CSC236 tutorial exercises, Week #7

Sample Solutions

1. Consider the recurrence relation

$$T(n) = \begin{cases} 1 & n = 1 \\ 1 + T\left(\left\lceil \frac{n}{2} \right\rceil\right) & n > 1 \end{cases}$$

Prove that $T(n)$ is non-decreasing.

Sample solution.

Let $P(n)$ denote $T(n) \leq T(n+1)$.

Proof by strong induction.

Basis step.

$P(1)$ holds since $T(1) = 1 \leq T(2) = 1 + T\left(\left\lceil \frac{2}{2} \right\rceil\right) = 1 + 1 = 2$.

Inductive step.

Assume $P(i)$ holds where $1 \leq i < k$ for an arbitrary $> 1 \in \mathbb{N}$; i.e., $T(i) \leq T(i+1)$.

We must show $T(k) \leq T(k+1)$.

There are two cases: either k is odd or k is even.

Case 1. When k is odd, $\left\lceil \frac{k+1}{2} \right\rceil = \left\lceil \frac{k}{2} \right\rceil$ by definition of ceiling.

$T(k) = 1 + T\left(\left\lceil \frac{k}{2} \right\rceil\right)$	by definition of T as $k > 1$
$T(k) = 1 + T\left(\left\lceil \frac{k+1}{2} \right\rceil\right)$	as k is odd and by IH, since $1 \leq \left\lceil \frac{k}{2} \right\rceil < k$
$T(k) = T(k+1)$	by definition of T as $k + 1 > 1$

Case 2. When k is even, $\left\lceil \frac{k+1}{2} \right\rceil = \left\lceil \frac{k}{2} \right\rceil + 1$ by definition of ceiling

$T(k+1) = 1 + T\left(\left\lceil \frac{k+1}{2} \right\rceil\right)$	by definition of T as $k > 1$
$T(k+1) = 1 + T\left(\left\lceil \frac{k}{2} \right\rceil + 1\right)$	as k is even
$T(k+1) \geq 1 + T\left(\left\lceil \frac{k}{2} \right\rceil\right)$	by IH since $1 \leq \left\lceil \frac{k}{2} \right\rceil < k \ \ \forall k > 1 \in \mathbb{N}$
$T(k+1) \geq T(k)$	by definition of T as $k > 1$ $\square$

2. Use repeated substitution (unwinding) to find a closed form for the recurrence S when n is a power of 3.

$$S(n) = \begin{cases} 1 & n < 3 \\ n^2 + a_1 S\left(\left\lceil \frac{n}{3} \right\rceil\right) + a_2 S\left(\left\lfloor \frac{n}{3} \right\rfloor\right) & n > 2 \end{cases}$$

where $a_1, a_2 \geq 0 \in \mathbb{N}$ and $a_1 + a_2 = 3$.

Sample Solution.

Assume $\hat{n} = 3^k$ *where* $k \in \mathbb{N}$. Since $\left\lceil \frac{\hat{n}}{3} \right\rceil = \left\lfloor \frac{\hat{n}}{3} \right\rfloor = \frac{\hat{n}}{3}$, and $a_1 + a_2 = 3$

$$S(\hat{n}) = \begin{cases} 1 & \hat{n} = 1 \\ \hat{n}^2 + 3S\left(\frac{\hat{n}}{3}\right) & \hat{n} > 1 \end{cases}$$

$S(\hat{n}) = \hat{n}^2 + 3S\left(\dfrac{\hat{n}}{3}\right)$

$= \hat{n}^2 + 3\left((\frac{\hat{n}}{3})^2 + 3S\left(\frac{\hat{n}}{3^2}\right)\right)$ by plugging in $\frac{\hat{n}}{3}$ to $S(\hat{n})$

$= \hat{n}^2 + 3(\frac{\hat{n}}{3})^2 + 3^2 S\left(\frac{\hat{n}}{3^2}\right)$

$= \hat{n}^2 + 3\left(\frac{\hat{n}}{3}\right)^2 + 3^2\left(\left(\frac{\hat{n}}{3^2}\right)^2 + 3S\left(\frac{\hat{n}}{3^3}\right)\right)$ by plugging in $\frac{\hat{n}}{3^2}$ to $S(\hat{n})$

$= \hat{n}^2 + 3\left(\frac{\hat{n}}{3}\right)^2 + 3^2\left(\frac{\hat{n}}{3^2}\right)^2 + 3^3 S\left(\frac{\hat{n}}{3^3}\right)$

