Differential Calculus

Derivatives

Definition 0.1. one variable differentiability A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$ if there exists an $m \in \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - mh}{h} = 0$$

where $m = f'(a)$.

Remark.
The idea is that f is differentiable at a if it can be well-approximated by a linear function m,

$$f(a+h) = f(a) + mh + error(h)$$

such that the error go to zero faster than linearly in h.

$$\lim_{h \to 0} \frac{error(h)}{h} = 0$$

Also we can calculate derivative by evaluating

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Note. Example of function continuous but not differentiable at 0

$$f(x) = \begin{cases} x \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Example of differentiable function whose derivative is not continous

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Definition 0.2. Differentiability of vector valued function A function $\gamma : \mathbb{R} \to \mathbb{R}^n$ is differentiable if at t_0,

$$\begin{aligned} \gamma'(t_0) &= \lim_{h \to 0} \frac{\gamma(t_0+h) - \gamma(t_0)}{h} \\ &= \big(\lim_{h \to 0} \frac{\gamma_1(t_0+h) - \gamma_1(t_0)}{h}, \ldots, \lim_{h \to 0} \frac{\gamma_2(t_0+h) - \gamma_2(t_0)}{h} \big) \end{aligned}$$

exists. γ is differentiable if all of its component functions are differentiable.

Proposition 0.2.1. Properties of vector valued function Let $f, g : \mathbb{R} \to \mathbb{R}^n$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be differentiable functions.

1. $(\varphi f)' = \varphi' f + \varphi f'$
2. $(f \cdot g)' = f' \cdot g + f \cdot g'$
3. $(f \times g)' = f' \times g + f \times g'$ (if $n = 3$)

Definition 0.3. Multivariable differentiability A function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}^n$ if there exists $c \in \mathbb{R}^n$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - c \cdot h}{||h||} = 0$$

where c if exists is called the **gradient** of f, denoted as $\nabla f(a)$

Theorem 0.1. *If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a then f is continous at a.*

Proof.

$$\begin{aligned}
\lim_{x \to a} f(a) &= \lim_{h \to 0} f(a+h) - f(a) \\
&= \lim_{h \to 0} [f(a+h) - f(a) - \nabla f(a) \cdot h] + \nabla f(a) \cdot h \\
&= \lim_{h \to 0} f(a+h) - f(a) - \nabla f(a) \cdot h + \lim_{h \to 0} \nabla f(a) \cdot h \\
&= 0 + 0 = 0
\end{aligned}$$

□

Definition 0.4. If $f : \mathbb{R}^n \to \mathbb{R}$. we define **partial derivatives** of f with respect to x_i at $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ as

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(a_1, \ldots a_1 + h, \ldots a_n) - f(a_1, \ldots, a_n)}{h}$$

That is $\dfrac{\partial f}{\partial x_i}$ is the one varirable derivative of $f(x_1, \ldots, x_n)$ with respect to x_i where all other variables are held constant.

Theorem 0.2. *If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a then the partials of f exist at a and*

$$\nabla f(a) = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$$

Remark.
Example of function where **partials exist** but function **not differentiable**. This is reasonable bcause partials only measure differentiability in finitely many directions that the converse direction does not hold.

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Proof. Function is not continous at $(x,y) = (0,0)$ (prove this by taking a path and show limit is depends on the path) and therefore not differentiable. However partials exists at $(0,0)$ by the limit definition.

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

Note this could be explained by the fact that partials of f near zero is not continous

$$\frac{\partial f}{\partial x} = \frac{y^3 - x^2 y}{(x^2+y^2)^2}$$

Partials does not exist as $(x,y) \to (0,0)$ $\square$

Definition 0.5. **Continously differentiable** functions is in the collection of C^1 function on U,

$$C^1(\mathbb{R}^n, \mathbb{R}) = \left\{ f : \mathbb{R}^n \to \mathbb{R} : \partial_i f \text{ exists and is continous for } i \in (1, \dots, n) \right\}$$

Theorem 0.3. ***C^1 functions are differentiable*** *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$, If $\partial_i f(x)$ all exists and are continous in an open neighborhood of a, then f is differentiable at a*

Remark. Example of function differentiable but not C^1

$$f(x,y) = \begin{cases} (x^2+y^2)\sin(\frac{1}{\sqrt{x^2+y^2}}), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

We can see that that derivative not continous at 0.

