# CSC321 Lecture 3: Linear Classifiers
# – or –
# What good is a single neuron?

Roger Grosse

# Overview

- Classification: predicting a discrete-valued target
- In this lecture, we focus on binary classification: predicting a binary-valued target
- Examples
  - predict whether a patient has a disease, given the presence or absence of various symptoms
  - classify e-mails as spam or non-spam
  - predict whether a financial transaction is fraudulent

# Overview

**Design choices so far**

- **Task:** regression, classification
- **Model/Architecture:** linear
- **Loss function:** squared error
- **Optimization algorithm:** direct solution, gradient descent, perceptron

# Overview

**Binary linear classification**

- **classification:** predict a discrete-valued target
- **binary:** predict a binary target $t \in \{0, 1\}$
  - Training examples with $t = 1$ are called positive examples, and training examples with $t = 0$ are called negative examples. Sorry.
- **linear:** model is a linear function of $\mathbf{x}$, followed by a threshold:

$$z = \mathbf{w}^T \mathbf{x} + b$$

$$y = \begin{cases} 1 & \text{if } z \geq r \\ 0 & \text{if } z < r \end{cases}$$

# Some simplifications

**Eliminating the threshold**

- We can assume WLOG that the threshold $r = 0$:

$$\mathbf{w}^T\mathbf{x} + b \geq r \quad \iff \quad \mathbf{w}^T\mathbf{x} + \underbrace{b - r}_{\triangleq b'} \geq 0.$$

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**Eliminating the bias**

- Add a dummy feature $x_0$ which always takes the value 1. The weight $w_0$ is equivalent to a bias.

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compensated by the bias

**Eliminating the bias**

- Add a dummy feature $x_0$ which always takes the value 1. The weight $w_0$ is equivalent to a bias.

do we update x_0?

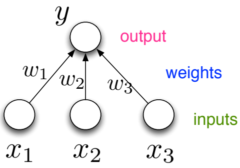
**Simplified model**

$$z = \mathbf{w}^T\mathbf{x}$$

$$y = \begin{cases} 1 & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 \end{cases}$$

# As a neuron

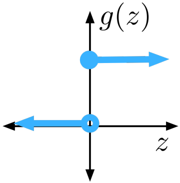
- This is basically a special case of the neuron-like processing unit from Lecture 1.




output bias i'th weight

$$y = g\left(b + \sum_i x_i w_i\right)$$

nonlinearity i'th input




- Today's question: what can we do with a single unit?

# Examples

**NOT**

x_0 = dummy

| $x_0$ | $x_1$ | t |
|---|---|---|
| 1 | 0 | 1 |
| 1 | 1 | 0 |

# Examples

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| $x_0$ | $x_1$ | t |
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$$b > 0$$
$$b + w < 0$$

$$b = 1,\ w = -2$$

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# Examples

**AND**

| $x_0$ | $x_1$ | $x_2$ | t |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

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$b < 0$

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$$b < 0$$
$$b + w_2 < 0$$
$$b + w_1 < 0$$

# Examples

**AND**

g(less than zero) = 0
g(greater than zero) = 1

| $x_0$ | $x_1$ | $x_2$ | t |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

$$b < 0$$
$$b + w_2 < 0$$
$$b + w_1 < 0$$
$$b + w_1 + w_2 > 0$$

# Examples

**AND**

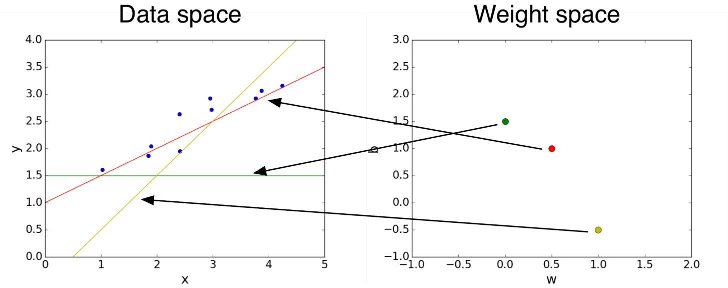
| $x_0$ | $x_1$ | $x_2$ | t |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

$$\begin{aligned} b &< 0 \\ b + w_2 &< 0 \\ b + w_1 &< 0 \\ b + w_1 + w_2 &> 0 \end{aligned}$$

$$b = -1.5, \; w_1 = 1, \; w_2 = 1$$

# The Geometric Picture

Recall from linear regression:

