table of contents flow chart

Contents

1. **(Taylor Expansion)** Given $f \in C^\infty(\mathbb{R})$, the Taylor expansion of f at a is given by

$$\begin{aligned}T(a) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots\end{aligned}$$

2. **(Taylor's Theorem)** Let $k \geq 1$ and function $f : \mathbb{R} \to \mathbb{R}$ be k times differentiable at a point $a \in \mathbb{R}$, then exists $h_k : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n + h_k(x)(x-a)^k$$

and $\lim_{x\to a} h_k(x) = 0$, i.e. the reminder term $R_k(x) = f(x) - P_k(x)$ is asymptotically trivial. If f is $k+1$ times differentiable on the open interval and f^k continous on the closed interval $[a, x]$, then the Lagrange remind is given by

$$R_k(x) = \frac{f^{k+1}(\zeta)}{(k+1)!}(x-a)^{k+1}$$

for soem $\zeta \in [a, x]$ by the mean value theorem

3. **($\mathcal{O}$ notation)** $f(x) = \mathcal{O}(g(x))$ describes asymptotic behavior of function f
   (a) (as $x \to \infty$) if there exists $M \geq 0$ and $x_0 \in \mathbb{R}$ such that $|f(x)| \leq Mg(x)$ for all $x > x_0$.
   (b) (as $x \to a$) if there exists $M \geq 0$ and $\delta \in \mathbb{R}$ such that $|f(x)| \leq Mg(x)$ when $0 < |x-a| < \delta$. Alternatively we can say

$$\limsup_{x\to a} \left| \frac{f(x)}{g(x)} \right| < \infty$$

4. **(power series expansions)**

$$\begin{aligned}\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \\ e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \\ (1+x)^\alpha &= \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \\ \ln(1+x) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} && \text{(convergent if } |x| < 1) \\ \ln(1+x) &= -\sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} && \text{(convergent if } |x| < 1)\end{aligned}$$

1 Euler's Method and Beyond

1.1 Ordinary differential equations and Lipschitz condition

1. **(Goal)** Approximate solution to

$$\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y}) \quad \text{with initial condition} \quad \boldsymbol{y}(t_0) = \boldsymbol{y}_0$$

where $t > t_0$ and $\boldsymbol{f} : [t_0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is a sufficiently well behaved function

2. **(Lipschitz condition)** Given $\boldsymbol{f}$ and norm $\|\cdot\|$, the Lipschitz condition is defined by

$$\|\boldsymbol{f}(t,\boldsymbol{x}) - \boldsymbol{f}(t,\boldsymbol{y})\| \leq \lambda \|\boldsymbol{x} - \boldsymbol{y}\| \qquad \text{for all} \quad \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^d \ , \ t > t_0$$

where $\lambda \in \mathbb{R}$ is called Lipschitz constant.

3. **(Picard Lindelof theorem)** Consider initial value problem

$$y'(t) = f(t, y(t)) \quad y(t_0) = y_0$$

If f is uniformly Lipschitz continous in y and continous in t, then for some $\epsilon > 0$, there exists unique solution $y(t)$ to the initial value problem on the interval $[t_0 - \epsilon, t_0 + \epsilon]$

4. **(Analytic function)** A function $\boldsymbol{f}$ is an analytic function if it is a function that is locally given by a convergent power series, i.e. an infinitely differentiable function such that at any point $(t, \boldsymbol{y}_0) \in [0,\infty) \times \mathbb{R}^d$ in its domain, the Taylor series converges to $\boldsymbol{f}(\boldsymbol{x})$ for $\boldsymbol{x}$ in a neighborhood of $(t, \boldsymbol{y}_0)$.

