Contents

1. **Functions**
   - **injective** $f : X \to Y$ injective or one-to-one if $\forall a, b \in X,\ f(a) = f(b) \Rightarrow a = b$
   - **surjective** $f : X \to Y$ is surjective if any $y \in Y \Rightarrow y = f(x)$ for some $x \in X$
   - composition of injective/surjective/bijective functions are injective/surjective/bijective
   - If f is injective with range Y, then its inverse function $f^{-1} : Y \to X$ is a bijective function
2. **Set Relations**
   - **De Morgan's Law**

$$X \setminus \left(\bigcup_{\alpha \in I} A_\alpha \right) = \bigcap_{\alpha \in I} (X \setminus A_\alpha) \qquad X \setminus \left(\bigcap_{\alpha \in I} A_\alpha \right) = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$$

   - **how functions acts on sets** Let $A, B \subseteq X$ and $C, D \subseteq Y$
     - f well-behaved for union

$$f(A \cup B) = f(A) \cup f(B)$$

     - f not well-behaved for intersection, difference

$$\begin{aligned} f(A \cap B) &\subseteq f(A) \cap f(B) \\ f(A) \setminus f(B) &\subseteq f(A \setminus B) \end{aligned}$$

     - f^{-1} well-behaved with union, intersection, difference, and set complement

$$\begin{aligned} f^{-1}(C \cup D) &= f^{-1}(C) \cup f^{-1}(D) \\ f^{-1}(C \cap D) &= f^{-1}(C) \cap f^{-1}(D) \\ f^{-1}(C \setminus D) &= f^{-1}(C) \setminus f^{-1}(D) \\ f^{-1}(Y \setminus C) &= X \setminus f^{-1}(C) \end{aligned}$$

     - f and f^{-1} mixed

$$A \subseteq f^{-1}(f(A)) \qquad \text{(with equality if } f \text{ is injective)}$$
$$f(f^{-1}(C)) \subseteq C \qquad \text{(with equality if } f \text{ surjective)}$$

Remark. Read

1. chapter 1, 1-8
2. chapter 2, 12-22
3. chapter 3, 23-29
4. chapter 4. 30-35

1 General Topology

1.0.1 3 Relation

1. A relation on a set A is a subset C of cartesian product $A \times A$. xCy means $(x, y) \in C$
2. **equivalence relation** is a relation if it satisfies reflexivity, symmetry, transitivity
3. **equivalence class** a subset of A determined by some $x \in A$, i.e. $E = \{y \mid y \sim x\}$
4. **partition** of a set A is a collection of disjoint nonempty subsets of A whose union is all of A
5. **order relation** is a relation if it satisfies comparability (any $x \neq y \in A$ either xCy or yCx but not both) nonreflexivity (xCx does not hold for any $x \in A$) and transitivity
6. **dictionary order relation** Let A, B bet sets and $<_A$ and $<_B$ be order relations. The order relation on $A \times B$ is defined by $a_1 <_A a_2$ or if $a_1 = a_2$ and $b_1 <_B b_2$
7. **least upper bound property** An ordered set A has the property if every nonempty subset A_0 of A that is bounded above ($\exists b \in A$ s.t. $x \leq b$ for all $x \in A_0$) has a least upper bound (all bounds of A_0 has a smallest element)
   - $\mathbb{R}$ and $(-1, 1)$ has least upper bounde property
   - $B = (-1, 0) \cup (0, 1)$ does not heave least upper bound property, $\{-{}^1\!/_{2n} \mid n \in \mathbb{Z}_+\}$ is bounded above by any $b \in (0, 1)$ but its least upper bound $0 \notin B$

1.0.2 5 Cartesian Product

1. **indexed family of sets** Let $\mathcal{A}$ be nonempty collection of sets, let $f : J \to \mathcal{A}$ be a surjective indexing function. $(\mathcal{A}, f)$ is called indexed family of sets, denoted by $\{\mathcal{A}_\alpha\}_{\alpha \in J} = \{\mathcal{A}_\alpha\}$ where $f(\alpha) = \mathcal{A}_\alpha$
2. **m-tuple** Let $m \in \mathbb{Z}_+$, Given a set X, define m-tuple of X to be a function $\mathbf{x} : \{1, \cdots, m\} \to X$ and denote $\mathbf{x} = (x_1, \cdots, x_m)$
3. **cartesian product** Let $\mathcal{A} = \{A_1, \cdots, A_m\}$ be indexed family of sets, let $X = \cup_{i=1}^m A_i$. Then cartesian product of $\mathcal{A}$ is
$$X^m = \prod_{i=1}^m A_i \qquad A_1 \times \cdots \times A_m$$
to be the set of all m-tuples $\mathbf{x}$ of elements of X such that $x_i \in A_i$ for each i
4. **ω-tuple** Given a set X, define ω-tuple of elements of X be a function $\mathbf{x} : \mathbb{Z}_+ \to X$. $\mathbf{x}$ is an *infinite sequence*, of elements of X. Denote $x_i = \mathbf{x}(i)$ as i-th coordinate of $\mathbf{x}$. Denote $\mathbf{x}$ itself by $(x_1, x_2, \cdots)$ or $(x_n)_{n \in \mathbb{Z}_+}$
5. **cartesian product (infinite)** Let $\mathcal{A} = \{A_1, A_2, \cdots\}$ be indexed family of sets and X be union of sets in $\mathcal{A}$, the cartesian product of $\mathcal{A}$
$$X^\omega = \prod_{i \in \mathbb{Z}_+} A_i \qquad A_1 \times A_2 \times \cdots$$
is defined to be the set of all ω-tuples $(x_1, x_2, \cdots)$ of elements of X such that $x_i \in A_i$ for each i

1.0.3 11 The Maximum Principle

Definition. ***(Strict Partial Ordering)*** *Given a set A, and a relation $\prec$ on A is a strict partial order if*

1. *(nonreflexivity) $a \prec a$ never holds*
2. *(tranasitivity) If $a \prec b$ and $b \prec c$, then $a \prec c$*

Idea is not all $x, y \in A$ is comparable, i.e. either $x \prec y$ or $y \prec x$. Although a subset $S \subset A$ can be simply ordered.

Theorem. ***(The Maximum Principle)*** *Let A be a set and $\prec$ be a strict partial order on A. Then exists a maximal simply ordered subset B of A, i.e. does not exsits $B' \subset A$ such that $B \subsetneq B'$ and B' simply ordered.*

Definition. ***(upper bound and maximal element)*** *Let A be a set and $\prec$ be a strict partial order on A. If $B \subset A$, an* ***upper bound*** *on B is $c \in A$ such that $b = c$ or $b \prec c$ for all $b \in B$. A* ***maximal element*** *of A is an element $m \in A$ such that for no element $a \in A$ does $m \prec a$ hold.*

Lemma. ***(Zorn's Lemma)*** *Let A be a strictly partially ordered set. If every ordered subset of A has an upper bound in A, then A has a maximal element.*

2 Topological Spaces and Continuous Functions

2.0.1 12 Topological Spaces

Definition. *(**Topology**) Topology on a set X is a collection $\mathcal{T}$ of subsets of X having properties*

1. $\emptyset, X \in \mathcal{T}$
2. *Arbitrary union of subcollection of $\mathcal{T}$ is in $\mathcal{T}$ (If $\forall \alpha \in I$, $U_\alpha \in \mathcal{T}$, then $\cap_{\alpha \in I} U_\alpha \in \mathcal{T}$)*
3. *Finite intersection of subcollection of $\mathcal{T}$ is in $\mathcal{T}$ (If $\forall 1 \leq i \leq n$, $U_i \in \mathcal{T}$, then $\cap_{i=1}^n U_i \in \mathcal{T}$)*

A topological space is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ are the open sets.

- *Standard topology on $\mathbb{R}^n$ is $\mathcal{T}_0 = \mathcal{T}_{std} = \{U \subset \mathbb{R}^n \mid \forall x \in U, \exists \epsilon > 0\, B_\epsilon(x) \subset U\}$*
- *Standard topology on $\mathbb{R}$ is generated by $\mathcal{B}_{std} = \{(a,b) = \{x \mid a < x < b\}$ the open intervals*
- *Lower limit topology on $\mathbb{R}$ is generated by $\mathcal{B}_{l.l.} = \{x \mid a \leq x < b\}$ the half-open intervals*
- *Discrete topology $\mathcal{T}_1 = \mathcal{T}_{disc} = \mathcal{P}(X)$ all subsets are open*
- *Trivial topology $\mathcal{T}_2 = \mathcal{T}_{triv} = \{\emptyset, X\}$ only empty set and X are open*
- *Finite complement topology $\mathcal{T}_{f.c.} = \{U \subseteq X \mid X - U$ is finite or all of $X\}$*
- *Countable complement topology $\mathcal{T}_c = \{U \subseteq X \mid X - U$ is countable or all of $X\}$*

Lemma. *(**Arbitrary intersection of topologies is a topology**) $\forall \alpha \in I$ $\mathcal{T}_\alpha$ is a topology, so is $\cap_{\alpha \in I} \mathcal{T}_\alpha$*

Definition. *(**Compare topology**) If $\mathcal{T}' \subset \mathcal{T}$, then $\mathcal{T}'$ is coarser / weaker / smaller than $\mathcal{T}$, $\mathcal{T}$ is finer / stronger / larger than $\mathcal{T}'$. $\mathcal{T}$ and $\mathcal{T}'$ are comparable if $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$*

$$\mathcal{T}_{triv} \subset \mathcal{T}_{f.c.} \subset \mathcal{T}_{std} \subset \mathcal{T}_{disc} \qquad \mathcal{T}_{std} \subset \mathcal{T}_{l.l.}$$

2.0.2 13 Basis for a Topology

A terser representation of $\mathcal{T}$

Definition. *(**Basis**) If X is a set, a basis for $(X, \mathcal{T})$ is a collection $\mathcal{B}$ (basis elements) of subsets of X s.t.*

1. *For each $x \in X$, exists at least one basis element $B \in \mathcal{B}$ containing it*
2. *If $x \in B_1 \cap B_2$, then exists $B_3 \in \mathcal{B}$ s.t. $x \in B_3 \in B_1 \cap B_2$*

*A topology $\mathcal{T}_\mathcal{B}$ **generated by** $\mathcal{B}$ is defined as*

$$\mathcal{T}_\mathcal{B} = \{U \subset X \mid \forall x \in U,\ \exists B \in \mathcal{B},\ x \in B \subset U\}$$

- *$\mathcal{T}_\mathcal{B}$ is the unique minimal topology containing $\mathcal{B}$*

$$\mathcal{T}_\mathcal{B} = \bigcap\nolimits_{\mathcal{T} \in \mathbb{T}} \mathcal{T}$$

  where $\mathbb{T} = \{\mathcal{T} \mid \mathcal{T} \supset \mathcal{B}$ and $\mathcal{T}$is a topology$\}$

- *For any X, all one point sets of X is a basis for $\mathcal{T}_{disc}$*

Lemma. *($\mathcal{B} \to \mathcal{T}_\mathcal{B}$) $\mathcal{T}_\mathcal{B}$ equals the collections of all unions of elements of $\mathcal{B}$*

$$\mathcal{T}_\mathcal{B} = \left\{\bigcup\nolimits_{\alpha \in I} B_\alpha \mid B_\alpha \in \mathcal{B} \quad \forall \alpha \in I\right\}$$

Lemma. *($\mathcal{T}_\mathcal{B} \to \mathcal{B}$) Let $(X, \mathcal{T})$ be topological space. Let $\mathcal{C}$ be a collection of open sets of X such that*

$$\forall U \subset X, \quad \forall x \in U, \quad \exists C \in \mathcal{C} \text{ s.t. } x \in C \subset U$$

Then $\mathcal{C}$ is a basis for $\mathcal{T}$ (handy in deciding $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for $Y \subset X$)

Lemma. *(**compare topology by basis**) Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then following equivalent*

1. *$\mathcal{T}' \supset \mathcal{T}$ ($\mathcal{T}'$ is finer than $\mathcal{T}$)*
2. *For each $x \in X$ and each $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset \mathcal{B}$*

2.0.3 14 Order Topology

Definition. *(**Order Topology**) Let X be a set with simple order relation. Let $\mathcal{B}$ be a collection of all sets of the following type*

$$\begin{aligned}\mathcal{B} = &\{(a,b) \mid a < b \quad a,b \in X\} \\ &\bigcup \{[a_0, b) \mid a_0 \text{ is minimal element (if any) of } X \ b \in X \quad b \neq a_0\} \\ &\bigcup \{(a, b_0] \mid b_0 \text{ is maximal element (if any) of } X \ a \in X \quad a \neq b_0\}\end{aligned}$$

Generated topology $\mathcal{T}_{\mathcal{B}}$ is called order topology

- *In $\mathbb{R}$, $\mathcal{T}_{ord} = \mathcal{T}_{std}$*
- *In $\mathbb{Z}_+$, $\mathcal{T}_{ord} = \mathcal{T}_{disc}$ (since any $\{n\} = (n-1, n+1) \in \mathcal{T}_{ord}$)*
- *In $\{1,2\} \times \mathbb{Z}_+$ in $\mathcal{T}_{dict}$ is not in $\mathcal{T}_{disc}$ (althogh most single point set are open, 2×1 is not open)*
- *In $\mathbb{R}^2$, both $\mathcal{B}$ and $\mathcal{B}'$ generates $\mathcal{T}_{dict}$*

$$\mathcal{B} = \{(a \times b, c \times d) \mid a < c \vee (a = c \wedge b < d)\} \quad \mathcal{B}' = \{(a \times b, a \times d) \mid b < d\}$$

2.0.4 15 Product Topology

Definition. *(**Product Topology**) Let X, Y be topological spaces, the product topology on $X \times Y$ is generated by the basis*

$$\mathcal{B} = \{U \times V \mid U \in \mathcal{T}_X \ V \in \mathcal{T}_Y\}$$

Alternatively define product topology with basis. If $\mathcal{B}$, $\mathcal{C}$ are basis for X and Y respectively. Then

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \ C \in \mathcal{C}\}$$

is a basis for topology of $X \times Y$

- *$X \times \{y\} \cong X$*
- *$X \times Y \cong Y \times X$*
- *$(X \times Y) \times Z \cong X \times (Y \times Z)$*
- *Product spaces does not work well with order topology.*
  - *Consider $X = \mathbb{R}^2$ and $Y = [0,1] \times [0,1]$, then $\{0.5\} \times [0,1]$ is not open in $\mathcal{T}_{ord}$ but is open in $\mathcal{T}_{subspace}$*

Definition. *(**Projection**) Let $\pi_1 : X \times Y \to X$ be defined by $\pi_1(x,y) = x$. π_1 is a projection of $X \times Y$ **onto** the first factor. (note projections are surjective)*

Definition. *(**Product Topology by Continuity of Functions**) Given X, Y, there exists unique topology on $X \times Y$ such that*

1. *projections π_X and π_Y are continuous*
2. *If $f : Z \to X$ and $g : Z \to Y$, hen $f \times g : Z \to X \times Y$ is continuous*

Proof. Define $\mathcal{B} = \{U \times V \mid U \subset X \text{ open } \ V \subset Y \text{ open }\}$. Show $\mathcal{T}_{\mathcal{B}}$ satisfies the above 2 conditions. Prove uniqueness by showing $id : (X, \mathcal{T}') \to (X, \mathcal{T}'')$ is a homeomorphism utilizing the above 2 conditions. □

2.0.5 16 Subspace Topology

Definition. *(**Subspace Topology**) Let $(X, \mathcal{T})$. Let $Y \subset X$, then*

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is the subspace topology on Y. Alternatively, define using basis. If $\mathcal{B}$ generates $\mathcal{T}$, then

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for $\mathcal{T}_Y$.

