MAT247HS **MIDTERM** **February 15, 2018**

Please write clearly and show all of your work.
No notes or calculators are allowed.
Each problem is worth 25 points.

1. Let

$$A = \begin{pmatrix} 3 & 1 & a \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

(i) Find the characteristic polynomial of A.
(ii) Find the eigenvalues of A.
(iii) For what value a is A diagonalizable? Justify you answer.
(iv) For a found in part (iii), find a matrix Q such that $Q^{-1}AQ$ is a diagonal matrix D and write down the matrix D.

2. Find the solution of the linear system of differential equations

$$x_1'(t) = 18\,x_1(t) - 15\,x_2(t)$$
$$x_2'(t) = 20\,x_1(t) - 17\,x_2(t)$$

with $x_1(0) = 2$ and $x_2(0) = 3$. Use the eigenvalues/eigenvectors method described in the book and in lecture.

3. Let

$$f(t) = t^8 + 6t^6 + 11t^4 + t^2 + 15.$$

(i) Write down a matrix A such that

$$P_A(t) = f(t).$$

(ii) Prove that A is not diagonalizable.

4. Let $V = M_2(\mathbb{C})$ and, for X and $Y \in V$, let

$$\langle X, Y \rangle = \mathrm{tr}(X^t \bar{Y}).$$

(i) Prove that $\langle\ ,\ \rangle$ defines an inner product on V.
(ii) Let

$$W = \mathrm{span}\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ i & 0 \end{pmatrix} \} \subset V.$$

Find an orthonormal basis for W.
(iii) Find the orthogonal projection to W of the vector

$$Y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(iv) Find the distance from Y to W.

5. Let V, $\langle\ ,\ \rangle$ be an inner product space. State and prove the Cauchy-Schwartz inequality.

6. Suppose that $T : V \longrightarrow V$ is a linear transformation of a finite dimensional vector space and let $W =< v >$ be the T-cyclic subspace generated by a non-zero vector v. Let $T_W : W \longrightarrow W$ be the linear transformation obtained by restricting T to W.
(i) Prove that the characteristic polynomial $P_{T_W}(t)$ of T_W divides the characteristic polynomial $P_T(t)$ of T.
(ii) Suppose that there is a factorization

$$P_{T_W}(t) = (t - a)g(t)$$

for some $a \in F$ and let

$$u = g(T)(v).$$

Show that u is an eigenvector of T with eigenvalue a.
Hint: Determine the T-cyclic subspace $U =< u >$.