Problem Set 8

You are strongly encouraged to solve the following exercises before next week's tutorial:

- Starting on page 362 (end of Chapter 9): 42 (a-b), 43, 44, 45
- Starting on page 459 (end of Chapter 11): 15, 36, 37.

Additional Exercise:

Let $X_1, \ldots, X_n$ be a random sample from a distribution with pdf $f(x|\theta) = 3\theta x^2 \mathrm{e}^{-\theta x^3}$, $x \geqslant 0$, and consider the problem of testing

$$\begin{cases} \mathcal{H}_0 : \theta = \theta_0 \\ \mathcal{H}_1 : \theta \neq \theta_0 \end{cases}$$

at level α.

(a) Find $\widehat{\theta}$, the MLE of θ, and its asymptotic distribution. What is the asymptotic distribution of $\mathcal{X}^2 = \dfrac{n}{\widehat{\theta}^2}(\widehat{\theta} - \theta_0)^2$ under $\mathcal{H}_0$?

(b) Calculate the generalized likelihood ratio statistic $\Lambda(\underline{X})$ and show that

$$2 \log \Lambda(\underline{X}) = 2n \left\{ \log \widehat{\theta} - \log \theta_0 + \frac{\theta_0}{\widehat{\theta}} - 1 \right\}.$$

(c) Use the second order Taylor approximation

$$\log \theta_0 \approx \log \widehat{\theta} + \frac{\theta_0 - \widehat{\theta}}{\widehat{\theta}} - \frac{(\theta_0 - \widehat{\theta})^2}{2\widehat{\theta}^2}$$

to show that $2 \log \Lambda(\underline{X}) \approx \mathcal{X}^2$ from (a).

(d) For $n = 50$ and $\sum_{i=1}^{50} X_i = 36$, test $\mathcal{H}_0 : \theta = 1$ vs. $\mathcal{H}_1 : \theta \neq 1$ at the 5% level, using

i. an asymptotic test based on $2\log\Lambda$, and

ii. an asymptotic test based on $\mathcal{X}^2$.

Solution:

(a) Here

$$\mathcal{L}(\theta) = 3^n\theta^n \prod x_i^2 \exp\left\{-\theta \sum x_i^3\right\} \quad , \quad \ell(\theta) = n\log(\theta) + 2\sum \log x_i - \theta \sum x_i^3 + \text{const},$$

and by solving $\ell'(\theta) = \dfrac{n}{\theta} - \sum x_i^3 = 0$, we learn that

$$\boxed{\widehat{\theta} = \frac{n}{\sum X_i^3}.}$$

In addition, $\ell''(\theta) = -\dfrac{n}{\theta^2}$, hence it is indeed a maximum, and $\mathcal{I}(\theta) = \dfrac{\theta^2}{n}$, thus the asymptotic distribution of $\widehat{\theta}$ is $\mathcal{N}(\theta, \theta^2/n)$, and using the plug-in principle,

$$\frac{\sqrt{n}}{\widehat{\theta}}(\widehat{\theta} - \theta) \xrightarrow{D} \mathcal{N}(0,1).$$

Therefore,

$$\mathcal{X}^2 = \frac{n}{\widehat{\theta}^2}(\theta_0 - \widehat{\theta})^2 \xrightarrow[\mathcal{H}_0]{\mathcal{D}} \mathcal{N}(0,1)^2 = \chi_1^2.$$

(b) Plugging $\widehat{\theta}$ in the likelihood, we obtain $\mathcal{L}(\widehat{\theta}) = 3^n\widehat{\theta}^n \prod X_i^2 \mathrm{e}^{-n}$, and

$$\Lambda(\underline{X}) = \frac{\mathcal{L}(\widehat{\theta})}{\mathcal{L}(\theta_0)} = \left(\frac{\widehat{\theta}}{\theta_0}\right)^n \exp\left\{\theta_0 \sum X_i^3 - n\right\} = \left(\frac{\widehat{\theta}}{\theta_0}\right)^n \exp\left\{n\left(\frac{\theta_0}{\widehat{\theta}} - 1\right)\right\},$$

and the result follows.

(c) Using the approximation, we have

$$2\log\Lambda(\underline{X}) = 2n\left\{\log\widehat{\theta} - \log\theta_0 + \frac{\theta_0}{\widehat{\theta}} - 1\right\}$$

$$\approx 2n\left\{\log\widehat{\theta} - \log\widehat{\theta} + \frac{\theta_0 - \widehat{\theta}}{\widehat{\theta}} + \frac{(\theta_0 - \widehat{\theta})^2}{2\widehat{\theta}^2} + \frac{\theta_0}{\widehat{\theta}} - 1\right\} = \frac{n}{\widehat{\theta}^2}(\theta_0 - \widehat{\theta})^2 = \mathcal{X}^2.$$

(d) Here $\widehat{\theta} = \dfrac{50}{36} = 1.389$.

both are asymptotic methods
1. 2 log \Lambda ==> use asymptotic property of mle
2. X^2 ==> approximation of 2 log \Lambda using taylor approximation

The rejection region for this test is

$$\mathcal{C} = \left\{2\log\Lambda \geqslant \chi^2_{1,1-\alpha}\right\} = \left\{2\log\Lambda \geqslant \underbrace{\chi^2_{1,0.95}}_{3.84}\right\},$$

df = 1 - 0 = 1

and the test statistic evaluated at the data is

$$2\log\Lambda = 2\cdot 50\left\{\log 1.389 - 0 + \frac{1}{1.389} - 1\right\} = 4.85,$$

and we reject $\mathcal{H}_0$ with

$$0.025 < \text{p-value} = \mathbb{P}\left(\chi^2_1 \geqslant 4.85\right) < 0.05$$

(exact value is 0.0276).

ii. Similarly, the rejection region is $\mathcal{C} = \{\mathcal{X}^2 \geqslant 3.84\}$, and the test statistic evaluated at the data is

$$\mathcal{X}^2 = \frac{n}{\widehat{\theta}^2}(\widehat{\theta} - \theta_0)^2 = \frac{50}{1.389^2}(1.389 - 1)^2 = 3.92,$$

and again we reject $\mathcal{H}_0$, this time with p-value = 0.0477.