- ***(lemma)** $Y \subset X$. U open in Y and Y open in X, then U open in X*
- ***(lemma)** A subspace of a subspace is a subspace*
- *<u>(theorem)</u> A product of subspaces is a subspace of the product (subspace and product topology work well)*

- *X ordered and $Y \subset X$. order topology on Y may not be same as order topology of Y inherited as a subspace of X (subspace and order topology does not work well)*
  - *Let $Y = [0,1] \subset \mathbb{R}$. $\mathcal{B}_Y$ are of the form $(a,b) \cap [0,1]$. Note*
    1. *$[0,b)$ where $b \notin [0,1]$ is open in $[0,1]$ but not in $\mathbb{R}$*
    2. *$\mathcal{T}_{ord} \cong \mathcal{T}_{subspace}$ since the basis elements of the same form*
  - *Let $Y = [0,1) \cup \{2\} \subset \mathbb{R}$.*
    1. *$\{2\}$ open in $\mathcal{T}_{subspace}$ since $(1.5, 2.5) \cap Y = \{2\}$.*
    2. *$\{2\}$ not open in order topology since any basis of the form $\{x \in Y \mid a < x \leq 2 \ a \in Y\}$ contains some other point other than $\{2\}$*
    3. *$\mathcal{T}_{ord} \ncong \mathcal{T}_{subspace}$*
- *(theorem) subspace and order topology works well if the subspace is convex*

Definition. ***(convex)*** *Given ordered X and subset $Y \subset X$ is convex if $(a,b) \subset X$ lies in Y completely*

Theorem. ***(subspace and order topology works well if the subspace is convex)***
Let X be ordered and $Y \subset X$ be convex. Then order topology on Y same as topology Y inherits as a subspace of X.

Definition. ***(Subspace Topology by Continuity of Functions)*** *Let X be topological space, $Y \subset X$, there exists unique topology on Y such that*

1. *inclusion $i_Y : Y \hookrightarrow X$ is continuous*
2. *If $f : Z \to Y$ such that $i_Y \circ f : Z \to X$ is continuous, then f is continuous*

2.0.6 17 Closed Sets and Limit Points

Definition. ***(Closed)*** *A subset A of X is closed if $X - A$ is open.*

- *In $\mathcal{T}_{f.c.}$, all finite subsets and X are closed*
- *In $\mathcal{T}_{disc}$ every set is closed.*
- *In $Y = [0,1] \cup (2,3)$, $[0,1]$ and $(2,3)$ are open and closed in subspace topology of Y*
- *C closed in Y does not imply C is closed in X. However a closed set in a closed subspace is closed overall, i.e. in X (Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.)*

Theorem. ***(Topology by closed sets)***

1. *$\emptyset$ and X are closed*
2. *Arbitrary intersection of closed sets are closed*
3. *Finite unions of closed sets are closed*

Theorem. ***(Closedness in Subspace)*** *Let Y be a subspace of X. Then A is closed in Y if and only if it equals the intersection of a closed set X with Y ($Y \subset X$, then $A \subset Y$ closed in Y if exists $K \subset X$ s.t. $A = K \cap Y$)*

Definition. ***(Closure and Interior)*** *If $A \subset X$*

1. ***Interior*** *$Int_X A = \mathring{A}$ is*
   - *the union of all open sets in X contained in A, i.e. $Int_X A = \cup_{U \in \mathcal{T}_X : U \subset A} U$*
   - *maximal open subests of A in X*
2. ***Closure*** *$Cl_X A = \overline{A}$ is*
   - *the intersection of all closed sets containing A, i.e. $Cl_X A = \cap_{U \in \mathcal{T}_X : U \supset A} U$*
   - *minimal closed set containing A*

- *Set relationship*
$$IntA \subset A \subset \overline{A}$$
- *(theorem). Let $A \subset Y \subset X$, Let $\overline{A}$ denote closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$*
  - *In $\mathbb{R}$, $Y = (0,1]$, let $A = (0, 0.5) \subset Y$. $Cl_{\mathbb{R}} A = [0, 0.5]$, $Cl_Y A = Cl_{\mathbb{R}} A \cap Y = (0, 0.5]$*
- *If A open, then $Int(A) = A$; If A closed, then $\overline{A} = A$*
- *If $A \subset X$, then $(\mathring{A})^c = \overline{(A^c)}$ (Complement of interior is closure of the complement)*

- *In $\mathbb{R}$, let $A = \mathbb{Q}$ or $A = \mathbb{R} - \mathbb{Q}$, $int(A) = \emptyset$ and $\overline{A} = \mathbb{R}$ ($int(A) = \emptyset$ since no (a, b) contained fully in $\mathbb{Q}$ or $\mathbb{R} - \mathbb{Q}$)*

Definition. ***(neighborhood)*** *U is an open set containing x is equivalent to U is a neighborhood of x*

Definition. ***(intersects)*** *A intersects B if and only if $A \cap B \neq \emptyset$*

Theorem. ***(define closure using neighborhoods)*** *Let A be a subset of X,*

1. *then $x \in \overline{A}$ if and only if every neighborhood of x intersects A*
2. *Let $\mathcal{B}$ be basis of X, then $x \in \overline{A}$ if and only if every basic neighborhood B of x intersects A*

proof by contraposition. Following are examples which uses this theorem to test/determine the closure

- *If $A = (0, 1]$ then $\overline{A} = [0, 1]$ (since every neighborhood of $\{0\}$ intersects A)*
- *If $B = \{1/n \mid n \in \mathbb{Z}_+\}$, then $\overline{B} = \{0\} \cup B$*
- *If $C = \{0\} \cup (1, 2)$ then $\overline{C} = \{0\} \cup [1, 2]$*
- *In $\mathbb{R}$, $\overline{\mathbb{Q}} = \mathbb{R}$ since every neighborhood of $x \in \mathbb{R}$ contains some rational number, so intersects $\mathbb{Q}$*
- *In $\mathbb{Z}_+$, $\overline{\mathbb{Z}_+} = \mathbb{Z}_+$*

Definition. ***(Limit Point)*** *Let $A \subset X$ and $x \in X$, x is a limit point of A if every neighborhood of x intersects A in some point other than x itself. In other words,*

$$x \in \overline{A - \{x\}}$$

- *In $\mathbb{R}$, $A = (0, 1]$, then any $x \in [0, 1]$ is a limit point of A and no other point in $\mathbb{R}$ is a limit point*
- *In $\mathbb{R}$, $B = \{1/n \mid n \in \mathbb{Z}_+\}$, 0 is the only limit point of B (any other $x \in \mathbb{R}$ has neighborhood that does not intersect B or intersects at x itself)*
- *In $\mathbb{R}$, $C = \{0\} \cup (1, 2)$, all $x \in [1, 2]$ are limit points of C*
- *In $\mathbb{R}$, every $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$*
- *In $\mathbb{R}$, no point is a limit point of $\mathbb{Z}_+$*

Theorem. ***(define closure using limit point)*** *Let $A \subset X$, let A' be set of all limit points, then*

$$\overline{A} = A \cup A'$$

- *(corollary) $A \subset X$ is closed if and only if it contains all its limit points, i.e. $A' \subset A$*

Remark. ways to prove A is closed

1. show A^c is open
2. show $\overline{A} = A$ by proving that every $x \in A^c$ has open neighborhood that does not intersect A
   - $A = \{x_0\} \subset \mathbb{R}$ closed since every point different from x_0 has neighborhood not intersecting $\{x_0\}$

Definition. ***(Sequence Convergence)*** *A sequence (x_n) converges to a point $x \in X$, denoted $x_n \to x$ or $\lim_{n\to\infty} x_n = x$ if*

$$\forall \text{ neighborhood } U \text{ of } x \; \exists N \in \mathbb{N} \text{ such that } \forall n \geq N \; x_n \in U$$

For metric spaces

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N \; |x_n - x| < \epsilon$$

Definition. ***(Separated)*** *Let $x, y \in X$, x and y can be separated if each lies in a neighborhood which does not contain the other point. (neighborhood not necessarily disjoint)*

Definition. ***(T_1 Space)*** *X is T_1 if any two distinct points in X are separated.*

- *(theorem) Let $A \subset X$ T_1. $x \in A'$ if and only if every neighborhood of x contains infinitely many points of A.*
- *(theorem) Every finite point set, specifically one-point set, in a T_1 space is closed*

Definition. ***(T_2 Hausdorff Space)*** *A topological space X is called Hausdorff space if for each pair x_1, x_2 of distinct points of X, there exists neighborhoods U_1 and U_2 of x_1 and x_2, respectively that are disjoint.*

$$\forall x \neq y \in X \; \exists \text{ neighborhoods } U, V \text{ of } x \text{ and } y \text{ respectively s.t. } U \cap V = \emptyset$$

- ***(motivation)*** *Generally, one point set not always closed; Sequences converges can converge to more than limit.*

- *(theorem) Every finite point set, specifically one-point set, in a Hausdorff space is closed*
- *(theorem) If X is T_2, then a sequence of points of X converges to at most 1 point of X*
- *(theorem) Every simply ordered set is T_2 in the order topology. Product of two T_2 space is T_2; Subspace of a T_2 space is T_2 (order/product/subspace topology well behaved with T_2)*
- ***(examples)***
  - $\mathbb{R}^n_{std}$, X_{disc} *are* T_2
  - X_{triv} *not* T_2 *except when* $|X_{triv}| = 1$
  - $X_{f.c.}$ *not* T_2 *when* X *is infinite (since any* $x, y \in X_{f.c.}$ *are infinite and intersects)*

2.0.7 18 Continuous Functions

Definition. ***(Continuous)*** *A function $f : X \to Y$ is continuous if each open subset $V \subset Y$, the set $f^{-1}(V)$ is open. Alternatively formulated with basis, f is continuous if every basis element $B \in \mathcal{B}$, $f^{-1}(B)$ is open*

- $id : \mathbb{R}_{std} \to \mathbb{R}_{l.l}$ *is not continuous;* $id : \mathbb{R}_{l.l.} \to \mathbb{R}_{std}$ *is continuous*

Theorem. ***(TFAE for continuous function)*** $f : X \to Y$

1. f *is continuous*
2. *For every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$ (convergence: $x_n \to x \Rightarrow f(x_n) \to f(x)$ to $x \in \overline{A} \Rightarrow f(x) \in \overline{f(A)}$)*
3. *For every closed set B of Y, the set $f^{-1}(B)$ is closed*
4. *(generalized ϵ-δ) $\forall x \in X$ and $\forall$ neighborhood V of $f(x)$, $\exists$ neighborhood U of x such that $f(U) \subset V$*

- *In metric space, 4 can be reformulated with ϵ-δ definition. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if*

$$\forall x_0 \in \mathbb{R}^n, \quad \forall \epsilon > 0 \quad \exists \delta > 0 \text{ s.t. } |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$
$$\text{i.e. } (x \in B_\delta(x_0) \Rightarrow f(x) \in B_\epsilon(f(x_0)))$$

Definition. ***(Homeomorphism)*** *Let $f : X \to Y$ be a bijection If f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a homeomorphism. (continuous bijection)*

- *(theorem) If $X \cong Y$, then X and Y share topological property, i.e. property expressed in terms of topology only.*
- *(example) $(-1, 1) \cong \mathbb{R}$ since $f : (0, 1) \to \mathbb{R}$ by $f(x) = \tan(x)$ is a homeomorphism*
- *(example) A function can be continuous but not homeomorphic. Consider $S^1 = \{x \times y \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$ the unit circle. Let $f : [0, 1) \to S^1$ by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. f is bijective and continuous but f^{-1} not continuous*
- *If $\mathcal{T}_1 \subset \mathcal{T}_2$, then $Id : (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ is continuous*
- *If $\mathcal{T}_1 = \mathcal{T}_2$, then $Id : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is a homeomorphism*

Definition. ***(Imbedding)*** *An injective map $f : X \to Y$ is a topological imbedding of X in Y if $f' : X \to Z$ is a homeomorphism (note the image set $Z = f(X)$ carries subspace topology inherited from Y)*

- *Intuitively, an imbedding $f : X \to Y$ let us treat X as a subspace of Y.*
- *(theorem) Every map that is injective, continuous, and either open or closed is am imbedding.*
- *(example) $f : [0, 1) \to \mathbb{R}$ by $f(t) = (\cos(2\pi t), \sin(2\pi t))$ maps to S^1. f is a continuous injective map but not am imbedding.*

