Linear Programming

Definition. ***General Linear Programs***

1. ***Linear Function*** *Given* $a_1, \cdots, a_n \in \mathbb{R}$*, and variables* $x_1, \cdots, x_n$*, define a linear function* f *of those variables by*

$$f(x_1, \cdots, x_n) = \sum_{j=1}^{n} a_j x_j$$

2. ***Linear equality & inequalities*** *If* $b \in \mathbb{R}$ *and* f *is a linear function, then*

$$f(x_1, \cdots, x_n) = b$$

   is a linear equality and the inequaliie

$$f(x_1, \cdots, x_n) \leq b \qquad f(x_1, \cdots, x_n) \geq b$$

   are linear inequalities

3. ***Linear Constraints*** *are either linear equalities or linear inequalities*

4. ***Linear programming problem*** *Either minimizing or maximizing a linear fucntion subject to a finite set of linear constraints. If want to minimize, then linear program is a* ***minimization linear problem****, otherwise its called a* ***maximization linear problem***

5. ***Feasible solution*** *Any setting of variable* $x_1, \cdots, x_n$ *that satisifies all constraints a feasible solution to the linear program*

6. ***Feasible Region*** *a convex set of feasible solutions for which we which to maximize the objective function*

7. ***Objective value*** *value of the objective function at a particular point in the feasible solution*

8. ***Graphical solution*** *If 2 variables, then we can use the let* z *be the objective. Such curve have the property that the intersection between the curve and the feasible solution is the set of feasible solutions with objective value* z*. A optimal solution to linear program occurs at a vertex of a feasible region, since the curve that intersect the feasible region for which maximum* z *is obtained is on the boundary of the feasible region. This holds for higher dimension curves as well*

9. ***Simplex*** *For* n *variables, each constraint defines a half-space in* n*-dimensional space, the feasible region formed by the intersection of these half spaces is a simplex. The objective function is a hyperplane, and because of convexity, an optimal solution still occurs at a vertex of the simplex*

10. ***Simplex alogrithm*** *takes as input a linear program and returns an optimal solution. It starts as some vertex of the simplex and performs a sequence of iterations. In each iteration, it moves along an edge of the simplex from a current vertex to a neighboring vertex whose objective value is no smaller than that of the current vertex. The algorithm terminates when it reaches a local minimum, i.e. all neighboring vertices have a smaller objective value.*

    Lemma. ***Duality*** *Since a feasible region is convex and objective function is linear, a local optimum from a simplex algorithm is a global optimum*

    (a) *Write linear program in slack form*

    (b) ***Pivot*** *Make one variable basic and another nonbasic*

Definition. ***Standard form***

1. ***Specification*** *Given n real number $c_1, \cdots, c_n \in \mathbb{R}$ and m real number $b_1, \cdots, b_m \in \mathbb{R}$ and mn real number a_{ij} for $i = 1, \cdots, m$ and $j = 1, \cdots, n$. We wish to find n real numbers $x_1, \cdots, x_n$ such that*

$$\begin{aligned} \text{Maximize } & \sum_{k=1}^{n} c_j x_j \\ \text{subject to } & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \text{ for } i = 1, \cdots, m \\ & x_j \geq 0 \text{ for } j = 1, \cdots, n \end{aligned}$$

    Note standard form requires the n nonnegative constraints on $x_1, \cdots, x_n$. Alternatively, let $A = (a_{ij})$ be $m \times n$ matrix, $b = (b_i)$ a m-vector, $c = (c_j)$ a n-vector, and $x = (x_j)$ an n-vector. Then

$$\begin{aligned} \text{Maximize } & c^T x \\ \text{subject to } & Ax \leq b \\ & x \geq 0 \end{aligned}$$

    Therefore a standard form can be expressed with (A, b, c)

2. ***Re-definition***

    (a) ***Feasible & Infeasible solution*** *A setting of variable $\bar{x}$ satisfies all constraints a fesasible solution, whereas a setting of $\bar{x}$ that fails to satisfy at least one constraint if an infeasible solution .*

(b) ***Objective Value*** *A solution $\bar{x}$ has objective value $c^T\bar{x}$*

(c) ***Optimal solution & Optimal Objective Value*** *a feasible solution $\bar{x}$ whose objective value is maximum over all feasible solutions is an optimal solution, its objective value $c^T\bar{x}$ is the optimal objective vlaue*

(d) ***Feasible & Unfeasible LP*** *If a linear program has no feasible solution, then it is infeasible, otherwise it is feasible*

(e) ***Unbounded LP*** *If a linear program has some feasible solution but does not have a finite optimal objective value, then LP is unbounded*

3. ***Converting linear program (4 types) to standard form***

(a) ***Equivalent LP***

i. *Two maximization linear programs L and L" are equivalent if for each feasible solution $\bar{x}$ to L with objetive value z, there is a corresponding solution $\bar{x}'$ to L' with objective value z, and vice versa*

ii. *A minimization linear program L and a maximization linear program L' are equivalent if for each feasible solution $\bar{x}$ to L with objective value z, there is a corresponding feasible solution $\bar{x}$ to L' with objective value $-z$, and vice versa*

(b) ***Objective function is a minimization rather than a maximization***

Negate coefficients ($c' = -c$) in the objective function.

