Chapter 3 Elementary Matrix Operation and System of Linear Equations

Definition. ***Elementary Row(Column) Operations*** *Let A be $m \times n$ matrix Any of 3 operations on A is called an elementary row (column) operation*

1. *Interchanging any two rows (column) of A*
2. *Multiplying any row (column) of A by a nonzero scalar*
3. *Adding any scalar multiple of a row (column) of A to another row (column)*

Definition. ***Elementary Matrix*** *An $n \times n$ elemntary matrix is a matrix obtained by performing an elementary operation on I_n. The elementary matrix is said to be of type 1,2,3 according to whether the elementary operation performed on I_n is a type 1,2,3 operation, respectively.*

Theorem. ***3.1 Existence of Elementary Matrix that Act as Elementary Operations*** *Let $A \in M_{n\times n}(F)$ and suppose B is obtained from A by performing an elementary row (column) operation. Then there exists an $m \times m$ ($n \times n$) elementary matrix E such that $B = EA$ ($B = AE$), where E is obtained from I_m (I_n) by performing the same elementary row (colunn) operation as that which was performed on A to obtained B. Conversely, if E is an elementary matrix, then EA is the matrix obtained from A by performing the same elementary row (column) operation as that which produces E from I_m (I_n)*

Theorem. ***3.2 Elementary Matrix are Invertible*** *Elementary matrices are invertible, and the inverse of an elementary matrix is an elementary matrix of the same type*

3.1 The Rank of a Matrix and Matrix Inverses

Definition. ***Rank*** *If $A \in M_{m\times n}(F)$, the rank of A, denoted as $rank(A)$ to be the rank of the linear transformation $L_A : F^n \to F^m$*

Definition. ***Properties for Rank***

1. *$n \times n$ matrix is invertible if and only if its rank is n*
2. *Every matrix A is the matrix representation of linear transformation L_A with respect to appropriate standard ordered bases. So the rank of a L_A is the same as the rank of **one of** its matrix representations, namely A.*

Theorem. ***3.3 Equivalence of Rank for Linear Transformation and Matrix*** *Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces, and let β and γ be ordered bases for V and W, respectively. Then*

$$rank(T) = rank([T]_\beta^\gamma)$$

Theorem. ***3.4 Rank Preserving Operations*** *Let A be $m \times n$ matrix. If P and Q are invertible $m \times m$ and $n \times n$ matrices, respectively, then*

1. $rank(AQ) = rank(A)$
2. $rank(PA) = rank(A)$
3. $rank(PAQ) = rank(A)$

Corollary. ***Elementary Operation is Rank Preserving*** *Elementary row and column operations on a matrix are rank-preserving*

Theorem. ***3.5 Rank is Number of Linearly Independent Columns***
The rank of any matrix equals the maximum number of its linearly independent columns; that is, the rank of a matrix is the dimension of the subspace generated by the columns

Theorem. ***3.6 Existence Matrix Transformation to a Specific Form***
Let A be an $m \times n$ matrix of rank r. Then $r \leq m$, $r \leq n$, and, by means of a finite number of elementary row and column operations, A can be transformed into the matrix

$$D = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}$$

where O_1, O_2, O_3 are zero matrices. Thus $D_{ii} = 1$ for $i \leq r$ and $D_{ij} = 0$ otherwise (proof with induction on number of row of A)

Corollary. ***1 Elementary Decomposition*** *Let A be $m \times n$ matrix of rank r. Then there exist invertible matrices B and C of sizes $m \times m$ and $n \times n$, respectively, such that $D = BAC$, where*

$$D = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}$$

is the $m \times n$ matrix in which O_1, O_2, O_3 are zero matrices.