Theorem. ***(Constructing Continuous Functions)*** *Given X, Y, Z*

1. *(constant function) If $f : X \to Y$ by $f(X) = \{y_0\}$, then f is continuous*
2. *(inclusion) If $A \subset X$ with subspace topology, inclusion $i_A : A \hookrightarrow X$ is continuous*
3. *(composition) If $f : X \to Y$ and $g : Y \to Z$ are continous, then $g \circ f : X \to Z$ is continuous*
4. *(restricting domain) If $f : X \to Y$ is continous and $A \subset X$, then $f|_A : A \to Y$ is continuous ($f|_A = f \circ i_A$)*
5. *(restricting or expanding range) Let $f : X \to Y$ continuous. If $f(X) \subset A \subset Y \subset B$, then $g : X \to A$ obtained by restricting range of f is continuous. $h : X \to B$ obtained by expanding range of f also continuous ($h = i_Y \circ f$)*

6. *(local formulation of continuity)* $f : X \to Y$ *is continuous if* $X = \cup_{\alpha \in I} U_\alpha$ *such that* $f|_{U_\alpha}$ *is continuous for each* $\alpha \in I$

7. *(map to products) Let* $f : A \to X \times Y$ *given by* $f(t) = (f_1(t), f_2(t))$ *and* $f_1 : A \to X$, $f_2 : A \to Y$. *Then* f *continuous if and only if* f_1 *and* f_2 *are continuous*

8. *(Algebraic Operations) If* $f, g : X \to \mathbb{R}$ *continous, then* $f + g$, $f - g$, $f \cdot g$, f/g $(g(x) \neq 0)$ *all continous*

9. *(Uniform Limit Theorem) If a sequence of continuous real-valued function of a real variable converges* ***uniformly*** *to a limit function, then the limit function is continuous*

Theorem. ***(the pasting lemma)*** *Let* $X = A \cup B$, *where* A *and* B *are both closed or both open in* X. *Let* $f : A \to Y$ *and* $g : B \to Y$ *be continuous. If* $f(x) = g(x)$ *for every* $x \in A \cap B$, *set* $h : X \to Y$

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

then h *is continuous*

- $h(x) = x$ *for* $x \leq 0$ *and* $h(x) = x/2$ *for* $x \geq 0$ *is continous*

2.0.8 18 The Product Topology

Definition. ***(J-tuple)*** *Let* J *be index set. Define* ***J-tuple*** *of elements of* X *be a function* $\mathbf{x} : J \to X$. *Given* $\alpha \in J$, *denote* α***th coordinate*** *as* $\mathbf{x}$ *at* α *by* x_α *instead of* $\mathbf{x}(\alpha)$. *Denote* $\mathbf{x}$ *as* $(x_\alpha)_{\alpha \in J}$. *Denote the* ***set of all*** J***-tuples*** *of elements of* X *by* X^J

Definition. ***(Cartesian Product)*** *Let* $\{A_\alpha\}_{\alpha \in J}$ *be an indexed family of sets; Let* $X = \cup_{\alpha \in J} A_\alpha$. *The cartesian product of this indexed family is given by*

$$\prod_{\alpha \in J} A_\alpha = \{(x_\alpha)_{\alpha \in J} \in X \mid x_\alpha \in A_\alpha\} = \{\mathbf{x} : J \to \bigcup_{\alpha \in J} X \mid \forall \alpha \in J \; \mathbf{x}(\alpha) \in A_\alpha\}$$

is the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ *of elements of* X *such that* $x_\alpha \in A_\alpha$ *for each* $\alpha \in J$. *Equivalently, the set of all functions* $\mathbf{x} : J \to \cup_{\alpha \in J} X$ *such that* $\mathbf{x}(\alpha) = A_\alpha$ *for each* $\alpha \in J$

Definition. ***(Projection)*** *The projection mapping associated with index* β *is defined by*

$$\pi_\beta = \prod_{\alpha \in J} X_\alpha \to X_\beta \qquad \pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta$$

Definition. ***(Box Topology)*** *Let* $\{X_\alpha\}_{\alpha \in J}$ *be indexed family of topological spaces. Basis*

$$\mathcal{B}_{box} = \left\{ \prod_{\alpha \in J} U_\alpha \mid U_\alpha \text{ open in } X_\alpha \; \forall \alpha \in J \right\}$$

generates box topology

- *(theorem) Given basis* $\mathcal{B}_\alpha$ *for each* X_α, $\prod_{\alpha \in J} \mathcal{B}_\alpha$ *where* $B_\alpha \in \mathcal{B}_\alpha$ *is a basis for* $\prod_{\alpha \in J} X_\alpha$

Definition. ***(Product Topology)*** *Let* $\{X_\alpha\}_{\alpha \in J}$ *be indexed family of topological spaces. Basis*

$$\begin{aligned} \mathcal{B}_{prod} &= \left\{ \pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n}) \mid U_{\beta_i} \in \mathcal{T}_{X_{\beta_i}} \; \beta_1, \cdots, \beta_n \in J \right\} \\ &= \left\{ \prod_{\alpha \in J} U_\alpha \mid \forall \alpha \in J \; U_\alpha \in \mathcal{T}_{X_\alpha} \text{ for almost all } \alpha \; U_\alpha = X_\alpha \right\} \end{aligned}$$

generates product topology

- *(theorem) Let* $f : A \to \prod_{\alpha \in J} X_\alpha$ *defined by* $f(a) = (f_\alpha(a))_{\alpha \in J}$ *where* $f_\alpha : A \to X_\alpha$ *for each* α, *Then* f *is continuous if and only if each function* f_α *is continuous*

- *(theorem) For finite products,* $\mathcal{T}_{box} = \mathcal{T}_{prod}$

- *(example where product topology works while box topology does not) In* $\mathbb{R}^\omega = \prod_{n \in \mathbb{Z}_+} \mathbb{R}$, *countably infinite product of* $\mathbb{R}$. *Define* $f : \mathbb{R} \to \mathbb{R}^\omega$ *by* $f(t) = (t, t, \cdots)$. *Each* $f_\alpha = \pi_\alpha \circ f$ *continuous so* f *continuous in product topology. However* f *not continuous in box topology. Consider* $B = \{(-1/n, 1/n) \mid n \in \mathbb{Z}_+\} \in \mathcal{B}_{box}$ *open,*

  $$f^{-1}(B) = \{t \mid (t, t, \cdots) \in B\} = \bigcap_{n \in \mathbb{Z}_+} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$$

  not open in $\mathbb{R}$

Definition. ***(product topology by continuity of functions)*** *There is a unique topology $\mathcal{T}_{prod}$ on $X = \prod_{\alpha \in J} X_\alpha$ with*

1. *$\pi_\beta : X \to X_\beta$ continuous for every $\beta \in J$*
2. *Given $f : Z \to \prod_{\alpha \in J} X_\alpha$. If $f_\alpha = \pi_\alpha \circ f$ is continuous for every α, then f is continuous*

Note $\mathcal{B}_{prod}$ satisfies the two condition

Proof. 1. by design. proof for 2 as follows

$$f^{-1}\left(\bigcap_{i=1}^{n} \pi_{\beta_i}^{-1}(U_{\beta_i})\right) = \{z \in Z \mid \forall i \; f_{\beta_i} \in U_{\beta_i}\} = \bigcap_{i=1}^{n} f_{\beta_i}^{-1}(U_{\beta_i})$$

is open as finite intersection of open sets □

Theorem. ***(Both box and product topology works well with subspace/T_2/closure)***

1. *Let $A_\alpha \subset X_\alpha$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$*
2. *If X_α is T_2, then $\prod X_\alpha$ is T_2*
3. *Let $A_\alpha \subset X_\alpha$, then $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$*

Proof. **proof of 2.** Let $\mathbf{x} = (x_\alpha)$ and $\mathbf{y} = (y_\alpha)$ where $\mathbf{x} \neq \mathbf{y}$. So exists $\beta \in J$ such that $x_\beta \neq y_\beta$. Since X_β is T_2, exsits neighborhoods U_β, V_β for x_β and y_β s.t. $U_\beta \cap V_\beta = \emptyset$. Note $\pi^{-1}(U_\beta)$ and $\pi^{-1}(V_\beta)$ are disjoint neighborhoods of $\mathbf{x}, \mathbf{y}$. **proof of 3.** Mainly use the definition of closure using neighborhoods. ($\Rightarrow$) Let $\mathbf{x} = (x_\alpha) \in \prod \overline{A_\alpha}$. Need to show $\mathbf{x} \in \overline{\prod A_\alpha}$. Let $U = \prod U_\alpha$ be basic neighborhood of $\mathbf{x}$ in $\mathcal{T}_{box}$ or $\mathcal{T}_{prod}$. For each α, since $x_\alpha \in A_\alpha$, can find $y_\alpha \in U_\alpha \cap A_\alpha$. Note $\mathbf{y} = (y_\alpha) \in U \cap \prod A_\alpha$. Since U arbitrary,$\mathbf{x} \in \overline{\prod A_\alpha}$. ($\Leftarrow$) Let $\mathbf{x} = (x_\alpha) \in \overline{\prod A_\alpha}$. Want to show $x_\beta \in \overline{A_\beta}$ for all β. Let V_β be arbitrary neighborhood of x_β. Consider $\pi^{-1}(V_\beta)$, which is open in both $\mathcal{T}_{box}$ and $\mathcal{T}_{prod}$. By definition of closure, exsits $\mathbf{y} = (y_\alpha) \in \pi^{-1}(V_\beta) \cap \prod A_\alpha$. Hence $y_\beta \in V_\beta \cap A_\beta$. Hence, $x_\beta \in \overline{A_\beta}$ □

2.0.9 19 The Metric Topology

Definition. ***(Metric)*** *A metric on X is a function*

$$d : X \times X \to \mathbb{R}$$

with properties

1. *(non-negativity) $d(x,y) \geq 0$ for all $x, y \in X$, wth equality if $x = y$*
2. *(symmetry) $d(x,y) = d(y,x)$ for all $x, y \in X$*
3. *(triangle inequality) $d(x,y) + d(y,z) \geq d(x,z)$ for all $x, y, z \in X$*

Definition. ***(ϵ-ball centered at x)***

$$B_d(x,\epsilon) = \{y \mid d(x,y) < \epsilon\}$$

Definition. ***(norm)*** *Given $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$, the norm of $\mathbf{x}$ defined by $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$.*

Definition. ***(Metric Topology)*** *Given X and a metric d, basis*

$$\mathcal{B}_d = \{B_d(x,\epsilon) \mid x \in X \; \epsilon > 0\}$$

generates the metric topology $\mathcal{T}_d$ induced by d. Therefore

$$\mathcal{T} = \{U \subset X \mid \forall x \in U \; \exists \epsilon > 0 \; B(x,\epsilon) \subset U\}$$

1. ***discrete metric*** *d_{disc}, defined by $d_{disc}(x,y) = 1$ if $x \neq y$ and $d_{disc}(x,y) = 0$ if $x = y$ induces $\mathcal{T}_{disc}$*
2. ***standard metric*** *on $\mathbb{R}$ defined by $d(x,y) = |x-y|$ induces $\mathcal{T}_{ord}$*
3. ***(diamond)*** *$d_1 = \sum_i |x_i - y_i|$*
4. ***euclidean metric (circle)*** *on $\mathbb{R}^n$, $d_2 = d(\mathbf{x},\mathbf{y}) = \|x-y\| = \sqrt{(x_1-y_2)^2 + \cdots + (x_n-y_n)^2}$*
5. ***square metric (square)*** *on $\mathbb{R}^n$, $d_\infty = \rho(\mathbf{x},\mathbf{y}) = max\{|x_1-y_1|, \cdots, |x_n-y_n|\}$ (furthest coordinate within ϵ)*
6. *(theorem) In $\mathbb{R}^n$, $\mathcal{T}_{d_1} = \mathcal{T}_{d_2} = \mathcal{T}_\infty$ induces same topology $\mathcal{T}_{std}$ (proof by showing basis elements nests!)*
7. *(theorem) If X is metrizable, then X is T_2*
8. *(theorem) Subspaces of metric space behaves well $d|_{A \times A}$ induces subspace topology for $A \subset X$.*

Definition. *(Metrizable) If X is topological space, X is* ***metrizable*** *if there exists metric d that induces the topology of X. A* ***metric space*** *is a metrizable space X together with a specific metric d that gives the topology of X*

- *Metrizability is a topological property*
- *(not every topology comes with a metric) consider X_{triv} where $|X| \geq 2$, X not metrizable since it's not T_2*
- *$\mathbb{R}^n$ is metrizable (d_1, d_2, d_∞ induces $\mathbb{R}^n_{std}$)*
- *$\mathbb{R}^\omega$ is metrizable under product topology but not box topology*
- *$\mathbb{R}^J$ where J uncountable is not metrizable*
- *(theorem) countable products of metrizable spaces is metrizable*
- *(by sequence lemma) sequences are sufficient to describe metrizable spaces*

Definition. *(Bounded and Diameter) Given (X, d), $A \subset X$ is* ***bounded*** *if there is some M such that*

$$d(a_1, a_2) \leq M$$

for all $a_1, a_2 \in A$. The ***diameter*** *of A is defined*

$$diam\,(A, d) = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$$

- *boundedness is not a topological property, since it depends on a specific d (consider d and $\overline{d}$)*

Definition. *(Standard Bounded Metric) Given (X, d), define $\overline{d} : X \times X \to \mathbb{R}$ by*

$$\overline{d}(x, y) = \min\{d(x, y), 1\}$$

to be the standard bounded metric corresponding to d

- *(theorem) d and $\overline{d}$ induces the same topology, i.e. $\mathcal{T}_d = \mathcal{T}_{\overline{d}}$*
- *(trick) by above theorem, we can say diam$(A) \leq 1$ without loss of generality by replacing d with $\overline{d}$*

Definition. *(Uniform Metric and Uniform Topology) Generalize square metric to $\mathbb{R}^J$. Given $\mathbf{x} = (x_\alpha)_{\alpha \in J}$, $\mathbf{y} = (y_\alpha)_{\alpha \in J} \in \mathbb{R}^J$. Define metric $\overline{\rho}$ by*

$$\overline{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\overline{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$$

is called ***uniform metric*** *on $\mathbb{R}^J$ inducing* ***uniform topology****.*

Theorem. *(Relationship of Topologies on $\mathbb{R}^J$)*

$$\mathcal{T}_{prod} \subset \mathcal{T}_{uniform} \subset \mathcal{T}_{box}$$

where the three topologies are all different if J is infinite.

