

22 Elementary Graph Algorithms

22.1 Representations of graphs

Definition. *Representations of graphs*

1. **Adjacency List**

- (a) An array of $|V|$ lists, one for each vertex in V . For each $u \in V$, $Adj[u]$ contains all the vertices v such that $(u, v) \in E$ (i.e. all vertices adjacent to u in G)
- (b) compact for **sparse** graphs ($|E| \ll |V|^2$)
- (c) For **directed graph**, the sum of lengths of all adjacency list is $|E|$, since edge of form (u, v) is represented as having v appearing in $Adj[u]$. (i.e. $u \rightarrow v$)
- (d) For **undirected graph**, the sum of lengths of all the adjacency lists is $2|E|$, since if (u, v) is an undirected edge, then u appears in v 's adjacency list and vice versa
- (e) Memory: $\Theta(V + E)$
- (f) Search: $\Theta(E)$ Have no quick way of determining if a given edge (u, v) is present in the graph than to search for v in the adjacency list $Adj[u]$ (have to go through the list)

2. **Adjacency Matrix**

- (a) Assume vertices numbered $1, 2, \dots, |V|$ arbitrarily. We have a $|V| \times |V|$ matrix $A = (a_{ij})$ such that

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

- (b) good for **dense** graphs ($|E| = |V|^2$) or if need to tell if there is an edge between two vertices quickly
- (c) Memory: $\Theta(V^2)$
- (d) Search: $\Theta(1)$

BFS $\Theta(V + E)$

Lemma. *For Proof of correctness*

- 1. Let $G = (V, E)$ be directed or undirected graph, let $s \in V$ be an arbitrary vertex, for any edge $(u, v) \in E$,

$$\delta(s, v) = \delta(s, u) + 1$$

Proof. The shortest path from s to v cannot be longer than shortest path from s to u followed by (u, v) , since otherwise we just take shortest path from s to v and (u, v) which will be a shorter path. \square

2. Upon termination, for each vertex $v \in V$, the value $v.d$ computed by BFS satisfies $v.d \geq \delta(s, v)$

Proof. Induction on the number of enqueue operations. Inductive hypothesis is that $v.d \geq \delta(s, v)$ for all $v \in V$. Before s enqueued, I.H. holds since $v.d = \infty \geq 0 = s.d = \delta(s, s)$. Now consider a white vertex v that is just being discovered and we search $Adj[u]$. I.H. implies $u.d \geq \delta(s, u)$. By assignment of $v.d = u.d + 1$, and previous lemma (since $u \rightarrow v$)

$$v.d = u.d + 1 \geq \delta(s, u) + 1 \geq \delta(s, v)$$

□

3. suppose queue Q contains vertices $\langle v_1, \dots, v_r \rangle$, where v_1 is the head of Q and v_r is the tail. Then $v_r.d \leq v_1.d + 1$ and $v_i.d \leq v_{i+1}.d$ for $i = 1, \dots, r - 1$

Proof. Proof by induction on number of queue operations. Initially, queue contains s only, lemma holds. Now we prove in inductive step that lemma holds after both dequeuing and enqueueing a vertex. If head v_1 is dequeued, v_2 is the new head. By inductive hypothesis $v_1.d \leq v_2.d$. But then we have $v_r.d \leq v_1.d + 1 \leq v_2.d + 1$, the remaining inequalities remain unaffected, so lemma holds with after dequeue of v_1 . During an enqueue, say v , it becomes v_{r+1} . At that time, we just moved u from the queue. By inductive hypothesis, the new head v_1 satisfies $v_1.d \geq u.d$. We have

$$v_{r+1}.d = v.d = u.d + 1 \leq v_1.d + 1$$

Now to prove inequalities holds, by I.H. $v_r.d \geq u.d + 1$ and so $v_r.d \leq u.d + 1 = v.d = v_{r+1}.d$ and the remaining inequalities remain unaffected. So lemma holds during enqueue □

