Basics

1. **Functions**
   - **injective** $f : X \to Y$ injective or one-to-one if $\forall a, b \in X,\ f(a) = f(b) \Rightarrow a = b$
   - **surjective** $f : X \to Y$ is surjective if any $y \in Y \Rightarrow y = f(x)$ for some $x \in X$
   - composition of injective/surjective/bijective functions are injective/surjective/bijective
   - If f is injective with range Y, then its inverse function $f^{-1} : Y \to X$ is a bijective function
2. **Set Relations**
   - **De Morgan's Law**

$$X \setminus \left(\bigcup_{\alpha\in I} A_\alpha\right) = \bigcap_{\alpha\in I} (X \setminus A_\alpha) \qquad X \setminus \left(\bigcap_{\alpha\in I} A_\alpha\right) = \bigcup_{\alpha\in I} (X \setminus A_\alpha)$$

   - **how functions acts on sets** Let $A, B \subseteq X$ and $C, D \subseteq Y$
     - f well-behaved for union
$$f(A \cup B) = f(A) \cup f(B)$$
     - f not well-behaved for intersection, difference
$$\begin{aligned} f(A \cap B) &\subseteq f(A) \cap f(B) \\ f(A) \setminus f(B) &\subseteq f(A \setminus B) \end{aligned}$$
     - f^{-1} well-behaved with union, intersection, difference, and set complement
$$\begin{aligned} f^{-1}(C \cup D) &= f^{-1}(C) \cup f^{-1}(D) \\ f^{-1}(C \cap D) &= f^{-1}(C) \cap f^{-1}(D) \\ f^{-1}(C \setminus D) &= f^{-1}(C) \setminus f^{-1}(D) \\ f^{-1}(Y \setminus C) &= X \setminus f^{-1}(C) \end{aligned}$$
     - f and f^{-1} mixed
$$A \subseteq f^{-1}(f(A)) \qquad \text{(with equality if } f \text{ is injective)}$$
$$f(f^{-1}(C)) \subseteq C \qquad \text{(with equality if } f \text{ surjective)}$$

Remark. Read

1. chapter 1, 1-8
2. chapter 2, 12-18

chapter 1

Relation

1. A relation on a set A is a subset C of cartesian product $A \times A$. xCy means $(x, y) \in C$
2. **equivalence relation** is a relation if it satisfies reflexivity, symmetry, transitivity
3. **equivalence class** a subset of A determined by some $x \in A$, i.e. $E = \{y \mid y \sim x\}$
4. **partition** of a set A is a collection of disjoint nonempty subsets of A whose union is all of A
5. **order relation** is a relation if it satisfies comparability (any $x \neq y \in A$ either xCy or yCx but not both) nonreflexivity (xCx does not hold for any $x \in A$) and transitivity
6. **dictionary order relation** Let A, B bet sets and $<_A$ and $<_B$ be order relations. The order relation on $A \times B$ is defined by $a_1 <_A a_2$ or if $a_1 = a_2$ and $b_1 <_B b_2$
7. **least upper bound property** An ordered set A has the property if every nonempty subset A_0 of A that is bounded above ($\exists b \in A$ s.t. $x \leq b$ for all $x \in A_0$) has a least upper bound (all bounds of A_0 has a smallest element)
   - $\mathbb{R}$ and $(-1, 1)$ has least upper bounde property
   - $B = (-1, 0) \cup (0, 1)$ does not heave least upper bound property, $\{-{}^1/_{2n} \mid n \in \mathbb{Z}_+\}$ is bounded above by any $b \in (0, 1)$ but its least upper bound $0 \notin B$

