# STA302/STA1001, Weeks 9-10

Mark Ebden, 14 & 16 November 2017

With grateful acknowledgment to Alison Gibbs

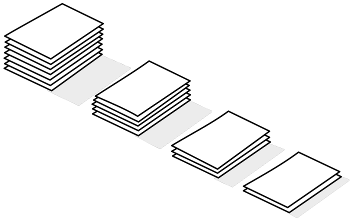
# This week's lectures

- Poll regarding the Exam Jam on 8 December
- Chapter 5:
  - Matrix version of SLR
  - Multiple linear regression (MLR)

# If you haven't retrieved your midterm

To pick up your test at SS 6027 CLTA:

1. Book an office-hours slot online for any Wednesday until 6 December, or
2. Drop in on Tuesday 14 November, 2-2:30 pm, or
3. Drop in on Thursday 16 November, 1-1:30 pm



# To appeal/discuss a recent TA decision on regrading

Please express your appeal via your section's regrading email address.


If the TA who had examined your work doesn't reply within a week then please notify me.

## Recap of our recent studies

The SLR model in matrix form is $\mathbf{Y} = \mathbf{X}\beta + \mathbf{e}$, in which:

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}, \qquad \mathbf{X} = \begin{pmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{pmatrix}, \qquad \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}, \qquad \mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$$

Setting the derivative of $\text{RSS}(\beta)$ to zero yielded $\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ when $\text{rank}(\mathbf{X}) = 2$. This plus the fact that $\mathrm{E}(\mathbf{e}) = \mathbf{0}$ gives

$$\widehat{\mathbf{Y}} = \mathbf{X}\hat{\beta} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} = \mathbf{H}\mathbf{Y}$$

the hat matrix again…

We can write the residuals in terms of idempotent matrix $\mathbf{I} - \mathbf{H}$ as

$$\widehat{\mathbf{e}} = \mathbf{Y} - \widehat{\mathbf{Y}} = \mathbf{Y} - \mathbf{H}\mathbf{Y} = (\mathbf{I} - \mathbf{H})\,\mathbf{Y}$$

We found $\mathrm{E}(\widehat{\mathbf{e}}) = \mathbf{0}$ and were about to try $\text{var}(\widehat{\mathbf{e}})$, requiring the notion of a covariance matrix: $\text{var}(\mathbf{X}) = \mathrm{E}\left[(\mathbf{X} - \mathrm{E}(\mathbf{X})\,(\mathbf{X} - \mathrm{E}(\mathbf{X}))'\right]$.

# Five facts about idempotent matrices *(Weeks 8–9, slide 18)*

1. A square matrix $\mathbf{A}$ is idempotent iff $\mathbf{A}^2 = \mathbf{A}$
2. If $\mathbf{A}$ is idempotent then $\text{trace}(\mathbf{A}) = \text{rank}(\mathbf{A})$
3. $\mathbf{A}$ is idempotent iff $\text{rank}(\mathbf{A}) + \text{rank}(\mathbf{I} - \mathbf{A}) = n$ where the dimensions of $\mathbf{A}$ are $n \times n$ and $\mathbf{I}$ is the $n \times n$ identity matrix
4. For hat matrix $\mathbf{H}$ and matrix of all 1's $\mathbf{J}$, the following matrices are idempotent:

$$\mathbf{H} \qquad \mathbf{I} - \mathbf{H} \qquad \frac{1}{n}\mathbf{J} \qquad \mathbf{H} - \frac{1}{n}\mathbf{J}$$

5. If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are idempotent and $\mathbf{A} = \mathbf{B} + \mathbf{C}$, then $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) + \text{rank}(\mathbf{C})$

# Variance of the residuals, in matrix form

note off-diagonal elements might be zero,
which is fine, since they are not necessarily independent
but their sum equates to 1

$$\begin{aligned}\text{var}(\widehat{\mathbf{e}}) &= \mathrm{E}\left\{[\hat{\mathbf{e}} - \mathrm{E}(\hat{\mathbf{e}})]\,[\hat{\mathbf{e}} - \mathrm{E}(\hat{\mathbf{e}})]'\right\} \\ &= \mathrm{E}\left\{(\mathbf{I} - \mathbf{H})\,\mathbf{Y}\mathbf{Y}'\,(\mathbf{I} - \mathbf{H})\right\} \\ &= \underline{(\mathbf{I} - \mathbf{H})}\,\mathrm{E}\underline{(\mathbf{Y}\mathbf{Y}')}\,(\underline{\mathbf{I} - \mathbf{H}})\end{aligned}$$

basically var(AX) = AVar(X)A’