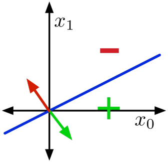


# The Geometric Picture

decision boundary always pass through zero
since it represent linear contraint offset by zero
due to the fact that the threshold = 0

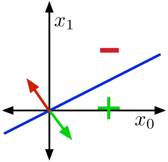
**Input Space, or Data Space**

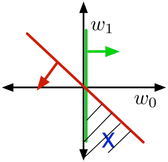
equation of line
determined by X^Tw = 0




- Here we're visualizing the **NOT** example
- Training examples are points
- Hypotheses are half-spaces whose boundaries pass through the origin
- The boundary is the decision boundary
  - In 2-D, it's a line, but think of it as a hyperplane
- If the training examples can be separated by a linear decision rule, they are linearly separable.

## Weight Space



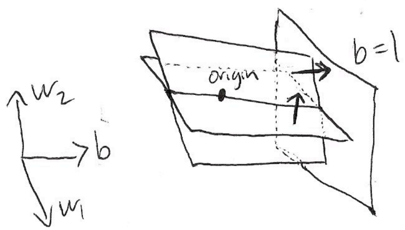

$w_0 > 0$

$w_0 + w_1 < 0$

- Hypotheses are points
- Training examples are half-spaces whose boundaries pass through the origin
- The region satisfying all the constraints is the feasible region; if this region is nonempty, the problem is feasible

# The Geometric Picture

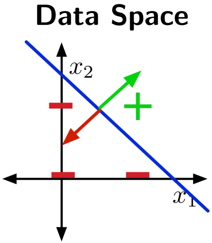
- The **AND** example requires three dimensions, including the dummy one.
- To visualize data space and weight space for a 3-D example, we can look at a 2-D slice:



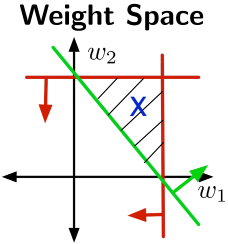

- The visualizations are similar, except that the decision boundaries and the constraints need not pass through the origin.

# The Geometric Picture

Visualizations of the **AND** example




Slice for $x_0 = 1$

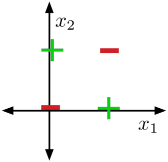



Slice for $w_0 = -1$

What happened to the fourth constraint?

# The Geometric Picture

Some datasets are not linearly separable, e.g. **XOR**



# The Perceptron Learning Rule

- Let's mention a classic classification algorithm from the 1950s: the perceptron


- Frank Rosenblatt, with the image sensor (left) of the Mark I Perceptron40

# The Perceptron Learning Rule

**The idea:**

- If $t = 1$ and $z = \mathbf{w}^\top \mathbf{x} > 0$
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$$\mathbf{w}' \leftarrow \mathbf{w} + \mathbf{x}$$

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  - Update:

$$\mathbf{w}' \leftarrow \mathbf{w} + \mathbf{x}$$

  - Justification:

$$\begin{aligned} \mathbf{w}'^T \mathbf{x} &= (\mathbf{w} + \mathbf{x})^T \mathbf{x} \\ &= \mathbf{w}^T \mathbf{x} + \mathbf{x}^T \mathbf{x} \\ &= \mathbf{w}^T \mathbf{x} + \|\mathbf{x}\|^2. \end{aligned}$$

# The Perceptron Learning Rule

For convenience, let targets be $\{-1, 1\}$ instead of our usual $\{0, 1\}$.

**Perceptron Learning Rule:**

Repeat:

For each training case $(\mathbf{x}^{(i)}, t^{(i)})$,

$z^{(i)} \leftarrow \mathbf{w}^T \mathbf{x}^{(i)}$

If $z^{(i)} t^{(i)} \leq 0$,

$\mathbf{w} \leftarrow \mathbf{w} + t^{(i)} \mathbf{x}^{(i)}$

Stop if the weights were not updated in this epoch.

