Chapter 2 Subgroups

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1 Definition and Examples

Definition. ***(Subgroup)***

1. ***(subgroup)*** *Let G be a group. The subset H of G is a subgroup of G, denoted as $H \leq G$ if*
   (a) *H is nonempty*
   (b) *H is closed under products and inverses, i.e. $x, y \in G$ implies $x^{-1}, xy \in H$*

   If $H \leq G$ and $H \neq G$, then $H < G$. $H \leq G$ implies operation on H is the operation on G restricted to H. So any equation in H can also be viewed as equation in G

2. ***(The Subgroup Criterion)*** *$H \subset G$ is a subgroup if and only if*
   (a) *$H \neq \emptyset$*
   (b) *for all $x, y \in H$, $xy^{-1} \in H$*

   Furthermore, if H is finite, then suffice to check H is nonempty and closed under multiplication

- *(examples)*
  - $G \leq G$ *and* $\{1\} \leq G$ *(latter is called the trivial subgroup)*
  - $\mathbb{Z} \leq \mathbb{Q} \leq \mathbb{R}$ *under operation of addition*
  - $\left\{1, r, r^2, \cdots, r^{n-1}\right\} \leq D_{2n}$
  - $2\mathbb{Z} \leq \mathbb{Z}$
  - $(\mathbb{Q}^\times, \times) \not\leq (\mathbb{R}, +)$ *(operation are different)*
  - $\mathbb{Z}^+ \leq \mathbb{Z}$ *and* $(\mathbb{Z}^+)^\times \not\leq \mathbb{Q}^\times$ *(not closed under inverses and does not contain identity)*
  - $D_6 \not\leq D_8$ $(D_6 \not\subset D_8)$
- ***(theorem)*** *subgroup is a transitive relation, i.e. $K \leq H, H \leq G$, then $K \leq G$*

2 Centralizers and Normalizers, Stabilizers and Kernels