# Problem Set 5

You are strongly encouraged to solve the following exercises before next week's tutorial: Starting on page 362 (end of Chapter 9): 1, 3, 5, 7, 9.

## Additional Exercises:

1. Suppose that you meet a man whom you know to be either from the UK or France. He has no accent and you feel uncomfortable asking him directly about his origin. Instead you offer him a drink: beer, brandy/cognac, whisky, or wine, and try to guess his origin according to his choice of drink. You want to find the most powerful (MP) test that will identify the mans origin, where

$$\begin{cases} \mathcal{H}_0 : \text{France} \\ \mathcal{H}_1 : \text{UK} \end{cases}$$

Suppose it is known that the distributions of alcohol consumption in the UK and France are as follows:

| State | Beer | Brandy | Whisky | Wine |
|---|---|---|---|---|
| France | 10% | 20% | 10% | 60% |
| UK | 50% | 10% | 20% | 20% |

(a) What rejection regions here correspond to tests at the significance level of (at most) $\alpha = 0.25$?

(b) Among the tests you found in (a), which one is the MP test?

(c) For each $x \in \{\text{Beer}, \text{Brandy}, \text{Whisky}, \text{Wine}\}$ calculate the likelihood ratio

$$\lambda(x) = \frac{\mathbb{P}\left(x \middle| \text{UK}\right)}{\mathbb{P}\left(x \middle| \text{France}\right)}.$$

and show that the Likelihood Ratio Test (LRT) at level $\alpha \leqslant 0.25$ is the same one you found in $(b)$.

2. Suppose that we observe $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Exp}(\lambda)$.

   (a) Find the most powerful test at level $\alpha$ to test $\mathcal{H}_0 : \lambda = \lambda_0$ vs. $\mathcal{H}_1 : \lambda = \lambda_1$ (for $\lambda_1 > \lambda_0$).

   (b) Calculate the power of the test you derived in part (a).

   (c) Test $\mathcal{H}_0 : \lambda = 1$ vs. $\mathcal{H}_1 : \lambda = 2$ at the 5% level, based on $\sum_{i=1}^{15} X_i = 9$.

## Solutions:

1. (a) The rejection regions with $\alpha = \mathbb{P}\left(\begin{array}{c}\text{type I}\\ \text{error}\end{array}\right) \leqslant 0.25$ here are –

   i. $\mathcal{C}_1 = \{\text{Beer}\}$ with $\alpha_1 = \mathbb{P}\left(\mathcal{C}_1 \middle| \text{France}\right) = 0.1$

   ii. $\mathcal{C}_2 = \{\text{Brandy}\}$ with $\alpha_2 = \mathbb{P}\left(\mathcal{C}_2 \middle| \text{France}\right) = 0.2$

   iii. $\mathcal{C}_3 = \{\text{Whisky}\}$ with $\alpha_3 = \mathbb{P}\left(\mathcal{C}_3 \middle| \text{France}\right) = 0.1$

   iv. $\mathcal{C}_4 = \{\text{Beer}, \text{Whisky}\}$ with $\alpha_4 = \mathbb{P}\left(\mathcal{C}_4 \middle| \text{France}\right) = 0.1 + 0.1 = 0.2$

   (b) The power of each of the above tests is –

   i. $\pi_1 = \mathbb{P}\left(\mathcal{C}_1 \middle| \text{UK}\right) = 0.5$

   ii. $\pi_2 = \mathbb{P}\left(\mathcal{C}_2 \middle| \text{UK}\right) = 0.1$

   iii. $\pi_3 = \mathbb{P}\left(\mathcal{C}_3 \middle| \text{UK}\right) = 0.2$

   iv. $\pi_4 = \mathbb{P}\left(\mathcal{C}_4 \middle| \text{UK}\right) = 0.5 + 0.2 = 0.7$

   Hence, at the desired significance level, the test defined by the rejection region $\mathcal{C}_4 = \{\text{Beer}, \text{Whisky}\}$ is the most powerful one.

   (c) Clearly

| $x$ | $\lambda(x)$ |
|---|---|
| Beer | 5 |
| Whisky | 2 |
| Brandy | 1/2 |
| Wine | 1/3 |

and for each calculated value $c \in \{5, 2, 1/2, 1/3\}$ we can identify the rejection region corresponding to $\lambda(x) \geqslant c$ –

| **Critical value $c$** | **Rejection Region** | $\boldsymbol{\alpha}$ |
|---|---|---|
| 5 | {Beer} | 0.1 |
| 2 | {Beer, Whisky} | 0.2 |
| 1/2 | {Beer, Whisky, Brandy} | 0.4 |
| 1/3 | {Beer, Whisky, Brandy, Wine} | 1 |

and the LRT with the largest significance level to still satisfy $\alpha \leqslant 0.25$ is the one whose rejection region is $\mathcal{C} = \left\{\lambda(x) \geqslant 2\right\} = \{\text{Beer}, \text{Whisky}\}$.