Definition 0.6. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$. If $u \in \mathbb{R}^n$ is a unit vector ($||u|| = 1$) then the **directional derivative** of f in the direction of u at a is

$$\partial_u f(a) = \lim_{t \to 0} \frac{f(a+tu) - f(a)}{t} = \frac{d}{dt}|_{t=0} f(a+tu)$$

Theorem 0.4. *If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a, then for any unit vector u, $\partial_u f$ exists. Moreover,*

$$\partial_u f(a) = \nabla f(a) \cdot u$$

Remark. To ways to compute partial derivatives.

1. compute using limit definition
2. compute partials first and then $\partial_u f(a) = \nabla f(a) \cdot u$

Definition 0.7. Generalized differentiability A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$ if there exists an $m \times n$ matrix A such that

$$\lim_{h \to 0} \frac{||f(a+h) - f(a) - Ah||_{\mathbb{R}^m}}{||h||_{\mathbb{R}^2}} = 0$$

Here $Df(a) = A$, the **Jacobian Matrix**

Proposition 0.7.1. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is given by $f(x) = (f_1(x), \dots, f_m(x))$, then f is differentiable if and only if each of the $f_i : \mathbb{R}^n \to \mathbb{R}$ is differentiable, that is

$$Df(a) = \begin{bmatrix} \nabla f_1(a) \\ \nabla f_2(a) \\ \vdots \\ \nabla f_m(a) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Theorem 0.5. ***Chain Rule*** *Let $g : \mathbb{R}^k \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$. If g is differentiable at $a \in \mathbb{R}^k$ and f is differentiable at $g(a) \in \mathbb{R}^n$, then $f \circ g$ is differentiable at a, and*

$$D(f \circ g)(a) = Df(g(a))Dg(a)$$

Remark. Note that the gradient of a function $\mathbb{R}^n \to \mathbb{R}$ is a row vector and the derivative of a function $\mathbb{R} \to \mathbb{R}^n$ is a column vector.
special case 1, When $g : \mathbb{R} \to \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$, so $f \circ g : \mathbb{R} \to \mathbb{R}$. Let $y = f(x)$ and let $(x_1, \dots, x_n) = g(t) = (g_1(t), \dots, g_n(t))$ so,

$$\frac{d}{dt}(f \circ g) = \frac{\partial y}{\partial x_1}\frac{\partial x_1}{\partial t} + \cdots + \frac{\partial y}{\partial x_n}\frac{\partial x_n}{\partial t}$$

special case 2, When $g : \mathbb{R}^n \to \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}$ so that $f \circ g : \mathbb{R}^n \to \mathbb{R}$. if $y = f(x)$ and $x = g(t)$ then

$$\frac{\partial}{\partial t_i}(f \circ g)(x) = \frac{\partial y}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \cdots + \frac{\partial y}{\partial x_m}\frac{\partial x_m}{\partial t_i}$$

Another way of putting is

$$\partial_i(f \circ g)(a) = \nabla f(g(a))Dg(a) = \nabla f(g(a)) \begin{pmatrix} \nabla g_i(a) \\ \vdots \\ \nabla g_m(a) \end{pmatrix} = \sum_{j=1}^{m} \partial_j f(g(a)) \cdot \partial_i g_j(a)$$

where $1 \leq j \leq m$ and $q \leq i \leq n$ and g_i is i-th component function of g
In summary we compute derivatives either with direct substitution or with the chain rule, where we compute jacobian matrix and compose them.