Theorem. *(Metric inducing product topology) If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^\omega$, define*

$$D(x, y) = \sup\left\{\frac{\overline{d}(x_i, y_i)}{i}\right\}$$

is a metric that induces product topology on $\mathbb{R}^\omega$

Proof. Show D is a metric. Then show D gives product topology. Now show $\mathcal{T}_D = \mathcal{T}_{prod}$. To show $\mathcal{T}_{prod}$ is finer, show there exists basis element in product topology that contains in a basis element of the metric topology. Let $B_D(\mathbf{x}, \epsilon)$ be basic neighborhood of $\mathbf{x}$. Let N be such that $1/N < \epsilon$. Consider $V \subset \mathcal{T}_{prod}$ defined as

$$V = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \mathbb{R} \cdots$$

Show $V \subset B_D(\mathbf{x}, \epsilon)$. Note for any $\mathbf{y} \in \mathbb{R}^\omega$, $\overline{d}(x_i, y_i)/i \leq 1/N$ for $i \geq N$. Therefore,

$$D(\mathbf{x}, \mathbf{y}) = \sup\left\{\frac{\overline{d}(x_1, y_1)}{1}, \cdots, \frac{\overline{d}(x_N, y_N)}{N}, \frac{1}{N}\right\}$$

If $\mathbf{y} \in V$, then $\overline{d}(x_i, y_i) < \epsilon$ for all $i < N$. So $D(\mathbf{x}, \mathbf{y}) < \epsilon$. Hence $V \subset B_D(\mathbf{x}, \epsilon)$. Conversely, want to show $\mathcal{T}_D$ is finer. The key here is recognize that product topology $\prod U_i$ where each component is metrizable and induced by d. Let $U = \prod_{i \in I} U_i \times \prod_{i \notin I} \mathbb{R}$ be a basis element in $\mathcal{T}_{prod}$ where I is finite. Let $\mathbf{x} \in U$, want to find a basic neighborhood $V \in \mathcal{T}_D$ such that $\mathbf{x} \in V \subset U$. For each $i \in I$, find an interval $(x_i - \epsilon_i, x_i + \epsilon_i)$ such that $i\epsilon < \epsilon_i$ and $\epsilon \leq 1$. We can achieve this by setting $\epsilon = \min\{\epsilon_i/i \mid i \in I\}$. Now we claim that $\mathbf{x} \in B_D(\mathbf{x}, \epsilon) \subset U$. Let $\mathbf{y} \in B_D(\mathbf{x}, \epsilon)$, then for all i,

$$\frac{\overline{d}(x_i, y_i)}{i} \leq D(\mathbf{x}, \mathbf{y}) < \epsilon$$

We need to show that $\mathbf{y} \in U$. We only care about $i \in I$ since $y_i \in \mathbb{R}$ for all $i \notin I$. When $i \in I$, $\overline{d}(x_i, y_i) < i\epsilon < \epsilon_i \leq 1$. Therefore, $d(x_i, y_i) < \epsilon_i$ implying $\mathbf{y} \in U$ $\square$

2.0.10 21 The Metric Topology Continued

Definition. ***(Continuity in Metric Spaces)*** *Let $f : X \to Y$ where (X, d_X) and (Y, d_Y) are metrizable. Then f continuous if and only if $\forall x \in X$ and $\forall \epsilon > 0$, $\exists \delta > 0$ such that*

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$$

Definition. ***(almost always)*** *means all but finitely many*

Definition. ***(Convergence)*** *$(x_n) \to x$ if for all neighborhood U of x, almost always $x_n \in U$.*

Definition. ***(Uniform Convergence)*** *Let $f_n : X \to Y$ be a sequence of functions where Y is metrizable. Let d be metric for Y. The sequence (f_n)* ***converges uniformly*** *to the function $f : X \to Y$ if given $\epsilon > 0$, there exists $N > 0$ such that*

$$d(f_n(x), f(x)) < \epsilon$$

for all $n > N$ and all $x \in X$.

- *depends on $\mathcal{T}_Y$ and metric d*
- *stronger than point-wise convergence*
- *(uniform limite theorem) Given $f_n : X \to Y$ where Y metrizable. If (f_n) converges uniformly to f, then f is continous.*

Definition. ***(Sequantial Closure)*** *Given $A \subset X$, sequential closure is given by*

$$seq\text{-}cl(A) = \{x \in X \mid \exists\, (x_n) \to x\ x_n \in A\}$$

- *(fact) $A \subset seq\text{-}cl(A) \subset \overline{A}$*
- *(the sequence lemma) $seq\text{-}cl(A) \subset cl(A)$ and with equality if X is metrizable.*

Lemma. ***(The sequence lemma)*** *Let X be a topological space and let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \overline{A}$. The converse holds if X is metrizable.*

Proof. $(\Rightarrow)$ If $x_n \to x$ where $x_n \in A$. Let U be any neighborhood of x. By definition of convergent sequence, $\exists N$ such that $\forall n \geq N$, $x \in U$. Since U arbitrary, every neighborhood of x contains some $x_n \in A$, hence $x \in \overline{A}$. Conversely, let d be metric inducing topology of X. Consider neighborhood $B_d(x, 1/n)$ for all $n \in \mathbb{Z}_+$, pick n-th term for the sequence as $x_n \in B_d(x, 1/n) \cap A$. We claim that $(x_n)_{n \in \mathbb{Z}_+}$ is convergent. Indeed, take $B_d(x, \epsilon)$ be arbitrary basic neighborhood of x, take $N > 0$ such that $1/N < \epsilon$. Therefore, for all $i \geq N$, $x_i \in B_d(x, \epsilon)$ by construction. $\square$

Theorem. ***(Sequence Continuity)*** *Let $f : X \to Y$. If f continuous, then every convergent sequence $x_n \to x$ in X, the sequence $f(x_n) \to f(x)$. The converse is true if X is metrizable.*

Definition. ***(First countability axiom)*** *A space X that has a countable basis at each point satisfies first countability axiom. A space X said to have* ***countable basis at the point*** *x if there is a countable collection $\{U_n\}_{n \in \mathbb{Z}_+}$ of neighborhoods of x such that any neighborhood U of x contains at least one of the sets U_n.*

- *used to prove the above lemma/theorem; metrizability is not necessary.*

Definition. ***Spaces that are not metrizable***

1. *$\mathbb{R}^\omega$ in box topology is not metrizable*
2. *$\mathbb{R}^J$ where J uncountable is not metrizable in product topology*

Proof. Generally, to show that a space X is not metrizable, we can show that the space does not satisfy the sequence lemma. Specifically, if X is metrizable, then $seq\text{-}cl(A) = cl(A)$, to prove X not metrizable, find $A \subset X$ and $x \in X$ such that $x \in cl(A)$ and $x \notin seq\text{-}cl(A)$ **(Point 1)** Consider $A \subset \mathbb{R}^\omega$ be points with only positive coordinate values, i.e $A = \{(x_1, x_2, \cdots) \mid x_i > 0\ i \in \mathbb{Z}_+\}$. Now we claim that $\mathbf{0} = (0, 0, \cdots)$ is in $cl(A)$ but not $seq\text{-}cl(A)$. $\mathbf{0} \in cl(A)$: any neighborhood $B = (a_1, b_1) \times (a_2, b_2) \times \cdots$ of $\mathbf{0}$ intersects A at point $({}^1\!/_2 b_1, {}^1\!/_2 b_2, \cdots)$. Now we show there is no sequence of A converging to $\mathbf{0}$. Assume for contradiction that $(\mathbf{x}_n) \to \mathbf{0}$ where $\mathbf{x}_n = (x_{n,k})_{k=1}^\infty$. Let $U = \prod_n (-1, x_{n,n}) \subset \mathcal{T}_{box}$ be a neighborhood of $\mathbf{0}$. However not almost always $\mathbf{x}_n \in U$: in fact U contains no elements of the sequence $(\mathbf{x}_n)$ since n-th coordinate $x_{n,n} \notin (-1, x_{n,n})$. Contradicts assumption that $(\mathbf{x}_n) \to \mathbf{0}$. Therefore $\mathbf{0} \notin seq\text{-}cl(A)$. **(Point 2)** Let $A = \{(x_\alpha) \mid x_\alpha = 1 \text{ except for finitely many } \alpha \in I\}$. Let $\mathbf{0} \in \mathbb{R}^J$ be origin. Now we show $\mathbf{0} \in cl(A)$. Let $U = \prod U_\alpha$ be neighborhood of $\mathbf{0}$ where $U_\alpha \neq \mathbb{R}$ for all $\alpha \in \{\alpha_1, \cdots, \alpha_n\}$ in product topology. Now we show $U \cap A \neq \emptyset$. Consider $\mathbf{y} = (y_\alpha) \in \mathbb{R}^J$ where $y_\alpha = 0$ for all $\alpha \in \{\alpha_1, \cdots, \alpha_n\}$ and $y_\alpha = 1$ otherwise. Note $\mathbf{y} \in A$ since all but finitely many y_α is 1; $\mathbf{y} \in U$ since $y_\alpha \in U_\alpha$ for $\alpha_1, \cdots, \alpha_n$ and $y_\alpha \in \mathbb{R} = U_\alpha$ otherwise. Hence $\mathbf{y} \in U \cap A$ and therefore $\mathbf{0} \in cl(A)$. Now we show $\mathbf{0} \notin seq\text{-}cl(A)$. Let $\mathbf{a}_n$ be a sequence of A. Let $J_n \subset J$ such that $\mathbf{a}_n(\alpha) \neq 1$ for all $\alpha \in J_n$. J finite by of A. Let $J' = \cup_{n \in \mathbb{Z}_+} J_n$. Note J' is a countable union of finite set and hence countable. Since J uncountable, exists $\beta \in J$ such that $\beta \notin J'$ and therefore every point in the sequence $\mathbf{a}_n(\beta) = 1$ for all $n \in \mathbb{Z}_+$. Let $U_\beta = (-1, 1) \in \mathbb{R}$. Consider a neighborhood $U = \pi^{-1}(U_\beta) \subset R^J$ of $\mathbf{0}$ that contains no points in $\mathbf{a}_n$. Therefore $\mathbf{a}_n$ does not converge to $\mathbf{0}$. Therefore $\mathbf{0} \notin seq\text{-}cl(A)$. $\square$

2.0.11 22 The Quotient Topology

Definition. ***(Quotient Map)*** *Given X, Y and $p : X \to Y$ be a surjective map. p is a quotient map if a subset $U \subset Y$ is open if and only if $p^{-1}(U)$ is open in X.*

- *(theorem) p is a surjective continuous map that is either open or closed, then p is a quotient map*
- *(example) projection maps $\pi_1 : X \times Y \to X$ is surjective, continuous, open and therefore a quotient map. However π_1 is not a closed map (since $\pi_1(\{x \times y \mid xy = 1\}) = \mathbb{R} - \{0\}$ not closed)*

Definition. ***(Quotient Topology and Quotient Space)*** *Let X be a space and A be a set. If $p : X \to A$ is a surjective map, then there uniquely exists one topology $\mathcal{T}$ on A relative to which p is a quotient map; $\mathcal{T}$ is called the* ***quotient topology*** *induced by p.*

$$\mathcal{T} = \{U \subset Y \mid p^{-1}(U) \in \mathcal{T}_X\}$$

As a special case. Givecn $\sim$ be an equivalence relation on X and $Y = {}^{X}/_{\sim} = \{\{y : y \sim x_0\} \mid x_0 \in X\} \subset \mathcal{P}(X)$ are equivalence classes of X. Then $p : X \to Y$ exists and is a surjection. The space Y with the quotient topology is called a ***quotient space*** *of X.*

Definition. ***define quotient topology with continuity of functions*** *Given topological space X and $\pi : X \to Y$ a surjection, there is a unique topology on Y satisfying*

1. *$\pi : X \to Y$ continuous*
2. *If Z is a topological space, $g : Y \to Z$ is a function. If $g \circ \pi$ is continuous, then g is continuous.*

3 Connectedness and Compactness

theorems about continuous functions

- (intermediate value theorem) If $f : [a, b] \to \mathbb{R}$ continuous and $f(a) < r < f(b)$, then exists $c \in [a, b]$ s.t. $f(c) = r$
- (maximum value theorem) If $f : [a, b] \to \mathbb{R}$ continuous then exists $c \in [a, b]$ such that $f(x) \le f(c)$ for all $x \in [a, b]$
- (uniform continuity theorem) If $f : [a, b] \to \mathbb{R}$ continuous, then given $\epsilon > 0$ exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \epsilon$ for every pair $x_1, x_2 \in [a, b]$ for which $|x_1 - x_2| < \delta$

3.0.1 23 Connected Spaces

Definition. ***(Separation and Connected)*** *Given topological space X. A* ***separation*** *of X is a pair U, V where $U \cap V = \emptyset$ and $X = U \cup V$ and U, V both nonempty. The space X is* ***connected*** *if there does not exists a separation of X. Equivalently, X is connected if the only clopen sets are $\emptyset$ and X.*

- *(fact) connectedness is a topological property*
- *(theorem) Image of a connected space under a continuous map is connected*
- *(theorem) Finite product $\prod X_\alpha$ connected if and only if X_α connected for all α*