Cartesian Product

1. **indexed family of sets** Let $\mathcal{A}$ be nonempty collection of sets, let $f : J \to \mathcal{A}$ be a surjective indexing function. $(\mathcal{A}, f)$ is called indexed family of sets, denoted by $\{\mathcal{A}_\alpha\}_{\alpha \in J} = \{\mathcal{A}_\alpha\}$ where $f(\alpha) = \mathcal{A}_\alpha$
2. **m-tuple** Let $m \in \mathbb{Z}_+$, Given a set X, define m-tuple of X to be a function $\mathbf{x} : \{1, \cdots, m\} \to X$ and denote $\mathbf{x} = (x_1, \cdots, x_m)$
3. **cartesian product** Let $\mathcal{A} = \{A_1, \cdots, A_m\}$ be indexed family of sets, let $X = \cup_{i=1}^m A_i$. Then cartesian product of $\mathcal{A}$ is
$$X^m = \prod_{i=1}^m A_i \qquad A_1 \times \cdots \times A_m$$
to be the set of all m-tuples $\mathbf{x}$ of elements of X such that $x_i \in A_i$ for each i
4. **ω-tuple** Given a set X, define ω-tuple of elements of X be a function $\mathbf{x} : \mathbb{Z}_+ \to X$. $\mathbf{x}$ is an *infinite sequence*, of elements of X. Denote $x_i = \mathbf{x}(i)$ as i-th coordinate of $\mathbf{x}$. Denote $\mathbf{x}$ itself by $(x_1, x_2, \cdots)$ or $(x_n)_{n \in \mathbb{Z}_+}$
5. **cartesian product (infinite)** Let $\mathcal{A} = \{A_1, A_2, \cdots\}$ be indexed family of sets and X be union of sets in $\mathcal{A}$, the cartesian product of $\mathcal{A}$
$$X^\omega = \prod_{i \in \mathbb{Z}_+} A_i \qquad A_1 \times A_2 \times \cdots$$
is defined to be the set of all ω-tuples $(x_1, x_2, \cdots)$ of elements of X such that $x_i \in A_i$ for each i

chapter 2

Topological Spaces

Definition. *(**Topology**) Topology on a set X is a collection $\mathcal{T}$ of subsets of X having properties*

1. $\emptyset, X \in \mathcal{T}$
2. *Arbitrary union of subcollection of $\mathcal{T}$ is in $\mathcal{T}$ (If $\forall \alpha \in I$, $U_\alpha \in \mathcal{T}$, then $\cap_{\alpha \in I} U_\alpha \in \mathcal{T}$)*
3. *Finite intersection of subcollection of $\mathcal{T}$ is in $\mathcal{T}$ (If $\forall 1 \le i \le n$, $U_i \in \mathcal{T}$, then $\cap_{i=1}^n U_i \in \mathcal{T}$)*

A topological space is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ are the open sets.

- *Standard topology on $\mathbb{R}^n$ is $\mathcal{T}_0 = \mathcal{T}_{std} = \{U \subset \mathbb{R}^n \mid \forall x \in U, \exists \epsilon > 0\, B_\epsilon(x) \subset U\}$*
- *Standard topology on $\mathbb{R}$ is generated by $\mathcal{B}_{std} = \{(a,b) = \{x \mid a < x < b\}$ the open intervals*
- *Lower limit topology on $\mathbb{R}$ is generated by $\mathcal{B}_{l.l.} = \{x \mid a \le x < b\}$ the half-open intervals*
- *Discrete topology $\mathcal{T}_1 = \mathcal{T}_{disc} = \mathcal{P}(X)$ all subsets are open*
- *Trivial topology $\mathcal{T}_2 = \mathcal{T}_{triv} = \{\emptyset, X\}$ only empty set and X are open*
- *Finite complement topology $\mathcal{T}_{f.c.} = \{U \subseteq X \mid X - U$ is finite or all of $X\}$*
- *Countable complement topology $\mathcal{T}_c = \{U \subseteq X \mid X - U$ is countable or all of $X\}$*

Lemma. *(**Arbitrary intersection of topologies is a topology**) $\forall \alpha \in I$ $\mathcal{T}_\alpha$ is a topology, so is $\cap_{\alpha \in I} \mathcal{T}_\alpha$*

Definition. *(**Compare topology**) If $\mathcal{T}' \subset \mathcal{T}$, then $\mathcal{T}'$ is coarser / weaker / smaller than $\mathcal{T}$, $\mathcal{T}$ is finer / stronger / larger than $\mathcal{T}'$. $\mathcal{T}$ and $\mathcal{T}'$ are comparable if $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$*

$$\mathcal{T}_{triv} \subset \mathcal{T}_{f.c.} \subset \mathcal{T}_{std} \subset \mathcal{T}_{disc} \qquad \mathcal{T}_{std} \subset \mathcal{T}_{l.l.}$$

