proof for matrix form of $y_i = \beta_0 x + \beta_1 + e_i$

$$\hat{\beta} = (X^T X)^{-1} X^T Y, \qquad X = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$= [\quad] \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$= [\quad] \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sum (x_i^2) \bar{y}}{S_{xx}} - \frac{\bar{x} \sum x_i y_i}{S_{xx}} \\ \frac{-n\bar{x}\bar{y}}{S_{xx}} + \frac{\sum x_i y_i}{S_{xx}} \end{bmatrix}$$

$$E(\hat{\beta}) = E[(X^T X)^{-1} X^T y] \qquad y = X\beta + e$$

$$= \underbrace{(X^T X)^{-1} X^T \underbrace{E(y)}_{X\beta}}_{I}$$

$$= \beta$$

$$Var(\hat{\beta}) = Var[(X^T X)^{-1} X^T y]$$

$$= \underbrace{(X^T X^{-1}) X^T}_{A} \underbrace{Var(y)}_{\delta^2 I} \underbrace{X (X^T X)^{-1}}_{A^{-1}}$$

$$= \delta^2 \underbrace{(X^T X)^{-1} X^T X}_{I} (X^T X)^{-1}$$

$$= \delta^2 (X^T X)^{-1}$$

$2 \times n \; n \times 2 = 2 \times 2$ dim

$$cov(\hat{\beta}) = \delta^2 \begin{bmatrix} \frac{\sum x_i^2}{n S_{xx}} & \frac{-\bar{x}}{S_{xx}} \\ \frac{-\bar{x}}{S_{xx}} & \frac{1}{S_{xx}} \end{bmatrix}$$

RHS

Show = $\quad cov(\hat{\beta}_1, \hat{\beta}_0) \checkmark \quad Var(\hat{\beta}_1) \checkmark$

$$LHS = \delta^2 \left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)$$

$$RHS = \frac{\sum (x_i - \bar{x})^2}{n S_{xx}} + \frac{\sum (2x_i \bar{x} - \bar{x}^2)}{n S_{xx}}$$

$$= \frac{1}{n} + \frac{\sum (2x_i \bar{x} - 2\bar{x}^2)}{n S_{xx}} + \frac{\sum \bar{x}^2}{n S_{xx}}$$

$$= \frac{1}{n} + \underbrace{\frac{2\bar{x} \sum (x_i - \bar{x})}{n S_{xx}}}_{0} + \frac{\cancel{n} \bar{x}^2}{\cancel{n} S_{xx}}$$

$$= \frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}$$