Definition. ***(Separation and Connected for subspaces)*** *Given $Y \subset X$. A* ***separation*** *of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other (i.e. $cl_Y(A) \cap B = \overline{A} \cap B = \emptyset$). The space Y is connected if there exits no separation of Y*

Proof. Suppose A, B forms a seperation of Y. Then $cl_Y(A) = \overline{A} \cap Y$. Since A closed in Y, $A = cl_Y(A) = \overline{A} \cap Y$. Since A, B disjoint, $\overline{A} \cap B = \emptyset$. Since $\overline{A}$ contains all its limit point, B has no limit points of A. Conversely, given assumption we have $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$. Hence $\overline{A} \cap Y = A$ and $\overline{B} \cap Y = B$. Then A, B are closed in Y. Since A, B partitions Y, A, B are open in Y as well. □

- *(examples)*
  - *$\{a, b\}$ with $\mathcal{T}_{triv}$ is connected*
  - *$Y = [-1, 0) \cup (0, 1] \subset \mathbb{R}$ is connected since $[-1, 0)$ and $(0, 1]$ is a seperation. (0 is a limit point to both, but does not matter since 0 is not contained in $[-1, 0)$ or $(0, 1]$)*
  - *$X = [-1, 1] \subset \mathbb{R}$. $[-1, 0]$ and $(0, 1]$ is not a seperation (0 is a limit point of $(0, 1]$ but contained in $[-1, 0]$)*
  - *$\mathbb{Q}$ not connected. Only one point subsets of $\mathbb{Q}$ are connected.*
  - *$X = \{(x, y) \mid y = 0\} \cup \{(x, y) \mid y = 1/x\} \subset \mathbb{R}^2$ not connected (neither contain a limit point of each other)*
- *(lemma) If C, D forms a separation of X and Y is a connected subspace of X, then Y lies entirely in C or D*
- *(theorem) Union of connected subspaces of X with a commont point is connected ($\cap A_\alpha \neq \emptyset$ where A_α connected for all α then $\cup A_\alpha$ connected)*
- *(theorem) If $A \subset X$ be a connected subspace. If $A \subset B \subset \overline{A}$, then B also connected*

Definition. ***(Connectedness for Infinite Products)***

1. *$\mathbb{R}^\omega$ is not connected in box topology*
2. *$\mathbb{R}^\omega$ is connected in product topology*

Proof. **(Point 1)** Enough to find a separation of $\mathbb{R}^\omega$. Interprete $\mathbb{R}^\omega$ as the collection of all real numbered sequences and is partitioned by the set of all bounded sequences of real numbers A and the set of all unbounded sequences of real numbers B. Now we show A, B both open. Consider $\mathbf{a} \in \mathbb{R}^\omega$, we can find a neighborhood of $\mathbf{a}$ in the box topology by $U = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \cdots$. If $\mathbf{a}$ is bounded, U consists of only bounded sequencs so $\mathbf{a} \in U \subset A$. If $\mathbf{a}$ is unbounded, then U consists of only unbounded sequences and $\mathbf{a} \in U \subset B$. Therefore $\mathbb{R}^\omega$ not connected in box topology. **(Point 2)** To show $\mathbb{R}^\omega$ in product topology is connected, we find some connected $C \subset \mathbb{R}^\omega$ where $C \subset \mathbb{R}^\omega \subset \overline{C}$ and use the lemma to show that $\mathbb{R}^\omega$ is connected. Consider $\tilde{\mathbb{R}}^n \subset \mathbb{R}^\omega$ defined to be the set of all sequences fixed to 0 beyond n: $\mathbf{x} \in \mathbb{R}^\omega$ such that $x_i > 0$ for all $i > n$. Since $\tilde{\mathbb{R}}^n \cong \mathbb{R}^n$ and $\mathbb{R}^n$ is connected, $\tilde{\mathbb{R}}^n$ is connected. Since each $\tilde{\mathbb{R}}^n$ is connected and $\cap_n \tilde{\mathbb{R}}^n = \{0, 0, \cdots\} = \mathbf{0}$, we have $\mathbb{R}^\infty = \cup_n \tilde{\mathbb{R}}^n$ connected. To complete the proof, we show $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$. Consider $\mathbf{a} = (a_1, a_2, \cdots) \in \mathbb{R}^\omega$ and $U = \prod U_\alpha$ be neighborhood of $\mathbf{a}$ in box topology. We show $U \cap \mathbb{R}^\infty \neq \emptyset$. Let N be such that $U_i = \mathbb{R}$ for all $i > N$. Consider $\mathbf{x} = \{a_1, a_2, \cdots, a_N, 0, 0, \cdots\} \in \mathbb{R}^\infty$. $\mathbf{x} \in U$ since $x_i \in U_i$ for all $i \le N$ and $x_i \in \mathbb{R} = U_i$ for all $i > N$. Therefore $\mathbf{x} \in U \cap \mathbb{R}^\infty$ hence $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$ □

3.0.2 24 Connected Subspaces of the Real Line

Definition. ***(convex)*** *$Y \subset X$ is convex if every $a < b \in Y$, $[a,b] \in Y$*

Definition. ***(Linear Continuum)*** *A simply ordered set L having more than 1 eleent is called a linear continuum if*

1. *L has least upper bound property*
2. *If $x < y$, there exists z such that $x < z < y$*

- *(fact) condition for connectedness to hold on $\mathbb{R}$*
- *(theorem) If L is linear continuum in order topology, then L is connected, and so are intervals and rays in L*
- *(theorem) If $Y \subset \mathbb{R}$, then Y is connected if and only if Y is convex and nonempty*
- *(corollary) $\mathbb{R}$, intervals and rays in $\mathbb{R}$ are all connected*

Theorem. ***(Unit interval in $I = [0,1] \subset \mathbb{R}$ is connected)***

Proof. Idea is to show any separation (A, A^c) gives $A = I$. Let $A \subset I$ be clopen. without loss of generality let $0 \in A$. Define $G = \{x \in I \mid [0,x] \in A\} \subset A$ and $g = \sup G$. Goal is to show $1 = g \in G$ such that $A = G$ and therefore $I - A = \emptyset$ which contradicts assumption of separation. **First** we show $g > 0$, note $0 \in A$ where A is open, hence $[0,\epsilon) \in A$ and therefore $\epsilon/2 \in G$. So $g = \sup G \geq \epsilon/2 > 0$. **Second** we show $g \not< 1$. We first show $g \in A$. Since $G \subset A$, we have $\overline{G} \subset \overline{A} = A$. Then $g = \sup G = \overline{G} \in A$. Hence $g \in A$. Since A open, can find $(g-\epsilon, g+\epsilon) \in A$. Easily, $[0, g+\epsilon/2] \in A$ and hence $g + \epsilon/2 \in G$ which contradicts $g = \sup G$. **Third** we show $g \in G$. Same as before we show $g \in A$. Can find open neighborhood $(1-\epsilon, 1] \in A$. Easily $[0,1] \in A$ hence $A = I$. To conclude (A, A^c) is not a separation. □

Theorem. ***(Intermediate Value Theorem)*** *Let $f : X \to Y$ be continuous map, where X is connected and Y is in order topology. If $a, b \in X$ and $r \in Y$ such that $f(a) < r < f(b)$, then there exists $c \in X$ such that $f(c) = r$.*

Proof. Let $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, \infty)$. Note $A \cap B = \emptyset$ and neither are empty since $f(a) \in A$ and $f(b) \in B$. Note $A, B \subset f(X)$ are open in the subspace topology by definition. If no $c \in X$ such that $f(c) = r$, then $f(X) = A \cup B$ then (A, B) constitutes a separation of $f(X)$, contradicting the fact that image of a connected space under a continuous map $f(X)$ is connected. □

Definition. ***(Path Connectedness)*** *Let $x, y \in X$, a **path** in X from x to y is a continuous map $f : [a,b] \to X$ such that $f(a) = x$ and $f(b) = y$. A space X is **path-connected** if every pair of points in X can be joined by a path.*

$$\forall x, y \in X\ \exists \text{ continuous } f : [0,1] \to X\ f(0) = x\ f(1) = y$$

- *(theorem) continuous image of path-connected space is path-connected*
- *(theorem) path-connected space is connected. (converse not always true: see topologist's sine curve)*
- *(proposition) connectedness and path-connected subsets of $\mathbb{R}$ are the same*
- *(theorem) If X_α path-connected, then $\prod X_\alpha$ is also path-connected (in product topology)*
- *(example) unit ball $B^n = \{\mathbf{x} \mid \|\mathbf{x}\| \leq 1\} \subset \mathbb{R}^n$ is path-connected ($f : [0,1] \to \mathbb{R}^n$ $t \to (1-t)\mathbf{x} + t\mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$)*
- *(example) punctured euclidean space $\mathbb{R} - \{0\}$ is path-connected*
- *(example) unit sphere $S^{n-1}\{\mathbf{x} \mid \|\mathbf{x}\| = 1\} \subset \mathbb{R}^n$ is path-connected ($g : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ $\mathbf{x} \to \mathbf{x}/\|\mathbf{x}\|$ is continuous)*
- *(example) Let $S = \{x \times \sin(1/x) \mid 0 < x \leq 1\} \subset \mathbb{R}^2$. The **topologist's sine curve** $\overline{S}$ is connected but not path-connected*
$$\overline{S} = (\{0\} \times [-1,1]) \bigcup \{x \times \sin(1/x) \mid 0 < x \leq 1\} \subset \mathbb{R}^2$$

Theorem. ***(Topologist's sine curve is connected but not path-connected.)***

Proof. Let $S' = (\{0\} \times [-1,1])$ and let $S = \{x \times \sin(1/x) \mid 0 < x \leq 1\} \subset \mathbb{R}^2$ and hence $\overline{S} = S' \cup S$. Since S is connected and $\overline{S}$ is image of a connected set $(0,1]$ under continuous map, $\overline{S}$ is also connected. Now we show $\overline{S}$ is not path-connected. Let $f : [a,c] \to \overline{S}$, $a < 0$ be a path connecting $(0,0)$ and $(1,0)$. Since S' is closed, $f^{-1}(S')$ is closed and has a largest element b. Therefore $f' : [b,c] \to \overline{S}$ where f' maps b to S' and rest of points to S. Replace $[b,c]$ with $[0,1]$ for convenience. Let $f(t) = (x(t), y(t))$ which has to be continuous. Then $x(0) = 0$ and $x(t) > 0$ and $y(t) = \sin(1/x(t))$ for $t > 0$. There exists t_n a sequence such that $y(t_n) = (-1)^n$ does not converge, contradicting continuity of f. We construct t_n as follows. For each n, pick u in range $0 < u < x(1/n)$ such that $\sin(1/u) = (-1)^n$. Use intermediate value theorem to find t_n suc that $x(t_n) = u$. □

3.0.3 26 Compact Spaces

Definition 3.1. ***(Cover and Compact)*** *A collection $\mathcal{A} \subset \mathcal{P}(X)$* ***cover*** *$X$, or be a* ***covering*** *of X, if $\cup_{A\in\mathcal{A}} A = X$. $\mathcal{A}$ is open cover, if it is a cover and all $A \in \mathcal{A}$ are open. A space X is said to be* ***compact*** *if every open cover $\mathcal{A}$ contains a finite subcollection that also covers X, i.e. if $\{A_\alpha\}$ is an open cover, exists $I = \{\alpha_1, \cdots, \alpha_n\}$ such that $\cup_{\alpha\in I} A_\alpha = X$*

- *(examples)*
  - $\mathbb{R}$ *is not compact ($\mathcal{A} = \{(n, n+2) \mid n \in \mathbb{Z}_+\}$ does not have a finite subcover)*
  - $(0,1]$ *(and similarly $(0,1)$) is not compact ($\mathcal{A} = \{(1/n, 1] \mid n \in \mathbb{Z}_+\}$ emits no finite cover)*
  - $X = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\} \subset \mathbb{R}$ *is compact. (idea: $U \in \mathcal{A}$ covering 0 covers all but finitely many points of $1/n$. Take U and $U_i \in \mathcal{A}$ for all $1/i$ not in U forms a finite cover for X)*
  - *X with finitely many points is compact (all covers are finite)*
  - $[0,1]$ *is compact*
- *(theorem) Image of a compact space under a continuous map is compact*
- *(theorem) Let $f : X \to Y$ be bijective continuous map. If X is compact and Y is T_2, then f is a homeomorphism (idea: $K \subset X$ closed. Since X compact, K also compact, $f(K)$ is then compact, which is closed in T_2 space Y)*
- *(theorem) Product of finitely many compact spaces is compact (tube lemma)*
- *(theorem) Product of infinitely many compact spaces in product topology is compact (Tychnoff theorem)*
- *(theorem) A continuous function $f : X \to \mathbb{R}$ where X is compact is bounded. (idea: $X = \cup_{i=1}^{\infty} f^{-1}((-\epsilon_i, \epsilon_i))$, by compactness $X = \cup_{i=1}^{n}((-\epsilon_i, \epsilon_i)) = f^{-1}((-M, M))$ where $M = \max_{i=1}^{n} \epsilon_i$)*

Definition. ***(Compactness for subspace)*** *Let $Y \subset X$. Y is compact (i.e. every covering of Y by open sets in Y has a finite subcover) if and only if every covering of Y by sets open in X contains a finite subcollection covering Y*

- *(theorem) Closed subspace of a compact space is compact, i.e. $Y \subset X$ where X is compact and Y is closed, then Y is compact (idea: Given covering $\mathcal{A}$ of Y by open sets in X, $\mathcal{A} \cup \{X - Y\}$ emits finite subcover for X, and therefore for Y by definition of compactness for subspace)*
- *(example) Given $[a, b] \in \mathbb{R}$ compact, any closed subset of $[a, b]$ is also compact.*
- *(theorem) Compact subspace of a T_2 space is closed, i.e. $Y \subset X$ where X is T_2 and Y is compact, then Y is closed (idea: fix $x_0 \in X \setminus Y$, find U s.t. $x_0 \in U \subset X \setminus Y$. By T_2, exists disjoint open pair U_y, V_y separating x_0, y for all $y \in Y$. By compactness of Y, $\{V_y \mid y \in Y\}$ emits finite cover $V = V_{y_1} \cup \cdots \cup V_{y_n}$ which are disjoint from $U = U_{y_1} \cap \cdots \cap U_{y_n}$. $X \setminus Y$ open hence Y closed)*
- *(lemma) If X is T_2 and $Y \subset X$ is compact and $x_0 \notin Y$, exists disjoint $U, V \in \mathcal{T}_X$ that covers x_0 and Y, respectively*
- *(example) $(a, b] \in \mathbb{R}$ not compact since its not closed. (In T_2, closedness is a necessary condition for compactness)*