   (a) (example) polynomial, exponential, trigonometric, logarithm, power function

   (b) (note) if $\boldsymbol{f}$ is analytic, solution $\boldsymbol{y}$ to the initial value problem is also analytic

1.2 Euler's method

Definition. ***(Euler's Method)*** *Given initial value problem $\boldsymbol{y}' = \boldsymbol{f}(t,\boldsymbol{y})$ for $t \geq t_0$ and initial value $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$. If we assume $\boldsymbol{y}'(t) = \boldsymbol{f}(t,\boldsymbol{y}(t)) \approx \boldsymbol{f}(t_0,\boldsymbol{y}(t_0))$ for $t \in [t_0, t_0+h)$ (i.e. deriviative in $[t_n, t_{n+1}]$ is approximated by value of derivative at t_n) for some sufficiently small time step $h > 0$, we can approximate the value of $\boldsymbol{y}(t)$ by*

$$\begin{aligned}\boldsymbol{y}(t) &= \boldsymbol{y}(t_0) + \int_{t_0}^{t} \boldsymbol{f}(\tau, \boldsymbol{y}(\tau)) d\tau \\ &\approx \boldsymbol{y}_0 + (t - t_0)\boldsymbol{f}(t_0, \boldsymbol{y}_0)\end{aligned}$$

Given a sequence of times $(t_n)_{n\in\mathbb{N}} = (t_0, t_0 + h, \cdots)$ we have numerical approximation $(\boldsymbol{y}_n)_{n\in\mathbb{N}}$ by

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{f}(t_n, \boldsymbol{y}_n)$$

1. *(intuition) euler's method is a time-stepping numerical method that covers interval by an equidistant grid and produe numerical solution at the grid points. we can show that euler's method is convergent, i.e. as $h \to 0$, grid is refined, the numerical solution tends to exact solution*

Definition. ***(convergent method)*** *Given a time-stepping numerical method on a compact inteval $[t_0, t_0 + t^*]$, we can compute numerical solutions dependent upon h*

$$\boldsymbol{y}_n = \boldsymbol{y}_{n,h} \quad \textit{for} \quad n = 0, 1, \cdots, \lfloor t^*/h \rfloor$$

A method is said to be convergent if for every ODE with Lipschitz function $\boldsymbol{f}$, the numerical solution tends to the true solution as the grid becomes increasingly fine. More rigorously, if every ODE with Lipschitz function $\boldsymbol{y}$ and for every $t^* > 0$, then following holds

$$\lim_{h\to 0^+} \max_{n=0,1,\cdots,\lfloor t*/h \rfloor} \|\boldsymbol{y}_{n,h} - \boldsymbol{y}(t_n)\| = 0$$

Theorem. ***(Euler's method is convergent)***

Proof. Assume $\boldsymbol{f}$ and therefore also $\boldsymbol{y}$ is analytic, i.e. convergent Taylor expansion. Let $\boldsymbol{e}_{n,h} = \boldsymbol{y}_{n,h} - \boldsymbol{y}(t_n)$ be the numerical error. Show $\lim_{h\to 0^+} \max_n \|\boldsymbol{e}_{n,h}\| = 0$. By Taylor's theorem

$$\boldsymbol{y}(t_{n+1}) = \boldsymbol{y}(t_n) + h\boldsymbol{y}'(t_n) + \mathcal{O}(h^2) = \boldsymbol{y}(t_n) + h\boldsymbol{f}(t_n, \boldsymbol{y}(t_n)) + \mathcal{O}(h^2)$$

given $\boldsymbol{y}$ continously differentiable, $\mathcal{O}(h^2)$ can be bounded uniformly for all $h > 0$ by a term ch^2 for some $c > 0$. Subtract previous from iterative formula of euler's method

$$\boldsymbol{e}_{n+1,h} = \boldsymbol{e}_{n,h} + h\left(\boldsymbol{f}(t_n, \boldsymbol{y}(t_n) + \boldsymbol{e}_{n,h}) - \boldsymbol{f}(t_m, \boldsymbol{y}(t_n))\right) + \mathcal{O}(h^2)$$

