Chapter 2 Linear Transformations and Matrices

2.1 Linear Transformations, Null Spaces, and Ranges

Definition. ***Linear Transformation***
Let V and W be vector spaces (over F). We call a function $T : V \to W$ a linear transformation from V to W if, for all $x, y \in V$ and $c \in F$, we have

1. $T(x + y) = T(x) + T(y)$
2. $T(cx) = cT(x)$

T is called linear, with properties

1. *If T is linear $T(0) = 0$*
2. *T is linear if and only if $T(cx + y) = cT(x) + T(y)$ for all $x, y \in V$ and $c \in F$* ***(For proving a transformation is linear)***
3. *If T is linear, then $T(x - y) = T(x) - T(y)$ for all $x, y \in V$*
4. *T is linear if and only if, for $x_1, x_2, \cdots, x_n \in V$ and $a_1, a_2, \cdots, a_n \in F$, we have*
$$T(\sum_i a_i x_i) = \sum_i a_i T(x_i)$$

Some examples of linear transformations

1. ***Rotation*** *For any angle θ, define $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$. $T_\theta(a_1, a_2)$ is the vector obtained by rotating (a_1, a_2) counterclockwise by θ if $(a_1, a_2) \neq (0, 0)$, and $T_\theta = (0, 0)$. Then T_θ is a linear transformation called rotation by θ, Let α be angle that (a_1, a_2) makes with the positive axis. Note $a_1 = r \cos \alpha$ and $a_2 = r \sin \alpha$, and suppose $r = \sqrt{a_1^2 + a_2^2}$*
$$T_\theta(a_1, a_2) = (r \cos \alpha + \theta, r \sin \alpha + \theta) = (a_1 \cos \theta - a2 \sin \theta, a_2 \sin \theta + a_2 \cos \theta)$$
2. ***Reflection*** *Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1, a_2) = (a_1, -a_2)$. T is called the reflection about the x-axis*
3. ***Projection*** *Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1, a_2) = (a_1, 0)$. T is called the projection on the x-axis*
4. ***Taking transpose is linear*** *Define $T : M_{m\times n}(F) \to M_{n\times m}(F)$ by $T(A) = A^t$ (by $(A + B)^t = A^t + B^t$ and $(cA)^t = cA^t$)*
5. ***Taking derivative is linear*** *Define $T : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ by $T(f(x)) = f'(x)$, where $f'(x)$ denotes the derivative of $f(x)$. Let $g(x), h(x) \in P_n(\mathbb{R})$ and $a \in \mathbb{R}$,*
$$T(ag(x) + h(x)) = (ag(x) + h(x))' = ag'(x) + h'(x) = aT(g(x)) + T(h(x))$$
so T is linear.

6. ***Taking integral is linear*** *Let $V = C(\mathbb{R})$, the set of continuous real-valued functions on $\mathbb{R}$., Let $a, b \in \mathbb{R}$, $a < b$. Define $T : V \to \mathbb{R}$ by*

$$T(f) = \int_a^b f(t)dt$$

for all $f \in V$. THen T is linear because the definite integral of a linear combination of functions is same as combination of the detinite integrals of the functions.

Definition. ***Identity and Zero Transformation*** *For vector spaces V and W (over F), define identity transformation $I_V : V \to V$ by $I_V(x) = x$ for all $x \in V$ and the zero transformation $T_0 : V \to W$ by $T_0(x) = 0$ for all $x \in V$.*

Definition. ***Null Space and Range*** *Let V and W be vector spaces, and let $T : V \to W$ be linear. We define the null space (or kernel) $N(T)$ of T to be the set of all vectors $x \in V$ such that $T(x) = 0$; that is $N(T) = \{x \in V : T(x) = 0\}$. We define the range (or image) $R(T)$ of T to be the subset of W consisting all images (under T) of vectors in V; that is $R(T) = \{T(x) : x \in V\}$*

1. ***identity and zero transformation*** *$N(I) = \{0\}$ and $R(I) = V$, $R(T_0) = V$ and $R(T_0) = \{0\}$*

Theorem. ***2.1 Range and null space are subspaces***
Let V and W be vector spaces and $T : V \to W$ be linear. Then $N(T)$ and $R(T)$ are subspaces of V and W, respectively.

