Problem Set 6

You are strongly encouraged to solve the following exercises before next week's tutorial: Starting on page 362 (end of Chapter 9): 8, 12, 13 (a-c), 17, 21, 28, 32 (c-e).

Additional Exercises:

1. Let $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$, where σ^2 is known.

   Find a UMP test at level α to test $\mathcal{H}_0 : \mu \geqslant \mu_0$ vs. $\mathcal{H}_1 : \mu < \mu_0$ (a left-tailed alternative), explain why it is UMP and plot its power $\pi(\mu^*)$.

2. (a) Use the approximation to the power function of the two-tailed test for the Normal mean with known variance (slide 34 of Lecture 7) to derive an expression for the minimum sample size required for the test at level α to have a Type II error probability of at most β, for a given *effect size* $\delta = |\mu - \mu_0|$.

   (b) Fined the smallest n to ensure that when testing $\mathcal{H}_0 : \mu = 175$ vs. $\mathcal{H}_1 : \mu \neq 175$ at the 5% level, we will reject the null hypothesis 90% of the time when the true mean deviates from 175 by 2, knowing that $\sigma^2 = 25$.

Solutions:

1. Repeat everything that we did in class to find the right-tailed UMP, starting from Lecture 6. The test is based on the rejection region

$$\mathcal{C} = \left\{ \overline{X} < \mu_0 - \frac{\sigma}{\sqrt{n}} z_{1-\alpha} \right\}.$$

2. (a) In class we showed that for large n and small α,

$$\pi(\mu) \approx 1 - \Phi\left(-\frac{\sqrt{n}(\mu^* - \mu_0)}{\sigma} + z_{1-\alpha/2} \right)$$

hence

$$\pi(\mu) \geqslant 1-\beta \Longrightarrow \Phi\left(-\frac{\sqrt{n}(\mu-\mu_0)}{\sigma} + z_{1-\alpha/2}\right) \leqslant \beta$$

$$\Longrightarrow -\frac{\sqrt{n}(\mu-\mu_0)}{\sigma} + z_{1-\alpha/2} \leqslant z_\beta = -z_{1-\beta}$$

$$\Longrightarrow n \geqslant \left\{\frac{\sigma(z_{1-\alpha/2} + z_{1-\beta})}{\mu_1 - \mu_0}\right\}^2.$$

(b) Simply substitute $\alpha = 0.05$, $\beta = 0.1$, $\sigma = 5$ and $\delta = |\mu - \mu_0| = 2$, to obtain

$$n \geqslant \left\{\frac{5(z_{0.975} + z_{0.9})}{2}\right\}^2 = \left\{\frac{5(1.96 + 1.645)}{2}\right\}^2 = 81.23,$$

thus we will need $n \geqslant 82$.