Chapter 6 Inner Product Spaces

6.1 Inner Products and Norms

Definition. ***Inner Product*** *Let V be a vector space over F. An inner product on V is a function that assigns, to every ordered pair of vectors x and y in V, a scalar in F, denoted $\langle x, y\rangle$, such that for all $x, y, z \in VE$ and all $c \in F$,*

1. $\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$
2. $\langle cx, y\rangle = c\langle x, y\rangle$
3. $\overline{\langle x, y\rangle} = \langle y, x\rangle$
4. $\langle x, x\rangle > 0$ *if* $x \neq 0$

First two condition requires inner product be linear in the first component. Also

$$\langle \sum_i a_i v_i, y\rangle = \sum_i a_i \langle v_i, y\rangle$$

Definition. ***Conjugate Transpose or Adjoint of a Matrix*** *Let $A \in M_{m\times n}(F)$, the conjugate transpose or adjoint of A is an $n \times m$ matrix A^* such that $(A^*)_{ij} = \overline{A_{ji}}$ for all i, j. For $F = \mathbb{R}$, $A^* = A^T$*

Definition. ***Inner Product Definition Example***

1. ***Standard Inner Product on F^n*** *For $x = (a_1, a_2, \cdots, a_n)$ and $y = (b_1, b_2, \cdots, b_n)$ in F^n, the standard inner product on F^n is given by*

$$\langle x, y\rangle = \sum_{i=1}^{n} a_i \bar{b}_i$$

2. ***Inner Product for Real-valued Continuous Functions on $[0, 1]$*** *Let $V = C([0, 1])$, $f, g \in V$, define*

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt$$

3. ***Frobenius Inner Product for Matrices*** *Let $V = M_{n\times n}(F)$, $A, B \in V$, then*

$$\langle A, B\rangle = tr(B^*A) = \sum_{i=1}^{n} (B^*A)_{ii}$$

Definition. ***Inner Product Space*** *A vector space over F endowed with a specific inner product is called an inner product space. If $F = C$, V is a complex inner product space; if $F = \mathbb{R}$, then V is a real inner product space*

Theorem. ***6.1 Properties From Inner Product Conditions*** *Let V be an inner product space. Then for $x, y, z \in V$ and $c \in F$, the following statements are true*

1. $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$
2. $\langle x, cy \rangle = \overline{c}\langle x, y \rangle$
3. $\langle x, 0 \rangle = \langle 0, x \rangle = 0$
4. $\langle x, x \rangle = 0$ *if and only if* $x = 0$
5. *If* $\langle x, y \rangle = \langle x, z \rangle$ *for all* $x \in V$*, then* $y = z$

The inner product is conjugate linear in the second argument

Definition. ***Norm/Length*** *Let V be an inner product space. For $x \in V$, define norm or length of x by*

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Definition. ***6.2 Properties of Norm*** *Let V be an inner product space over F. Then for all $x, y \in V$ and $c \in F$, the following statements are true*

1. $\|cx\| = |c| \cdot \|x\|$
2. $\|x\| = 0$ *if and only if* $x = 0$*. In any case,* $\|x\| \geq 0$
3. ***Cauchy-Schwarz Inequality*** $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$
4. ***Triangular Inequality*** $\|x+y\| \leq \|x\| + \|y\|$

Definition. ***Angle*** *For $F = \mathbb{R}$, $x, y \neq 0$, and θ be angle between x and y*

$$\cos\theta = \frac{\langle x, y \rangle}{\|x\| \, \|y\|} \qquad\qquad \theta = \cos^{-1}\left(\frac{\langle x, y \rangle}{\|x\| \, \|y\|}\right)$$

Note

$$\left|\frac{\langle x, y \rangle}{\|x\| \, \|y\|}\right| \leq 1$$

So valid input to arccos function

Definition. ***Orthogonal Vectors*** *Let V be an inner product space. Vectors x and y in V are orthogonal (perpendicular) if $\langle x, y \rangle = 0$.*

Definition. ***Orthogonal Sets and Orthonormal Sets*** *A subset S of V is orthogonal if any two distinct vectors in S are orthogonal. A vector x in V is a unit vctor if $\|x\| = 1$. A subset S of V is orthonormal if S is orthogonal and consists entirely of unit vectors.*

