Definition.

1. ***directed graph*** *G is a pair (V, E), where V is a finite set and E is a binary relation on V. V is a* ***vertex set*** *of G, and its elements are called* ***vertices****. The set E is called the edge set of G and its elements are called edges*

   (a) *If (u, v) is an edge in a directed graph, (u, v) is* ***incident from*** *or* ***leaves*** *vertex u and is* ***incident to*** *or* ***enters*** *vertex v*

   (b) *In the case of directed graph, if v is* ***adjacent*** *to u, then $u \to v$*

   (c) *the* ***out-degree*** *of a vertex is the number of edges leaving it, and the* ***in-degree*** *of a vertex is the number of edges entering it. The* ***degree*** *of a vertex in a directed graph is its in-degree plus its out-degree*

   (d) *In directed graph, a path $\langle v_0, \cdots, v_k \rangle$ forms a* ***cycle*** *if $v_0 = v_k$ and the path contains at least one edges;*

      i. *The* ***cycle is simple*** *if, in addition, $v_1, \cdots, v_k$ are distinct*

      ii. *A self-loop is a cycle of length 1, i.e. (v, v)*

      iii. *A directed graph with no self-loop is* ***simple***

      iv. *A graph with no cycles is* ***acyclic***

   (e) *A directed graph is* ***strongly connected*** *if*

      i. *every two vertices are reachable from each other*

      ii. *has exactly one strongly connected component*

   (f) *the* ***strongly connected component*** *of a directed graph are equivalence classes of vertices under the 'are mutually reachable' relation*

   (g) *Given a directedf graph $G = (V, E)$, the* ***undirected version*** *of G is the undirected graph $G' = (V, E')$, where $(u, v) \in E'$ if and only if $u \neq v$ and $(u, v) \in E$ (that is, undirected version contains edges of G with directions removed and self-loop eliminated)*

   (h) *the* ***neighbor*** *of a vertex u is any vertex that is adjacent to u in the undirected version of G (that is, v is a neighbor of u if $u \neq v$ and either $(u, v) \in E$ or $(v, u) \in E$).*

2. ***undirected graph*** *$G = (V, E)$, the set E consists of unordered pairs of vertices, rather than ordered pairs, i.e. exists set $\{u, v\}$, where $u, v \in V$ and $u \neq v$. (by convention, we use $(u, v) = (v, u)$ to denote a set)*

   (a) *If (u, v) is an edge in an undirected graph, then (u, v) is* ***incident on*** *vertices u and v*

   (b) *If (u, v) is an edge in $G = (V, E)$, then vertex v is* ***adjacent*** *to vertex u*

   (c) *The* ***degree*** *of a vertex in an undirected graph is the number of edges incident on it; A vertex with degree 0 is* ***isolated***

(d) *In undirected graph, a path* $\langle v_0, \cdots, v_k \rangle$ *forms a* ***cycle*** *if* $k \geq 3$ *and* $v_0 = v_k$

   i. *The* ***cycle is simple*** *if* $v_1, \cdots, v_k$ *are distinct*

   ii. *A graph with no cycles is* ***acyclic***

(e) *An undirected graph is* ***connected*** *if*

   i. *every vertex is reachable from all other vertices*

   ii. *has exactly one connected component*

(f) *Given undirected* $G = (V, E)$*, the* ***directed version of*** G *is the directed graph* $G' = (V, E')$*, where* $(u, v) \in E'$ *if and only if* $(u, v) \in E$*. (replacing each undirected edge* (u, v) *by two directed edges* (u, v) *nad* (v, u)*)*

(g) u *and* v *are* ***neighbors*** *if they are adjacent*

3. *The* ***connected component*** *of a graph are equivalence classes of vertices under the 'is reachable from' relation*

   (a) *The* ***edges of a connected component*** *are those that are incident on only the vertices of the component, i.e.* (u, v) *is an edge of a connected component if both* u *and* v *are vertice of the component*

Definition. *A* ***path*** *of* ***length*** k *from a vertex* u *to a vertex* u' *in a graph* $G = (V, E)$ *is a sequence* $\langle v_0, v_1, \cdots, v_k \rangle$ *of vertices such that* $u = v_0$ *and* $u' = v_k$*, and* $(v_{i-1}, v_i) \in E$ *for* $i = 1, 2 \cdots k$

1. *the* ***length*** *of the path is the number of edges in the path*

2. *If there is a path from* u *to* u'*, we say that* u' *is* ***reachable*** *from* u *via* p*, which we sometimes write as* $u \overset{p}{\leadsto} u'$ *if* G *is directed*

3. *A path is* ***simple*** *if all vertices in the path are distinct*

4. *A* ***subpath*** *of path* $p = \langle v_0, v_1, \cdots, v_k \rangle$ *is a contiguous subsequence of its vertices . That is, for any* $0 \leq i \leq j \leq k$*, subsequence of vertices* $\langle v_i, \cdots, v_j \rangle$ *is a subpath of* p

Definition. ***Isomorphism and Subgraphs***

1. *Two graphs* $G = (V, E)$ *and* $G' = (V', E')$ *are* ***isomorphic*** *if there exists a bijection* $f : V \to V'$ *such that* $(u, v) \in E$ *if and only if* $(f(u), f(v)) \in E'$*. In other words, we can relable the vertices of* G *and be the vertices of* G'*, maintaining the corresponding edges in* G *and* G'

2. *A graph* $G' = (V', E')$ *is a* ***subgraph*** *of* $G = (V, E)$ *if* $V' \subseteq V$ *and* $E' \subseteq E$*. Given a set* $V' \subseteq V$*, the subgraph of* G ***induced*** *by* V' *is the graph* $G' = (V', E')$ *where*

$$E' = \{(u, v) \in E : u, v, \subseteq V'\}$$

Definition. ***Types of graphs***

1. *A **Complete graph** is an undirected graph in which every pair of vertices is adjacent*
2. *A **Bipartite graph** is an undirected graph* $G = (V, E)$ *in which* V *can be partitioned into two sets* V_1 *and* V_2 *such that* $(u, v) \in E$ *implies either* $u \in V_1$ *and* $v \in V_2$ *or* $u \in V_2$ *and* $v \in V_1$
3. *A **Weighted graph** is a graph for which each edge has an associated **weight**, given by a weight function* $w : E \to \mathbb{R}$.
4. *A **forest** is an acyclic, undirected graph*
5. *A **tree** is a connected, acyclic, undirected graph*
6. *A **directed acyclic graph (DAG)** is as its name suggests*