Compare to our previous work: $\text{var}(\hat{e}_i) = \sigma^2(1 - h_{ii})$. Does the above match?

yes, but matrix form gives more information given off-diagonal elements

NB: As before, the "$|\mathbf{X}$" is implicit — e.g. $\text{var}(\hat{\mathbf{e}}|X)$ is abbreviated as $\text{var}(\hat{\mathbf{e}})$.

## Variance of the residuals, in matrix form

pretty convoluted, there is a simpler method in typed notes

The middle factor is

$$\begin{aligned}
\mathrm{E}(\mathbf{YY}') &= \mathrm{E}\left\{(\mathbf{X}\beta+\mathbf{e})(\mathbf{X}\beta+\mathbf{e})'\right\} \\
&= \mathrm{E}\left\{(\mathbf{X}\beta+\mathbf{e})(\beta'\mathbf{X}'+\mathbf{e}')\right\} \\
&= \mathrm{E}\left\{\mathbf{X}\beta\beta'\mathbf{X}'+\mathbf{X}\beta\mathbf{e}'+\mathbf{e}\beta'\mathbf{X}'+\mathbf{ee}'\right\} \\
&= \mathrm{E}\left\{\mathbf{X}\beta\beta'\mathbf{X}'\right\}+\mathbf{0}+\mathbf{0}+\mathrm{E}(\mathbf{ee}') \\
\mathrm{E}(\mathbf{YY}') &= \mathbf{X}\beta\beta'\mathbf{X}'+\sigma^2\mathbf{I}
\end{aligned}$$

Inserting the above into $\mathrm{var}(\widehat{\mathbf{e}})$ gives

$$\begin{aligned}
\mathrm{var}(\widehat{\mathbf{e}}) &= (\mathbf{I}-\mathbf{H})(\mathbf{X}\beta\beta'\mathbf{X}'+\sigma^2\mathbf{I})(\mathbf{I}-\mathbf{H}) \\
&= \left[\mathbf{I}\left(\mathbf{X}\beta\beta'\mathbf{X}'+\sigma^2\mathbf{I}\right)-\mathbf{H}\left(\mathbf{X}\beta\beta'\mathbf{X}'+\sigma^2\mathbf{I}\right)\right](\mathbf{I}-\mathbf{H}) \\
&= \left[\mathbf{X}\beta\beta'\mathbf{X}'+\sigma^2\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\left(\mathbf{X}\beta\beta'\mathbf{X}'\right)-\sigma^2\mathbf{H}\right](\mathbf{I}-\mathbf{H}) \\
&= \sigma^2(\mathbf{I}-\mathbf{H})(\mathbf{I}-\mathbf{H}) \\
\mathrm{var}(\widehat{\mathbf{e}}) &= \sigma^2(\mathbf{I}-\mathbf{H})
\end{aligned}$$

by indempotency

# What's the rank of $\text{var}(\widehat{\mathbf{e}})$?

Recall the fifth of our *Five facts about idempotent matrices*:

$$\text{If } \mathbf{A} = \mathbf{B} + \mathbf{C}, \text{ then } \text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) + \text{rank}(\mathbf{C}).$$

Put another way, $\text{rank}(\mathbf{B}) = \text{rank}(\mathbf{A}) - \text{rank}(\mathbf{C})$.

Therefore, for example $\text{rank}(\mathbf{I} - \mathbf{H}) = \text{rank}(\mathbf{I}) - \text{rank}(\mathbf{H}) = n - 2$. We'll do other similar calculations when considering ANOVA in matrix terms.

