An example where the MLE does not follow an asymptotically normal distribution

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Let $X_1, \ldots, X_n$ be a sample from a truncated exponential distribution with a density

$$f(x\big|\theta) = \begin{cases} \mathrm{e}^{-(x-\theta)} & x > \theta \\ 0 & \text{otherwise} \end{cases}$$

1. find the MLE for θ
2. show that $n(\hat{\theta} - \theta)$ is exponentially distributed
3. find the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$
4. why does the asymptotic normality of the MLEs not hold in this case?

<u>Solution</u>:

1. Writing

$$\mathcal{L}(\theta) = \prod_{i=1}^{n} \mathrm{e}^{-(x_i-\theta)} I_{[\theta,\infty]}(x_i) = \begin{cases} \mathrm{e}^{-n\overline{x}}\mathrm{e}^{n\theta} & x_{\min} > \theta \\ 0 & \text{otherwise} \end{cases}$$

we see that for a given sample (fixed $\overline{x}$) the likelihood is monotonically increasing in θ before being truncated at $x_{\min}$. Figure 1 makes it clear that the MLE is $\widehat{\theta} = X_{\min}$.

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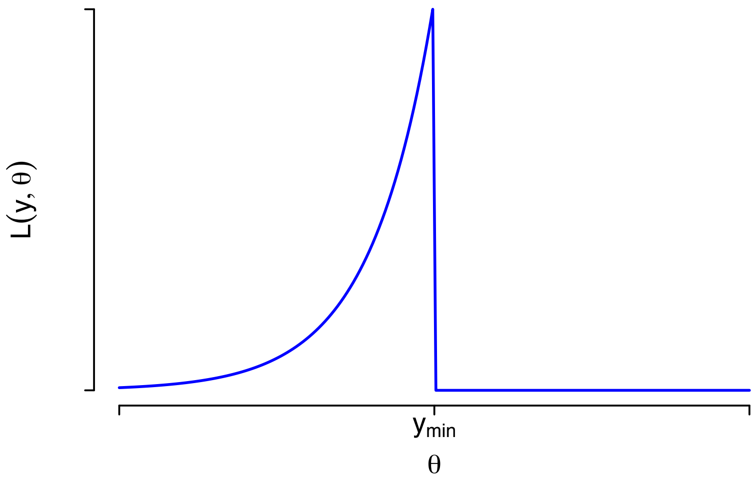

Figure 1: A sketch of the likelihood

2. First, for $x > \theta$

$$F(x \big| \theta) = \mathbb{P}(X \leq x) = \int_{\theta}^{x} \mathrm{e}^{-(t-\theta)} \mathrm{d}t = 1 - \mathrm{e}^{-(t-\theta)} \quad .$$

Now,

$$\begin{aligned}
\mathbb{P}\left(n(\widehat{\theta} - \theta) \leq t\right) &= \mathbb{P}\left(X_{\min} \leq \frac{t}{n} + \theta\right) \\
&= 1 - \mathbb{P}\left(X_{\min} > \frac{t}{n} + \theta\right) = 1 - \prod_{i=1}^{n} \left[1 - F\left(\theta + \frac{t}{n}\right)\right] \\
&= 1 - \begin{cases} \exp\left\{-\sum_{i=1}^{n}\left(\theta + \frac{t}{n} - \theta\right)\right\} & \theta + \frac{t}{n} > \theta \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 - \mathrm{e}^{-t} & t > 0 \\ 0 & \text{otherwise} \end{cases} \quad ,
\end{aligned}$$

and we see that $n(\widehat{\theta} - \theta) \sim \mathrm{Exp}(1)$.

3. To derive the desired distribution, note that

$$\mathbb{P}\left(\sqrt{n}(\hat{\theta}-\theta) \le t\right) = \mathbb{P}\left(n(\hat{\theta}-\theta) \le \sqrt{n}t\right)$$

$$\overset{(2)}{=} \begin{cases} 1-\mathrm{e}^{-\sqrt{n}t} & \sqrt{n}t > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1-\mathrm{e}^{-\sqrt{n}t} & t > 0 \\ 0 & \text{otherwise} \end{cases} \xrightarrow[n \longrightarrow \infty]{} \begin{cases} 1 & t > 0 \\ 0 & \text{otherwise} \end{cases},$$

therefore the cdf of $\sqrt{n}(\hat{\theta}-\theta)$ approaches the cdf of the constant random variable 0 instead of that of a normal distribution.

4. Note that the likelihood is not even differentiable once at the MLE, deeming the proof of the asymptotic normality invalid.