Chapter 5 Diagonalization

5.1 Eigenvalues and Eigenvectors

Definition. ***Diagonalizable*** *A linear operator T on a finite-dimensional vector space V is called diagonalizable if there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix. A square matrix A is called diagonalizable if L_A is diagonalizable.*

Remark. Want to determine if an linear operator T is diagonalizable and if so, ways to obtain the basis $\beta = \{v_1, v_2, \cdots, v_n\}$ for V such that $[T]_\beta$ is a diagonal matrix. Note that if $D = [T]_\beta$ is a diagonal matrix, i.e. $D_{ij} = 0$ for $i \neq j$, then for each $v_j \in \beta$, we have

$$T(v_j) = \sum_i^n D_{ij} v_i = D_{jj} v_j = \lambda_j v_j$$

where $\lambda_j = D_{jj}$. Conversely, if $\beta = \{v_1, v_2, \cdots, v_n\}$ is an ordered basis for V such that $T(v_j) = \lambda_j v_j$ for some scalars $\lambda_1, \lambda_2, \cdots, \lambda_n$, then

$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Definition. ***Eigenvalue and Eigenvector (characteristic/proper value or vector)***
Let T be a linear operator on a vector space V. A nonzero vector $v \in V$ is called an eigenvector of T if there exists a scalar λ such that $T(v) = \lambda v$. The scalar λ is called the eigenvalue corresponding to the eigenvector v.
Let A be in $M_{n\times n}(F)$. A nonzero vector $v \in F^n$ is called an eigenvector of A if v is an eigenvector of L_A; that is, if $Av = \lambda v$ for some scalar λ. The scalar λ is the eigenvalue of A corresponding to the eigenvector v

Theorem. ***5.1 Diagonalizable and Eigenvalue/Eigenvector***
A linear operator T on a finite-dimensional vector space V is diagonalizable if and only if there exists an ordered basis β for V consisting of eigenvectors of T (i.e. $v \in V$ is eigenvector if exists λ such that $T(v) = \lambda v$). Furthermore, if T is diagonalizable, $\beta = \{v_1, v_2, \cdots, v_n\}$ is an ordered basis of eigenvectors of T, and $D = [T]_\beta$, then D is diagonal matrix and D_{jj} is the eigenvalue corresponding to v_j for $1 \leq j \leq n$

Remark. To diagonalize a matrix or linear operator is to find a basis of eigenvectors and the corresponding eigenvalues

Theorem. ***5.2 Computing Eigenvalues***
Let $A \in M_{n\times n}(F)$. Then a scalar λ is an eigenvalue of A if and only if $det(A - \lambda I_n) = 0$

Definition. ***Characteristic Polynomial of a Matrix*** *Let $A \in M_{n\times n}(F)$. The polynoimal $f(t) = det(A - tI_n)$ is called the characteristic polynomial of A*

1. *THe eigenvalues of a matrix are the zeros of its characteristic polynomial*
2. *To determine the eigenvalues of a matrix or linear operator, we normally compute its characteristic polynomial.*

Definition. ***Characteristic Polynomial of a Linear Operator*** *Let T be a linear operator on an n-dimensional vector space V with ordered basis β. We define the characteristic polynomial $f(t)$ of T to be the characteristic polynomial of $A = [T]_\beta$. That is,*

$$f(t) = det(A - tI_n)$$

We denote characteristic polynomial of an operator T by $det(T - tI)$. Note the definition is independent of the choice of ordered basis β, the resulting characteristic polynomial is the same regardless the choice of basis.

Theorem. ***5.3 Properties of Characteristic Polynomial***
Let $A \in M_{n\times n}(F)$

1. *The characteristic polynomial of A is a polynomial of degree n with leading coefficients $(-1)^n$*
2. *A has at most n distinct eigenvalues.*

Theorem. ***5.4 Computing Eigenvectors***
Let T be a linear operator on a vector space T, and let λ be an eigenvalue of T. A vector $v \in V$ is an eigenvector of T corresponding to λ if and only if $v \neq 0$ and $v \in N(T - \lambda I)$

Proposition. ***Equivalent problem of finding eigenvectors*** *Let $T : V \to V$ be a linear operator and β be an ordered basis for V. Let $A = [T]_\beta$. For all $v \in V$ an eigenvector of T corresponding to an eigenvalue λ if and only if $\phi_\beta(v)$ is an eigenvector of A corresponding to λ. We have reduced the problem of finding the eigenvectors of a linear operator on a finite-dimensional vector space to the problem of finding the eigenvectors of a matrix.*

Definition. ***Geometric Description of how a linear operator T acts on an eigenvector*** *in the context of a vector space V over $\mathbb{R}$. Let v be eigenvector of T and λ be corresponding eigenvalue. Let $W = span(\{v\})$, the one-dimensional subspace of V spanned by v, a line passing through 0 and v. For any $w \in W$, $w = cv$ for some $c \in \mathbb{R}$*

$$T(w) = T(cv) = cT(v) = c\lambda v = \lambda w$$

T acts on the vector in W by multiplying each such vector by λ