Chapter 6 Inner Product Spaces

Definition. ***Inner Product*** *Let V be a vector space over F. An inner product on V is a function that assigns, to every ordered pair of vectors x and y in V, a scalar in F, denoted $\langle x, y\rangle$, such that for all $x, y, z \in VE$ and all $c \in F$,*

1. $\langle x + z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$
2. $\langle cx, y\rangle = c\langle x, y\rangle$
3. $\overline{\langle x, y\rangle} = \langle y, x\rangle$
4. $\langle x, x\rangle > 0$ *if* $x \neq 0$

First two condition requires inner product be linear in the first component. Also

$$\langle \sum_i a_i v_i, y\rangle = \sum_i a_i \langle v_i, y\rangle$$

Definition. ***Conjugate Transpose or Adjoint of a Matrix*** *Let $A \in M_{m\times n}(F)$, the conjugate transpose or adjoint of A is an $n \times m$ matrix A^* such that $(A^*)_{ij} = \overline{A_{ji}}$ for all i, j. For $F = \mathbb{R}$, $A^* = A^T$*

Definition. ***Inner Product Definition Example***

1. ***Standard Inner Product on*** F^n *For $x = (a_1, a_2, \cdots, a_n)$ and $y = (b_1, b_2, \cdots, b_n)$ in F^n, the standard inner product on F^n is given by*

$$\langle x, y\rangle = \sum_{i=1}^{n} a_i \bar{b}_i$$

2. ***Inner Product for Real-valued Continuous Functions on*** $[0, 1]$ *Let $V = C([0, 1])$, $f, g \in V$, define*

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt$$

3. ***Frobenius Inner Product for Matrices*** *Let $V = M_{n\times n}(F)$, $A, B \in V$, then*

$$\langle A, B\rangle = B^* A = \sum_{i=1}^{n} (B^* A)_{ii}$$

Definition. ***Inner Product Space*** *A vector space over F endowed with a specific inner product is called an inner product space. If $F = C$, V is a complex inner product space; if $F = \mathbb{R}$, then V is a real inner product space*

Theorem. ***6.1 Properties From Inner Product Conditions*** *Let V be an inner product space. Then for $x, y, z \in V$ and $c \in F$, the following statements are true*

1. $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

2. $\langle x, cy \rangle = \bar{c}\langle x, y \rangle$

3. $\langle x, 0 \rangle = \langle 0, x \rangle = 0$

4. $\langle x, x \rangle = 0$ *if and only if* $x = 0$

5. *If* $\langle x, y \rangle = \langle x, z \rangle$ *for all* $x \in V$*, then* $y = z$

The inner product is conjugate linear in the second argument

Definition. ***Norm/Length*** *Let* V *be an inner product space. For* $x \in V$*, define norm or length of* x *by*

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Definition. ***6.2 Properties of Norm*** *Let* V *be an inner product space over* F*. Then for all* $x, y \in V$ *and* $c \in F$*, the following statements are true*

1. $\|cx\| = |c| \cdot \|x\|$

2. $\|x\| = 0$ *if and only if* $x = 0$*. In any case,* $\|x\| \geq 0$

3. ***Cauchy-Schwarz Inequality*** $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$

4. ***Triangular Inequality*** $\|x + y\| \leq \|x\| + \|y\|$

Definition. ***Angle*** *For* $F = \mathbb{R}$*,* $x, y \neq 0$*, and* θ *be angle between* x *and* y

$$\cos\theta = \frac{\langle x, y \rangle}{\|x\| \, \|y\|} \qquad \theta = \cos^{-1}\left(\frac{\langle x, y \rangle}{\|x\| \, \|y\|}\right)$$

Note

$$\left| \frac{\langle x, y \rangle}{\|x\| \, \|y\|} \right| \leq 1$$

So valid input to arccos function

Definition. ***Orthogonal Vectors*** *Let* V *be an inner product space. Vectors* x *and* y *in* V *are orthogonal (perpendicular) if* $\langle x, y \rangle = 0$*.*

Definition. ***Orthogonal Sets and Orthonormal Sets*** *A subset* S *of* V *is orthogonal if any two distinct vectors in* S *are orthogonal. A vector* x *in* V *is a unit vctor if* $\|x\| = 1$*. A subset* S *of* V *is orthonormal if* S *is orthogonal and consists entirely of unit vectors.*

1. $S = \{v_1, v_2, \cdots\}$*, then* S *is orthonormal if and only if* $\langle v_i, v_j \rangle = \delta_{ij}$

2. *We can* ***normalize*** *an orthogonal set* S*, by multiplying* $1/\|x\|$ *for each* $x \in S$

Definition. ***Orthonormal Set Property*** *Let V be inner product space and $S = \{s_1, s_2, \cdots\} \subseteq V$ be an orthonormal set. Let $v \in span(S)$, then $v = a_1 s_1 + \cdots + a_k s_k$. Then*

$$\langle v, s_j \rangle = a_j$$

by

$$\langle v, s_j \rangle = \langle \sum_i a_i s_i, s_j \rangle = \sum_i a_i \langle s_i, s_j \rangle = \sum_i a_i \delta_{ij} = a_j$$