Definition. *(T_3* ***Space****) A space X is T_3 if*

1. *it is T_1 (singletons are closed)*
2. *if $A \subset X$ is closed and $y \in A^c$, then there exists disjoint open $U, V \in \mathcal{T}_X$ such that $A \subset U$ and $y \in V$*

- *(theorem) X is compact and T_2, then X is T_3 (straight from previous lemma)*

Lemma. ***The Tube Lemma*** *Consider $X \times Y$, where Y is compact. If $N \subset X \times Y$ is open and contains the slice $\{x_0\} \times Y$, then N contains some* ***tube*** *$W \times Y$ about $\{x_0\} \times Y$, where W is a neighborhood of $x_0 \in X$*

Proof. Idea is try to cover the slice $\{x_0\} \times Y$ with basis elements $U \times V \in N$, which happens to cover some tube about the slice. Since $\{x_0\} \times Y \cong Y$ and Y compact, then the slice is compact. Hence can cover $\{x_0\} \times Y$ with finitely basis elements $\mathcal{A} = \{U_1 \times V_1, \cdots, U_n \times V_n\}$ where $U_i \times V_i \in N$. Note $W = U_1 \cap \cdots \cap U_n$ is open and contains x_0. We claim that $\mathcal{A}$ actually covers not only the slice but also the tube $W \times Y$. Let $x \times y \in W \times Y$. Some $V_i \supset y$ and since $x \in \cap_j U_j$, $x \in U_i$. Hence $x \times y \in U_i \times V_i$. □

- *(example) $N = \{x \times y \mid |x| < 1/(y^2 + 1)\}$ is an open set containing $\{0\} \times \mathbb{R}$ but it contains no tube about the slice. (since $\mathbb{R}$ is not compact, tube lemma does not hold)*

Theorem. ***(Product of finitely many compact spaces is compact)***

Proof. Show $X \times Y$ compact given X, Y compact and do induction from here. Let $\mathcal{A}$ be any open cover for $X \times Y$. For any $x_0 \in X$, $\{x_0\} \times Y \subset X \times Y$ is compact and therefore covered by finitely many open sets of $X \times Y$, specifically $\{A_1, \cdots, A_m\} \in \mathcal{A}$. Let $N = \{A_1, \cdots, A_m\}$. Since $N \supset \{x_0\} \times Y$ and N is open, there exists tube $W \times Y$ where W is a neighborhood of x_0. For each $x \in X$, we can find such neighborhood W_x of x such that the tube $W_x \times Y$ is covered by finitely many elements of $\mathcal{A}$. Since X is compact, there is a finite covering $\{W_1, \cdots, W_k\}$ of X. Union of the tubes covers $X \times Y$, i.e. $\cup_{j=1}^{k} W_j \times Y = X \times Y$. Since each tube covered by finitely many elements of $\mathcal{A}$ and there is finitely many tubes, we can cover $X \times Y$ with finitely many elements of $\mathcal{A}$ $\square$

Definition. ***(Finite intersection property - FIP)*** *A collection $\mathcal{C} \subset \mathcal{P}(X)$ is said to have **finite intersection property** if intersection of every finite subcollection is nonempty*

$$\{C_1, \cdots, C_n\} \subset \mathcal{C} \quad \Rightarrow \quad \bigcap_{i=1}^{n} C_i \neq \emptyset$$

Theorem. ***(define compactness using finite intersection property)*** *X is compact if and only if every collection of closed sets having the finite intersection property has nonempty intersection*

$$X \text{ compact} \quad \iff \quad \forall \mathcal{C} \text{ of closed sets in } X \text{ with FIP} \Rightarrow \cap_{C \in \mathcal{C}} C \neq \emptyset$$

Proof. Given $\mathcal{A} \subset \mathcal{P}(X)$, let $\mathcal{C} = \{X - A \mid A \in \mathcal{A}\}$. Then the following holds

1. $\mathcal{A}$ is a collection of open sets if and only if $\mathcal{C}$ is a collection of closed sets
2. $\mathcal{A}$ covers X if and only if intersection $\cap_{C \in \mathcal{C}} C$ of all elements of $\mathcal{C}$ is empty (by DeMorgan's Law)
3. $\{A_1, \cdots, A_n\} \subset \mathcal{A}$ covers X if and only if intersection of corresponding elements $C_i = X - A_i$ of $\mathcal{C}$ is empty

Idea is to take contraposiitive of normal definition of compactness: Given any $\mathcal{A}$ of open sets, if no finite subcollection of $\mathcal{A}$ covers X, then $\mathcal{A}$ does not cover X. Let $\mathcal{C}$ be defined as above, apply above 3 points. We get: Given any $\mathcal{C}$ of closed sets (1), if every finite collection of elements of $\mathcal{C}$ is nonempty (3), then the intersection of all the elements of $\mathcal{C}$ is nonempty (2) $\square$

- *(example) nested sequence of closed sets $C_1 \supset C_2 \supset \cdots$, where each $C_n \neq \emptyset$, then $\mathcal{C} = \{C_n\}_{n \in \mathbb{Z}_+}$ has finite intersection property and the intersection $\cap_{n \in \mathbb{Z}_+} C_n \neq \emptyset$*

3.0.4 27 Compact Subspaces of the Real Line

Theorem. ***(Compactness for Ordered Set with Least Upper Bound Property)*** *Let X be a simply ordered set having least upper bound property. In order topology, each closed interval in X is compact.*

Theorem. ***(Characterize Compact Subspace of $\mathbb{R}^n$)***

1. *($\mathbb{R}$) every closed interval in $\mathbb{R}$ is compact (follows from previous theorem)*
   - *$[0, 1] \in \mathbb{R}$ is compact*
2. *($\mathbb{R}^n$) $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded in euclidean d_2 or square d_∞ metric*
   - *(note) compact sets in metric space is **not** equivalent to the set of closed and bounded sets. boundedness depends on metric d whereas compactness is purely a topological property*
   - *Unit sphere and closed unit ball $S^{n-1}, B^n \subset \mathbb{R}^n$ are compact (closed and bounded)*
   - *$A = \{x \times 1/x \mid 0 < x \leq 1\} \subset \mathbb{R}^2$ is not compact (closed but not bounded)*
   - *$S = \{x \times \sin(1/x)\}$ is not compact (bounded but not closed since does not contain limit points $S' = \{0\} \times [-1, 1]$)*

Proof. **(Part 1)** proof similar to how we proved $I = [0, 1]$ is connected: 3 cases, by contradiction. **(Part 2)** Note $d_\infty(\mathbf{x}, \mathbf{y}) \leq d_2(\mathbf{x}, \mathbf{y}) \leq \sqrt{n} d_\infty(\mathbf{x}, \mathbf{y})$. So A is bounded under d_2 if and only if it is bounded under d_∞. So we need to consider square metric d_∞ only. ($\Rightarrow$) Assume A compact. Since $A \subset \mathbb{R}^n$ is compact and $\mathbb{R}^n$ is T_2, A is closed. To show it is bounded, consider an open cover $\mathcal{A} = \{B_{d_\infty}(\mathbf{0}, m) \mid m \in \mathbb{Z}_+\}$ for $\mathbb{R}^n$. Since $A \subset \mathbb{R}^n$ is compact, some finite collections covers A. So $A \subset B_{d_\infty}(\mathbf{0}, M)$ for some $M < \infty$. So any $\mathbf{x}, \mathbf{y} \in A$ has $d_\infty(\mathbf{x}, \mathbf{y}) \leq 2M$. Hence A is bounded. ($\Leftarrow$) Idea is to find a compact superset of A, when combined with the fact that A is closed, gives A is compact. Assume for any $\mathbf{x}, \mathbf{y} \in A$, $d_\infty(\mathbf{x}, \mathbf{y}) \leq N$. Let $\mathbf{x}_0 \in A$ and let $b = d_\infty(\mathbf{x}_0, \mathbf{0})$. Then for any $\mathbf{x} \in A$, we have $d_\infty(\mathbf{0}, \mathbf{x}) \leq d_\infty(\mathbf{0}, \mathbf{x}_0) + d_\infty(\mathbf{x}_0, \mathbf{x}) \leq N + b = P$. Therefore $A \subset \prod_{i=1}^{n} [-P, P]$ which is compact since $[-P, P] \subset \mathbb{R}$ is compact and finite product of compact set is compact. Since A is closed and $A \subset [-P, P]^n$, A is also compact. $\square$

Theorem. ***(Extreme Value Theorem)*** *Let $f : X \to Y$ be continuous, Y is in order topology. If X is compact, then f attains its maximum and minimum, i.e. $\exists c, d \in X$ such that $f(c) = \inf_{x \in X} f(x)$ and $f(d) = \sup_{x \in X} f(x)$*

Proof. Now show f attains its maximum, proof for minimum is similar. ($\Rightarrow$) Since f continuous and X compact, then $A = f(X)$ is compact. Enough to show that A contains its largest element M, and since $M \in A$, $M = f(d)$ for some $d \in X$. By contradiction assume A has no largest element, the $\mathcal{A} = \{(-\infty, a) \mid a \in A\}$ forms an open covering of A. Since A compact, $\{(-\infty, a_1), \cdots, (-\infty, a_n)\}$ covers A. Let $a = \max\{a_1, \cdots, a_n\}$. Note $a \notin (-\infty, a_i)$ for all $1 \le i \le n$. Since $a \in A$, a is not covered by $\mathcal{A}$ which is a contradiction. (Alternatively, simply use the fact that $f(X)$ is a compact subspace of T_2 space Y so closed, and therefore $\sup_{x \in X} f(x) \in f(X)$) $\square$

Definition. ***(point-set distance)*** *Let (X, d) be metric, let $A \subset X$ be nonempty. For each $x \in X$, define* ***distance from*** *x* ***to*** *A by $d(x, A) = \inf\{d(x, a) \mid a \in A\}$.*

- *(lemma) Fixing A, $d : X \to \mathbb{R}$ $x \mapsto d(x, A)$ is a continuous function*

Definition. ***(diameter)*** *For metric space (X, d) and $A \subset X$, $diam(A, d) = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$*

Lemma. ***(Lebesgue Number Lemma)*** *Let $\mathcal{A}$ be open covering of metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathcal{A}$ containing it. The number δ is a* ***Lebesgue number*** *for the covering $\mathcal{A}$. In other words,*

$$X \text{ compact} \quad \Rightarrow \quad \Big(\exists \delta > 0 \;\; \forall B \subset \mathcal{P}(X) \;\; (diam(B, d) < \delta \Rightarrow \exists A \subset \mathcal{A} \; B \subset A) \Big)$$

Proof. Let $\mathcal{A}$ be a cover for X. If $X \in \mathcal{A}$, then $\delta > 0$ could be any since X contains any subsets in X. Now assume $X \notin \mathcal{A}$. Choose finite $\{A_1, \cdots, A_n\} \subset \mathcal{A}$ covering X. For each i, let $C_i = X \setminus A_i$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i)$$

Key idea is $f(x) \ge 0$. Consider any $x \in X$, pick ϵ such that $B_\epsilon(x) \subset A_i$, then $d(x, C_i) \ge \epsilon$, so then $f(x) \ge \epsilon/n$. Note f is continuous and by extreme value theorem attains its minimum δ. Now we show δ is the Lebesgue number for $\mathcal{A}$. Let $B \subset X$ having diameter less than δ, let $x_0 \in B$ be arbitrary. Note $B \subset B_\delta(x_0)$. By definition of f as the average distance to all C_is, since $f(x_0) \ge \delta$, we have $d(x_0, C_m) \ge \delta$ for some C_m. Therefore $B \subset X \setminus C_m = A_m \subset \mathcal{A}$. $\square$

Definition. ***(Uniform Continuity)*** *Given (X, d_X) and (Y, d_Y) metric, then $f : X \to Y$* ***uniformly continuous*** *if*

$$\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x_0, x_1 \in X \qquad (d_X(x_0, x_1) < \delta \Rightarrow d_Y(f(x_0), f(x_1)) < \epsilon)$$

- *(note) Idea is one $\delta > 0$ works for all $x, y \in X$ nearby such that $f(x), f(y) \in Y$ are also nearby*
- *(theorem) Given continuous $f : X \to Y$, where X, Y are metric and X is compact, then f is uniformly continuous*

Definition. ***(Uniform Continuity Theorem)*** *continuous function on compact space is uniformly continuous*

Proof. Let $\epsilon > 0$. take open covering of Y by balls $B(y, \epsilon/2)$. Let $\mathcal{A} = \{f^{-1}(B(y, \epsilon/2)) \mid y \in Y\}$ be open covering for X. Let δ be Lebesgue Number for $\mathcal{A}$ on compact X. Let $x_0, x_1 \in X$ such that $d_X(x_0, x_1) \le \delta$. Then $diam(\{x_0, x_1\}) \le \delta$, and by Lebesgue Number Lemma, $\exists A \in \mathcal{A}$ such that $\{x_0, x_1\} \in A = f^{-1}(B(y, \epsilon/2))$ for some $y \in Y$. Therefore, $d_Y(f(x_0), f(x_1)) < \epsilon$ $\square$

3.0.5 28 Limit Point Compactness

4 Countability and Separation Axioms

4.0.1 30 The Countability Axioms

Definition. ***(First Countability Axiom*** α_1***)*** *A space X have a* ***countable basis at*** *x if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$. A space that has a countable basis at each of its points is said to satisfy the* ***first countability axiom****, or to be* ***first countable***