Theorem. Correctness of BFS Let $G = (V, E)$ be directed or undirected graph, suppose BFS is run on G given $s \in V$. during execution, BFS discovers every vertex $v \in V$ reachable from s and upon termination, $v.d = \delta(s, v)$ for all $v \in V$. Moreover, for any vertex $v \neq s$ reachable from s , one of the shortest paths from s to v is a shortest path from s to $v.\pi$ followed by $(v.\pi, v)$

Definition. Breadth-first Tree For graph $G = (V, E)$ with source s , a predecessor subgraph of G , $G_\pi = (V_\pi, E_\pi)$ where

$$V_\pi = \{v \in V : v.\pi \neq NIL\} \quad E_\pi = \{(v.\pi, v) : v \in V_\pi - \{s\}\}$$

The predecessor graph G_π is a Breadth first tree if V_π consists of vertices reachable from s , and for all $v \in V_\pi$, the subgraph G_π contains a unique simple path from s to v that is also the shortest path from s to v in G

Lemma. procedure BFS constructs π such that predecessor graph $G_\pi = (V_\pi, E_\pi)$ is a breadth-first tree

DFS $\Theta(V + E)$

Definition. Depth-first Tree For graph $G = (V, E)$ with source s , a predecessor subgraph of G , $G_\pi = (V, E_\pi)$ where

$$E_\pi = \{(v.\pi, v) : v \in V \text{ and } v.\pi \neq \text{NIL}\}$$

The predecessor subgraph of a depth-first search forms a **Depth-first forest** comprising several **Depth-first trees**. The edges in E_π are **tree edges** (Note how we are not restricting V since DFS will include vertices unreachable from source s)

Proposition. Timestamping

1. **Timestamp** $v.d$ records when v first discovered ($\text{WHITE} \rightarrow \text{GRAY}$) and $v.f$ records when finishes examining $\text{Adj}[v]$ ($\text{GRAY} \rightarrow \text{BLACK}$)
2. Vertex u is WHITE before time $u.d$, GRAY between $u.d$ and $u.f$ and BLACK thereafter
3. vertex v is a descendent of u in Depth-first forest if and only if v is discovered during the time in which u is gray

Theorem. Parenthesis theorem In DFS of G , for any two vertices u and v , exactly one of following holds

1. $[u.d, u.f]$ and $[v.d, v.f]$ are entirely disjoint, then neither u nor v is a descendent of the other in depth-first forest
2. $[u.d, u.f]$ contained entirely within $[v.d, v.f]$, and u is a descendent of v in the depth-first tree
3. $[v.d, v.f]$ is containedf entirely within $[u.d, u.f]$ and v is a descendent of u in depth-first tree

Corollary. Nesting of descendents' interval Vertex v is a proper descendent of u in depth-first forest for a graph G if and only if

$$u.d < v.d < v.f < u.f$$

Theorem. White-Path Theorem In depth-first forest of $G = (V, E)$, vertex v is a descendent of u if and only if at time $u.d$ that search discovers u , there is a path from u to v consisting entirely of white vertices

Proposition. classification of edges

1. **Tree Edges** edges in depth-first forest G_π . (u, v) is a tree edge if v was first discovered by exploring edge (u, v)

2. **Back Edges** are (u, v) connecting u to an ancestor v in a depth-first tree. Self-loos (in directed graph) is also back edge
3. **Forward Edges** are edges (u, v) connecting u to a descendent v in a depth-first tree
4. **Cross Edges** are all other edges. They go between vertice same vertices in the same depth-first tree, as long as one vertex is not an ancestor of the other, or they can go between vertices in different depth-first trees.

When (u, v) is first explored, the color of vertex v tells us about its edge category

1. *WHITE* indicate a tree edge
2. *GRAY* indicate a back edge
3. *BLACK* indicates a forward (if $u.d < v.d$) or cross edge (if $u.d > v.d$)

Theorem. In DFS of an undirected graph G , every edge of G is either a tree edge or a back edge

22.4 Topological Sort $\Theta(V + E)$

Definition. Topological sort A topological sort of a dag $G = (V, E)$ is a linear ordering of all its vertices such that if G contains an edge (u, v) , then u appears before v in the ordering. Can be viewed as ordering of vertices on a horizontal line so that all directed edges go from left to right

