Flow Network

Given a flow network, find a maximum flow, ie. flow f such that $|f|$ is maximum possible.

Definition. *A* ***Flow Network*** *is a directed graph $G = (V, E)$ in which each edge $(u, v) \in E$ has a capacity $c(u, v) \geq 0$ and two special vertices, called* **source** *s and* **sink** *t. Note every other vertex v lie on a path from s to t and there is no $(u, v) \in E$ such that $(v, u) \in E$, (no anti-parallel edges)*

1. *Every node other than s and t has at least one incoming edge and outgoing edge, hence $|E| \geq |V| - 1$*
2. *Usually when $c(u, v) = 0$, the edge is omitted since no flow is allowed*

Definition. *A* ***Flow*** *in G is a real-valued function $f : V \times V \to \mathbb{R}$ satisfying*

1. ***Capacity constraint*** *For all $v \in V$,*

$$0 \leq f(u, v) \leq c(u, v)$$

2. ***Conservation constraint*** *At any vertex $v \in V \setminus \{s, t\}$,*

$$\sum_{(u,v)\in E} f(u, v) = \sum_{(v,u)\in E} f(v, u)$$

In other words, sum of inward flows equates to outward flows, nothing stays at vertex v

The ***value of a flow*** *is defined to be*

$$|f| = \sum_{(s,v)\in E} f(s, v) - \sum_{(v,s)\in E} f(v, s) = \sum_{(s,v)\in E} f(s, v)$$

since assuming no edges doing into s

Definition. ***Residual capacity nad residual network***
Let f be flow of a flow network $G = (V, E)$ with source s and sink t, we define ***residual capacity*** *$c_f(u, v)$ as*

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \textit{if } (u, v) \in E \\ f(v, u) & \textit{if } (v, u) \in E \textit{(push back has limit of flow)} \end{cases}$$

The ***residual network*** *of a flow network G is a directed graph $G_f = (V, E_f)$ where*

$$E_f = \{(u, v) \in V \times V | c_f(u, v) > 0\}$$

where the edge capacity is defined by $c_f(u, v)$.

1. *Since at most 2 edges can be drawn in G_f given 1 edge in G.*

$$|E_f| = 2|E| \iff O(E_f) = O(E)$$

Solution. □

Let P be a simple path from s to t in G_f (no cycle) Define

$$bottleneck(P, f) = min\,\{c_f(u, v) \mid (u, v) \in P\}$$

The following Augment works on G not on G_f and f is the flow with respect to G

Function `Augment` (f, P)
    $b \leftarrow bottleneck(P, f)$
    for $e = (u, v) \in P$ **do**
        if *e is a forward edge (there is capacity left)* **then**
            $f(u, v) \leftarrow f(u, v) + b$
        else
            //(v, u) is a backward edge
            $f(v, u) \leftarrow f(v, u) - b$
Function `Ford-Folkerson` (G, s, t)
    for $e = (u, v) \in E$ **do**
        $f(u, v) \leftarrow 0$
    while $\exists P = \{s, \cdots, t\}$ *in* G_f **do**
        $f' \leftarrow$ `Augment`(f, P)
        Update $f \leftarrow f'$
        Update $G_f \leftarrow G'_f$
    return f

Proposition.

1. *Augment(f, P) is a new flow f' in G*

   Proof. We have to check that f' satisfies capacity and conservation constraint.

   (a) **capacity constraint**
   By definition, $b = bottleneck(P, f) \le c_f(u, v)$ for any $(u, v) \in P$.

   i. If (u, v) is a forward edge then

   $$0 \le f(u, v) \le f'(u, v) = f(u, v) + b \le f(u, v) + c_f(u, v) = f(u, v) + c(u, v) - f(u, v) = c(u, v)$$

   ii. If (u, v) is a backward edge, then $c_f(u, v) = f(v, u)$ by definition, then

   $$c(v, u) \ge f(v, u) \ge f'(v, u) = f(v, u) - b = f(v, u) - c_f(u, v) = f(v, u) - f(v, u) = 0$$

In both case, $0 \leq f'(u,v) \leq c(u,v)$ hence satisfy capacity constraint.

(b) **Conservation constraint** For any vertex v on P, disucss the incident edges to v depending on if they are forward/backward edges (3 cases)...