1 epoch = 1 pass through training set

# The Perceptron Learning Rule

**Compare:**

- SGD for linear regression

$$\mathbf{w} \leftarrow \mathbf{w} - \alpha(y - t)\,\mathbf{x}$$

- perceptron

happens if z and t are different, i.e. wrong

$$z \leftarrow \mathbf{w}^T \mathbf{x}$$

If $zt \leq 0$,

$$\mathbf{w} \leftarrow \mathbf{w} + t\mathbf{x}$$

# The Perceptron Learning Rule

- Under certain conditions, if the problem is feasible, the perceptron rule is guaranteed to find a feasible solution after a finite number of steps.
- If the problem is infeasible, all bets are off.
  - Stay tuned...

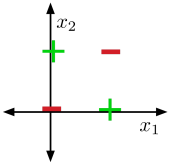
i.e. able to classify all samples correctly

- The perceptron algorithm caused lots of hype in the 1950s, then people got disillusioned and gave up on neural nets.
- People were discouraged about fundamental limitations of linear classifiers.

limitation = overfit, and
linear classifier can not separate linearly inseparable problems

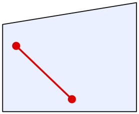
# Limits of Linear Classification

- Visually, it's obvious that **XOR** is not linearly separable. But how to show this?



# Limits of Linear Classification

## Convex Sets



- A set $\mathcal{S}$ is convex if any line segment connecting points in $\mathcal{S}$ lies entirely within $\mathcal{S}$. Mathematically,

$$\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{S} \implies \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in \mathcal{S} \quad \text{for } 0 \leq \lambda \leq 1.$$

- A simple inductive argument shows that for $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathcal{S}$, weighted averages, or convex combinations, lie within the set:

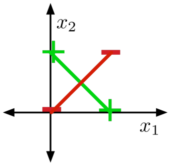
are linear combination s.t. coefficients > 0 and add up to 1

$$\lambda_1 \mathbf{x}_1 + \cdots + \lambda_N \mathbf{x}_N \in \mathcal{S} \quad \text{for } \lambda_i > 0, \ \lambda_1 + \cdots \lambda_N = 1.$$

# Limits of Linear Classification

**Showing that XOR is not linearly separable**

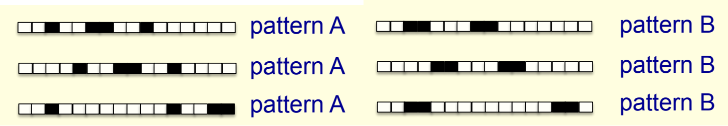
- Half-spaces are obviously convex.
- Suppose there were some feasible hypothesis. If the positive examples are in the positive half-space, then the green line segment must be as well.
- Similarly, the red line segment must line within the negative half-space.




- But the intersection can't lie in both half-spaces. Contradiction!

## Limits of Linear Classification

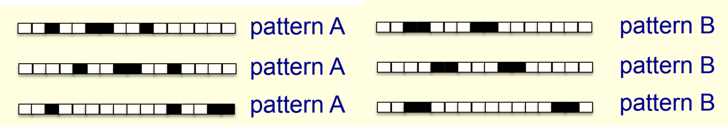
### A more troubling example




- These images represent 16-dimensional vectors. White $= 0$, black $= 1$.
- Want to distinguish patterns A and B in all possible translations (with wrap-around)
- Translation invariance is commonly desired in vision!

# Limits of Linear Classification

## A more troubling example




- These images represent 16-dimensional vectors. White $= 0$, black $= 1$.
- Want to distinguish patterns A and B in all possible translations (with wrap-around)

average=percentage of black

- Translation invariance is commonly desired in vision!
- Suppose there's a feasible solution. The average of all translations of A is the vector $(0.25, 0.25, \ldots, 0.25)$. Therefore, this point must be classified as A.
- Similarly, the average of all translations of B is also $(0.25, 0.25, \ldots, 0.25)$. Therefore, it must be classified as B. Contradiction!

Credit: Geoffrey Hinton

# Limits of Linear Classification

- Sometimes we can overcome this limitation using feature maps, just like for linear regression. E.g., for **XOR**:

$$\phi(\mathbf{x}) = \begin{pmatrix} x_1 \\ x_2 \\ x_1 x_2 \end{pmatrix}$$

| $x_1$ | $x_2$ | $\phi_1(\mathbf{x})$ | $\phi_2(\mathbf{x})$ | $\phi_3(\mathbf{x})$ | $t$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 0 |

- This is linearly separable. (Try it!)
- Not a general solution: it can be hard to pick good basis functions. Instead, we'll use neural nets to learn nonlinear hypotheses directly.