2 LP's are equivalent since we have the same feasible solution (constraints unchanged) and for each feasible solution, the objective value in L is the negative of the objective value in L' hence 2 linear programs are equivalent

(c) ***There might be variables without nonnegativity constraints***

Rerplace each occurrence of a variable variable x_j without nonnegativity constraint by $x'_j - x''_j$, and add the nonnegativity constraint $x'_j > 0$ and $x''_j > 0$

(d) ***There might be equality constraints***

Replace equality constraints with a pair of inequality constraints

$$f(x_1, \cdots, x_n) \leq b \qquad f(x_1, \cdots, x_n) \geq b$$

(e) ***There might be $\geq$ inequality constraints***

Multiple the greater than or equal to $\geq$ constraints to less than or equal to $\leq$ constraints by multiplying these constraints by -1

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i \quad \Longleftrightarrow \quad -\sum_{j=1}^{n} a_{ij}x_j \geq -b_i$$

Definition. ***Slack form***

1. ***Slack variable*** *Given inequality constraints* $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$, *we have*

$$s = b_i - \sum_{j=1}^{n} a_{ij}x_j$$
$$s \geq 0$$

   where s is a slack variable because it measures the slack, or difference, between left-hand and right-hand sides of equation. We can use this methods to convert from standard form to slack form, where the only inequality constraints are the nonnegativity constraints

2. ***Conversion from standard to slack form*** *Use x_{n+i} instead of s to denote the slack variable associated with the i-th inequality. The i-th constriant is therefore*

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij}x_j \qquad\qquad x_{n+i} \geq 0$$

3. ***Basic & Nonbasic variables*** *Given a slack form with a set of equality constriants, one of variables on left-hand side of equality and all others on the right-hand side. The variables on the left-hand side of equalities are basic variables, and those on the right-hand side are nonbasic variables. Nonbasic variables are the only varaibles that constitutes the objective function*

4. ***Slack Form*** *Let z be the value of the objective function and linear inequalities be converted to a set of slack variables. Omit the nonnegativity constraints since it is assumed that all variables are nonnegative. Let N be the set of indices of nonbasic variables, let B be set of indices of the basic variables, we always have $|N| = n$ and $|B| = m$, where $N \cup B = \{1, \cdots, n+m\}$.*

   (a) *equations are indexed by entries of B*

   (b) *variables on RHS of equation are index by entries of N*

   Let A, b, c, be constants and coefficients. Let v be the constant term in objective function. Therefore, we define a slack form by a tuple (N, B, A, b, c, v) where

$$z = v + \sum_{j \in N} c_j x_j$$
$$x_i = b_i - \sum_{j \in N} a_{ij}x_j \qquad \text{for } i \in B$$

   Note indices into A, b, c are no necessarily sets of contiguous integers, they depend on the index sets B and N

Formulating problems as Linear Programs

Definition. ***Shortest path*** *Given a weighte, directed graph $G = (V, E)$ with weights $w : E \to \mathbb{R}$ and source s and destination t. Wish to ompute the value d_t, i.e. the weight of a shortest path from s to t. We can formulate it as LP as follows*

$$\begin{aligned} &\textit{Maximize} && d_t \\ &\textit{Subject to} && d_v \le d_u + w(u,v) \qquad \textit{for each } (u,v) \in E \\ & && d_s = 0 \end{aligned}$$

The bellman-form algorithm sets source vertex distance $d_s = 0$ and never changes it. When the algorithm terminates, it has computed, for each v, a value d_v such that for each edge $(u,v) \in E$, we have $d_v \le d_u + w(u,v)$
*Note we are **maximizing** d_t for 2 reasons*

1. *setting $\bar{d}_v = 0$ for all $v \in V$ yields optimal solution without solving shortest-path problem*
2. *Maximize because an optimnal solution to shortest path problem sets each $\bar{d}_v$ to be $\underset{u:(u,v)\in E}{Min}\{d_u + w(u,v)\}$ (considers all incident edges to v) such that d_v is the maximum value that is less than or equal to values in the set $\{\bar{d}_u + w(u,v)\}$. We maximize d_v for all vertex v on a shortest path from s to t subject to constraints, and maximizing d_t achieves this...*

Definition. ***Maximum flow*** *Given directed graph $G = (V, E)$ with nonnegative capacity $c : E \to \mathbb{R}^+$ and two vertices, a source s and a sink t. A flow $f : V \times V \to \mathbb{R}$ satisfies capacity constraint and flow conservation. A maximum flow is a flow that satisfies these constraints and maximizes the flow value. Also we assume $c(u,v) = 0$ if $(u,v) \notin E$ and no antiparallel edges*

$$\begin{aligned} &\textit{Maximize} && \sum_{v\in V} f_{sv} - \sum_{v\in V} f_{vs} \ \textit{(Value of a flow)} \\ &\textit{Subject to} && f_{uv} \le c(u,v) \qquad \textit{for each } u,v \in V \ \textit{(capacity constraint)} \\ & && \sum_{v\in V} f_{vu} = \sum_{v\in V} f(u,v) \quad \textit{for each } u \in V \setminus \{s,t\} \ \textit{(flow conservation)} \\ & && f_{uv} \ge 0 \end{aligned}$$