Homework 8 Solutions

Chapter 3:

68. Length of time to complete a job of a server has exponential pdfand cdf $F_X(x)=1-e^{-\lambda x}$ Call Y the waiting time until service of the first job. Its distribution is distribution of the 1st order statistics.

$$f_Y(y)=f_1(y)=nf_X(y)(1-F_X(y))^{n-1}=n\lambda e^{-\lambda y}(1-(1-e^{-\lambda y}))^{n-1}=n\lambda e^{-\lambda y}e^{-\lambda y(n-1)}=n\lambda e^{-n\lambda y},\quad y\geqslant 0$$

Call Z the waiting time until service of the second job. Its distribution is distribution of the 2nd order statistics.

$$f_Z(z)=f_2(z)=n(n-1)f_X(z)(F_X(z))^{2-1}(1-F_X(z))^{n-2}=n(n-1)\lambda e^{-\lambda z}(1-e^{-\lambda z})(1-(1-e^{-\lambda z}))^{n-2}=$$
$$=n(n-1)\lambda e^{-(n-1)\lambda z}(1-e^{-\lambda z})=,\quad y\geqslant 0$$

Chapter 4:

6.Let X continuous random variable with pdf $f_X(x)=2\mathrm{x}, 0\leqslant x\leqslant 1$

a. Find $E(X)$:

Ans: $E(X)=\int_{-\infty}^{\infty} x f_X(x)\,dx=\int_0^1 x\,2\mathrm{x}\,dx=\int_0^1 2\mathrm{x}^2\,dx=[\frac{2\mathrm{x}^3}{3}]_{x=0}^{1}=\frac{2}{3}$

b. Let $Y=X^2$ Find pdf of Y and use it to find $E(Y)$.

Ans: $Y=X^2$ so $x=g^{-1}(y)=\sqrt{(y)}$, $\frac{d}{dy}g^{-1}(y)=\frac{d}{dy}\sqrt{y}=\frac{1}{2\sqrt{y}}$

Hence $f_Y(y)=f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right|=2\sqrt{y}\frac{1}{2\sqrt{y}}=1, 0\leqslant y\leqslant 1$ So

$E(Y)=\int_{-\infty}^{\infty} x f_Y(x)\,dx=\int_0^1 x\,dx=[\frac{x^2}{2}]_{x=0}^{1}=\frac{1}{2}$

c. $E(X^2)=\int_{-\infty}^{\infty} x^2 f_X(x)\,dx=\int_0^1 x^2 2\mathrm{x}\,dx=\int_0^1 2\mathrm{x}^3\,dx=[\frac{2\mathrm{x}^4}{4}]_{x=0}^{1}=\frac{1}{2}$

b. and c. give the same answer.

d. $Var(X)=E(X^2)-(E(X))^2=\frac{1}{2}-(\frac{2}{3})^2=\frac{1}{2}-\frac{4}{9}=\frac{1}{18}$

29. Prove that $E(g(X)h(Y))=E(g(X))E(h(Y))$

Ans: Continuous RVs

$$E(g(X)h(Y))=\iint_{-\infty}^{\infty} g(x)h(y)f_{X,Y}(x,y)\,dx\,dy=\iint_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y)\,dx\,dy=$$
$$=\int_{-\infty}^{\infty} h(y)f_Y(y)\int_{-\infty}^{\infty} g(x)f_X(x)\,dx\,dy=\int_{-\infty}^{\infty} h(y)f_Y(y)E(g(X))\,dy=E(g(X))\int_{-\infty}^{\infty} h(y)f_Y(y)dy=$$
$$=E(g(X))E(h(Y))$$

The second equality come from X,Y independent

30. Find $E(\frac{X}{X+1})$ where X is a Poisson random variable.

Ans: X is Poisson random variable so pmf of X is: $p_X(k)=\frac{\lambda^k}{k!}e^{-\lambda},\ k=0,1,2,\ldots$

$$E(\frac{1}{X+1})=\sum_{k=0}^{\infty}\frac{1}{k+1}p_X(k)=\sum_{k=0}^{\infty}\frac{1}{k+1}\frac{\lambda^k}{k!}e^{-\lambda}=\sum_{k=0}^{\infty}\frac{\lambda^k}{(k+1)!}e^{-\lambda}=\frac{1}{\lambda}\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{(k+1)!}e^{-\lambda}=\frac{1}{\lambda}\sum_{l=1}^{\infty}\frac{\lambda^l}{l!}e^{-\lambda}=$$
$$=\frac{1}{\lambda}(\sum_{l=0}^{\infty}\frac{\lambda^l}{l!}e^{-\lambda}-e^{-\lambda})=\frac{1}{\lambda}(1-e^{-\lambda})$$

The last equality come from “total sum of any pdf (here Poisson) is 1”.