1. $S = \{v_1, v_2, \cdots\}$*, then S is orthonormal if and only if* $\langle v_i, v_j \rangle = \delta_{ij}$

2. *We can* ***normalize*** *an orthogonal set S, by multiplying $1/\|x\|$ for each $x \in S$*

Definition. ***Orthonormal Set Property*** *Let V be inner product space and $S = \{s_1, s_2, \cdots\} \subseteq V$ be an orthonormal set. Let $v \in span(S)$, then $v = a_1 s_1 + \cdots + a_k s_k$. Then*

$$\langle v, s_j \rangle = a_j$$

by

$$\langle v, s_j \rangle = \langle \sum_i a_i s_i, s_j \rangle = \sum_i a_i \langle s_i, s_j \rangle = \sum_i a_i \delta_{ij} = a_j$$

Gram-Schmidt Orthogonalization Process and Orthogonal Complements

Definition. ***Orthonormal Basis*** *Let V be an inner product space. A subset of V is an orthonormal basis for V if it is an ordered basis that is orthonormal*

Definition. ***Every Inner Product Space has n Orthogonal Basis*** *Let V be an inner product space and $S = \{v_1, v_2, \cdots, v_k\}$ be an orthogonal subset of V consisting of nonzero vectors. If $y \in span(S)$, then*

$$y = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i$$

Corollary. ***Special case for Orthonormal Set*** *If, in addition to hypotheses of previous theorem, S is orthonormal and $y \in S$, then*

$$y = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$$

Corollary. ***Nonzero Orthonormal Set is Linearly Independent*** *Let V be an inner product space, andf let S be an orthogonal subset of V consisting of nonzero vectors. Then S is linearly independent*

Theorem. ***6.4 Gram-Schmidt Process*** *Let V be an inner product space and $S = \{w_1, w_2, \cdots, w_n\}$ be a linearly independent subset of V. Define $S' = \{v_1, v_2, \cdots, v_n\}$, where $v_1 = w_1$ and*

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j \qquad 2 \leq k \leq n$$

Then S' is an orthogonal set of nonzero vectors such that $span(S') = span(S)$

Theorem. ***6.5 Every Finite Dimensional I.P.S has an Orthonormal Basis*** *Let V be a nonzero finite-dimensional inner product space. Then V has an orthonormal basis β. Furthermore, if $\beta = \{v_1, v_2, \cdots, v_n\}$ and $x \in V$, then*

$$x = \sum_{x=1}^{n} \langle x, v_i \rangle v_i$$

Corollary. ***Expression for Matrix Representation of Transformation on Orthonormal Basis*** *Let V be a finite-dimensional inner product space with an orthonormal basis $\beta = \{v_1, v_2, \cdots, v_n\}$. Let T be a linear operator on V, and let $A = [T]_\beta$. Then for any i and j, $A_{ij} = \langle T(v_j), v_i \rangle$, i.e.*

$$T(v_j) = \sum_{i=1}^{n} \langle T(v_j), v_i \rangle v_i$$

Definition. ***Fourier Coefficients*** *Let β be an orthonormal subset (possibly infinite) of an inner product space V, and let $x \in V$. We define the Fourier coefficients of x relative to β to be the scalars $\langle x, y \rangle$, where $y \in \beta$*

Orthogonal Complements

Definition. ***Orthogonal Complements*** *Let S be a nonempty subset of an inner product space V. We define $S^\perp = \{x \in V : \langle x, y \rangle = 0$ for all $y \in S\}$. The set $S^\perp$ is called the orthogonal complement of S*

1. *$\{0\}^\perp = V$ and $V^\perp = \{0\}$*

Theorem. ***6.6 Finding Projection of a Vector onto a Subspace*** *Let W be a finite-dimensional subspace of an inner product space V, and let $y \in V$. Then there exist unique vectors $u \in W$ and $z \in W^\perp$ such that $y = u + z$. Furthermore, if $\{v_1, v_2, \cdots, v_k\}$ is an orthonormal basis for W, then*

$$u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$$

where u is the orthogonal projection of y on W.