# ANOVA in matrix terms

Recall from Week 3 that

$$\boxed{\text{SST} = \text{SSReg} + \text{RSS}}$$

where

$$\text{SST} = \sum_{i=1}^{n} (y_i - \bar{y})^2 = \sum_{i=1}^{n} y_i^2 - n\bar{y}^2$$

**Exercise:** Show that SST can be re-expressed as

$$\text{SST} = \mathbf{Y}'\mathbf{Y} - \frac{1}{n}\mathbf{Y}'\mathbf{J}\mathbf{Y}$$

where $\mathbf{J}$ is an $n \times n$ matrix of 1's. This means we can also write

$$\boxed{\text{SST} = \mathbf{Y}'\left(\mathbf{I} - \frac{1}{n}\mathbf{J}\right)\mathbf{Y}}$$

## Some properties of $\mathbf{I} - \frac{1}{n}\mathbf{J}$

1. Note that $\mathbf{I} - \frac{1}{n}\mathbf{J}$ is symmetric. For a vector $\mathbf{Y}$ and symmetric matrix $\mathbf{A}$, you may recall from other courses that $\mathbf{Y}'\mathbf{A}\mathbf{Y}$ is a *quadratic form* (second-degree polynomial).

2. Since $\mathbf{I}$ is idempotent and $\frac{1}{n}\mathbf{J}$ is idempotent (from the five facts), $\mathbf{I} - \frac{1}{n}\mathbf{J}$ is also idempotent.

3. The rank of $\mathbf{I} - \frac{1}{n}\mathbf{J}$ is $\text{rank}(\mathbf{I}) - \text{rank}\left(\frac{1}{n}\mathbf{J}\right) = n - 1$.

   - This is the number of degrees of freedom for SST

     rank of symmetric matrix in quadratic form = degrees of freedom

## Decomposing SST

Taking the first term of $\text{SST} = \mathbf{Y}'\mathbf{Y} - \frac{1}{n}\mathbf{Y}'\mathbf{J}\mathbf{Y}$,

$$
\begin{aligned}
\mathbf{Y}'\mathbf{Y} &= (\mathbf{Y} - \mathbf{Xb} + \mathbf{Xb})'(\mathbf{Y} - \mathbf{Xb} + \mathbf{Xb}) \\
&= (\mathbf{Y} - \mathbf{Xb})'(\mathbf{Y} - \mathbf{Xb}) + (\mathbf{Y} - \mathbf{Xb})'\mathbf{Xb} + (\mathbf{Xb})'(\mathbf{Y} - \mathbf{Xb}) + (\mathbf{Xb})'(\mathbf{Xb}) \\
&= \hat{\mathbf{e}}'\hat{\mathbf{e}} + \underbrace{\hat{\mathbf{e}}'\mathbf{Xb} + (\mathbf{Xb})'\hat{\mathbf{e}}}_{\text{equal scalars, both } 0} + \mathbf{b}'\mathbf{X}'\mathbf{Xb} \\
&= \hat{\mathbf{e}}'\hat{\mathbf{e}} + \mathbf{b}'\mathbf{X}'\mathbf{Xb}
\end{aligned}
$$

The middle terms were zero because

$$\mathbf{X}'\hat{\mathbf{e}} = \mathbf{X}'(\mathbf{I} - \mathbf{H})\mathbf{Y} = \mathbf{X}'\mathbf{Y} - \mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} = \mathbf{0}$$

So,

$$\text{SST} = \mathbf{Y}'\mathbf{Y} - \frac{1}{n}\mathbf{Y}'\mathbf{J}\mathbf{Y}$$

$$\text{SST} = \underbrace{\hat{\mathbf{e}}'\hat{\mathbf{e}}}_{\text{RSS}} + \underbrace{\mathbf{b}'\mathbf{X}'\mathbf{Xb} - \frac{1}{n}\mathbf{Y}'\mathbf{J}\mathbf{Y}}_{\text{SSReg}}$$

# A closer look at RSS

$$\boxed{\text{SST} = \text{SSReg} + \text{RSS}}$$

Making use of our expression $\hat{\mathbf{e}} = (\mathbf{I} - \mathbf{H})\mathbf{Y}$, we have

$$\begin{aligned} \text{RSS} &= \hat{\mathbf{e}}'\hat{\mathbf{e}} \\ &= \mathbf{Y}'(\mathbf{I} - \mathbf{H})'(\mathbf{I} - \mathbf{H})\mathbf{Y} \\ &= \mathbf{Y}'(\mathbf{I} - \mathbf{H})\mathbf{Y} \end{aligned}$$

This is another quadratic form in $\mathbf{Y}$. Also, $\text{rank}(\mathbf{I} - \mathbf{H}) = n - 2$ from earlier, the number of degrees of freedom for the error.

