Problem 6. Solution.

(i) [**8 points**] This is standard and only uses the fact that W is T-invariant. Take a basis β_W for W and **extend it** to a basis β for V. Then the matrix $[T]_\beta$ has the form

$$\begin{pmatrix} [T_W]_{\beta_W} & B \\ 0 & C \end{pmatrix},$$

and

$$\begin{aligned} P_T(t) &= \det(\begin{pmatrix} [T_W]_{\beta_W} - tI_m & B \\ 0 & C - tI_{n-m} \end{pmatrix}) \\ &= \det([T_W]_{\beta_W} - tI_m) \cdot \det(C - tI_{n-m}) \\ &= P_{T_W}(t) \cdot g(t) \end{aligned}$$

so that $P_{T_W}(t) \mid P_W(t)$.

(ii) [**7 points**] **Show that** $u \neq 0$**.** By the theorem in the book about cyclic subspaces,

$$\{v, T(v), \dots, T^{m-1}(v)\} \qquad (\star)$$

is a basis for W where $m = \dim W = \deg P_{T_W}(t)$ and the set

$$\{v, T(v), \dots, T^m(v)\}$$

is linearly dependent. Now

$$(-1)^m \, g(t) = a_0 + a_1 t + \cdots + t^{m-1}$$

and

$$(-1)^m \, u = a_0 v + a_1 T(v) + \cdots + T^{m-1}(v).$$

But then $u = 0$ would contradict the fact that the set $(\star)$ is linearly independent.

[**10 points**] **Show that** u **is an eigenvector.** The Hamilton-Cayley Theorem says that

$$P_{T_W}(T_W) = 0, \qquad \text{as a linear transformation of } W.$$

In particular, if x is any vector in W, then $P_{T_W}(T)(x) = 0$, because $T(x) = T_W(x)$. Thus

$$0 = P_{T_W}(T)(v) = ((T - aI)g(T))(v) = (T - aI)(g(T)(v)) = (T - aI)(u) = T(u) - au$$

i.e., $T(u) = au$. Since $u \neq 0$, this says that u is an eigenvector with eigenvalue a.