2. (a) The likelihood in this case is given by

$$\mathcal{L}(\lambda) = \lambda^n \exp\left\{-\lambda \sum_{i=1}^{n} x_i\right\},$$

and from the Neyman–Pearson Lemma, the most powerful test at level $\alpha$ for these simple hypotheses will be the Likelihood Ratio Test (LRT). Calculating

$$\lambda(\underline{x}) = \frac{\mathcal{L}(\lambda_1)}{\mathcal{L}(\lambda_0)} = \frac{\lambda_1^n \exp\left\{-\lambda_1 \sum_{i=1}^{n} x_i\right\}}{\lambda_0^n \exp\left\{-\lambda_0 \sum_{i=1}^{n} x_i\right\}} = \left(\frac{\lambda_1}{\lambda_0}\right)^n \exp\left\{-(\lambda_1 - \lambda_0)\sum_{i=1}^{n} x_i\right\},$$

hence

$$\lambda(\underline{x}) \geqslant c \iff \exp\left\{-(\lambda_1 - \lambda_0)\sum_{i=1}^{n} x_i\right\} \geqslant c_1 \iff -(\lambda_1 - \lambda_0)\sum_{i=1}^{n} x_i \geqslant c_2$$

$$\iff \sum_{i=1}^{n} x_i \leqslant c_3,$$

here the switch of sign

thus the rejection region will be of the form

$$\mathcal{C} = \left\{\sum_{i=1}^{n} X_i \leqslant c\right\} \quad \text{subject to} \quad \mathbb{P}\left(\underline{X} \in \mathcal{C} \middle| \mathcal{H}_0 : \lambda = \lambda_0\right) = \alpha.$$

To find the critical value $c$, recall that $\sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \lambda)$, and thus

$$2\lambda \sum_{i=1}^{n} X_i \sim \chi^2_{2n}.$$

Now,

$$\begin{aligned}
\alpha &= \mathbb{P}\left(\underline{X} \in \mathcal{C} \middle| \mathcal{H}_0 : \lambda = \lambda_0\right) = \mathbb{P}\left(\sum_{i=1}^{n} X_i \leqslant c \middle| \mathcal{H}_0 : \lambda = \lambda_0\right) \\
&= \mathbb{P}\left(2\lambda_0 \sum_{i=1}^{n} X_i \leqslant 2\lambda_0 c \middle| \mathcal{H}_0 : \lambda = \lambda_0\right) \Longrightarrow 2\lambda_0 c = \chi^2_{2n,\alpha} \Longrightarrow c = \frac{\chi^2_{2n,\alpha}}{2\lambda_0},
\end{aligned}$$

and so the rejection region for this test is $\mathcal{C} = \left\{\sum_{i=1}^{n} X_i \leqslant \frac{\chi^2_{2n,\alpha}}{2\lambda_0}\right\}$.

(b) The power of the test is

$$\begin{aligned}
\pi &= \mathbb{P}\left(\underline{X} \in \mathcal{C} \middle| \mathcal{H}_1 : \lambda = \lambda_1\right) = \mathbb{P}\left(\sum_{i=1}^{n} X_i \leqslant \frac{\chi^2_{2n,\alpha}}{2\lambda_0} \middle| \mathcal{H}_1 : \lambda = \lambda_1\right) \\
&= \mathbb{P}\left(2\lambda_1 \sum_{i=1}^{n} X_i \leqslant \frac{\lambda_1 \chi^2_{2n,\alpha}}{\lambda_0} \middle| \mathcal{H}_0 : \lambda = \lambda_1\right) = F_{\chi^2_{2n}}\left(\frac{\lambda_1 \chi^2_{2n,\alpha}}{\lambda_0}\right),
\end{aligned}$$

where $F_{\chi^2_{2n}}(\cdot)$ denotes the cdf of a $\chi^2_{2n}$ random variable. Note that for $\lambda_1 = \lambda_0$ we get $\underline{\pi = F_{\chi^2_{2n}}\left(\chi^2_{2n,\alpha}\right) = \alpha}$. Explain to yourselves why this is exactly what we would expect.

(c) For $\lambda_0 = 1$, $\lambda_1 = 2$, $\alpha = 0.05$ and $n = 15$, the rejection region is

$$\mathcal{C} = \left\{\sum_{i=1}^{n} X_i \leqslant \frac{\chi^2_{30,0.05}}{2}\right\} = \left\{\sum_{i=1}^{n} X_i \leqslant \frac{18.49}{2} = 9.245\right\},$$

thus, for an observed $\sum_{i=1}^{15} X_i = 9$, we reject $\mathcal{H}_0$ at the 5% level.