- *(theorem) Every metrizable space is first countable*
- *(observation) convergent sequences are adequate to detect limit points of sets and to check continuity of functions*

Theorem. ***(Convergent Sequences in First Countable Space)*** *Let X be a space*

1. *(the sequence lemma) Let $A \subset X$, if there is a sequence of points of A converging to x, then $x \in \overline{A}$ (seq-cl$(A) \subset$ cl(A)); Converse holds if X is first-countable*
2. *(sequence continuity) Let $f : X \to Y$. If f is continuous, then very convergent sequence $x_n \to x$ in X, the sequence $f(x_n) \to f(x)$; Converse holds if X is first-countable*

Definition. ***(Second Countability Axiom*** α_2***)*** *If a space X has a countable basis for its topology, then X is said to satisfy the* ***second countability axiom****, or to be* ***second-countable***

- *(theorem) second countable implies first countable*
- *(fact) not all metrizable space is second countable (this axiom is very strong!)*
- *(examples)*
  - $\mathbb{R}$ *is second countable* ($\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{Q}\}$ *countable)*
  - $\mathbb{R}^n$ *is second countable* ($\mathcal{B} = \{\prod_{i=1}^n (a_i, b_i) \mid a_i, b_i \in \mathbb{Q}\}$ *countable)*
  - $\mathbb{R}^\omega$ *is second countable in product topology* ($\mathcal{B} = \{\prod_{i \in I}(a_1, b_1) \times \prod_{i \notin I} \mathbb{R} \mid a_i, b_i \in \mathbb{Q}$ *I countable* $\}$ *countable)*
  - $\mathbb{R}^\omega$ *is not second countable in box topology* ($\mathcal{B} = \{\prod_{i=1}^\infty \mid a_i, b_i \in \mathbb{Q}\}$ *is not countable)*
  - $\mathbb{R}^\omega$ *is first countable (being metrizable) but not second countable under uniform topology*
- *(theorem) Countability Axioms behaves well for subspaces or countable products*
- *(theorem) Every open cover in a second countable space contains a countable subcover*
- *(theorem) Second countable space has a countable dense subset, i.e. X second countable, $\exists A \subset X$ s.t. $\overline{A} = X$*

Definition. ***(Dense)*** *A subset $A \subset X$ is* ***dense*** *in X if $\overline{A} = X$*

- *(example) $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are dense in $\mathbb{R}$*

4.0.2 31 The Separation Axioms

Definition. *(T_0* ***Space****) X is T_0 if for $x, y \in X$ where $x \neq y$, then either*

1. *exists open U such that $x \in U$ and $y \notin U$*
2. *exists open U such that $y \in U$ and $x \notin U$*

Definition. ***(Separated)*** *Let $x, y \in X$, x and y can be separated if each lies in a neighborhood which does not contain the other point. (neighborhood not necessarily disjoint)*

$$\forall x \neq y \in X \qquad \exists \text{ neighborhoods } U, V \text{ for } x, y \text{ respectively s.t. } y \notin U \text{ and } x \notin V$$

Definition. *(T_1* ***Space****) X is T_1 if any two distinct points in X are separated*

- *<u>(theorem)</u> A space is T_1 if and only if singleton/finite sets are closed*

Definition. *(T_2* ***Hausdorff Space****) A topological space X is called Hausdorff space if X is T_1 and for each pair x_1, x_2 of distinct points of X, there exists neighborhoods U_1 and U_2 of x_1 and x_2, respectively that are disjoint.*

$$\forall x \neq y \in X \qquad \exists \text{ neighborhoods } U, V \text{ of } x \text{ and } y \text{ respectively s.t. } U \cap V = \emptyset$$

- *<u>(theorem)</u> $T_2 \Rightarrow T_1$*
- *<u>(theorem)</u> If X is T_2 and compact, then X is T_3*
- *<u>(theorem)</u> If X is T_2, then a sequence of points of X converges to at most 1 point of X*

- *(theorem) T_2 is well behaved under order/product/subspace topology*
- *(examples)*
  - $\mathbb{R}^n_{std}$, X_{disc} *are* T_2
  - X_{triv} *not* T_2 *except when* $|X_{triv}| = 1$
  - $X_{f.c.}$ *not* T_2 *when* X *is infinite (since any* $x, y \in X_{f.c.}$ *are infinite and intersects)*

Definition. *(T_3 **Regular Space**) X is **regular** if X is T_1 and for each pair consisting of a point x and a closed set B disjoint from x, there exists disjoint open sets containing x and B, respectively.*

$$\forall x \in X \ B \subset X \text{ closed s.t. } x \notin B \qquad \exists\, U, V \text{ open s.t. } x \in U \ \ B \subset V \ \ U \cap V = \emptyset$$

- *(lemma) X regular iff given $x \in X$ and neighborhood U of x, exists neighborhood V of x such that $\overline{V} \subset U$*
- *(theorem) T_3 is well behaved under subspace/product*

Definition. *(T_4 **Normal Space**) X is **normal** if X is T_1 and for each pair A, B of disjoint closed sets of X, there exists disjoint open sets containing A and B, respectively.*

$$\forall A, B \subset X \text{ closed s.t. } A \cap B = \emptyset \qquad \exists\, U, V \text{ open s.t. } A \subset U \ \ B \subset V \ \ U \cap V = \emptyset$$

- *(lemma) X normal iff given $A \subset X$ and neighborhood U of A, exists neighborhood V of A such that $\overline{V} \subset U$*
- *(observation) T_4 is not well behaved under subspace/product*
- *(theorem) $T_3 + \alpha_2 \Rightarrow T_4$*
- *(theorem) metrizable $\Rightarrow T_4$*
- *(theorem) compact $+T_2 \Rightarrow T_4$*

Definition. *($T_{3.5}$ **Completely Regular Space**) X is completely regular if X is T_1 and for all $x \in X$, $B \subset X$ closed, $x \notin B$, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ and $f|_B = \{0\}$*

- *(theorem) X is $T_{3.5}$ if and only if $\{f^{-1}((0,\infty)) \mid f : X \to \mathbb{R}$ continuous $\}$ is a basis for $\mathcal{T}_X$ ($\Rightarrow$ check 2 axioms for a basis. (1) Let U open and let $x \in U$, can find $f : X \to [0,1]$ such that $f|_{U^c} = \{0\}$ and $f(x) = 1$, hence $x \in f^{-1}((0,\infty)) \subset U$. Let U, V open and let $x \in U \cap V$, can find $f : X \to [0,1]$ such that $f|_{(U \cap V)^c} = \{0\}$ and $f(x) = 1$, hence $x \in f^{-1}((0,\infty)) \subset U \cap V$. $\Leftarrow$ Let $x \in X$ and $B \in X$ closed s.t. $x \notin B$. Let $U = X - B$ so $x \in U$. Hence can find continuous $f : X \to [0,1]$ such that $f(x) = 1$ and $f(B^c) = \{0\}$. Hence $T_{3.5}$*
- *(theorem) $T_{3.5}$ well behaved with subspace/product (subspace easy. product tricky. Let $\mathbf{a} = (a_\alpha)$ disjoint from closed set $A \subset X = \prod X_\alpha$. Consider basis element $\prod U_\alpha$ containing $\mathbf{a}$ disjoint from A. Since each X_α $T_{3.5}$ can find $f_i : X_{\alpha_i} \to [0,1]$ for $\{a_1, \cdots, a_n\}$ where $X_{\alpha_i} \neq U_{\alpha_i}$ such that $f_i(a_{\alpha_i}) = 1$ and $f_i(X - U_{\alpha_i}) = \{0\}$. Let $\phi_i(\mathbf{x}) = (f_i \circ \pi_{\alpha_i})(\mathbf{x})$; ϕ_i maps X continuously into $\mathbb{R}$ and vanishes outside $\pi^{-1}_{\alpha_i}(U_{\alpha_i})$. Hence $f(\mathbf{x}) = \phi_1(\mathbf{x}) \cdot \phi_2(\mathbf{x}) \cdot \cdots \cdot \phi_n(\mathbf{x})$ is continuous on X and equal 1 at $\mathbf{a}$ and vanishes outside $\prod U_\alpha$)*
- *(theorem) $T_4 \Rightarrow T_{3.5}$ ($T_4 \iff T_{4.5} \Rightarrow T_{3.5}$)*
- *(theorem) $T_{3.5} \Rightarrow T_3$ (by $f^{-1}([0,0,5)$ and $f^{-1}((0,5,1])$ separates B and $\{x\}$)*

Definition. *($T_{4.5}$ **Completely Normal Space**) X is completely normal if X is T_1 and for all $A, B \subset X$ closed $A \cap B = \emptyset$, there exists a continuous function $f : X \to [0,1]$ such that $f|_A = \{0\}$ and $f|_B = \{1\}$. We say A and B can be separated by a continuous function*

- *(theorem) $T_{4.5} \iff T_4$ ($\Rightarrow$ by Urysohn lemma; $\Leftarrow$ by $f^{-1}([0,0,5)$ and $f^{-1}((0,5,1])$ separates A and B)*

4.0.3 32 Normal Spaces

Theorem. ***(Every regular space with a countable basis is normal)*** *$T_3 + \alpha_2 \Rightarrow T_4$*

Proof. Let A, B disjoint closed subsets of X. For each $x \in A$, exists open neighborhood U s.t. $U \cap B = \emptyset$. By regularity, exists neighborhood V of x such that $\overline{V} \subset U$. Now since there is a countable basis, pick an element from $U_x \in \mathcal{B}$ such that $x \in U_x \subset V$. Pick such U_x for every $x \in A$, we find a countable covering for A, i.e. $\{U_n\}$, such that each $\overline{U}_n \cap B = \emptyset$. Similarly choose $\{V_n\}$ as a covering for B where each $\overline{V}_n \cap A = \emptyset$. But note that $U = \cup_n U_n$ and $V = \cup_n V_n$ not necessarily disjoint. Use trick

$$U_n' = U_n - \bigcup_{i=1}^n \overline{V}_i \qquad V_n' = V_n - \bigcup_{i=1}^n \overline{U}_i \qquad\qquad U' = \bigcup_{n \in \mathbb{Z}_+} U_n' \quad V' = \bigcup_{n \in \mathbb{Z}_+} V_n'$$

to avoid U_ns and V_ns from potential intersections. Each U_n', V_n' are open since each is difference of an open set with a closed set. Note $\{U_n'\}$ covers A, since any $x \in A$ is in some U_n and in no $\overline{V}_i$. Similarly $\{V_n'\}$ covers B. Now we show $U' \cap V' = \emptyset$. Suppose $x \in U' \cap V'$. Then $x \in U_j' \cap V_k'$ for some j, k. Suppose $j \leq k$, then by definition of U_j', $x \in U_j$ and by definition of V_k', $x \notin U_j$. here is a contradiction. □

Theorem. ***(Every metrizable space is normal)*** *metrizable $\Rightarrow T_4$*

Proof. Given (X, d). Let $A, B \subset X$ disjoint and closed. For each $a \in A$, find ϵ_a s.t. $B_\epsilon(a) \cap B = \emptyset$. Let $U = \cup_{a \in A} B_{\epsilon_a/2}(a)$, similarly for $V = \cup_{b \in B} B_{\epsilon_b/2}(b)$. Note $A \subset U$ and $B \subset V$. Now claim they are disjoint. Suppose $x \in U \cap V$. then $x \in B_{\epsilon_a}(a) \cap B_{\epsilon/b}(b)$ for some $a \in A$ and $b \in B$. Then $d(a, b) \leq d(a, x) + d(x, y) \leq (\epsilon_a + \epsilon_b)/2$, implying either $d(a, b) < \epsilon_a$ or ϵ_b. If former, then $b \in B_{\epsilon_a}(a)$, which is a contradiction. Same for the latter case. □

Theorem. ***(Compact Hausdorff space is normal)*** *compact $+T_2 \Rightarrow T_4$*

Proof. Let A, B be disjoint closed subsets of X. For each $a \in A$, find U_a, V_a disjoint containing a and B respectively, by regularity of X. $\{U_a\}$ covers A. Since A compact, $\{U_i \mid i \in I\}$ is a finite cover for A. Let $U = \cup_{i \in I} U_i$ and $V = \cap_{i \in I} V_i$ are disjoint open sets containing A and B respectively. □

Theorem. ***(Every order topology is normal)*** *order $\Rightarrow T_4$*

4.0.4 33 The Urysohn Lemma

Theorem. ***(Urysohn Lemma** $T_4 \Rightarrow T_{4.5}$**)*** *Let X be T_4; let A and B be disjoint subsets of X. There exsits continuous map $f : X \to [a, b]$ such that $f(x) = a$ and $f(B) = \{b\}$*

Proof. Proof sketch

1. Construct a simply ordered, countable set $\{U_p \mid p \in [0, 1] \cap \mathbb{Q}\}$ such that for all $p < q$, $\overline{U}_p \subset U_q$.
2. Extend U_p for $p \in \mathbb{Q}$ by defining $U_p = \emptyset$ if $p < 0$ and $U_p = X$ if $p > 1$
3. Consider $f(x) = \inf\{p \mid x \in U_p\}$ and show f is the desired function
   (a) $x \in A$, then $x \in U_p$ for all $p \geq 0$, hence $f(x) = \inf\{p \in \mathbb{Q} \mid p \geq 0\} = 0$
   (b) $x \in B$, then $x \in U_p$ for no $p \leq 1$, hence $f(x) = \inf\{p \in \mathbb{Q} \mid p > 1\} = 1$
   (c) f continuous
   - (fact) $x \in \overline{U}_r \Rightarrow f(x) \leq r$ (since $x \in U_s$ for all $s > r$, $f(x) = \inf\{p \in \mathbb{Q} \mid p > r\} \leq r$)
   - (fact) $x \notin U_r \Rightarrow f(x) \geq r$ (since $x \notin U_s$ for any $s < r$, $f(x) = \inf\{p \in \mathbb{Q} \mid p \geq r\} \geq r$)

   Let $x_0 \in X$ and $(c, d) \in \mathbb{R}$, want to find neighborhood U of x_0 such that $f(U) \in (c, d)$. Pick $p, q \in \mathbb{Q}$ such that $c < p < f(x_0) < q < d$. Let $U = U_q - \overline{U}_p$. Let $y \in U$. By $y \in U_q \in \overline{U}_q$, we have $f(y) \leq q$. By $y \notin \overline{U}_p$ implying $y \notin U_p$, we have $f(y) \geq p$. Therefore $f(y) \in [p, q] \subset (c, d)$

□

4.0.5 34 The Urysohn Metrization Theorem

Theorem. ***(Imbedding Theorem)*** *Let X be T_1. Suppose $\{f_\alpha\}_{\alpha \in J}$ is an indexed family of continuous functions $f_\alpha : X \to \mathbb{R}$ satisfying requiremen that for each $x_0 \in X$ and each neighborhood U of x_0, there is an index α such that $f_\alpha(x_\alpha) > 0$ and $f_\alpha(X - U) = \{0\}$. Then the function $F : X \to \mathbb{R}^J$ defined by*

$$F(x) = (f_\alpha(x))_{\alpha \in J}$$

is am imbedding of X in $\mathbb{R}^J$. If f_α maps X to $[0, 1]$ for each α, then F imbeds X in $[0, 1]^J$.