# A closer look at SSReg

$$\boxed{\text{SST} = \text{SSReg} + \text{RSS}}$$

Making use of $\hat{\beta} = \mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$,

$$\begin{aligned}
\text{SSReg} &= \mathbf{b}'\mathbf{X}'\mathbf{X}\mathbf{b} - \mathbf{Y}'\frac{1}{n}\mathbf{J}\mathbf{Y} \quad \text{from slide 12} \\
&= \mathbf{Y}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\cancel{\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}}\mathbf{X}'\mathbf{Y} - \mathbf{Y}'\frac{1}{n}\mathbf{J}\mathbf{Y} \\
&= \mathbf{Y}'\underbrace{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'}_{\mathbf{H}}\mathbf{Y} - \mathbf{Y}'\frac{1}{n}\mathbf{J}\mathbf{Y} \\
&= \mathbf{Y}'\left(\mathbf{H} - \frac{1}{n}\mathbf{J}\right)\mathbf{Y}
\end{aligned}$$

This is again a quadratic form in $\mathbf{Y}$, since the middle matrix is symmetric. Also, $\text{rank}(\mathbf{H} - \frac{1}{n}\mathbf{J}) = \text{rank}(\mathbf{H}) - \text{rank}\left(\frac{1}{n}\mathbf{J}\right) = 2 - 1 = 1$, the number of degrees of freedom for SSReg.

## Using RSS to estimate $\sigma^2$

In $S^2 = \text{RSS}/(n-2)$, we have an unbiased estimator for $\sigma^2$. We can show it's unbiased using matrices by showing that $\mathrm{E}(\text{RSS}) = (n-2)\sigma^2$ as we did without matrices — i.e. when we considered $\mathrm{E}(\text{RSS}) = \mathrm{E}\left(\sum_{i=1}^{n} \hat{e}_i^2\right)$.

$$
\begin{aligned}
\mathrm{E}(\text{RSS}) &= \mathrm{E}\left(\hat{\mathbf{e}}'\hat{\mathbf{e}}\right) \\
&= \mathrm{E}\left\{\mathbf{Y}'(\mathbf{I}-\mathbf{H})\mathbf{Y}\right\} && \text{from slide 13} \\
&= \mathrm{E}\left\{\text{trace}\left[\mathbf{Y}'(\mathbf{I}-\mathbf{H})\mathbf{Y}\right]\right\} && \text{since } \mathbf{Y}'(\mathbf{I}-\mathbf{H})\mathbf{Y} \text{ is a scalar} \\
&= \mathrm{E}\left\{\text{trace}\left[(\mathbf{I}-\mathbf{H})\mathbf{Y}\mathbf{Y}'\right]\right\} && \text{since trace}(\mathbf{AB}) = \text{trace}(\mathbf{BA}) \\
&= \text{trace}\left[(\mathbf{I}-\mathbf{H})\,\mathrm{E}(\mathbf{Y}\mathbf{Y}')\right] \\
&= \text{trace}\left[(\mathbf{I}-\mathbf{H})\,(\sigma^2\mathbf{I} + \mathbf{X}\beta\beta'\mathbf{X}')\right] && \text{from slide 8} \\
&= \text{trace}\left[(\mathbf{I}-\mathbf{H})\,\sigma^2 + \mathbf{X}\beta\beta'\mathbf{X}' - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\beta\beta'\mathbf{X}'\right] \\
&= \text{trace}(\mathbf{I}-\mathbf{H})\,\sigma^2 \\
\mathrm{E}(\text{RSS}) &= (n-2)\sigma^2 && \text{so unbiased estimator}
\end{aligned}
$$

where we have used $\text{trace}(\mathbf{A}+\mathbf{B}) = \text{trace}(\mathbf{A}) + \text{trace}(\mathbf{B})$, and thus $\text{trace}(\mathbf{I}-\mathbf{H}) = \text{trace}(\mathbf{I}) - \text{trace}(\mathbf{H}) = n - \sum_{i=1}^{n} h_{ii} = n-2$.