Theorem. ***2.2 Transformation on basis yields a spanning set for the range***
Let V and W be vector spaces, and let $T : V \to W$ be linear. If $\beta = \{v_1, \cdots, v_n\}$ is a basis for V, then

$$R(T) = span(T(\beta)) = span(\{T(v_1), \cdots, T(v_n)\})$$

So we simply transform the original basis to find the generating set for the range of a transformation, then reduce the generating set to a linearly independent set to find the basis.

Definition. ***Nullity and Rank*** *Let V and W be vector spaces, and let $T : V \to W$ be linear. If $N(T)$ and $R(T)$ are finite-dimensional, then we define the nullity of T, denoted by $nullity(T)$, and the rank of T, denoted $rank(T)$, to be the dimensions of $N(T)$ and $R(T)$, respectively.*

Theorem. ***2.3 Rank-Nullity (Dimension) Theorem***
Let V and W be vector spaces, and let $T : V \to W$ be linear. If V is finite-dimensional, then

$$nullity(T) + rank(T) = dim(V)$$

In the context of matrcies, the rank and the nullity of a matrix add up to the number of columns of the matrix.

Theorem. ***2.4 One-to-One Transformation***
Let V and W be vector spaces, and let $T : V \to W$ be linear. Then T is one-to-one if and only if $N(T) = \{0\}$, or $nullity(T) = 0$

Theorem. ***2.5 One-to-One and Onto Equivalence***
Let V and W be vector spaces of equal (finite) dimension, and let $T : V \to W$ be linear. Then the following are equivalent

1. *T is one-to-one*
2. *T is onto*
3. *$rank(T) = dim(V)$*

If not a special case to see if a transformation is onto we verify that $R(T) = W$

Theorem. ***2.6 Uniqueness Linear Transformation***
Let V and W be vector spaces over F, and suppose that $\{v_1, v_2, \cdots, v_n\}$ is a basis for V. For $w_1, w_2, \cdots, w_n \in W$, there exists exactly one linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for $i = 1, 2, \cdots, n$

Remark. Given $x \in V$, we write x as a linear combination of the basis, i.e. $x = \sum_{i=1}^{n} a_i v_i$ where $a_i \in F$s are unique scalars. Then we can specify such transformation as

$$T : V \to W \qquad T(x) = T(\sum_i a_i v_i) = \sum_i a_i w_i$$

We can prove that T is linear, unique, and follows $T(v_i) = w_i$

Corollary. ***Transformation is determined completely by action on a basis***
Let V and W be vector spaces, and suppose that V has a finite basis $\{v_1, v_2, \cdots, v_n\}$. If $U, T : V \to W$ are linear and $U(v_i) = T(v_i)$ for $i = 1, 2, \cdots, n$, then $U = T$.

2.2 The Matrix Representation of Linear Transformation

Definition. ***Ordered Basis*** *Let V be a finite-dimensional vector space. An ordered basis for V is a basis for V endowed with a specific order; that is an ordered basis for V is a finite sequence of linearly independent vectors in V that generates V.*

1. ***Standard ordered basis*** *$\{e_1, e_2, \cdots, e_n\}$ is the standard ordered basis for F^n and $\{1, x, \cdots, x^n\}$ is the standard ordered basis for $P_n(F)$*
2. *In F^3, $\beta = \{e_1, e_2, e_3\}$ and $\gamma = \{e_2, e_1, e_3\}$ are 2 different ordered basis, i.e. $\beta \neq \gamma$*

Definition. ***Coordinate Vector*** *Let $\beta = \{u_1, u_2, \cdots, u_n\}$ be an ordered basis for a finite-dimensional vector space V. For $x \in V$, let $a_1, a_2, \cdots, a_n$ be the unique scalars such that*

$$x = \sum_{i=1}^{n} a_i u_i$$

We define the coordinate vector of x relative to β, denoted $[x]_\beta$, by

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

1. $[u_i]_\beta = e_i$
2. $x \to [x]_\beta$ *is a transformation that maps from V to F^n*
3. *Let $V = P_2(\mathbb{R})$, let $\beta = \{1, x, x^2\}$ be standard ordered basis for V. If $f(x) = 4 + 6x - 7x^2$, then*

$$[f]_\beta = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}$$

Definition. ***Matrix*** *Let V and W be finite-dimensional vector spaces with ordered basis $\beta = \{v_1, v_2, \cdots, v_n\}$ and $\gamma = \{w_1, w_2, \cdots, w_m\}$, respectively. Let $T : V \to W$ be linear. Then for each $j, 1 \leq j \leq n$, there exist unique scalar $a_{ij} \in F$, $1 \leq i \leq m$, such that*

