of Sets*** *If S_1 and S_2 are nonemtpy subsets of a vector space V, then the **sum** of S_1 and S_2, denoted $S_1 + S_2$, is the set*

$$\{x + y : x \in S_1 \text{ and } y \in S_2\}$$

1. *$W_1 + W_2$ is a subspace of V containing both W_1 and W_2*
2. *If for a subset $S \subseteq V$, $W_1 \subseteq S$ and $W_2 \subseteq S$, then $W_1 + W_2 \subseteq S$*

Definition. ***Direct Sum*** *A vector space V is called the direct sum of W_1 and W_2, denoted as $V = W_1 \oplus W_2$, if W_1 and W_2 are subspaces of V such that*

1. *$V = W_1 + W_2$*
2. *$W_1 \cap W_2 = \{0\}$ (implies uniqueness)*

More generally, assume $W_1, \cdots, W_k$ are subspaces of V, then $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ if

1. *$V = W_1 + \cdots + W_k$*
2. *$W_i \bigcap (\sum_{j \neq i} W_j) = \{0\}$*

1. *Direct sum of the set of upper triangular like matrices and lower triangular matrices is $M_{m \times n}(F)$*
2. *The trick of decomposing vector space into direct sums is that the intersection of subsets yield the zero vector*

5.4 Invariant Subspaces and Direct Sum

Chapter 7 Canonical Forms

7.1 The Jordan Canonical Form

Definition. ***Jordan Block and Jordan Canonical Form*** *Select ordered basis whose union is an ordered basis β, the Jordan caconical basis for T, for V such that*

$$[T]_\beta = \begin{pmatrix} A_1 & O & \cdots & O \\ O & A_2 & \cdots & O \\ \vdots & \vdots & & \vdots \\ O & O & \cdots & A_k \end{pmatrix}$$

where A_i are jordan block corresponding to λ

$$A_i = (\lambda) \qquad or \qquad A_i = \begin{pmatrix} \lambda & 1 & O & \cdots & O & O \\ O & \lambda & 1 & \cdots & O & O \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ O & O & O & \cdots & \lambda & 1 \\ O & O & O & \cdots & O & \lambda \end{pmatrix}$$

Definition. ***Generalized Eigenvector*** *Let T be a linear operator on a vector space V, and let λ be a scalar. A nonzero vector x in V is called a generalized eigenvector of T corresponding to λ if $(T-\lambda I)^p(x) = 0$ for some positive integer p*

1. *For v in a Jordan canonical basis for T, $(T-\lambda I)^p(v) = 0$ for sufficiently large p. Eigenvectors satisfy this condition for $p = 1$*

2. *If x is a generalized eigenvector of T corresponding to λ, and p is smallest positive integer for which $(T-\lambda I)^p(x) = 0$, then $(T-\lambda I)^{p-1}(x) = 0$ is an eigenvector of T corresponding to λ*

$$(T-\lambda I)(v) = 0 \qquad \text{where eigenvector } v = (T-\lambda I)^{p-1}(x) \neq 0$$

Definition. ***Generalized Eigenspace*** *Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. The generalized eigenspace of T corresponds to λ, denoted K_λ, is the subset of V defined by*

$$K_\lambda = \{x \in V : (T-\lambda I)^p(x) = 0 \text{ for some positive integer } p\} = \bigcup_{p\geq 1} N((T-\lambda I)^p)$$

1. *Note*
$$N(U) \subseteq N(U^2) \subseteq \cdots \subseteq N(U^k) \subseteq N(U^{k+1}) \subseteq \cdots$$

Theorem. ***7.1 Properties of Generalized Eigenspace*** *Let T be linear operator on a vector space V, and let λ be an eigenvalue of T. Then*

1. *K_λ is a T-invariant subspace of V containing E_λ (the eigenspace of T corresponding to λ)*

2. *For any scalar $\mu \neq \lambda$, the restiction $T - \mu I$ to K_λ is one-to-one.*
   (a) *$E_\mu = N(T-\mu I) = 0$ for all $\mu \neq \lambda$, so λ is the only eigenvalue of T_{λ_k}*
   (b) *For $\mu \neq \lambda$, $K_\lambda \cap K_\mu = 0$.*

Proof. Prove 2.2
Suppose $\mu \neq \lambda$, $T - \mu I|_{K_\lambda}$ is invertible on K_λ by property 2. Then let

$$x \in K_\lambda \cap N(T-\mu I)$$

then $x = 0$. Similarly, consider large enough q, such that $K_\mu = (T-\mu I)^q$, which is invertible as it is a composition of invertible transformation. So then

$$K_\mu \cap K_\lambda = \emptyset$$

■

Theorem. ***7.2 Property of Generalized Eigenspace When Characteristic Polynomial Splits*** *Let T be a linear operator on a finite-dimensional vector space V such that the characteristic polynomial of T splits. Suppose that λ is an eigenvalue of T with multiplicity m. Then*

