STA247 - Probability with Computer Applications
Midterm 1 Fall 2016
Nov. 18, 2016
50 Minutes

Name (Print): Solutions Student Number: ____________

This test contains 6 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and write your name on the top of every page, in case the pages become separated.

This is a closed book test. You are allowed a **non-programmable** calculator on this test

- **Show all your work**. Answers, correct or not, unsupported by calculations, algebraic work, or explanation will not earn any marks.
- **Clearly state and define any variables and/or distributions**. These are part of the problem solving process and are worth marks. Include concluding statements where appropriate.
- **Organize your work** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page that cannot be understood will not earn full marks.
- You may use the back pages for rough work. Any work/steps to be graded **must** be written in the space provided. **Only work written in the space provided will be graded.**
- Do your best! If you get stuck, there's no need to panic. Take a few slow, deep breaths, look at another question before returning to your problem. You got this!

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	12	
4	7	
Total:	39	

1. (10 points) The operator of a pumping station has observed that demand for water during early afternoon hours has an approximate exponential distribution with mean 100 cfs (cubic feet per second).

   (a) (5 points) Find the probability that the demand will exceed 200 cfs during the early afternoon on a randomly selected day.

Let W = demand for water (cfs), $W \sim exp(100)$

0.5 $P(W>200) = \int_{200}^{\infty} \frac{1}{100} e^{-w/100}\, dw$ 0.5

check that their limits are correct

$= 1 - e^{-w/100}\Big|_{w=200}^{\infty}$

$= 1 - e^{-2}$

$= 0.1353$

$\therefore$ 13.53% that water demand will exceed 200 cfs on any given day.

   (b) (5 points) Approximately what range of water demand (around the mean) will occur at least 90% of the time during the early afternoon hours? *Hint: Use Chebyshev's Inequality.*

$P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$

$\mu = 100$

$\sigma^2 = \theta^2 = 100^2$

$P(|W-100| > 100k) \geq 1 - \frac{1}{k^2} = 0.9$

either or, some statement of Chebyshev's in context

① Find k

$1/k^2 = 0.1$

$\boxed{k = \sqrt{10}}$

②

$\therefore P(-100\sqrt{10} < W - 100 < 100\sqrt{10}) \geq 0.9$

③ Interval (approximate)

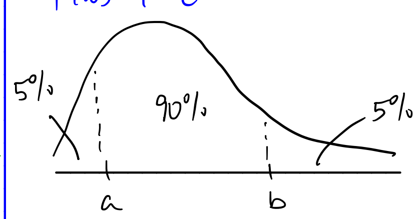
$(100 - 100\sqrt{10},\ 100 + 100\sqrt{10})$

$= (-216.23, 416.23) \rightarrow$ can't have $\ominus$ demand

$= (0, 416.23)$

$\therefore$ Demand is between 0 & 416.23 cfs at least 90% of the time.

$F(w) = 1 - e^{-w/100}$



$F(a) = 0.05$

$0.05 = 1 - e^{-a/100}$

$e^{-a/100} = 0.95$

$a = -100 \ln(0.95)$

$= 5.13$

$F(b) = 0.95$

$0.95 = 1 - e^{-b/100}$

$e^{-b/100} = 0.05$

$b = -100 \ln(0.05)$

$= 299.57$

$\therefore$ 90% interval is (5.13, 299.57) cfs.

90% of all days will have a demand between 5.13 and 299.57 cfs.

2. (10 points) The number of customers that call in to complain can be modeled as a Poisson process with $\lambda = 2$ per 15 minutes.

   (a) (6 points) If the customer service representative steps out to grab coffee for 10 minutes, what is the probability the phone rings during the time he is gone?

starting distⁿ

t = # of 15 mins intervals. – t = 2/3 here

let X = # of phone calls during 10 mins ✓ $X \sim \text{Poisson}(\overset{\lambda t}{2 \times \tfrac{2}{3}}) = \text{Poisson}(4/3)$ ✓

Finding new λ

$P(X \geq 1) = 1 - P(X=0)$ ✓

$= 1 - \dfrac{(4/3)^0 e^{-4/3}}{0!}$

$= 1 - 0.2636$

$= 0.7364$ ✓

∴ 73.64% probability the phone rings while he is gone.

   (b) (4 points) How long can the customer service representative be gone for if he wants the probability of receiving no phone calls during that time to be less than 0.5?

t = # of 15 mins intervals. Let X = # of phone calls during 15t minutes.