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \qquad \text{for } 1 \leq j \leq n$$

The $m \times n$ matrix A defined by $A_{ij} = a_{ij}$ the matrix representation of T in the ordered basis β and γ and write $[T]_\beta^\gamma$. If $V = W$ and $\beta = \gamma$, then write $A = [T]_\beta$

1. *jth column of A is simply $[T(v_j)]_\gamma$*
2. ***Equal Linear Transformation has Equivalent Matrices*** *Observe if $U : V \to W$ is a linear transformation such that $[U]_\beta^\gamma = [T]_\beta^\gamma$, then $U = T$ by previous corollary*
3. *Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ with $T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$. Let β and γ be standard ordered bases for $\mathbb{R}^2$ and $\mathbb{R}^3$. Now*

$$T(1,0) = (1,0,2) = 1e_1 + 0e_2 + 2e_3 \qquad T(0,1) = (3,0,-4) = 3e_1 + 0e_2 - 4e_3$$

hence

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}$$

Definition. ***Addition and Scalar Multipliation Operations for Function*** *Let $T, U : V \to W$ be arbitrary functions, where V and W are vector spaces over F, and let $a \in F$. We define $T + U : V \to W$ by $(T + U)(x) = T(x) + U(x)$ for all $x \in V$, and $aT : V \to W$ by $(aT)(x) = aT(x)$ for all $x \in V$.*

Theorem. *2.7*
Let V and W be vector spaces over a field F, and let $T, U : V \to W$ be linear.

1. ***Sums/Scalar Multiples of Linear Transformation are Linear*** *For all $a \in F$, $aT + U$ is linear (Prove $(aT + U)(cx + y) = c(aT + U)(x) + (aT + U)(y)$)*

2. ***The Collection of Linear Transformation from V to W is a vector space*** *Using the operations of addition and scalar multiplication in the preceding definition, the collection of all linear transformations from V to W is a vector space over F. (With T_0 the zero transformation as the zero vector)*

Definition. ***Vector space of Linear Transformations*** *Let V and W be vector spaces over F. We denote the vector space of all linear transformations from V into W by $\mathcal{L}(V, W)$. In the case that $V = W$, we write $\mathcal{L}(V)$ instead of $\mathcal{L}(V, W)$*

Theorem. ***2.8 Linearity of Matrix Representations*** *Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively, and let $T, U : V \to W$ be linear transformations. Then*

1. $[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

2. $[aT]_\beta^\gamma = a\,[T]_\beta^\gamma$

Intuitively, the matrices are defined such that sum and scalar multiples of matrices are associated with the corresponding sum and scalar multiples of the transformation

2.3 Composition of Linear Transformations and Matrix Multiplication

Theorem. ***2.9 Composition of Linear Transformation is Linear***
Let V, W, and Z be vector spaces over the same field F, and let $T : V \to W$ and $U : W \to Z$ be linear. Then $UT : V \to Z$ is linear. (Prove $UT(ax + y) = a(UT)(x) + UT(y)$)

Theorem. ***2.10 Properties of Composition of Linear Transformations***
Let $T, U_1, U_2 \in \mathcal{L}(V)$. Then

1. $T(U_1 + U_2) = TU_1 + TU_2$ *and* $(U_1 + U_2)T = U_1T + U_2T$

2. $T(U_1U_2) = (TU_1)U_2$

3. $TI = IT = T$

4. $a(U_1U_2) = (aU_1)U_2 + U_1(aU_2)$

Definition. ***Matrix Product*** *Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. We define the product of A and B, denoted AB, to be the $m \times p$ matrix such that*

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} \qquad \textit{for } 1 \le i \le m, 1 \le j \le p$$

Note

1. $(AB)_{ij}$ *is sum of products of corresponding entries from* i*th row of* A *and* j*th column of* B*.*