1. *$dim(K_\lambda) \leq m$*
2. *$K_\lambda = N((T - \lambda I)^m)$*

For proofs

1. *Use theorem 5.21 T-invariant $W \subseteq V$ have $P_{T_W}(t) \mid P_T(t)$, we have $[T_W]_\beta$ in the form of a Jordan Block, therefore*

$$h(t) = P_{T_W}(t) = (-1)^d (t - \lambda)^d$$

2. *Prove forward direction ($\Rightarrow$) Use theorem 5.23 Cayley-Hamilton $f(T) = T_0$, i.e. linear operator satisfies its characteristic equation, on T_W*

$$h(T_W) = (-1)^d (T - \lambda I)^d = T_0$$

   So $(T - \lambda I)^d(x) = 0$ for all $x \in W$ where $d \leq m$, so $K_\lambda \subseteq N((T - \lambda I)^m)$

Definition. ***Nilpotent*** *A linear operator T on a vector space V is called nilpotent if $T^p = T_0$ for some positive p. An $n \times n$ matrix A is called nilpotent if $A^p = 0$ for some positive integer p*

Lemma. ***Fitting Decomposition*** *For $S \in L(V)$, there is a unique decomposition*

$$V = W \oplus U$$

where W, U are S-invariant, and

1. *$S|_W$ invertible*
2. *$S|_U$ nilpotent that is, if $(S|_U)^q = 0$ for some $q > 0$*

Proof. Note

$$N(S) \subseteq N(S^2) \subseteq \cdots \qquad R(S) \supseteq R(S^2) \supseteq \cdots$$

have to stablize for nilpotent $S = (T - \lambda I)$, that is there exists $p > 0$, such that

$$N(S^p) = N(S^{p+1}) \qquad R(S^p) = R(S^{p+1})$$

Now let $U = N(S^p)$ and $W = R(S^p)$, both S-invariant. It is obvious that $S|_V$ is nilpotent. Also $S(W) = S(R(S^p)) = R(S^{p+1}) = R(S^p) = W$, i.e. on-to. so $S|_W$ invertible. Claim $V = W \oplus U$, let $x \in V$, then

$$S^p x \in R(S^p) = R(S^{2p})$$

So exists $y \in V$, such that $S^p x = S^{2p} y$, so then $S^p(x - S^p y) = 0$, then $x - S^p y \in N(S^p) = U$. So then

$$x = x_1 + x_2 \qquad x_1 = S^p y \in W \quad x_2 = x - x_1 \in U$$

■

Theorem. ***7.3 Generalized Eigenspace Decomposition***
Let T be a linear operator on a finite-dimensional vector space V such that the characteristic polynomial of T splits, and let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be the distinct eigenvalues of T. Then for every $x \in V$, there exists vectors $v_1 \in K_{\lambda_i}$, $1 \leq i \leq k$, such that

$$x = v_1 + v_2 + \cdots + v_k$$

Proof. Cayley-Hamilton theorem works on some special case of characteristic polynomial of the form $(t-\lambda)^d$ yields the zero transformation, which makes some subset of the vector space satisfy condition for generalized eigenspace, i.e. $(T - \lambda I)(x) = 0$ ∎

Theorem. ***7.4 Basis for Generalized Eigenspace***
Let T be a linear operator on a finite-dimensional vector space V sucht hat the characteristic polynomial of T splits, and let $\lambda_1, \cdots, \lambda_k$ be the distinct eigenvalues of T with corresponding multiplicity $m_1, \cdots, m_k$. For $1 \leq i \leq k$, let β_i be an ordered basis for K_{λ_i}. Then the following statements are true

1. *$\beta_i \cap \beta_j = \emptyset$ for $i \neq j$*
2. *$\beta = \beta_1 \cap \beta_2 \cap \cdots \cap \beta_k$ is an ordered basis for V*
3. *$dim(K_{\lambda_i}) = m_i$ for all i*

Corollary. ***Assumption for Diagonalizability*** *Let T be a linear operator on a finite-dimensional vector space V sucht hat the characteristic polynomial of T splits. Then T is diagonalizable if and only if $E_\lambda = K_\lambda$ for every eigenvalue λ of T*

Definition. ***Cycle of Generalized Eigenvectors*** *Let T be a linear operator on a vector space V, and let x be a generalized eigenvector of T corresponding to the eigenvalue λ. Then x is a generalized eigenvector of **height** p if p is the smallest positive integer for which $(T - \lambda I)^p(x) = 0$ but $(T - \lambda I)^{p-1}(x) \neq 0$. Then the ordered set,*

$$\{(T - \lambda I)^{p-1}(x), (T - \lambda I)^{p-2}(x), \cdots, (T - \lambda I)(x), x\}$$

