Answer for Week 8, slide 10

We should set $\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ when $(\mathbf{X}'\mathbf{X})$ is *invertible*.

This means:

- $\det(\mathbf{X}'\mathbf{X}) \neq 0$
- The rows/columns of $\mathbf{X}'\mathbf{X}$ are linearly independent
- The rank of $\mathbf{X}'\mathbf{X}$ is 2

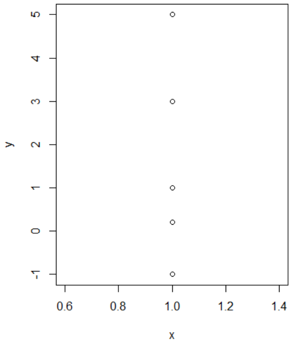
If the rank of $\mathbf{X}'\mathbf{X}$ is 2, then the rank of $\mathbf{X}$ is also 2.

Question: What do our data look like when $\text{rank}(\mathbf{X}) \neq 2$?

Answer to what the data might look like

Note that $\text{rank}(\mathbf{AB}) \leq \min(\text{rank}\mathbf{A}, \text{rank}\mathbf{B})$ and $\mathbf{X} = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$

If $\text{rank}(\mathbf{X}) \neq 2$, then $x_1 = x_2 = \ldots = x_n$. Example:



We'll assume this isn't the case. i.e., $\text{rank}(\mathbf{X}) = 2$, $\text{rank}(\mathbf{X}'\mathbf{X}) = 2$.

Answer to slide 11

Evaluating $\mathbf{X}'\mathbf{X}$ gives:

$$\mathbf{X}'\mathbf{X} = \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} = \begin{pmatrix} n & n\bar{x} \\ n\bar{x} & \sum x_i^2 \end{pmatrix}$$

Answer to slide 12

$(\mathbf{X}'\mathbf{X})^{-1}$ simplifies to:

$$(\mathbf{X}'\mathbf{X})^{-1} = \frac{1}{nS_{xx}} \begin{pmatrix} \sum_{i=1}^{n} x_i^2 & -n\bar{x} \\ -n\bar{x} & n \end{pmatrix} = \frac{1}{S_{xx}} \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} x_i^2 & -\bar{x} \\ -\bar{x} & 1 \end{pmatrix}$$