2. $(AB)^t = B^t A^t$

Remark. The motivation is as follows. Let $T : V \to W$ and $U : W \to Z$ be linear transformations, and let $A = [U]_\beta^\gamma$ and $B = [T]_\alpha^\beta$ where $\alpha = \{v_1, \cdots, v_n\}$, $\beta = \{w_1, \cdots, w_n\}$, and $\gamma = \{z_1, \cdots, z_p\}$ are ordered bases for V, W, and Z, respectively. We would like to define the product AB of two matrices such so that $AB = [UT]_\alpha^\gamma$. Consider for $1 \leq j \leq n$, we have

$$\begin{aligned}(UT)(v_j) = U(T(v_j)) = U\left(\sum_k^m B_{kj} w_k\right) &= \sum_k^m B_{kj} U(w_k) \\ = \sum_k^m \left(\sum_i^p A_{ik} z_i\right) = \sum_i^p \left(\sum_k^m A_{ik} B_{kj}\right) z_i &= \sum_i^p C_{ij} z_p\end{aligned}$$

Theorem. ***2.11 Composition of Linear Transformation***
Let V*,* W*, and* Z *be finite-dimensional vector spaces with ordered bases* α*,* β*,* γ*, respectively. Let* $T : V \to W$ *and* $U : W \to Z$ *be linear transformations. Then*

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$$

Proof. Proof directly result from definition of matrix product. Given $T, U, \alpha, \beta, \gamma$ defined above, we have

$$(UT)(v_j) = \sum_{i=1}^p C_{ij} z_i \qquad C_{ij} = \sum_{k=1}^m A_{ik} B_{kj}$$

$$([UT]_\alpha^\gamma)_{ij} = C_{ij} = \sum_{k=1}^m A_{ik} B_{kj} = (AB)_{ij} = ([U]_\beta^\gamma [T]_\alpha^\beta)_{ij} \quad \to \quad [UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$$

□

Corollary. ***Special Case When*** U, T ***are Linear Operators***
Let V *be finite-dimensional vector space with ordered basis* β*. Let* $T, U \in \mathcal{L}(V)$*. Then* $[UT]_\beta = [U]_\beta [T]_\beta$

Definition. ***Identity Matrix*** *We define the Kronecker delta* δ_{ij} *by* $\delta_{ij} = 1$ *if* $i = j$ *and* $\delta_{ij} = 0$ *if* $i \neq j$*. The* $n \times n$ *identity matrix* I_n *is defined by* $(I_n)_{ij} = \delta_{ij}$

Theorem. ***2.12 Properties of Composition of Matrices (Analogous to 2.10 acd)***
Let A *be* $n \times n$ *matrix,* B *and* C *be* $n \times p$ *matrices,* D *and* E *be* $q \times m$ *matrices. Then*

1. $A(B + C) = AB + AC$ *and* $(D + E)A = DA + EA$

2. $a(AB) = (aA)B = A(aB)$ *for* $a \in F$

3. $I_m A = A = A I_n$ *(identity matrix as multiplicative identity in* $M_{n\times n}(F)$*)*

4. If V is an n-dimensional vector space with ordered basis β, then $[I_V]_\beta = I_n$ (identity transformation)

Proved using definition of matrix product

Proof. Proving number 3

$$(I_m A)_{ij} = \sum_k^m (I_m)_{ik} A_{kj} = \sum_k^m \delta_{ik} A_{kj} = A_{ij}$$

□

Corollary. *Let A be an $m \times n$ matrix, $B_1, B_2, \cdots, B_k$ be $n \times p$ matrices, $C_1, C_2, \cdots, C_k$ be $q \times m$ matrices, and $a_1, a_2, \cdots, a_k$ be scalars. Then*

$$A\left(\sum_i^k a_i B_i\right)\sum_i^k = a_i A B_i \qquad \textit{and} \qquad \left(\sum_i^k a_i C_i\right) A = \sum_i^k a_i C_i A$$

Proof by a.b. of previous theorem

Definition. ***Matrix Exponentials*** *Define $A^0 = I_n$ and $A^k = A^{k-1}A$ for $k > 1$.*

Theorem. ***2.13 Regarding columns in matrix multiplication***
Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. For each j $(1 \leq j \leq p)$ let u_j and v_j denote the jth columns of AB and B, respectively. Then

1. $u_j = Av_j$

2. $v_j = Be_j$, where e_j is the jth standard vector of F^p

Proof. We have

$$u_j = \begin{pmatrix} (AB)_{1j} \\ \vdots \\ (AB)_{mj} \end{pmatrix} = \begin{pmatrix} \sum_k^n A_{1k} B_{kj} \\ \vdots \\ \sum_k^n A_{mk} B_{kj} \end{pmatrix} = A \begin{pmatrix} B_{1j} \\ \vdots \\ B_{nj} \end{pmatrix} = Av_j$$

