1 Open, closed and everything in between

Definition 1.1. *Let $x \in \mathbb{R}^n$ and $r > 0$ a real number. We define the **open ball** of radius r at the point x as*

$$B_r(x) := \{y \in \mathbb{R}^n : d(x,y) < r\} = \{y \in \mathbb{R}^n : \|x - y\| < r\}$$

Remark. Here $d(x,y)$ means distance, i.e. $d(x,y) = \|x-y\|$. The ball is therefore a collection (set) of points which are a distance at most r from x. In $\mathbb{R}^1$, $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\} = (x - r, x + r)$. In $\mathbb{R}^2$, $B_r(x) = \{(x,y) \in \mathbb{R}^n : x^2 + y^2 < r^2\}$

Definition 1.2. *The **boundary of an open ball** is,*

$$\partial B = \{y \in \mathbb{R}^n : d(y,x) = r\}$$

Definition 1.3. *The **closed ball**, denoted $\overline{B}$, is defined as*

$$\begin{aligned} \overline{B} &= B \cup \partial B \\ &= \{y \in \mathbb{R}^n : d(y,x) \le r\} \end{aligned}$$

Definition 1.4. *A set $S \subseteq \mathbb{R}^n$ is **bounded** if there exitsts a large enough ball $B \subseteq \mathbb{R}^n$ such that $\exists r > 0, S \subseteq B_r(0)$*

Definition 1.5. *Let $S \subseteq \mathbb{R}^n$,*

1. *We say that $x \in \mathbb{R}^n$ is an **interior point** of S if $\exists$ an $r > 0$ such that $B_r(x) \subseteq S$; that is, x is an interior point if we can enclose it in an open ball which is completely contained in S*

2. *We say that $x \in \mathbb{R}^n$ is a **boundary point** of S if for every $r > 0$, $B_r(x) \cap S \neq \emptyset$ and $B_r(x) \cap S^c \neq \emptyset$; that is, x is a boundary point if no matter how small the ball we place around x, that ball meets or lives both inside and outside of S.*

Definition 1.6.
If $S \subseteq \mathbb{R}^n$

1. *The **interior** of S, denoted $intS$, is the set of all interior points of S*

2. *The **boundary** of S, denoted ∂S, is the set of all boundary points of S*

There are some properties,

$$\text{If } x \notin \partial S \text{ , then either } x \in intS \vee x \in intS^c \tag{1}$$
$$\partial(S) = \partial(S^c) \qquad \text{(by definition1.6)}$$
$$x = intS \sqcup intS^c \sqcup \partial S \qquad \text{(by \# 1)}$$
$$\partial S \cap intS = \emptyset \qquad \text{(by definition1.6)}$$
$$S \subseteq \partial S \sqcup intS \qquad \text{(by \# 1)}$$
$$intS \text{ is an open set} \qquad \text{(by definition1.7)}$$

Note. $A \sqcup B$ means $A \cup B \wedge A \cap B = \emptyset$

Remark. As an example, $(-1, 1]$, has interior points $(-1, 1)$ and boundary points $1, -1$

Definition 1.7.

1. *A set $S \subseteq \mathbb{R}^n$ is said to be **open** if every point of S is an interior point; that is, S is open if for every $x \in S$, there exists an $r > 0$ such that $B_r(x) \subseteq S$.*

$$\forall x \in S, \exists r > 0, B_r(x) \subseteq S$$

2. *The set S is **closed** if S^c is open.*

   Remark. The contrapositive is also true: If set S is not closed then S^c is not open.

   Proposition 1. *A set $S \subseteq \mathbb{R}^n$ is closed if and only if $\partial S \subseteq S$*

Remark.
An open ball is open.
A closed ball B is closed, which is equivalent to proving B^c is open, or that every point in B^c is an interior point.
The entire space and $\emptyset$ are both open and closed.
If U is open, then U^c is closed

Remark. To solve that a set is open, we prove all points in the set are interior points or there exists a ball centered at every point that contains completely in the set. We set an arbitrary element within a set and construct a ball centered at this point with radius satisfying the contraints of the set, so that the ball is contained in the set. Then we choose an arbitrary point within the ball and prove that it always become contained in the set. In this process, we can use the fact that the metric of this arbitrary point and the center of the open ball to be less than the radius chosen. Triangular inequality becomes handy here.

Definition 1.8. *Let $S \subseteq \mathbb{R}^n$. The closure of the set S is the set $\overline{S} = S \cup \partial S$ Closure follows properties,*

1. *The closure is always a closed set*
2. *S is closed if and only if $S = \overline{S}$*
3. *S is the smallest closed set containing S*

Remark. The closure of open interval is closed interval. The closure of the open ball is

$$B_r(x) = \{y \in \mathbb{R}^n : \|x - y\| \leq r\}$$

Closure simply means that the entirety of the boundary is delegated to any set S. Therefore S is always closed by proposition in 1.7