By triangle inequality and Lipschitz condition

$$\begin{aligned}\|\boldsymbol{e}_{n+1,h}\| &\leq \|\boldsymbol{e}_{n,h}\| + h\left\|\boldsymbol{f}(t_n, \boldsymbol{y}(t_n) + \boldsymbol{e}_{n,h}) - \boldsymbol{f}(t_n, \boldsymbol{y}(t_n))\right\| + ch^2 \\ &\leq (1 + h\lambda)\|\boldsymbol{e}_{n,h}\| + ch^2\end{aligned}$$

for $n = 0, 1, \cdots, \lfloor t^*/h \rfloor - 1$. By induction on n, we can show $\|\boldsymbol{e}_{n,h}\| \leq \frac{c}{\lambda} h\left((1 + h\lambda)^n - 1\right)$. Since $1 + h\lambda < e^{h\lambda}$ we have $(1 + h\lambda)^n < e^{nh\lambda} < e^{\lfloor t*/h \rfloor h\lambda} \leq e^{t^*\lambda}$. Therefore

$$\|\boldsymbol{e}_{n,h}\| \leq \frac{c}{\lambda}(e^{t^*\lambda} - 1)h$$

for $n = 0, 1, \cdots, \lfloor t * /h \rfloor$. This is an upper bound on the error that is independent of h, hence $\lim_{h\to 0}\|\boldsymbol{e}_{n,h}\| = 0$. from which we can infer that error decays globally as $\mathcal{O}(h)$ $\square$

Definition. ***(order** p **method)*** *Given arbitrary time-stepping method*

$$\boldsymbol{y}_{n+1} = \boldsymbol{\mathcal{Y}}_n(\boldsymbol{f}, h.\boldsymbol{y}_0, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_n) \qquad n = 0, 1, \cdots$$

for initial value problem, it is of order p if

$$\boldsymbol{y}(t_{n+1}) - \boldsymbol{\mathcal{Y}}_n(\boldsymbol{f}, h, \boldsymbol{y}(t_0), \boldsymbol{y}(t_1), \cdots, \boldsymbol{y}(t_n)) = \mathcal{O}(h^{p+1})$$

for every analytic $\boldsymbol{f}$ and $n = 0, 1, \cdots$. Intuitively, a method is of order p if it recovers exactly every polynomial oslution of degrees p or less.

1. *(intuition) order of a method gives information about local behavior, i.e. advancing from t_n to t_{n+1} where $h > 0$ is sufficiently small, we are incurring an error of $\mathcal{O}(h^{p+1})$. Generally want the the global (convergence) behavior of the method instead.*
2. ***(fact)*** *euler's method is order 1*

   Proof. Euler's method can be written as $\boldsymbol{y}_{n+1} - (\boldsymbol{y}_n + h\boldsymbol{f}(t_n, \boldsymbol{y}_n)) = 0$. Replace $\boldsymbol{y}_k$ by $\boldsymbol{y}(t_k)$ and expand terms of Taylor series about t_n we have

   $$\boldsymbol{y}(t_{n+1}) - (\boldsymbol{y}(t_n) + h\boldsymbol{f}(t_n, \boldsymbol{y}(t_n))) = \left(\boldsymbol{y}(t_n) + h\boldsymbol{y}'(t_n) + \mathcal{O}(h^2)\right) - \left(\boldsymbol{y}(t_n) + h\boldsymbol{y}'(t_n)\right) = \mathcal{O}(h^2)$$

   $\square$

1.3 The trapezoidal rule

Definition. ***(Trapezoidal Rule)*** *Instead of approximating derivative by a constant in $[t_n, t_{n+1}]$, namely by its value at t_n, the trapezoidal rule approximates the value of the derivate by average of values at the endpoints. We can approximate solution $\boldsymbol{y}(t)$ by*

$$\begin{aligned}\boldsymbol{y}(t) &= \boldsymbol{y}(t_n) + \int_{t_n}^{t} \boldsymbol{f}(\tau, \boldsymbol{f}(\tau))d\tau \\ &\approx \boldsymbol{y}(t_n) + \frac{1}{2}(t - t_n)\left(\boldsymbol{f}(t_n, \boldsymbol{y}(t_n)) + \boldsymbol{f}(t, \boldsymbol{y}(t))\right)\end{aligned}$$

Given a sequence of times $(t_n)_{n\in\mathbb{N}} = (t_0, t_0 + h, \cdots)$ we have numerical approximation $(\boldsymbol{y}_n)_{n\in\mathbb{N}}$ by

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \frac{1}{2}h\left(\boldsymbol{f}(t_n, \boldsymbol{y}_n) + \boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})\right)$$

