Linear Programming

Definition. ***Linear Programming Problem (LPP)***

1. ***Linear Function*** *Given a set of real numbers* $a_1, \cdots a_n$ *and a set of variables* $x_1, \cdots, x_n$, *we define a linear function* f *on these variables as*

$$f(x_1, \cdots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

2. ***Linear equality & inequality*** *If* b *is a real number, then*

$$f(x_1, \cdots, x_n) = b$$

is called a linear equality and

$$f(x_1, \cdots, x_n) \geq b \qquad f(x_1, \cdots, x_n) \leq b$$

are called linear inequalities

3. ***Linear Constraint*** *Combined they are called linear constraint*

4. ***Linear Programming Problem (LPP)*** *is a problem of minimizing or maximizing a linear function subject to a finite set of linear constraint*

5. ***Feasible Solution & Feasible Region*** *Any value* $(\overline{x}_1, \cdots, \overline{x}_n)$ *of variables that satisfies the constriant is called a feasible solution. The set of all feasible solutions is called feasible region of the LPP*

6. ***Infeasible & Feasible LPP*** *A LLP that has no feasible solution is called infeasible, otherwise it is called feasible*

7. ***Unbounded LPP*** *A LPP has feasible solution but no finite optimal objective value, we say the LPP is unbounded*

8. ***Objective Function & Object Value*** *the function used to maximize or minimize is called the objective function, and its value* $(\overline{x}_1, \cdots, \overline{x}_n)$ *is the objective value of that point*

9. ***Optimal Point*** *A point in the feasible region that has the maximum or miminium objective value is an optimal point*

Example. Maximize $(x_1, x_2) \xrightarrow{f} 2x_1 + 3x_2$ subject to

$$\begin{aligned} 3x_1 - 2x_2 &\leq 6 \\ x_1 + x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution. □

Can plot a 2-D plot representing a feasible region, calculate the function at endpoints and find the maximizing value. But why at endpoints... We calculate the gradient

$$\partial f(x_1, x_2) = [2, 3] \neq 0$$

hence no critical points in interior of the feasible region. Must be at boundary

Definition. ***Representation of LP problem***

1. ***Standard Form***

   (a) *Maximize* $a_1x_2 + a_2x_2 + \cdots + c_nx_n$ *subject to*

   $$\begin{aligned}&a_{11}x_1 + a_{12}x_n + \cdots + a_{1n}x_n \leq b_1\\&\cdots\\&a_{m1}x_1 + a_{m_2}x_n + \cdots + a_{mn}x_n \leq b_m\end{aligned}$$

   (b) *Maximize* $\sum_{j=1}^{n} c_jx_j$ *subject to* $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ *for* $i = 1, \cdots, m$ *where* $x_i \geq 0$ *for all* $i = 1, \cdots, m$

   (c) *Maximize* $c^T x$ *subject to* $Ax \leq b$ *and* $x \geq 0$ *where*

   $$c = (c_j)_{n\times 1} \quad b = (b_i)_{m\times 1} \quad n = (x_j)_{n\times 1} \quad A = (a_{ij})_{m\times n}$$

2. ***Transformation to Standard form***

   (a) *To minimize* $c^T x$*, we maximize* $-c^T x$

   (b)
   $$\sum_{j=1}^{n} a_{ij}x_j = b_i \text{ for some } i \quad \iff \quad \sum a_{ij}x_j \leq b_i \text{ and } \sum a_{ij}x_j \geq b_i$$

   (c)
   $$\sum_{j=1}^{n} a_{ij}x_j \geq b_i \quad \iff \quad -\sum_{j=1}^{n} a_{ij}x_j \leq -b_i$$

   (d) x_i *is unbounded for some c*

   $$x_i = x_i' - x_i'' \qquad x_i', x_i'' \geq 0'$$

3. ***Slack Form*** $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ *Define*

$$s = b_i - \sum_{j=1}^{n} a_{ij}x_j$$

A slack form is defined by (N, B, A, b, c, v) *where* N *is the set of nonbasic variables,* B *is the set of basic variables,* A, b, c *are coefficients and* v *is the optimal constant*

4. ***Transformation from Standard Form***

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij}x_j \quad i = 1, \cdots, n \quad x_{n+i} \geq 0$$

Remove MAXIMIZING *and* SUBJECT TO *and introduce a new variable* z *as follows*

$$z = \sum_{j=1}^{n} c_j x_j + v$$

where v *is some optimal constraint coefficient*

Example. maximize $3x_1 + 4x_2$ subject to

$$\begin{aligned} &2x_1 - 3x_2 \leq 5 \\ &x_1 + x_2 \leq 6 \\ &x_1, x_2 \geq 0 \end{aligned}$$

Slack form

$$\begin{aligned} &z = 3x_1 + 4x_2 \\ &x_3 = 5 - 2x_1 - 3x_2 \\ &x_4 = 6 - x_1 - x_2 \\ &x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The variables on the left are **basic variables** B and the variables on the right are called **nonbasic variables** N

Example. Given

$$\begin{aligned} &z = 5 - \frac{x_1}{7} - \frac{x_3}{8} + \frac{x_4}{10} \\ &x_2 = 7 + \frac{x_1}{8} + \frac{x_3}{7} - 2x_4 \\ &x_5 = 10 - \frac{x_1}{9} - \frac{2x_3}{3} + 3x_4 \\ &x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

where $N = \{x_1, x_3, x_4\}$ $B = \{x_2, x_5\}$,

$$A = \begin{bmatrix} -\frac{1}{8} & -\frac{1}{7} & 2 \\ \frac{1}{9} & \frac{2}{3} & -3 \end{bmatrix}$$
$$b = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$$
$$c = \begin{bmatrix} -\frac{1}{7} \\ -\frac{1}{8} \\ \frac{1}{10} \end{bmatrix}$$