Corollary. ***2 Column Space and Rank***

1. $rank(A^t) = rank(A)$
2. *The rank of any matrix equals the maximum number of its linearly independent rows; that is, the rank of a matrix is the dimension of the subspace generated by its rows*
3. *The rows and columns of any matrix generate subspaces of the same dimension, numerically equal to the rank of the matrix (use previous 2 corollaries)*

Corollary. *3 Every invertible matrix is a product of elementary matrices*

Theorem. ***3.7 L.T. Composition / Matrix Multiplication Always Reduces Rank***
Let $T : V \to W$ and $U : W \to Z$ be linear transformations on finite-dimensional vector spaces V, W, and Z, and let A and B be matrices such that the product AB is defined. Then

1. $rank(UT) \leq rank(U)$

2. $rank(UT) \leq rank(T)$
3. $rank(AB) \leq rank(A)$
4. $rank(AB) \leq rank(B)$

Definition. ***Augmented Matrix*** *Let A and B be $m \times n$ and $m \times p$ matrices, respectively. By the augmented matrix $(A|B)$, we mean the $m \times (n+p)$ matrix $(A\ B)$, that is the matrix whose first n columns are the columns of A, and whose last p columns are the columns of B*

Definition. ***Computing Matrix Inverse*** *If A is $n \times n$ matrix, and the matrix $(A|I_n)$ is transformed into a matrix of the form $(I_n|B)$ by means of a finite number of elementary row operations, then $B = A^{-1}$. If A is an $n \times n$ matrix tha tis not invertible, then any attempt to transform $(A|I_n)$ into a matrix of the form $(I_n|B)$ produces a row whose first n entries are zeros.*
Convenient for computing inverse of linear transformation by computing inverse of matrix representation of the transformation

3.3 Systems of Linear Equations – Theoretical Aspects

Definition. ***System of Linear Equations***

1. *The system of equations S where a_{ij} and b_i are scalars in field F and $x_1, x_2, \cdots, x_n$ are n variables taking values in F, is called the system of m linear equations in n unknowns over field F*

$$Ax = b \qquad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

2. *A **solution** to the system is an n-tuple*

$$x = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} \in F^n$$

*The set of all solutions to S is called **solution** set of the system. System is **consistent** if its solution set is nonempyty, otherwise **inconsistent***

Definition. ***Homogenous/Nonhomogeneous System*** *A system $Ax = b$ of m linear equations in n unknowns is said to be homogeneous if $b = 0$. Otherwise the system is said to be nonhomogeneous*

Theorem. ***3.8 Solution Set to a Homogeneous System Is a Nullspace***
Let $Ax = 0$ be a homogeneous system of m linear equations in n unknowns over a field F (A is $m \times n$). Let K denote the set of all solutions to $Ax = 0$. Then $K = N(L_A)$. Hence K is a subspace of F^n of dimension $n - rank(L_A) = n - rank(A)$

Corollary. *If $m < n$, then the system $Ax = 0$ has a nonzero solution*

Remark. Solving homogeneous system of equations
For computing solution to a system of equations $Ax = 0$. Use $rank(K = N(L_A)) = n - rank(A)$ to compute the dimension of subspace consisting of solutions. Then describe solution set K as span of linearly independent basis to describe

Theorem. ***3.9 Express Solutions to Nonhomogeneous Systems as Solutions to the Corresponding Homogeneous System***
Let K be the solution set of a system of linear equations $Ax = b$, and let K_H be the solution set of the corresponding homogeneous system $Ax = 0$. Then for any solution s to $Ax = b$

$$K = \{s\} + K_H = \{s + k : k \in K_H\}$$

Theorem. ***3.10 Square Invertible Matrix as a Condition for Exactly One Solution***
Let $Ax = b$ be a system of n linear equations in n unknowns. If A is invertible, then the system has exactly one solution, namely, $A^{-1}b$. Conversely, if the system has exactly one solution, then A is invertible (invertibility from $K_H = \{0\}$)

Remark. Idea is if we can find one solution to a given system and a basis for solution set of corresponding homogeneous system then we have all solution to a given system