# The big picture

We have expressed the ANOVA identity in matrix form:

$$\text{SST} = \text{SSReg} + \text{RSS}$$

$$\underbrace{\mathbf{Y}'\left(\mathbf{I} - \frac{1}{n}\mathbf{J}\right)\mathbf{Y}}_{\text{SST}} = \underbrace{\mathbf{Y}'\left(\mathbf{H} - \frac{1}{n}\mathbf{J}\right)\mathbf{Y}}_{\text{SSReg}} + \underbrace{\mathbf{Y}'\left(\mathbf{I} - \mathbf{H}\right)\mathbf{Y}}_{\text{RSS}}$$

What does R have to say about this?

# ANOVA in R

The anova command is one way in R to produce an ANOVA table (see Week 3, slide 42), in addition to analysing it. For example, for the 654-point SLR problem in Assignment 2, question 1:

```
a2 = read.table("DataA2.txt",sep=" ",header=T) # Load the data set
fev <- a2$fev; age <- a2$age
mod1 = lm(fev~age)
anova(mod1)
```

```
## Analysis of Variance Table
##
## Response: fev
##            Df Sum Sq Mean Sq F value    Pr(>F)
## age         1 280.92 280.919  872.18 < 2.2e-16 ***
## Residuals 652 210.00   0.322
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# ANOVA in R

The $p$-value will match that obtained from the `summary(lm...)` command:

```r
summary(mod1)
```

```
##
## Call:
## lm(formula = fev ~ age)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.57539 -0.34567 -0.04989  0.32124  2.12786
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 0.431648   0.077895   5.541 4.36e-08 ***
## age         0.222041   0.007518  29.533  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.5675 on 652 degrees of freedom
## Multiple R-squared:  0.5722, Adjusted R-squared:  0.5716
## F-statistic: 872.2 on 1 and 652 DF,  p-value: < 2.2e-16
```

same as ANOVA p-value

# This week's lectures

- Poll regarding the Exam Jam on 8 December
- Chapter 5:
  - Matrix version of SLR
  - **Multiple linear regression (MLR)**

# Multiple regression

Multiple regression is used when we have more than one explanatory variable. Multiple $x$'s can arise naturally. In addition, sometimes we want to:

- Control for some $x$'s to consider the effect on $y$ of other $x$'s over and above the control variables
- Fit a polynomial
- Compare the regression line for two or more groups

1. handles more than 1 explanatory variable
2. fitting curvlinear model
3. compare regression lines with multiple groups

In multiple linear regression (MLR), generally we let $p$ represent the number of explanatory variables in the model, i.e.

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_p x_{ip} + e_i$$

for $i \in \{1, \ldots, n\}$. How many parameters do we need to estimate? p+2

And therefore, how many observations do we need at a minimum?

p+2

# Matrix version of MLR

each column is an explanatory variable,
each row is an observation

Our main equation is unchanged: $\mathbf{Y} = \mathbf{X}\beta + \mathbf{e}$

However, the **design matrix** $\mathbf{X}$ and $\beta$ are bigger:

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} 1 & X_{11} & X_{12} & \cdots & X_{1p} \\ 1 & X_{21} & X_{22} & & X_{2p} \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & X_{n1} & X_{n2} & \cdots & X_{np} \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$$

A design matrix gives the explanatory variables (often without the column of 1's). Each row is an observation and each column corresponds to a different kind of variable.

# Gauss-Markov assumptions for MLR

The key equations are unchanged:

$$\mathrm{E}(\mathbf{e}) = \mathbf{0} \quad \text{and} \quad \mathrm{var}(\mathbf{e}) = \sigma^2 \mathbf{I}$$

For our inference methods (CIs etc), we need $\mathbf{e}$ to have a multivariate normal distribution as before.

The expression for residuals is still $\widehat{\mathbf{e}} = \mathbf{Y} - \widehat{\mathbf{Y}} = \mathbf{Y} - \mathbf{X}\mathbf{b} = (\mathbf{I} - \mathbf{H})\mathbf{Y}$, where now we have

$$\mathbf{b} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_p \end{pmatrix}$$

# Estimating $\sigma^2$ in MLR

Recall that

$$S^2 = \text{MSE} = \frac{\sum_{i=1}^{n} \hat{e}_i^2}{\text{d.f. of error}} = \frac{\widehat{\mathbf{e}}'\widehat{\mathbf{e}}}{\text{d.f. of error}}$$

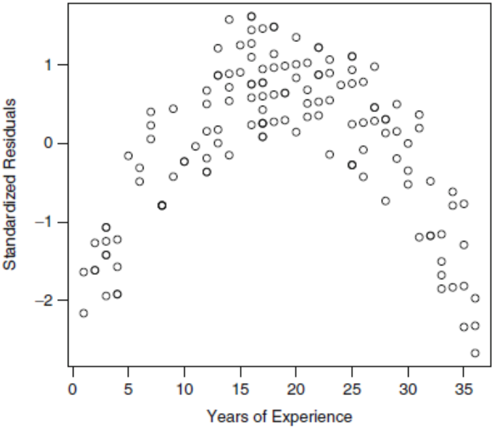
The degrees of freedom was $n-2$ in SLR, and is $n-p-1$ in MLR. To see this, recall that $\text{RSS} = \widehat{\mathbf{e}}'\widehat{\mathbf{e}} = \mathbf{Y}'(\mathbf{I}-\mathbf{H})\mathbf{Y}$. Using our five key properties of idempotent matrices again, $\text{rank}(\mathbf{I}-\mathbf{H}) = \text{rank}(\mathbf{I}) - \text{rank}(\mathbf{H}) = n-(p+1)$ assuming that the columns of $X$ are linearly independent.

To show that $S^2$ is unbiased in MLR, similar to before we can show $\mathrm{E}(\text{RSS}) = (n-p-1)\sigma^2$. The proof is akin to the SLR proof except that:

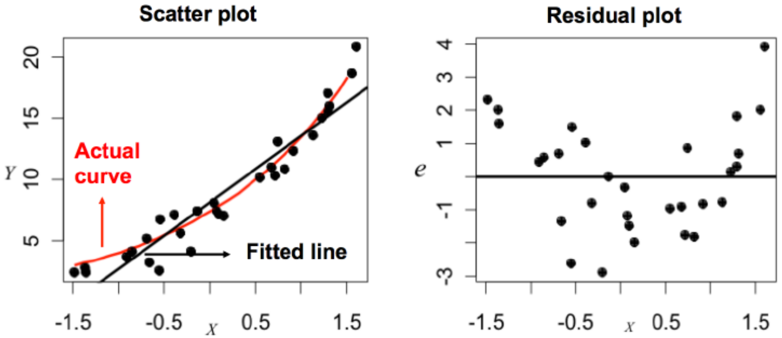
$$\begin{aligned}
\text{trace}(\mathbf{I}-\mathbf{H}) &= \text{trace}(\mathbf{I}) - \text{trace}(\mathbf{H}) \\
&= n - \text{trace}\left[\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\right] \\
&= n - \text{trace}\left[\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\right] \\
&= n - \text{trace}(\mathbf{I}_{p+1}) \\
&= n - (p+1)
\end{aligned}$$

# Example of MLR: Fitting a polynomial

A professional-salary database contains 143 ordered pairs: (years of experience, salary). Generally, but not monotonically, salary increases with years of experience. Using SLR, our model is $Y_i = \beta_0 + \beta_1 x_i + e_i$. After fitting a straight line, this is the plot of standardized residuals:



# Example of a nonlinear relationship *(Weeks 4–5, Slide 41)*

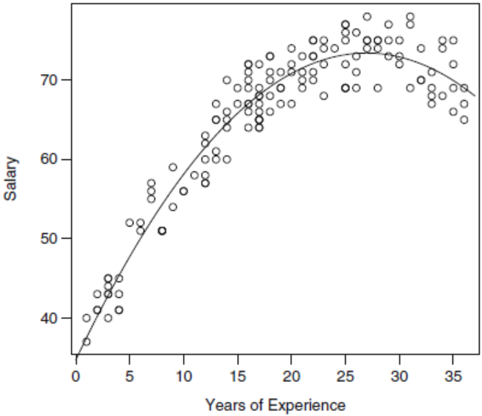



Remedial measure: If the regression function isn't linear,

- In some cases, a variable transformation can make the data "more linear"
- Otherwise, a different (e.g. nonlinear) model might be better

## Back to our salary database

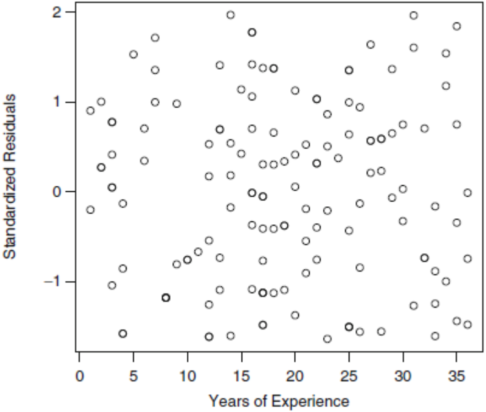
A simple nonlinear model is MLR in which we fit a parabola, i.e. incorporate $x$ and $x^2$. The model is $Y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + e_i$ and the plot is:




Here, $\beta_0 \approx 35$, $\beta_1 \approx 2.87$, and $\beta_2 \approx -0.053$, each with $p < 2 \times 10^{-16}$.

# MLR example: fitting a polynomial

The residuals no longer have a pattern:



# R code for MLR

person#, salary, experience