□

Corollary. *The jth column of AB is a linear combination of the columns of A with the coefficients in the linear combination being the entries of the column j of B. Analogously, row i of AB is a linear combination of the rows of B with the coefficients in the linear combination being the entries of the row i of A.*

Theorem. ***2.14 Evaluate Transformation For a Vector***
Let V and W be finite-dimensional vector spaces having ordered bases β and γ, respectively, and let $T : V \to W$ be linear. Then, for each $u \in V$, we have

$$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$$

Proof. Fix $u \in V$, define linear transformations $f : F \to V$ by $f(a) = au$ and $g : F \to W$ by $g(a) = aT(u)$ for all $a \in F$. Let $\alpha = \{1\}$ be standard ordered basis for F. Note $g = Tf$. Identify column vectors as matrices, i.e. column vector $[g(1)]_\gamma$ is simply the matrix representing transformation g, $[g]_\alpha^\gamma$, since the transformation is determined by operation on the basis, which is a set of size 1.

$$[T(u)]_\gamma = [g(1)]_\gamma = [g]_\alpha^\gamma = [Tf]_\alpha^\gamma = [T]_\beta^\gamma [f]_\alpha^\beta = [T]_\beta^\gamma [f(1)]_\beta = [T]_\beta^\gamma [u]_\beta$$

□

As an example, Let $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be linear transformation defined by $T(f(x)) = f'(x)$, and let β and γ be standard ordered bases for $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$. If $A = [T]_\beta \gamma$, then, we have

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

We verify the theorem. Let $p(x) \in P_3(\mathbb{R})$ be $p(x) = 2 - 4x + x^2 + 3x^3$, let $q(x) = T(p(x))$, then $q(x) = p'(x) = -4 + 2x + 9x^2$. So

$$[T(p(x))]_\gamma = [q(x)]_\gamma = \begin{pmatrix} -4 \\ 2 \\ 9 \end{pmatrix} \qquad [T]_\beta^\gamma [p(x)]_\beta = A [p(x)]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -4 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 9 \end{pmatrix}$$

Definition. ***Left-multiplication Transformation*** *Let A be $m \times n$ matrix with entries from a field F. We denote by L_A by mapping $L_A : F^n \to F^m$ defined by $L_A(x) = Ax$ (the matrix product of A and x) for each column vector $x \in F^n$. We L_A a left-multiplication transformation*

Theorem. ***2.15 Properties of Left-multiplication Transformation***
Let A be $m \times n$ matrix with entries from F. Then the left-multiplication transformation $L_A : F^n \to F^m$ is ***linear****. Furthermore, if B is any other $m \times n$ matrix (with entries from F) and β and γ are the standard ordered bases for F^n and F^m, respectively, then we have the following properties*

1. *$[L_A]_\beta^\gamma = A$*
2. *$L_A = L_B$ if and only if $A = B$*
3. *$L_{A+B} = L_A + L_B$ and $L_{aA} = aL_A$ for all $a \in F$*

4. If $T : F^n \to F^m$ is linear, then there exists a unique $m \times n$ matrix C such that $T = L_C$. In fact, $C = [T]_\beta^\gamma$

5. If E is an $n \times p$ matrix, then $L_{AE} = L_A L_E$

6. If $m = n$, then $L_{I_n} = I_{F^n}$

Theorem. ***2.16 Matrix Multiplication is Associative***
Let A, B, and C be matrices such that $A(BC)$ is defined. Then $(AB)C$ is also defined and $A(BC) = (AB)C$; that is, the matrix multiplication is associative.

Proof.

$$L_{A(BC)} = L_A L_{BC} = L_A(L_B L_C) = (L_A L_B)L_C = L_{AB}L_C = L_{(AB)C}$$

implies $A(BC) = (AB)C$ by 5th point in previous theorem. □

Definition. ***Incident Matrices*** *An incident matrix is a square matrix in which all the entries are either zero or one, and for convenience, all diagonal entries are zero. $A_{ij} = 1$ if i is related to j, and $A_{ij} = 0$ otherwise.*

2.4 Invertibility and Isomorphisms

Definition. ***Function Invertibility*** *Let V and W be vector spaces, and let $T : V \to W$ be linear. A function $U : W \to V$ is said to be an inverse of T if $TU = I_W$ and $UT = I_V$. If T has an inverse, then T is said to be invertible. If T is invertible, the inverse of T is unique and is denoted by T^{-1}. The following holds for invertible functions T and U*