Definition. ***Simplex Algorithm*** *maximize* $5x_1 - 3x_2$ *subject to*

$$x_1 - x_2 \leq 1$$
$$2x_1 + x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

1. *Convert the problem into a slack form*

$$z = 5x_1 - 3x_2$$
$$x_3 = 1 - x_1 + x_2 \qquad x_4 = 2 - 2x_1 - x_2$$
$$x_1, x_2, x_3, x_4 \geq 0$$

   Find ***Basic solution*** *by setting all nonbasic variables to zero*

$$x_1 = x_2 = 0 \quad x_3 = 1 \quad x_4 = 2$$

   Since feasible, it is a ***Basic feasible solution***

2. *Find a variable whose coefficient in the objective function is positive. This variable is called the* ***Leaving variable****. Finds a variable with a positive coefficient in the objective function. Restricts the increase of* x_i *to 1.*

3. *Find* ***Entering variable*** *Choose* x_3 *as the leaving variable* $x_1 = 1 + x - 2 - x_3$*. Now update slack form*

$$z = 5x_1 - 3x_2 = 5(1 + x_2 - x_3) - 3x_2 = 5 + 2x_2 - 5x_3$$
$$x_1 = 1 + x - 2 - x_3$$
$$x_4 = 2 - 2x_1 - x_2 = 2 - 2(1 + x_2 - x_3) - x_2 = -3x_2 - x_3$$

4. *Find basic solution again* $x_2 = x_3 = 0$, $x_1 = 1$, $x_4 = 0$. *The objective value* $z = 5 + 0 = 5$

Function `Pivot` (N, B, A, b, c, v, l, e)
  // Compute the coefficients of the equations for new basic variables
  $\hat{A} \leftarrow n \times n$ matrix
  $\hat{b}_e \leftarrow \frac{b_l}{a_{le}}$
  for $j \in N \setminus \{e\}$ **do**
    $\hat{a}_{ej} \leftarrow \frac{a_{lj}}{a_{le}}$
  $\hat{a}_{el} \leftarrow \frac{1}{a_{le}}$
  // Compute the coefficients of the remaining constraints
  for $i \in B \setminus \{l\}$ **do**
    $\hat{b}_i = b_i - a_{ie}\hat{b}_e$
    for $j \in N \setminus \{e\}$ **do**
      $\hat{a}_{ij} \leftarrow a_{ij} - a_{ie}\hat{a}_{ej}$
    $\hat{a}_{il} \leftarrow -a_{ie}\hat{a}_{el}$
  // Compute objective function
  $\hat{v} \leftarrow v + c_e\hat{b}_e$
  for $j \in N \setminus \{e\}$ **do**
    $\hat{c}_j \leftarrow c_j - c_e\hat{a}_{ej}$
  $\hat{c}_l \leftarrow -c_e\hat{e}_{el}$
  // Compute the new set of basic and nonbasic variables
  $\hat{N} = N \setminus \{e\} \cup \{l\}$
  $\hat{B} = B \setminus \{l\} \cup \{e\}$
  return $(\hat{N}, \hat{B}, \hat{A}, \hat{b}, \hat{c}, \hat{v})$
Function `Simplex` (A, b, c)
  $(N, B, A, b, c, v) \leftarrow$ `Initialize-Simplex`(A, b, c)
  $\triangle \leftarrow$ a new vector of length m
  while $j \in N$ *where* $c_j > 0$ **do**
    Choose index $e \in N$ such that $c_e > 0$
    for $i \in B$ **do**
      if $a_{ie} > 0$ **then**
        $\triangle_i \leftarrow \frac{b_i}{a_{ie}}$
      else
        $\triangle_i = \infty$
    Choose index $l \in B$ that minimizes $\triangle_l$
    if $\triangle_l == \infty$ **then**
      return *Unbounded*
    else
      // l is leaving variable
      $(N, B, A, b, c, v) =$ `Pivot`(N, B, A, b, c, v, l, e)
  for $i = 1 \rightarrow n$ **do**
    if $i \in B$ **then**
      $\bar{x}_i = b_i$
    else
      $\bar{x}_i = 0$
  return $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$

Theorem. ***Proof of correctness***

Proof. Proof by induction.

1. The slack form in every iteration is equivalent to the slack form retured by INITIALIZE-SIMPLEX
2. For each $i \in B$, we have $b_i \geq 0$
3. The basic solution as associated with the slack form is feasible

Termination

1. let (A, b, c) be a LPP given a set B of basic variables, the associated slack form is unique.

   Proof. For contradiction, assume 2 slack forms L and L' with same set of basic variables.

$$L : z = v + \sum_{j \in N} c_j x_j$$
$$x_i = b_i - \sum a_{ij} x_j \text{for } i \in B$$

$$L' : z = v' + \sum_{j \in N} c'_j x_j$$
$$x_i = b'_i - \sum a'_{ij} x_j \text{for } i \in B$$

$$L - L' : \sum_{j \in N} a_{ij} x_j = (b_i - b'_i) + \sum_{j \in N} a'_{ij} x_j$$

□

2. The number of unique slack forms is equal to number of ways of choosing B rom $\{x_1, x_2, \cdots, x_{n+m}\}$ which is $\binom{n+m}{m}$

3. If SIMPLEX fails to terminate in $\binom{n+m}{m}$ iterations, then it must cycle. There are techniques to avoid cycles which implies SIMPLEX terminates in less than $\binom{n+m}{m}$ steps
4. Hence runtime is exponential. but in practice, it is a very fast algorithm

$\square$