```
X <- read.csv("profsalary.txt",sep="\t")
mod1 <- lm(Salary ~ Experience + I(Experience^2), data=X)
summary(mod1)
```

Typing `I(.)` is a way to express formulae within a call to `lm`.

The + sign indicates that more than one explanatory variable is being used. To have four variables, use e.g. `y ~ x1 + x2 + x3 + x4`

# R output for MLR

```
##
## Call:
## lm(formula = Salary ~ Experience + I(Experience^2), data = X)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -4.5786 -2.3573  0.0957  2.0171  5.5176
##
## Coefficients:
##                  Estimate Std. Error t value Pr(>|t|)
## (Intercept)     34.720498   0.828724   41.90   <2e-16 ***
## Experience       2.872275   0.095697   30.01   <2e-16 ***
## I(Experience^2) -0.053316   0.002477  -21.53   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.817 on 140 degrees of freedom
## Multiple R-squared:  0.9247, Adjusted R-squared:  0.9236
## F-statistic: 859.3 on 2 and 140 DF,  p-value: < 2.2e-16
```

# Using the R model

Interpolate at 5 years of experience:

```
e <- 5; mod1$coefficients%*%c(1,e,e^2)
```

matrix multiplication

```
##          [,1]
## [1,] 47.74897
```

Alternatively, use the `predict` command:

```
predict(mod1,data.frame(Experience=5))
```

```
##        1
## 47.74897
```

The data frame passed to `predict` names and initializes all of the information used towards making the predictor variables. Another example would be:

```
predict(lm(y~x1+x2),data.frame(x1=5,x2=3))
```

# Interpreting MLR coefficients

How should we interpret $\beta_j$, or similarly their estimates $b_j$ — i.e. what's the meaning of the coefficients of MLR predictor variables?

In general, $\beta_j$ is the change in the mean value of $Y$ associated with a one-unit change in the predictor variable $x_j$, with *all other variables held constant*.

For our salary database example, this is impossible. The closest interpretations we can make are of this sort: x cant change when x^2 held constant

- If Experience increases from 5 years to 6 years, the estimated change in mean Salary is $2.87 - 0.053(36 - 25) \approx 2.3$
- If Experience increases from 35 years to 36 years, the estimated change in mean Salary is $2.87 - 0.053(36^2 - 35^2) \approx -0.9$

# Do we need a polynomial fit?

We can quantify whether the quadratic term is 0 or not using familiar hypothesis testing:

$$H_0 : \beta_2 = 0 \quad \text{vs} \quad H_a : \beta_2 \neq 0$$

**Exercise:** Try this on the salary database. What do you find?

quantify if MLR is valid

# Do we need the $j$th predictor?

dropping some predictor variables, the betas' generally go from significant to not significant

In general, a test of $H_0 : \beta_j = 0$ gives an indication of whether or not the $j$th predictor variable statistically significantly contributes to the estimation/prediction of $Y$ *over and above* the other predictor variables.

That is, the test assumes that the other variables are in the model.

# Next steps

- Solutions to **HW2** were posted during the Study Break
- Complete Chapter 5's **question 1**
- The lecture on Tuesday **21 November** will start at 11:10 am (not 10:10 am) and finish at the usual time. Sections 1 and 2 will then be resynchronized
- **Further ahead:**
  - In Chapter 6, we won't cover Marginal Model Plots, Inverse Response Plots, or Box-Cox transformations
  - In Chapter 7, mainly we'll cover §7.2.3 and p. 252