*is called a cycle of generalized eigenvectors of T corresponding to λ. The vectors $(T - \lambda I)^{p-1}(x)$ and x are called the **initial vector** and the **end vector** of the cycle, respectively. We say that the **length** of the cycle is p (number of vectors).*

1. *The elements of a cycle are linearly independent*

Proof. Given
$$a_1(T - \lambda I)^{p-1}x + \cdots + a_{p-1}(T - \lambda I)x + a_p x = 0$$
Apply $(T - \lambda I)^{p-i}$ for $i = 1, \cdots, p$ times. For $i = 1$, we have

$$0 + \cdots + 0 + a_p(T - \lambda I)^{p-1}x = 0$$

Note $(T - \lambda I)^{p-1}x \neq 0$, so then $a_p = 0$. We can deduce $a_1 = \cdots = a_p = 0$ ∎

Theorem. ***7.5 Disjoint Union of Cycles of Generalized Eigenvectors as Jordan Canonical Basis***
Let T be a linear operator on a finite-dimensional vector space V whose characteristic polynomial splits, and suppose that β is a basis for V such that β is a disjoint union of cycles of generalized eigenvectors of T. Then the following are true

1. *For each cycle γ of generalized eigenvectors contained in β, $W = span(\gamma)$ is T-invariant and $[T_W]_\gamma$ is a Jordan block*

2. *β is a Jordan canonical basis for V*

Theorem. ***7.6 Existence Condition for a Disjoint Union of Cycles***
Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. Suppose that $\gamma_1, \gamma_2, \cdots, \gamma_q$ are cycles of generalized eigenvectors of T corresponding to λ such that the initial vectors of the γ_i's are

1. ***distinct**, and*

2. *form a **linearly independent set***

Then the γ_i's are

1. ***disjoint**, i.e. $\gamma_i \cap \gamma_j = \emptyset$ for $i \neq j$, and*

2. *their union $\gamma = \bigcup_{i=1}^{q} \gamma_i$ is **linearly independent***

Proof. To prove cycles disjoint. Assume there exits $x \in K_\lambda$ with height p, such that $x \in \gamma_1$ and $x \in \gamma_2$, without loss of generality of choice of γs'. Then $(T - \lambda I)^{p-1}x \neq 0$ is initial vector for both γ_1 and γ_2. Contradiction as we assumed that initial vectors are all distinct.
Let $\gamma = \{x_1, \cdots, x_n\}$, suppose

$$a_1x_1 + \cdots + a_nx_n = 0$$

where a_j not all zero. Let k be such that $a_k \neq 0$ and that x_k has largest height p possible. Apply $(T - \lambda I)^{p-1}$ to the equation, we get

$$\cdots + 0 + a_k(T - \lambda I)^{p-1}x_k + 0 + \cdots = 0$$

since x_j with height less than p is killed by $(T - \lambda I)^{p-1}$ and x_j whose height is larger than p has $a_j = 0$ by the choice of k, so then, $a_k(T - \lambda I)^{p-1}x_k = 0$ implies $a_k \neq 0$, contradiction. ∎

Corollary. *Every cycle of generalized eigenvectors of a linear operator is linearly independent*

Theorem. ***7.7 Existence of Disjoint Union in Generalized Eigenspace***
Let T be a linear operator on a finite-dimensional vector space V, and let λ be an eigenvalue of T. Then K_λ has an ordered basis consisting of a union of disjoint cycles of generalized eigenvectors corresponding to λ,

$$\gamma = \gamma_1 \cup \cdots \cup \gamma_q$$

where the initial vectors of $r_1, \cdots, r_q$ are eigenvectors that form a basis of E_λ

Corollary. ***$P_T(t)$ Splits Ensures Existence of Jordan Canonical Form*** *Let T be a linear opertor on a finite-dimensional vector space V whose characteristic polynomial splits. Then T has a Jordan Canonical Form*

Definition. ***Jordan Canonical Form for Matrices*** *Let $A \in M_{n\times n}(F)$ be such that the characteristic polynomial of A (and hence of L_A) splits. Then the Jordan canonical form of A is defined to be the Jordan canonical form of the linear operator L_A on F^n*

Corollary. *Let A be $n \times n$ matrix whose characteristic polynomial splits. Then A has a Jordan canonical form J, and A is similar to J*

Definition. ***Finding Basis For JCF***

1. *Compute characteristic polynomial*
2. *Compute $dim(E_{\lambda_i})$, which is the number of disjoint cycles as basis for K_{λ_i}*
3. *Find proper end vector*
4. *Take union of vectors in the disjoint union of cycles of generalized eigenvectors*