$X \sim \text{Pois}(2t)$ ✓

$P(X=0) < 0.5$

$\dfrac{(2t)^0 e^{-2t}}{0!} < 0.5$ ✓

$e^{-2t} < 0.5$

$-2t < \ln(0.5)$

$t > \dfrac{\ln(0.5)}{2}$

$t > 0.3466$ ✓

∴ If he is gone for at least $(0.3466 \times 15) = 5.20$ minutes, the probability he receives no phone calls will be less than 50%. Or, if he is gone for more than 5.20 minutes, there is more than 50% chance he will miss a call.

3. (12 points) A random variable Y has moment-generating function:

$$f(y) = \begin{cases} e^y, & y < 0 \\ 0, & \text{elsewhere} \end{cases}$$

(a) (3 points) Find $E(e^{3Y/2})$

0.5

$$E(e^{3y/2}) = \int_{-\infty}^{0} e^{3y/2} \cdot e^y dy = \int_{-\infty}^{0} e^{5y/2} dy = \frac{2}{5} e^{5y/2} \Big|_{y=-\infty}^{0}$$

0.5 Limits of integration!

$$= \frac{2}{5} - 0$$

$$= 2/5$$

(b) (4 points) Find the moment-generating function for Y.

$$E(e^{ty}) = \int_{-\infty}^{0} e^{ty} \cdot e^y dy = \int_{-\infty}^{0} e^{(t+1)y} dy = \frac{1}{t+1} e^{(t+1)y} \Big|_{y=-\infty}^{0}$$

only if $(t+1) > 0$ or $t > -1$

$$= \frac{1}{t+1}$$

$$\therefore M_y(t) = (t+1)^{-1}, \quad t > -1$$

(c) (5 points) Find the variance of Y using the MGF found in (b). If you could not find it **(and ONLY if you did not answer b)**, you may use $M_Y(t) = \cancel{\frac{3}{9+t}}$ to do this question. → $\frac{1}{(3+t)}$

$$E(y) = M_y'(0) = -(t+1)^{-2} \Big|_{t=0}$$

$$= -1$$

$$E(y^2) = M_y''(0) = 2(t+1)^{-3} \Big|_{t=0}$$

$$= 2$$

$$V(y) = E(y^2) - E(y)^2 = 2 - 1^2 = 1$$

$$E(y) = -1(3+t)^{-2} \Big|_{t=0}$$

$$= -\frac{1}{9}$$

$$E(y^2) = 2(3+t)^{-3} \Big|_{t=0}$$

$$= \frac{2}{27}$$

$$V(y) = \frac{2}{27} - \left(-\frac{1}{9}\right)^2 = \frac{5}{81}$$

If students ended up with a different MGF but found V(y) correctly using their MGF, then award full points.

4. (7 points) **BONUS** Six percent of the apples in a large shipment are damaged. Before accepting each shipment, the quality control manager of a large store randomly selects 1000 apples. If forty or more are damaged, the shipment is rejected. What is the probability that this shipment is rejected? *Hint: Approximation.*

Let X = # of damaged apples. $X \sim Bin(1000, 0.06)$

Students must verify that a Normal approximation is appropriate before using it

$\hookrightarrow np = 1000 \times 0.06 = 60 > 10$ $\mu = E(X) = np = 60$

$n(1-p) = 1000 \times 0.94 = 940 > 10$ $\sigma^2 = V(X) = np(1-p) = 60 \times 0.94 = 56.4$

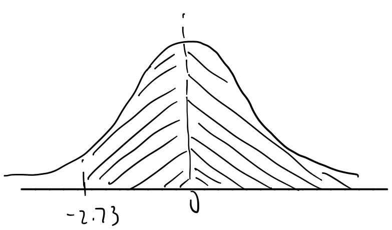
A Normal distribution would provide a good approximation.

$\therefore X \overset{\sim}{\sim} N(60, 56.4)$ State approximating distribution, namely it has the same μ & σ^2 as X.

$P(X \geq 40) \doteq P(X > 39.5) = P\left(\frac{X-\mu}{\sigma} > \frac{39.5 - 60}{\sqrt{56.4}}\right)$

(0.5) continuity correction (0.5) standardizing

$= P(Z > -2.73)$



$= 0.5 + P(0 \leq Z \leq 2.73)$

$= 0.5 + 0.4968$ correct use of Normal dist^{n} table

$= 0.9968$

$\therefore$ There is a 99.68% probability that the shipment is rejected.

concluding statement for probability calculated