Basis for a Topology

A terser representation of $\mathcal{T}$

Definition. *(**Basis**) If X is a set, a basis for $(X, \mathcal{T})$ is a collection $\mathcal{B}$ (basis elements) of subsets of X s.t.*

1. *For each $x \in X$, exists at least one basis element $B \in \mathcal{B}$ containing it*
2. *If $x \in B_1 \cap B_2$, then exists $B_3 \in \mathcal{B}$ s.t. $x \in B_3 \in B_1 \cap B_2$*

*A topology $\mathcal{T}_\mathcal{B}$ **generated by** $\mathcal{B}$ is defined as*

$$\mathcal{T}_\mathcal{B} = \{U \subset X \mid \forall x \in U,\ \exists B \in \mathcal{B},\ x \in B \subset U\}$$

- *$\mathcal{T}_\mathcal{B}$ is the unique minimal topology containing $\mathcal{B}$*

  $$\mathcal{T}_\mathcal{B} = \bigcap_{\mathcal{T} \in \mathbb{T}} \mathcal{T}$$

  where $\mathbb{T} = \{\mathcal{T} \mid \mathcal{T} \supset \mathcal{B}$ and $\mathcal{T}$ is a topology$\}$
- *For any X, all one point sets of X is a basis for $\mathcal{T}_{disc}$*

Lemma. *($\mathcal{B} \to \mathcal{T}_\mathcal{B}$) $\mathcal{T}_\mathcal{B}$ equals the collections of all unions of elements of $\mathcal{B}$*

$$\mathcal{T}_\mathcal{B} = \left\{\bigcup_{\alpha \in I} B_\alpha \mid B_\alpha \in \mathcal{B} \quad \forall \alpha \in I\right\}$$

Lemma. *($\mathcal{T}_\mathcal{B} \to \mathcal{B}$) Let $(X, \mathcal{T})$ be topological space. Let $\mathcal{C}$ be a collection of open sets of X such that*

$$\forall U \subset X, \quad \forall x \in U, \quad \exists C \in \mathcal{C} \text{ s.t. } x \in C \subset U$$

Then $\mathcal{C}$ is a basis for $\mathcal{T}$ (handy in deciding $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for $Y \subset X$)

Lemma. *(**compare topology by basis**) Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then following equivalent*

1. *$\mathcal{T}' \supset \mathcal{T}$ ($\mathcal{T}'$ is finer than $\mathcal{T}$)*
2. *For each $x \in X$ and each $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset \mathcal{B}$*

Order Topology

Definition. *(**Order Topology**) Let X be a set with simple order relation. Let $\mathcal{B}$ be a collection of all sets of the following type*

$$\begin{aligned}\mathcal{B} = &\{(a,b) \mid a < b \quad a,b \in X\}\\ &\bigcup\{[a_0,b) \mid a_0 \text{ is minimal element (if any) of } X \ b \in X \quad b \neq a_0\}\\ &\bigcup\{(a,b_0] \mid b_0 \text{ is maximal element (if any) of } X \ a \in X \quad a \neq b_0\}\end{aligned}$$

Generated topology $\mathcal{T}_{\mathcal{B}}$ is called order topology

- *In $\mathbb{R}$, $\mathcal{T}_{ord} = \mathcal{T}_{std}$*
- *In $\mathbb{Z}_+$, $\mathcal{T}_{ord} = \mathcal{T}_{disc}$ (since any $\{n\} = (n-1, n+1) \in \mathcal{T}_{ord}$)*
- *In $\{1,2\} \times \mathbb{Z}_+$ in $\mathcal{T}_{dict}$ is not in $\mathcal{T}_{disc}$ (althogh most single point set are open, 2×1 is not open)*
- *In $\mathbb{R}^2$, both $\mathcal{B}$ and $\mathcal{B}'$ generates $\mathcal{T}_{dict}$*

$$\mathcal{B} = \{(a \times b, c \times d) \mid a < c \vee (a = c \wedge b < d)\} \quad \mathcal{B}' = \{(a \times b, a \times d) \mid b < d\}$$