1. $(TU)^{-1} = U^{-1}T^{-1}$

2. $(T^{-1})^{-1} = T$, *in particular* T^{-1} *is invertible*

3. *Let $T : V \to W$ be a linear transformation, where V and W are finite-dimensional spaces of equal dimension. Then T is invertible if and only if $rank(T) = dim(V)$, i.e. T is one-to-one ($dim(N(T)) = 0$) and onto ($R(T) = W$)*

Theorem. ***2.17 Inverse of Transformation is Linear***
Let V and W be vector spaces, and let $T : V \to W$ be linear and invertible. Then $T^{-1} : W \to V$ is linear.

Proof. Let $y_1, y_2 \in W$ and $c \in F$. Since T is onto and one-to-one, there exists unique vectors x_1 and x_2 such that $T(x_1) = y_1$ and $T(x_2) = y_2$. So $x_1 = T^{-1}(y_1)$ and $x_2 = T^{-1}(y_2)$

$$T^{-1}(cy_1 + y_2) = T^{-1}(cT(x_1) + T(x_2)) = T^{-1}(T(cx_1 + x_2)) = cx_1 + x_2 = cT^{-1}(y_1) + T^{-1}(y_2)$$

□

Proposition. ***Equivalence of One-to-One, Onto, Invertible in Special Case***
If $dim(V) = dim(W)$ and let $T : V \to W$, then the following are equivalent by theorem 2.5

1. *T is invertible*
2. *T is one-to-one*
3. *T is onto*

Definition. ***Matrix Invertibility*** *Let A be $n \times n$ matrix. Then A is invertible if there exists an $n \times n$ matrix B such that $AB = BA = I$. Such matrix B is unique, called inverse of A and denoted by A^{-1}*

Lemma. ***Domain/Codomain of Invertible Transformation have Equal Dimension*** *Let T be an invertible linear transformation from V to W., Then V is finite-dimensional if and only if W is finite-dimensional. In this case, $dim(V) = dim(W)$*

Theorem. ***2.18 Matrix and Transformation Invertibility are Equivalent*** *Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively. Let $T : V \to W$ be linear. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible. Furthermore, $\left[T^{-1}\right]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$*

Corollary. ***1 Special case where*** $W = V$ *Let V be a finite-dimensional vector space with an ordered basis β, and let $T : V \to V$ be linear. Then T is invertible if and only if $[T]_\beta$ is invertible. Furthermore, $\left[T^{-1}\right]_\beta = ([T]_\beta)^{-1}$*

Corollary. *2 Let A be an $n \times n$ matrix. Then A is invertible if and only if L_A is invertible. Futhermore, $(L_A)^{-1} = L_{A^{-1}}$*

Definition. ***(Vector Space) Isomorphism*** *Let V and W be vector spaces. We say V is isomorphic to W if there exists a linear transformation $T : V \to W$ that is invertible. Such a linear transformation is called an isomorphism from V onto W.*

Theorem. ***2.19 Isomorphic vector space have equal dimensions***
Let V and W be finite-dimensional vector spaces (over the same field). Then V is isomorphic to W if and only if $dim(V) = dim(W)$. (Proof directly from lemma preceding theorem 2.18)

Corollary. *Let V be a vecctor space over F. Then V is isomorphic to F^n if and only if $dim(V) = n$ (finite)*

Theorem. ***2.20 Collection of all linear transformation may be identified with appropriate vector space of*** $m \times n$ ***matrices***
Let V and W be finite-dimensional vector spaces over F of dimensions n and m, respectively, and let β and γ be ordered bases for V and W, respectively. Then the function $\Phi : \mathcal{L}(V, W) \to M_{m\times n}(F)$, defined by $\Phi(T) = [T]_\beta^\gamma$ for $T \in \mathcal{L}(V, W)$, is an isomorphism

Corollary. *Let V and W be finite-dimensional vector spaces of dimensions n and m, respectively. Then $\mathcal{L}(V,W)$ is finite-dimensional of dimension mn (From the fact that $dim(M_{m\times n}(F)) = mn$)*