Theorem. ***3.11 Necessary and Sufficient Condition for a Consistent System***
Let $Ax = b$ be a system of linear equations. Then the system is consistent if and only if $rank(A) = rank(A|b)$ proof

3.4 Systems of Linear Equations – Computational Aspects

Definition. ***Equivalent System of Equations*** *Two systems of linear equations are called equivalent if they have the same solution set*

Theorem. ***3.13 Left Multiply an Invertible Matrix Yields an Equivalent System***
Let $Ax = b$ be a system of m linear equations in n unknowns, and let C be an invertible $m \times m$ matrix. Then the system $(CA)x = Cb$ is equivalent to $Ax = b$

Corollary. ***Elementary Operation Yields Equivalent System*** *Let $Ax = b$ be a system of m linear equations in n unknowns. If $(A'|b'|)$ is obtained from $(A|b|)$ by a finite number of elementary row operations, then the system $A'x = b'$ is equivalent to the original system.*

Definition. ***Reduced Row Echelon Form*** *A matrix is in reduced row echelon form if the following three conditions are satisfied*

1. *Any row containing a nonzero entry precedes any row in which all the entries are zero (if any)*
2. *The first nonzero entry in each row is the only nonzero entry in its column*
3. *The first nonzero entry in each row is 1 and it occurs in a column to the right of the first nonzero entry in the preceding row*

Definition. ***Gaussian Elimination*** *A method for reducing an augmented matrix to reduced row echelon form*

1. ***Forward Pass*** *The augmented matrix is transformed into an upper triangular matrix in which the first nonzero entry of each row is 1, and it occurs in a column to the right of the first nonzero entry of each preceding row*
2. ***Backward Pass/Substitution*** *The upper triangular matrix transformed into reduced row echelon form by making the first nonzero entry of each row the only nonzero entry of its column*

Theorem. ***3.14 Gaussian Elimination Guarantees*** *Gaussian Elimination transforms any matrix into its reduced row echelon form*

Definition. ***Methods for Solving a System***

1. *Transform $(A|b)$ into reduced row echelon form $(A'|b')$*
2. *If a row is obtained in which the only nonzero entry lies in the last column, then system is inconsistent. Otherwise discard any zero rows and write the corresponding system of equations*
3. *Solve the system in reduced row echelon form to yield a solution*

$$s = s_0 + t_1 u_1 + t_2 u_2 + \cdots + t_{n-r} u_{n-r}$$

*where r is number of nonzero rows in A'. s is a **general solution** to $Ax = b$*

Theorem. ***3.15 General Solution*** *Let $Ax = b$ be a system of r nonzero equations in n unknowns. Suppose $rank(A) = rank(A|b)$ and that $(A|b)$ is in reduced row echelon form. Then*

1. *$rank(A) = r$*
2. *If general solution in form $s = s_0 + t_1 u_1 + t_2 u_2 + \cdots + t_{n-r} u_{n-r}$, then $\{u_1, u_2, \cdots, u_{n-r}\}$ is a basis for the solution set of corresponding homogeneous system, and s_0 is a solution to the original system*

Theorem. ***3.16 Property of Reduced Row Echelon Form***
Let A be $m \times n$ matrix of rank r, where $r > 0$, and let B be the reduced row echelon form of A. Then

1. *The number of nonzero rows in B is r*
2. *For each $i = 1, 2, \cdots, r$, there is a column b_{j_i} of B such that $b_{j_i} = e_i$*
3. *The columns of A numbered $j_1, j_2, \cdots, j_r$ are linearly independent*
   (a) *can be used to find linearly independent subset (basis) of a generating set*
   (b) *can be used to extend a linearly independent subset to a basis*
4. *For each $k = 1, 2, \cdots, n$, if column k of B is $d_1 e_1 + d_2 e_2 + \cdots + d_r e_r$, then column k of A is $d_1 a_{j1} + d_2 a_{j2} + \cdots + d_r a_{j_r}$*