□

2. $|f'| \geq |f|$

Proof. P is a path in G_f from s to t. The first edge e in P is an edge out of s. In G there is no edge going into s. The new flow on e is $f'(e) = f(e) + b > f(e)$, since all other edges going out of s are unchanged, the total flow going out of s is incremented by b so

$$|f'| = |f| + b > |f|$$

□

Proposition. *The algorithm while loop will end*

Proof.

1. All capacities/flows are integers. Let f^* be a maximum flow on G satisfying

$$|f^*| \leq \sum_{i=1}^{n} c_i \qquad |f^*| \in \mathbb{I}$$

Since $f \in \mathbb{I}$ we have $f' > |f| + 1$ implies $f' \geq |f| + 1$. Hence we can always hit f^* by a finitely many iterations

2. If flows are fractions, then can multiple all capacity by a constant such that capacities and resulting flows are ints, use the previous argument, it will end in finitely many iterations

3. non-rationals? not so sure about..

□

The complexity is $O(E|f^*|)$, $O(|E|)$ for running BFS to find path from s to t and $O(|f^*|)$ for an upper bound on number of iterations for while loop

Definition.

*1. A **cut** $C = (S, T)$ of a flow network $G = (V, E)$ is a partition of V into S and $T = V \setminus S$ such that $s \in S$ and $t \in T$*

2. *A **net flow** $f(S,T)$ for a flow f across the cut (S,T) is defined as*

$$f(S,T) = \sum_{u\in S}\sum_{v\in T} f(u,v) - \sum_{u\in S}\sum_{v\in T} f(v,u)$$

3. *The **capacity** of the cut (S,T) is*

$$c(S,T) = \sum_{u\in S}\sum_{v\in T} c(u,v)$$

4. *The **Minimum cut** of a network is a cut where capacity is minimum across all possible cut*

Lemma. *Let f be a flow in a flow network G and let $C = (S,T)$ be any cut of G then*

$$|f| = f(S,T)$$

Corollary. *The value of any flow in a flow network G is bounded above by the capacity of any cut of G*

Proof. By lemma, $|f| = f(S,T)$ then

$$\begin{aligned}
|f| &= \sum_{u\in S}\sum_{v\in T} f(u,v) - \sum_{u\in S}\sum_{v\in T} f(v,u) \\
&\le \sum_{u\in S}\sum_{v\in T} f(u,v) \\
&\le \sum_{u\in S}\sum_{v\in T} c(u,v) \\
&= c(S,T)
\end{aligned}$$

□

Theorem. *Max-flow Min-cut Theorem*
If the flow f in a flow network $G = (V,E)$ with source s and sink t then the following conditions are equivalent

1. *f is a maximum flow in G*
2. *the residual network G_f contains no augmented paths*
3. *$|f| = c(S,T)$ for some cut (S,T) of G. In other words, max flow is equal to minimum cut capacity*

(Note 1 and 2 can be used to prove the algorithm...)

Proof.

- $1 \to 2$ Let there be a path P from s to t in G_f Define $f' = \textsc{Augment}(f, P)$ where $|f'| > |f|$. contradiction
- $2 \to 3$ Try to construct some cut... Let $S = \{v \in V \mid \exists P = \{s, \cdots, v\}\}$ and let $T = V \setminus S$. Clearly $s \in S$ and $t \in T$. Since there is no augmenting path, then $S \neq \emptyset \neq T$, $S \cap T = \emptyset$, $S \cup T = V$, hence (S, T) is a cut of G. Let $u \in S$ and $v \in T$. If $(u, v) \in E$ we must have $f(u, v) = c(u, v)$, since otherwise there will be a forward edge $(u, v) \in E_f$ such that $v \in S$ (not cut anymore). If $(u, v) \in E$, we must have $f(v, u) = 0$ otherwise $f(u, v) = f(v, u) = 0$ (for similar reason). By lemma $|f| = f(S, T)$, then

$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u) = \sum_{u \in S} \sum_{v \in T} c(u,v) = c(S,T)$$

- $3 \to 1$. By previous corollary, $|f| \leq c(S, T)$ for any cut. So, $|f| = c(S, T)$ implies f is maximum

□