Proof. Let $\mathbb{R}^\omega$ be in product topology. Recall $F(x) = (f_{\alpha_1}(x), f_{\alpha_2}(x), \cdots))$. Note F is continuous, since each f_α is continuous. F is injective. Indeed, any $x \neq y \in X$, can find neighborhood U of x such that $y \in U^c$, so we can find $\alpha \in J$ such that $f_\alpha(x) > 0$ and $f_\alpha(y) = 0$, implying $F(x) \neq F(y)$. To prove F is an imbedding, enough to show $X \cong F(X) \subset \mathbb{R}^J$. Only left to prove F is an open map. Let $U \in X$, want to show $F(U)$ is open in Z. Let $x_0 \in X$ and $y_0 = F(x_0) \in \mathbb{R}^J$. We want to find open W such that $y_0 \in W \subset F(U)$. We can pick α such that $f_\alpha(x_0) > 0$ and $f_\alpha(U^c) = \{0\}$. Let $W = F(X) \cap \pi_\alpha^{-1}((0, \infty)) \subset \mathbb{R}^J$. Show

1. ($y_0 \in W$) $y_0 \in F(X)$ by assumption. $\pi_\alpha(y_0) = \pi_\alpha(F(x_0)) = f_\alpha(x_0) > 0$. Therefore $y_0 \in \pi_\alpha^{-1}((0, \infty))$.
2. ($W \subset F(U)$) Let $y \in W$, exists $x \in X$ such that $f(x) = y$ and $\pi_\alpha(y) = (0, \infty)$. Since $\pi_\alpha(y) = \pi_\alpha(F(x)) = f_\alpha(x)$ and $f_\alpha(x) = 0$ if $x \in X - U$, then $x \in U$.

Therefore, F is an imbedding of X in $\mathbb{R}^J$ □

- *(theorem) X is $T_{3.5}$ if and only if X imbeds in $[0, 1]^J$ for some J (X is $T_{3.5}$ satisfies requirement for $\{f_\alpha\}_{\alpha \in J}$)*
- *(theorem) X is T_3 and has a countable basis, then T_3 imbeds in $[0, 1]^\omega$ (Urysohn Metriation Theorem)*

Theorem. ***(Urysohn Metrization Theorem** $T_3 + \alpha_2 \Rightarrow$ **metric)*** *Every regular space with a countable basis is metrizable*

Proof. Idea is to imbed X in a metrizable space Y, thereby proving X is metrizable by homeomorphism. To show $X \hookrightarrow [0,1]^\omega$, we need to find a countable collection of continuous functions $\{f_n : X \to [0,1] \mid n \in \mathbb{Z}_+\}$ that separates points from closed sets. The result follows by applying the imbedding theorem. Let $\{B_n\}$be a countable basis for X. Note X is $T_3 + \alpha_2 \Rightarrow T_2 + \alpha_2 \Rightarrow T_4 \Rightarrow T_{4.5}$ by Urysohn lemma. For each $\overline{B}_n \subset B_m$, pick a continuous function $g_{n,m} : X \to [0,1]$ such that $g_{n,m}(\overline{B}_n) = \{1\}$ and $g_{n,m}(X - B_m) = \{0\}$. We claim $\{g_{n,m}\}$ separates points and closed sets! Let $x_0 \in X$ and $B \subset X$ closed s.t. $x_0 \notin B$, we can find a basic element B_m of x contained in B. Since X is $T_{3.5}$, we can find B_n s.t. $x \in B_n \overline{B}_n \subset B_m$. By construction, $g_{n,m}(x_0) = 1$ and $g_{n,m}(X - U) = \{0\}$. Hence $\{g_{n,m}\}$ as desired. Note $\{g_{n,m}\}$ indexed by $\mathbb{Z}_+ \times \mathbb{Z}_+$ and hence countable. Reindex and this gives us a collection $\{f_{n,m}\}$ which satisfies condition for imbedding theorem. □

4.0.6 35 The Tietze Extension Theorem

Theorem. ***(Tietze Extension Theorem)*** *Let X be T_4; Let $A \subset X$ be closed. Any continuous function of A into closed $[a,b] \in \mathbb{R}$ ($\mathbb{R}$) maybe extended to a continuous function of all of X into $[a,b]$ ($\mathbb{R}$)*

$$X \text{ is } T_4 \quad A \subset X \text{ closed} \quad f : A \to \mathbb{R} \text{ continuous} \quad \Rightarrow \quad \exists\, \tilde{f} : X \to \mathbb{R} \text{ continuous} \quad \tilde{f}|_A = f$$

Proof. Idea is to find $\{g_n\} \to \tilde{f}$ a sequence of functions that uniformly converges to $\tilde{f}$ to ensure its continuity. Let $f :\to [-1,1]$, be continuous, we can use the fact that X is now $T_{4.5}$ (Urysohn lemma) and find $g_1 : X \to [-1/3, 1/3]$ where $g(B) = -1/3$ and $g(C) = 1/3$ where $B = f^{-1}([-1,-1/3])$ and $C = f^{-1}([1/3,1])$. This function satisfies

$$\begin{aligned} |g_1(x)| &\le 1/3 & x &\in X \\ |f(x) - g_1(x)| &\le 2/3 & x &\in A \end{aligned}$$

Consider $f - g_1 : X \to [-2/3, 2/3]$. We can iteratively define g_n and let $\tilde{f}(x) = \sum_{n \in \mathbb{Z}_+} g_n(x)$. g converges and g is uniformly continuous. □

- <u>*(theorem)*</u> *If X satisfies Tietze, then T is $T_{4.5}$ (let $B, C \subset X$ closed. Define $f : B \cup C \to \mathbb{R}$ by $f(x) = 0$ if $x \in B$ and $f(1) = 1$ if $x \in C$. f is continuous and use Tietze to find $\tilde{f} : X \to \mathbb{R}$, which is as required for $T_{4.5}$)*

5 The Tychonoff Theorem

5.0.1 37 The Tychonoff Theorem

Lemma. ***(Get Maximal $\mathcal{D} \subset \mathcal{P}(X)$ with FIP using Zorn's Lemma)*** *Let X be a set; $\mathcal{A} \subset \mathcal{P}(X)$ having finite intersection property. Then there exists a collection $\mathcal{D} \subset \mathcal{P}(X)$ such that*

1. *$\mathcal{D} \supset \mathcal{A}$*
2. *$\mathcal{D}$ has finite intersection property*
3. *no collection $\mathcal{D}' \subset \mathcal{P}(X)$ where $\mathcal{D}' \supsetneq \mathcal{D}$ has finite intersection property*

Call $\mathcal{D}$ as maximal with respect to FIP

Proof. Let $\mathbb{C}$ be a superset whose elements are collections of subsets of X, i.e. $\mathcal{A}, \mathcal{D} \in \mathbb{C}$. Let

$$\mathbb{A} = \{\mathcal{B} \subset \mathcal{P}(X) \mid \mathcal{B} \supset \mathcal{A} \text{ and } \mathcal{B} \text{ has FIP }\}$$

Use $\subsetneq$ as strict ordering on $\mathbb{A}$. In order to apply Zorn's lemma on $(\mathbb{A}, \subsetneq)$, we show if $\mathbb{B} \subset \mathbb{A}$ that is simply orderd by $\subsetneq$, then $\mathbb{B}$ has an upper bound in $\mathbb{A}$. Let $\mathcal{C} = \cup_{\mathcal{B} \in \mathbb{B}} \mathcal{B}$ and we show that $\mathcal{C}$ is an element of $\mathbb{A}$, and then it is the required upper bound for $\mathbb{B}$. To show $\mathcal{C} \in \mathbb{A}$. We show

1. ($\mathcal{A} \subset \mathcal{C}$) obvious since any $\mathcal{B} \in \mathbb{B}$ has $\mathcal{A} \subset \mathcal{B}$
2. ($\mathcal{C}$ has FIP) Let $C_1, \cdots, C_n \subset \mathcal{C}$ a finite subset, we want to show that their intersection is nonempty. Since $\mathcal{C} = \cup_{\mathcal{B} \in \mathbb{B}} \mathcal{B}$, $\mathcal{C}_i \in \mathcal{B}_i$ for each i. Hence, $\{\mathcal{B}_1, \cdots, \mathcal{B}_n\} \subset \mathbb{B}$. Since $\mathbb{B}$ is assumed to be simply ordered, there exists $\mathcal{B}_k$ such that $\mathcal{B}_i \subset \mathcal{B}_k$ for all $i \neq k$. Therefore $\{C_1, \cdots, C_n\} \subset \mathcal{B}_k$. Since $\mathcal{B}_k$ has FIP, $\cap_{i=1}^n C_i \neq \emptyset$ as desired.

Hence the upper bound $\mathcal{C}$ is inside $\mathbb{A}$ □

Lemma. ***(Intersection on Maximal $\mathcal{D}$ w.r.t FIP)*** *Let X be a set and $\mathcal{D}$ be a collection of subsets of X that is maximal with respect to the finite intersection property. Then,*

1. *Any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$*
2. *If $A \subset X$ that intersects every element of $\mathcal{D}$, then A is an element of $\mathcal{D}$*

Proof. **(Part 1)** Let B be intersection of finitely many elements of $\mathcal{D}$. Let $\mathcal{E} = \mathcal{D} \cup \{B\}$. To prove $B \in \mathcal{D}$, we show $\mathcal{E}$ has FIP so by maximality of $\mathcal{D}$, $\mathcal{E} = \mathcal{D}$ and $B \in \mathcal{D}$. Now take finitely many elements of $\mathcal{E}$, if none of them is B, then their intersection is nonempty since they are a subset of $\mathcal{D}$ which has FIP. Now if one of finitely many elements of $\mathcal{E}$ is B, then the intersection $D_1 \cap \cdots \cap D_m \cap B$ is nonempty since B is the intersection of finitely many elements of $\mathcal{D}$. **(Part 2)** Let $X \subset X$ s.t. $A \cap D \neq \emptyset$ for all $D \in \mathcal{D}$. Similar to idea of part 1, define $\mathcal{E} = \mathcal{D} \cup \{A\}$. We show that $\mathcal{E}$ has FIP. Now take finitely many elements of $\mathcal{E}$. If none of them is A, then they are nonempty by FIP of $\mathcal{D}$. Otherwise, $D_1 \cap \cdots \cap D_m \cap A$ is nonempty since $A \cap D_j$ for all $1 \leq j \leq m$ by assumption. □

Theorem. ***(Tychonoff Theorem)*** *Arbitrary product of compact spaces is compact in product topology*

Proof. Let $X = \prod_{\alpha \in J} X_\alpha$ where each space X_α is compact. Let $\mathcal{A} \subset \mathcal{P}(X)$ be arbitrary and have FIP, now we show that $\cap_{A \in \mathcal{A}} A$ is nonempty, thereby proving that X is compact. We apply lemma 1 to find a maximal collection $\mathcal{D}$ such that $\mathcal{A} \in \mathcal{D}$ and $\mathcal{D}$ has FIP. Now we show that $\cap_{D \in \mathcal{D}} \overline{D}$ is nonempty. Let $\pi_\alpha : X \to X_\alpha$ be projection map. For each α, consider $\{\pi_\alpha(D) \mid D \in \mathcal{D}\} \subset X_\alpha$ which has FIP since $\mathcal{D}$ does. By compactness, we can chooose $x_\alpha \in \cap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$, which is possible by FIP. Let $\mathbf{x} = (x_\alpha)_{\alpha \in J} \in X$. To complete the proof, we show that $\mathbf{x} \in \cap_{D \in \mathcal{D}} \overline{D}$. Consider any coordinate index β and any $D \in \mathcal{D}$, let $U_\beta \subset X_\beta$ be a neighborhood of x_β. Since $x_\beta \in \overline{\pi_\beta(D)}$, U_β intersects $\pi_\beta(D)$. Let $\pi_\beta(\mathbf{y}) \in U_\beta \cap \pi_\beta(D)$ for some $\mathbf{y} \in D$. Hence $\mathbf{y} \in \pi_{-1}(U_\beta) \cap D$. Since D is arbitrary, by lemma 2.2, $\pi^{-1}(U_\beta) \in \mathcal{D}$. Since β arbitrary, this holds for all $\beta \in J$. Then any basis element containing $\mathbf{x}$ as finite intersection of subbasis $\pi^{-1}(U_\beta) \in \mathcal{D}$, then by lemma 2.1, is also in $\mathcal{D}$. Now use $\mathcal{D}$ FIP again, $B \cap D \neq \emptyset$ where $D \in \mathcal{D}$ and $B \in \mathcal{D}$ is any basic element containing $\mathbf{x}$. $\mathbf{x} \in \cap_{D \in \mathcal{D}} \overline{D}$ □