7.2 The Jordan Canonical Form II

Definition. ***Get Away***

1. *T is unique up to an ordering of eigenvalues of T*
2. *β_i for β is not unique*
3. *for each i, the number n_i of cycles that form β_i, and length p_j of each cycle, is completely determined by T*

Definition. ***Dot Diagram*** *Use an array of dots called dot diagram of T_i, where T_i is restriction of T to K_{λ_i}, to visualize each of A_i and ordered basis β_i. Suppose β_i is a disjoint union of cycles of generalized eigenvectors $\gamma_1, \cdots, \gamma_{n_i}$ with lengths $p_1 \geq \cdots \geq p_{n_i}$, respectively. The dot diagram T_i contains one dot for each vector in β_i, and the dots are configured as follows*

1. *there are n_i columns (each representing a cycle or Jordan block)*

2. *j-th column consists of p_j dots that correspond to cycle γ_j starting with initial vector at the top and continuing down to the end vector*

Note

1. *Dot diagram has dimension $p_1 \times n_i$*
2. *Let r_j be number of dots in j-th row, then $r_1 \geq r_2 \geq \cdots \geq r_{p_1}$*
3. *Dot diagram is complete determined by T and λ_i*

Theorem. ***7.9 Dots in First r Rows are a Basis for $N((T - \lambda I)^r)$***
For any positive integer r, the vectors in β_i that are associated with the dots in the first r rows of the dot diagrams of T_i constitute a basis for $N((T - \lambda_i I)^r)$. Hence the number of dots in the first r rows of the dot diagram equals nullity$((T - \lambda_i I)^r)$

1. *Implies number of dots in a row (r_j), the dot diagram, and consequently the number of Jordan blocks (n_i columns) all does not depend on choice of basis*

Corollary. ***Number of Jordan Blocks is Dimension of Eigenspace***
The dimension of E_{λ_i} is n_i. Hence in Jordan canonical form of T, the number of Jordan blocks corresponding to λ_i equals the dimension of E_{λ_i}

Theorem. ***7.10 Supplementing Preivous Theorem***
Let r_j denote number of dots in jth row of dot diagram of T_i, the restriction of T to K_{λ_i}. Then the following statements are true

1. $r_1 = dim(V) = rank(()T - \lambda_i I)$
2. $r_j = rank((T - \lambda_i I)^{j-1}) - rank((T - \lambda_i I)^j)$ *if* $j > 1$

Corollary. ***Dot Diagram of $T_i = T|_{K_{\lambda_i}}$ is Unique***
For any eigenvalue λ_i of T, the dot diagram of T_i is unique. Thus, subject to the convention that the cycles of the generalized eigenvectors for the bases of each generalized eigenspace are listed in order of decreasing length, the Jordan canonical form of a linear operator or a matrix is unique up to the ordering of the eigenvalues

Theorem. ***7.11 Similar Matrix $\iff$ Same JCF***
Let A and B be $n \times n$ matrices, each having Jordan canonical forms computed according to the conventions of this section. Then A and B are similar if and only if they have (up to an ordering of their eigenvalues) the same Jordan canonical form.

Proof. A property: If A and B similar, then exists Q such that $A = Q^{-1}BQ$, then A and B have same eigenvalues. Specifically, if $Av = \lambda v$, then

$$Q^{-1}BQv = \lambda v \qquad \iff \qquad B(Qv) = \lambda(Qv)$$

∎

Definition. ***Steps for Finding Jordan Canonical Form/Basis***

1. *Determine the shape of Jordan Canonical Form J*
   (a) *Compute characteristic polynomial*
   (b) *Determine the dot diagram for each* K_{λ_i}
   (c) *Compute* $dim(N((T - \lambda I)^i))$ *for* $i = 1, \cdots, p$
   (d) *Compute* r_j *based on* $dim(N(T - \lambda I)^i)$
   (e) *Determine the shape of Jordan Canonical Form from dot diagrams for all* K_{λ_i}
2. *Find a Jordan Canonical Basis for each* K_{λ_i}
   (a) *Compute matrix* $(T - \lambda I)^i$ *for* $i = 1, \cdots, p$
   (b) *Find a basis for* $K_{\lambda_i} = N((T - \lambda I)^p)$ *and select an end vector for the first cycle*
   (c) *Compute the cycle*
   (d) *Compute other cycles, by selecting vectors that are linearly independent of the vectors already determined*
   (e) *In case when* $K_{\lambda_i} = E_{\lambda_i}$, *Jordan canonical basis for* T_i *is simply the basis for* E_{λ_i}