Definition 0.8. Some properties of multivariate differentiable function

1. If f is a constant function ($\exists y \in \mathbb{R}^m, f(x) = y$ for all $x \in \mathbb{R}^n$) then $Df(a) = T_o$ where $T_o = \vec{0}$

2. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $Df(a) = f$, i.e. the derivative is itself.

   Proof. Since f differentible, error approach 0 as $h \to 0$

   $$0 = error(h) = f(a+h) - f(a) - Ah = f(a) + f(h) - f(a) - Ah \Rightarrow f(h) = A(h)$$

   Meaning that the linear map $Df = A = f$ □

   As an example, If $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x + y = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x & y \end{smallmatrix}\right)$, i.e. f is linear, then $Df(a) = s$

   Proof.

   $$\lim_{h \to 0} \frac{f(a+h) - f(a) - \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} h_1 & h_2 \end{smallmatrix}\right)}{||h||} = 0$$

   □

3. $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a if and only if f_i, the i-th component function, is differentiable at a

4. $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto xy$, then $Df(a) : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto a_2 x + a_1 y$

   Theorem 0.6. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$, then*

1. ***Sum Rule:***

$$D(f + g)(a) = Df(a) + Dg(a)$$

2. ***Product Rule:***

$$D(f \cdot g)(a) = f(a)Dg(a) + g(a)Df(a)$$

Proof. Let s represent the summation and

$$\begin{aligned} D(f+g)(a) &= D(s \circ (f,g))(a) \\ &\overset{\text{chain rule}}{=} Ds(f(a), g(a)) \circ D(f,g)(a) \\ &= Ds(f(a), g(a)) \circ (Df(a), Dg(a)) = Df(a) + Dg(a) \end{aligned}$$

□

3. ***Quotient Rule:***

$$D(\frac{f}{g})(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{g(a)^2} \text{ if } g(a) \neq 0$$

Theorem 0.7. ***Mean Value Theorem for One Variable*** *In one variable, if $f : [a,b] \to \mathbb{R}$ is continous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that*

$$f(b) - f(a) = f'(c)(b-a)$$

Corollary 0.7.1. *A short list of propositions*

1. *There is a point such that the tangent line has the same slope as the secant between $(a, f(a))$ and $(b, f(b))$*
2. *If $f : [a,b] \to \mathbb{R}$ is differentiable with bounded derivative, say $f'(x) \leq M$ for all $x, y \in [a,b]$, then $|f(y) - f(x)| \leq M|y-x|$*
3. *If $f'(x) \equiv 0$ for all $x \in [a,b]$ then f is the constant function on $[a,b]$*
4. *If $f'(x) > 0$ for all $x \in [a,b]$ then f is an increasing (and hence injective) function*

Theorem 0.8. ***Mean Value Theorem for Multivariate Functions*** *Let $U \subseteq \mathbb{R}^n$ and let $a, b \in U$ be such that the straight line connecting them lives entirely within U. More precisely, the curve $\gamma : [0,1] \to \mathbb{R}^n$ given by $\gamma(t) = (1-t)a + tb$ satisfies $\gamma(t) \in U$ for all $t \in [0,1]$. If $f : U \to \mathbb{R}$ is a function such that $f \circ \gamma$ is continous on $[0,1]$ and differentiable on $(0,1)$, then there exists a $t_0 \in (0,1)$ such that $c = \gamma(t_0)$ and*

$$f(b) - f(a) = \nabla f(c) \cdot (b-a)$$

Corollary 0.8.1. *If $U \subseteq \mathbb{R}^n$ is convex and $f : U \to \mathbb{R}$ is a differentiable function such that $|\nabla f(x)| \leq M$ for all $x \in U$, then for every $a, b \in U$, we have*

$$|f(b) - f(a)| \leq M|b-a|$$

Corollary 0.8.2. *If $U \subseteq \mathbb{R}^n$ is convex and $f : U \to \mathbb{R}$ is a differentiable function such that $\nabla f(x) = 0$ for all $x \in U$, then f is a constant function on U*