Product Topology

Definition. *(**Product Topology**) Let X, Y be topological spaces, the product topology on $X \times Y$ is generated by the basis*

$$\mathcal{B} = \{U \times V \mid U \in \mathcal{T}_X \ V \in \mathcal{T}_Y\}$$

Theorem. *(**Product topology as basis**) If $\mathcal{B}$, $\mathcal{C}$ are basis for X and Y respectively. Then*

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \ C \in \mathcal{C}\}$$

is a basis for topology of $X \times Y$

Definition. *(**Projection**) Let $\pi_1 : X \times Y \to X$ be defined by $\pi_1(x,y) = x$. π_1 is a projection of $X \times Y$ **onto** the first factor*

- *projections are surjective*

Subspace Topology

Definition. *(**Subspace Topology**) Let $(X, \mathcal{T})$. Let $Y \subset X$, then*

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is the subspace topology on Y. Alternatively, define using basis. If $\mathcal{B}$ generates $\mathcal{T}$, then

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for $\mathcal{T}_Y$.

- ***(lemma)** $Y \subset X$. U open in Y and Y open in X, then U open in X*
- ***(theorem)** $A \subset X$ and $B \subset Y$, then $\mathcal{T}_{product\ A \times B} = \mathcal{T}_{subspace A \times B}$ (subspace and product topology work well)*
- *X ordered and $Y \subset X$. order topology on Y maynot be same as order topology of Y inherited as a subspace of X (subspace and order topology does not work well)*
  - *Let $Y = [0,1] \subset \mathbb{R}$. $\mathcal{B}_Y$ are of the form $(a,b) \cap [0,1]$. Note*
    1. *$[0,b)$ where $b \notin [0,1]$ is open in $[0,1]$ but not in $\mathbb{R}$*
    2. *$\mathcal{T}_{ord} \cong \mathcal{T}_{subspace}$ since the basis elements of the same form*
  - *Let $Y = [0,1) \cup \{2\} \subset \mathbb{R}$.*
    1. *$\{2\}$ open in $\mathcal{T}_{subspace}$ since $(1.5, 2.5) \cap Y = \{2\}$.*
    2. *$\{2\}$ not open in order topology since any basis of the form $\{x \in Y \mid a < x \leq 2 \ a \in Y\}$ contains some other point other than $\{2\}$*
    3. *$\mathcal{T}_{ord} \ncong \mathcal{T}_{subspace}$*
- ***(theorem)** subspace and order topology works well if the subspace is convex*

Definition. *(**convex**) Given ordered X and subset $Y \subset X$ is convex if $(a,b) \subset X$ lies in Y completely*

Theorem. *(**subspace and order topology works well if the subspace is convex**)*
Let X be ordered and $Y \subset X$ be convex. Then order topology on Y same as topology Y inherits as a subspace of X.

Closed Sets and Limit Points

Definition. ***(Closed)*** *A subset A of X is closed if $X - A$ is open.*

- *In $\mathcal{T}_{f.c.}$, all finite subsets and X are closed*
- *In $\mathcal{T}_{disc}$ every set is closed.*

Continuity

1. **Continuity**

   (a) A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if

   $$\forall x_0 \in \mathbb{R}^n, \quad \forall \epsilon > 0 \quad \exists \delta > 0 \text{ s.t. } |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$
   $$\text{i.e. } (x \in B_\delta(x_0) \Rightarrow f(x) \in B_\epsilon(f(x_0)))$$

   (b) A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if $f^{-1}(U)$ open for any open $U \subset \mathbb{R}^m$

2. **Homeomorphism** Topological spaces X, Y are homeomorphic if there is a homeomorphism between X and Y, i.e. a bijective function $f : X \to Y$ such that f and f^{-1} are continuous

   - $(-1, 1) \cong \mathbb{R}$ by using $f(x) = \tan(x)$
   - If $\mathcal{T}_1 \subset \mathcal{T}_2$, then $Id : (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ is continuous
   - If $\mathcal{T}_1 = \mathcal{T}_2$, then $Id : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is a homeomorphism