1. ***(theorem)*** *order of trapezoidal rule is 2*

Proof. Compute by performing Taylor expansion on $\boldsymbol{y}(t_{n+1})$ and $\boldsymbol{y}'(t_{n+1})$ about t_n

$$\boldsymbol{y}(t_{n+1}) - \left\{\boldsymbol{y}(t_n) + \frac{1}{2}h\left\{\boldsymbol{f}(t_n, \boldsymbol{y}(t_n)) + \boldsymbol{f}(t_{n+1}, \boldsymbol{y}(t_{n+1}))\right\}\right\} = \mathcal{O}(h^3)$$

□

2. ***(theorem)*** *trapezoidal rule is convergent*

   Proof. Detail of proof here . We can show error is bounded by

   $$\|\boldsymbol{e}_{n,h}\| \leq \frac{ch^2}{\lambda} exp\left(\frac{t^*\lambda}{1-\frac{1}{2}h\lambda}\right)$$

   from which we can infer that error decays globally as $\mathcal{O}(h^2)$ □

3. *(note) euler's method is explicit, since we can compute $\boldsymbol{y}_{n+1}$ with a few arithmetic operations by computing $\boldsymbol{f}$, a function of a known $\boldsymbol{y}_n$. Trapezoidal rule is implicit, i.e. finding $\boldsymbol{y}_{n+1}$ is not trivial and $\boldsymbol{f}$ is a function of both $\boldsymbol{y}_n$ and $\boldsymbol{y}_{n+1}$. We might need to solve a nonlinear equation of $\boldsymbol{y}_{n+1}$*

   $$\boldsymbol{y}_{n+1} - \frac{1}{2}h\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1}) = \boldsymbol{v}$$

   where $\boldsymbol{v} = \boldsymbol{y}_n + \frac{1}{2}h\boldsymbol{f}(t_n, \boldsymbol{y}_n)$ can be evaluated easily from assumptions.

1.4 The theta method

Definition. ***(theta method)*** *is a generalization of Euler's method ($\theta = 1$) and the trapezoidal rule ($\theta = 1/2$), whereby the derivates are assumed to be piecewise constant and provided by a linear combination of derivatives at the endpoints of each interval. The numerical approximates are,*

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\left(\theta\boldsymbol{f}(t_n, \boldsymbol{y}_n) + (1-\theta)\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})\right) \qquad n = 0, 1, \cdots$$

for some fixed $\theta \in [0, 1]$

1. *(fact) theta method is explicit for $\theta = 1$ and implicit otherwise*
2. ***(theorem)*** *theta method is of order 2 for $\theta = 1/2$ and order 1 otherwise.*
3. ***(theorem)*** *theta method is convergent for every $\theta \in [0, 1]$*

2 Multistep Method

3 8 Finite Differences Schemes

3.1 8.1 Finite differences

1. **(Finite difference operators)** Given real sequences $\boldsymbol{z} = \{z_k\}_{k\in\mathbb{Z}} = z(kh)$ for $k \in \mathbb{Z}$ as discrete sampling of a function z for some $h > 0$. Let $x_k = kh$. We can define finite difference operators mapping the space $\mathbb{R}^{\mathbb{Z}}$ of all such sequences to itself.

   $$(\boldsymbol{\mathcal{E}}\boldsymbol{z})_k = z_{k+1} \qquad \text{(shift)}$$
   $$(\boldsymbol{\Delta}_+\boldsymbol{z})_k = z_{k+1} - z_k \qquad \text{(forward difference)}$$
   $$(\boldsymbol{\Delta}_-\boldsymbol{z})_k = z_k - z_{k-1} \qquad \text{(backward difference)}$$
   $$(\boldsymbol{\Delta}_0\boldsymbol{z})_k = z_{k+\frac{1}{2}} - z_{k-\frac{1}{2}} \qquad \text{(central difference)}$$
   $$(\boldsymbol{\Upsilon}_0\boldsymbol{z})_k = \tfrac{1}{2}(z_{k-\frac{1}{2}} + z_{k+\frac{1}{2}}) \qquad \text{(averaging)}$$

   Finite difference operators are composed under function composition.