Corollary. ***Orthogonal Projection is Unique and Closest to Projected Vector*** *In the notation of previous theorem, the vector u the unique vector in W that is closest to y; that is, for any $x \in W$, $\|y - x\| \geq \|y - u\|$, and this inequality is an equality if and only if $x = u$*

Theorem. ***6.7 Orthonormal Basis and Subspaces*** *Suppose that $S = \{v_1, v_2, \cdots, v_k\}$ is an orthonormal set in an n-dimensional inner product space V.Then*

1. *S can be extended to an orthonormal basis $\{v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_n\}$ for V.*
2. *If $W = span(S)$, then $S_1 = \{v_{k+1}, \cdots, v_n\}$ is an orthonormal basis for $W^\perp$*
3. *If W is any subspace of V, then $dim(V) = dim(W) + dim(W^\perp)$*

6.3 The Adjoint of a Linear Operator

Definition. ***Dual Space*** *is a space of all linear transformations from a vector space V to its field F.*

Theorem. ***6.8 Every Linear Transformation from V to F Can Be Written as a Inner Product*** *Let V be a finite-dimensional inner product space over F, and let $g : V \to F$ be a linear transformation. Then there exists a unique vector $y \in V$ such that $g(x) = \langle x, y \rangle$ for all $x \in V$, where*

$$y = \sum_i \overline{g(v_i)} v_i \qquad \beta = \{v_1, \cdots, v_n\} \text{ is orthonormal basis}$$

Definition. ***Adjoint Linear Operator*** *Given inner product space V, let T be a linear operator on V. The adjoint of operator T, T^*, is the unique operator on V satisfying*

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle \qquad \text{for all } x, y \in V$$

Theorem. ***6.9 Adjoint of an Linear Operator Exist for f.d. Inner Product Space*** *Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Then there exists a unique function, called the adjoint of T, $T^* : V \to V$ such that*

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle$$

for all $x, y \in V$. Furthermore, T^* is linear.

Theorem. ***6.10 Adjoint of a Linear Operator in Matrix Form is the Adjoint of Matrix Form of that Linear Operator*** *Let v be a finite-dimensional inner product space. Let β be an orthonormal basis for V. If T is a linear operator on V, thern*

$$[T^*]_\beta = [T]_\beta^*$$

Corollary. ***For Left-Matrix Transformation*** *Let A be $n \times n$ matrix, then $L_{A^*} = (L_A)^*$. (theorem 2.16)*

Theorem. ***6.11 Properties of Adjoint of Linear Operators***
Let V bewr an inner product space, and let T, U be linear operators on V, then

1. $(T + U)^* = T^* + U^*$
2. $(cT)^* = \overline{c}T^*$ *for any* $c \in F$
3. $(TU)^* = U^*T^*$
4. $T^{**} = T$
5. $I^* = I$

assuming adjoints always exists.

Corollary. ***For Matrix***
Let A and B be $n \times n$ matrix, then

1. $(A+B)^* = A^* + B^*$
2. $(cA)^* = \overline{c}A^*$ *for all* $c \in F$
3. $(AB)^* = B^*A^*$
4. $A^{**} = A$
5. $I^* = I$

Least Squares Approximation

Definition. ***Some notation*** *Fort* $x, y \in F^n$

1. $\langle x, y \rangle_n$ *is the standard inner product of* x *and* y *in* F^n
2. *If* x *and* y *are column vectors, then* $\langle x, y \rangle_n = y^*x$

Lemma. *Let* $A \in M_{m\times n}(F)$, $x \in F^n$ *and* $y \in F^m$, *then*

$$\langle Ax, y \rangle_m = \langle x, A^*y \rangle_n$$

Lemma. *Let* $A \in M_{m\times n}(F)$. *Then* $rank(A^*A) = rank(A)$

Corollary. *If* A *is* $m \times n$ *matrix such that* $rank(A) = n$, *then* A^*A *is invertible*

Theorem. ***6.12 Close Form Solution for Least Squared Problem*** *Let* $A \in M_{m\times n}(F)$ *and* $y \in F^m$. *Then there exists* $x_0 \in F^n$ *such that* $(A^*A)x_0 = A^*y$ *and* $\|Ax_0 - y\| \leq \|Ax - y\|$ *for all* $x \in F^n$. *Furthermore, if* $rank(A) = n$, *then* $x_0 = (A^*A)^{-1}A^*y$