... after k steps

$= \hat{n}^2 + 3\left(\frac{\hat{n}}{3}\right)^2 + 3^2\left(\frac{\hat{n}}{3^2}\right)^2 + 3^3\left(\frac{\hat{n}}{3^3}\right)^2 + \cdots + 3^{k-1}\left(\frac{\hat{n}}{3^{k-1}}\right)^2 + 3^k S\left(\frac{\hat{n}}{3^k}\right)$

$= \hat{n}^2 + 3\left(\frac{\hat{n}}{3}\right)^2 + 3^2\left(\frac{\hat{n}}{3^2}\right)^2 + 3^3\left(\frac{\hat{n}}{3^3}\right)^2 + \cdots + 3^{k-1}\left(\frac{\hat{n}}{3^{k-1}}\right)^2 + \hat{n}$ since $\hat{n} = 3^k$, and $S(1) = 1$

$= \hat{n}^2(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{k-1}}) + \hat{n}$ factoring out $\hat{n}^2$

$= \hat{n}^2 \frac{1-(\frac{1}{3})^k}{1-\frac{1}{3}} + \hat{n}$ from geometric series

$= \hat{n}^2(\frac{1}{2/3} - \frac{1/3^k}{2/3}) + \hat{n} = \frac{3}{2}\hat{n}^2(1 - \frac{1}{\hat{n}}) + \hat{n} = \frac{3}{2}\hat{n}(\hat{n} - 1) + \hat{n}$

Now we should prove $S_r(\hat{n}) = S_c(\hat{n})$ where

$$S_r(\hat{n}) = \begin{cases} 1 & n < 3 \\ \hat{n}^2 + a_1 S_r\left(\left\lceil \frac{\hat{n}}{3} \right\rceil\right) + a_2 S_r\left(\left\lfloor \frac{\hat{n}}{3} \right\rfloor\right) & n > 2 \end{cases}$$

where $a_1 \,, a_2 \geq 0 \;\in\; \mathbb{N}$, $a_1 + \; a_2 = 3$, and $S_c(\hat{n}) = \frac{3}{2}\hat{n}(\hat{n}-1) + \hat{n}$ where $\hat{n} = 3^k$.

Proof by strong induction.

Basis step.

$$S_r(1) = 1 = S_c(1) = \frac{3}{2}1(1-1) + 1 = 1$$

Inductive step.

Assume $S_r(\hat{\imath}) = S_c(\hat{\imath})$ where $1 \leq \hat{\imath} < \hat{k}$ for $\hat{k} \geq 3 \in \mathbb{N}$ where $\hat{\imath}$ and $\hat{k}$ are powers of 3.

We must show $S_r(\hat{k}) = S_c(\hat{k})$

$S_r(\hat{k}) = \hat{k}^2 + a_1 S_r\left(\left\lceil\frac{\hat{k}}{3}\right\rceil\right) + a_2 S_r\left(\left\lfloor\frac{\hat{k}}{3}\right\rfloor\right)$ by definition of S_r when $\hat{k} > 2$

$S_r(\hat{k}) = \hat{k}^2 + (a_1 + a_1) S_r\left(\frac{\hat{k}}{3}\right)$ since $\hat{k}$ is a power of 3 $\left\lceil\frac{\hat{k}}{3}\right\rceil = \left\lfloor\frac{\hat{k}}{3}\right\rfloor = \frac{\hat{k}}{3}$

$S_r(\hat{k}) = \hat{k}^2 + 3 S_r\left(\frac{\hat{k}}{3}\right)$ since $a_1 + \; a_2 = 3$

$S_r(\hat{k}) = \hat{k}^2 + 3.S_c\left(\frac{\hat{k}}{3}\right)$ by IH, since $1 \leq \frac{\hat{k}}{3} < k$ when $\hat{k} \geq 3$

$S_r(\hat{k}) = \hat{k}^2 + 3.(\frac{3}{2}\frac{\hat{k}}{3}\left(\frac{\hat{k}}{3} - 1\right) + \frac{\hat{k}}{3})$ by definition of S_c

$S_r(\hat{k}) = \hat{k}^2 + \frac{\hat{k}^2}{2} - \frac{3}{2}\hat{k} + \hat{k} = \frac{3}{2}\hat{k}^2 - \frac{3}{2}\hat{k} + \hat{k} = \frac{3}{2}\hat{k}(\hat{k}-1) + \hat{k} = S_c(\hat{k})$

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