(a) **(fact)** $\mathcal{T} \in \{\boldsymbol{\mathcal{E}}, \boldsymbol{\Delta}_+, \boldsymbol{\Delta}_-, \boldsymbol{\Delta}_0, \boldsymbol{\Upsilon}_0, \boldsymbol{\mathcal{D}}\}$ are linear operators

$$\mathcal{T}(a\mathbf{w} + b\boldsymbol{z}) = a\mathcal{T}\mathbf{w} + b\mathcal{T}\boldsymbol{z} \quad \text{for} \quad \mathbf{w}, \boldsymbol{z} \in \mathbb{R}^{\mathbb{Z}}, \quad a, b \in \mathbb{R}$$

(b) **(convention)** $\mathcal{T}z_k$ stands for $(\mathcal{T}\boldsymbol{z})_k$

2. **(Differential operator)** The goal is to approximate derivatives $\boldsymbol{\mathcal{D}}$ by expresing it with a linear combination of values along the grid.

$$(\boldsymbol{\mathcal{D}}\boldsymbol{z})_k = z'(kh) \tag{differential}$$

3. **(Functions of operators)** Finite difference operators are functions of h. Given an analytic function as Taylor series, $g(x) = \sum_{j=0}^{\infty} a_j x^j$, we can expand g about $\{\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{I}}, \boldsymbol{\Upsilon}_0 - \boldsymbol{\mathcal{I}}, \boldsymbol{\Delta}_+, \boldsymbol{\Delta}_-, \boldsymbol{\Delta}_0, h\boldsymbol{\mathcal{D}}\}$,

$$g(\boldsymbol{\Delta}_+)\boldsymbol{z} = \left(\sum_{j=0}^{\infty} a_j \boldsymbol{\Delta}_+^j\right)\boldsymbol{z} = \sum_{j=0}^{\infty} a_j (\boldsymbol{\Delta}_+^j \boldsymbol{z})$$

4. **(Asymptotics of operators)**

$$\{\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{I}}, \boldsymbol{\Upsilon}_0 - \boldsymbol{\mathcal{I}}, \boldsymbol{\Delta}_+, \boldsymbol{\Delta}_-, \boldsymbol{\Delta}_0, h\boldsymbol{\mathcal{D}}\} \overset{h\to 0+}{\longrightarrow} O$$

(a) **(example)**

$$\boldsymbol{\Delta}_+ z_k = z_{k+1} - z_k = z(x_k + h) - z(x_k) = hz'(\eta_k) = \mathcal{O}(h)$$

by some $\eta_k \in [x_k, x_{k+1}]$ by mean value theorem

5. **(Operator $\boldsymbol{\mathcal{E}}^{1/2}$)**

$$(\boldsymbol{\mathcal{E}}^{1/2}\boldsymbol{z})_k = z((k + \frac{1}{2})h)$$

by defining it with a power series expansion of $g(x) = \sqrt{1 + x}$

$$\boldsymbol{\mathcal{E}}^{1/2} = (\boldsymbol{\mathcal{I}} + (\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{I}}))^{1/2} = \boldsymbol{\mathcal{I}} + \sum_{j=0}^{\infty} \frac{(-1)^{j-1}}{2^{2j-1}} \frac{(2j-2)!}{(j-1)!j!} (\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{I}})^j$$

6. **(Operator ommutativity)** Idea is all operator can be expressed w.r.t. $\boldsymbol{\mathcal{E}}$.

$$\begin{aligned} \boldsymbol{\Delta}_+ &= \boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{I}} \\ \boldsymbol{\Delta}_- &= \boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{E}}^{-1} \\ \boldsymbol{\Delta}_0 &= \boldsymbol{\mathcal{E}}^{1/2} - \boldsymbol{\mathcal{E}}^{-1/2} \\ \boldsymbol{\Upsilon}_0 &= \frac{1}{2}(\boldsymbol{\mathcal{E}}^{1/2} + \boldsymbol{\mathcal{E}}^{-1/2}) \\ \boldsymbol{\mathcal{I}} &= \boldsymbol{\mathcal{E}}^0 \end{aligned}$$