

1 Structures in \mathbb{R}^n

Definition 1.1. A **Vector Space** is a collection of objects called vectors, which may be added together and multiplied ("scaled") by numbers, called scalars in this context. The set V and the operations of addition and multiplication must adhere to a number of requirements called axioms. let u, v and w be arbitrary vectors in V , and a and b scalars in F . First of all $u + v \in V$ and $au \in V$ and

$$u + (v + w) = (u + v) + w \quad (1)$$

$$u + v = v + u \quad (2)$$

$$\exists 0 \in V, v + 0 = v \forall v \in V \quad (3)$$

$$\forall v, \exists -v, v + (-v) = 0 \quad (4)$$

$$a(bv) = (ab)v \quad (5)$$

$$1v = v \quad (6)$$

$$a(u + v) = au + av \quad (7)$$

$$(a + b)v = av + bv \quad (8)$$

Definition 1.2. The **Euclidean inner product**, or dot product given two vectors $x = (x_1, \dots, x_n)$, and $y = (y_1, \dots, y_n)$ in \mathbb{R}^n , which are two equal-length sequences of numbers, and returns a single number.

Algebraically, it is the sum of the products of the corresponding entries of the two sequences of numbers.

$$\langle x, y \rangle = x \cdot y := \sum_{i=1}^n x_i y_i = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

Geometrically, it is the product of the Euclidean magnitudes of the two vectors and the cosine of the angle between them.

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos(\theta)$$

where $\|x\|$ is the length of x , $\|y\|$ is the length of y , θ is the angle between x, y .

Note.

When $a \cdot b = 0$, a and b are **orthogonal**.

When $a \cdot b = \|a\| \|b\|$, a and b are **co-directional**.

Here is a list of properties of the **inner product space**. Given $a, b, c \in V$ and $r \in \mathbb{R}$

$a \cdot b = b \cdot a$	(Commutative)
$a \cdot (b + c) = a \cdot b + a \cdot c$	(Distributive over vector addition)
$a \cdot (rb + c) = r(a \cdot b) + (a \cdot c)$	(Bilinear)
$(c_1 a) \cdot (c_2 b) = c_1 c_2 (a \cdot b)$	(Scalar multiplication)
two non-zero vectors are orthogonal $\iff a \cdot b = 0$	(Orthogonality)
$a \cdot b = a \cdot c$ does not imply $b = c$	(No cancellation)
$a \cdot b \geq 0$ and is equal to zero if and only if $x = 0$	(Non-negative)

Remark. Mostly proved by using algebraic definition of inner dot product

Definition 1.3. The scalar projection is

$$\text{comp}_b a = \frac{a \cdot b}{\|b\|}$$

Proof. $b \cdot (a - b) = 0$ because they are orthogonal to each other. Arrange and with Bilinear property for inner dot product we arrive at $\text{comp}_b a = \frac{a \cdot b}{\|b\|}$ \square

Then the projection of a into b is the scalar projection multiply by the unit vector $\frac{b}{\|b\|}$

$$\text{proj}_b a = \frac{\langle a, b \rangle}{\|b\|^2} b$$

Definition 1.4. Norm is a way of measuring the length of a vector. Here we define $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}$ as the function

$$\|x\| := \sqrt{\langle x, x \rangle} = \left(\sum_{i=1}^n x_i^2 \right)^{1/2} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Proof. Use the algebraic definition of inner product $\mathbf{x} \cdot x = \|x\| \|x\| \cos(0) = \|x\|^2$ \square

The **normed space** has the following properties. Let $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

$\ x\ \geq 0$ with equality if and only if $x = 0$	(Non-degeneracy)
$\ cx\ = c \ x\ $	(Normality)
$\ x + y\ \leq \ x\ + \ y\ $	(Triangle Inequality)
$ \langle x, y \rangle \leq \ x\ \ y\ $	(Cauchy Schwarz Inequality)

Remark. proofs for Cauchy Schwarz Inequality can be derived from geometric definition of inner dot product on the condition that $\cos(x) \leq 1$. Proofs for Triangle Inequality requires Cauchy Schwarz Inequality.

Definition 1.5. Metric is a method for determining the distance between the two vectors.

$$d(x, y) = \|x - y\| = \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2} = \sqrt{(x_1 - x_2)^2 + \cdots + (x_n - y_n)^2}$$

The metric space satisfies the following properties. Let $x, y, z \in \mathbb{R}^n$

$$d(x, y) = d(y, x) \quad (\text{Symmetry})$$

$$d(x, y) \geq 0 \text{ with equality if and only if } x = y \quad (\text{Non-degeneracy})$$

$$d(x, z) \leq d(x, y) + d(y, z) \quad (\text{Triangle Inequality})$$

Definition 1.6. In \mathbb{R}^3 the cross product of two vectors is a way of determining a third vector which is orthogonal to the original two. If $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ then

$$v \times w = (v_2 w_3 - w_2 v_3, w_1 v_3 - v_1 w_3, v_1 w_2 - w_1 v_2)$$

or we can use determinants to solve $v \times w$. Here i, j, k represent standard unit vectors in \mathbb{R}^3

$$v \times w = \det \begin{pmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = i \begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix} - j \begin{pmatrix} v_1 & v_3 \\ w_1 & w_3 \end{pmatrix} + k \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}$$

Note. Determinants of 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad - bc$