Definition. ***Standard Representation of Vector Space is a Mapping*** $x \to [x]_\beta$
Let β be an ordered basis for an n-dimensional vector space V over the field F. The standard representation of V with respect to β is the function $\phi_\beta : V \to F^n$ defined by $\phi_\beta(x) = [x]_\beta$ for each $x \in V$

Theorem. ***2.21 Standard Representation is an Isomorphism***
For any finite-dimensional vector space V with ordered basis β, ϕ_β is an isomorphism

Definition. *Let V and W be vector spaces of dimensions n and m. let $T : V \to W$ be a linear transformation. Define $A = [T]_\beta^\gamma$, where β and γ are arbitrary ordered bases of V and W, respectively. We can use ϕ_β and ϕ_γ to study the relationship between linear transformations T and $L_A : F^n \to F^m$. We can use two composites of linear transformation to map V into F^m*

1. *Map V into F^n with ϕ_β and follow transformation with L_A, yielding $L_A\phi_\beta$*
2. *Map V into W with T and follow it by ϕ_γ to obtain the composite $\phi_\gamma T$*

Together, we can conclude that the two ways of composition commutes

$$L_A\phi_\beta = \phi_\gamma T$$

This allows us to transfer operations on abstract vector spaces to ones on F^n and F^m

2.5 The Change of Coordinate Matrix

Theorem. ***Coordinate Vector Change of Basis***
Let β and β' be two ordered basis for a finite-dimensional vector space V, and let $Q = [I_V]_{\beta'}^{\beta}$. Then

1. *Q is invertible*
2. *For any $v \in V$, $[v]_\beta = Q\,[v]_{\beta'}$*

where Q is called a ***change of coordinate matrix****. We say that Q changes β'-coordinates into β-coordinates. Observe that if $\beta = \{x_1, x_2, \cdots, x_n\}$ and $\beta' = \{x'_1, x'_2, \cdots, x'_n\}$, then*

$$I_V(x'_j) = x'_j = \sum_i^n Q_{ij}x_i$$

for $j = 1, 2, \cdots, n$ that is jth column of Q is $\left[x'_j\right]_\beta$ (by definition of coordinate vector)

Proof. For any $v \in V$

$$[v]_\beta = [I_V(v)]_\beta = [I_V]_{\beta'}^\beta [v]_{\beta'} = Q [v]_{\beta'}$$

$\square$

Definition. ***Linear Operator*** *A linear transformation that map a vector space V into itself*

Theorem. ***2.23 Linear Operator Change of Basis***
Let T be a linear operator on a finite-dimensional vector space V, and let β and β' be ordered bases for V. Suppose that Q is the change of the coordinate matrix that changes β'-coordinates into β-coordinates. Then

$$[T]_{\beta'} = Q^{-1} [T]_\beta Q$$

Proof. Let I be identity transformation on V. Then $T = IT = TI$ hence, by multiplication of linear transformations

$$Q [T]_{\beta'} = [I]_{\beta'}^\beta [T]_{\beta'}^{\beta'} = [IT]_{\beta'}^\beta = [TI]_{\beta'}^\beta = [T]_\beta^\beta [I]_{\beta'}^\beta = [T]_\beta Q$$

Therefore $[T]_{\beta'} = Q^{-1} [T]_\beta Q$ $\square$

Corollary. *Let $A \in M_{n\times n}(F)$, and let γ be an ordered basis for F^n. Then $[L_A]_\gamma = Q^{-1}AQ$, where Q is the $n \times n$ matrix whose jth column is the jth vectort of γ,*

Remark. This is true because

$$[L_A]_\beta (e_j) = Ae_j = \text{ j-th column of } A \quad \rightarrow \quad [L_A]_\beta = A$$

$$Q(v_j) = [I]_\gamma^\beta (v_j) = I(v_j) = v_j = \text{ j-th column of } Q$$

where β is the standard basis.
Note we make distinction between A and L_A. The former is a matrix, the latter is a function. They are not equivalent when represented as matrices since A is the same regardless but L_A is subject to a change of basis.

Definition. ***Similar Matrices*** *Let A and B be matrices in $M_{n\times n}(F)$. We say that B is similar to A if there exists an invertible matrix Q such that $B = Q^{-1}AQ$*

1. *If T is a linear operator on a finite dimensional vector space V, and if β and β' are any ordered bases for VB, then $[T]_{\beta'}$ is similar to $[T]_\beta$*

2. *If 2 matrices are similar, i.e. $A \sim B$, then $\det A = \det B$*