1. Call DFS to compute the finishing time $v.f$ for each vertex v
2. As each vertex is finished, insert it onto the front of a linked list

Lemma. A directed graph G is acyclic if and only if a depth-first search of G yields no back edges

Theorem. TOPOLOGICAL SORT produces a topological sort of the DAG provided as input

Proof. Suppose DFS run on a given DAG $G = (V, E)$. It suffices to show that for all $e = (u, v) \in E$, then $v.f < u.f$. Consider (u, v) being explored by DFS. v cannot be gray, since otherwise v is an ancestor of u and (u, v) would be a back edge, i.e. cycle (by previous lemma), which contradicts assumption that G is acyclic. Therefore v must be either white or black. If v is white, v is a descendent of u , so $v.f < u.f$ by the parenthesis lemma. If v is black, it has already finished, but we are still exploring from u , so u has yet to finish, hence $v.f < u.f$. This completes the proof \square

23 Minimum Spanning Tree

Definition. The MST Problem Given a connected, undirected, weighted graph $G = (V, E)$, with weight function $w : E \rightarrow \mathbb{R}$. Find an acyclic subset $T \subseteq E$ that connects all of the vertices and whose total weight

$$w(T) = \sum_{(u,v) \in T} w(u,v)$$

is minimized. Since T is acyclic and connects all of the vertices, we call it a **spanning tree**.

Generic Greedy solution to MST Generic greedy method for MST involves maintaining a loop invariant on a set $A \subseteq E$,

Prior to each iteration, A is a subset of some MST

At each step, we determine an edge (u,v) that we can add to A without violating the invariant. Assume $A \subseteq E$ satisfies the loop invariant, a **safe edge** (u,v) is an edge such that $A \cup \{(u,v)\}$ maintains the invariant

Definition. Cut

1. A **cut** $(S, V - S)$ of an undirected graph $G = (V, E)$ is a partition of V .
2. An edge $(u,v) \in E$ **crosses** the cut if one of its endpoints is in S and the other is in $V - S$
3. A cut **respects** a set $A \subseteq E$ if no edges in A crosses the cut
4. An edge is a **light edge** crossing a cut if its weight is the minimum of any edge crossing the cut (maybe ≥ 1 light edges in case of tie)
5. A **light edge satisfying a given property** if its weight is minimum of any edge satisfying the property

Proposition. MST

1. **Possible Multiplicity** If there are n vertices in the graph, then each spanning tree has $n-1$ edges.
2. **Cycle property** For any cycle C in the graph, if the weight of an edge e of C is larger than the individual weights of all other edges of C , then this edge cannot belong to a MST.

Theorem. The greedy choice (light edge) is optimal (safe) Let $G = (V, E)$ be connected, undirected graph with $w : E \rightarrow \mathbb{R}$. Let $A \subseteq E$ be included in some MST for G , let $(S, V - S)$ be any cut of G that respects A , and let (u, v) be a light edge crossing $C = (S, V - S)$. Then edge (u, v) is safe for A , i.e. $A \cup \{(u, v)\}$ is also in a subset of some MST

Proof. Let T be a MST such that $A \subseteq T$. Assume T does not contain light edge (u, v) , since otherwise $A \cup \{(u, v)\} \subseteq T$, done. Otherwise, $(u, v) \notin T$. Prove using cut-and-paste that (u, v) is safe. In the context of MST, inclusion of (u, v) in T forms a cycle, (u, v) along with path p , s.t. $u \overset{p}{\rightsquigarrow} v$. Since T is by definition simple, p is also simple. Since u and v are on opposite sides of the cut C , let (x, y) be an edge that crosses C . Note $(x, y) \notin A$ since C respects A . Let $T' = T \cup \{(u, v)\} \setminus \{(x, y)\}$. T is connected since removal of (x, y) breaks T into 2 components, and inclusion of (u, v) joins the components together. Now we show that T' is a MST. Since (u, v) is a light edge crossing C and (x, y) also crosses the cut, we have $w(u, v) \leq w(x, y)$, hence

$$w(T') = w(T) + w(u, v) - w(x, y) \leq w(T)$$

Since T already a MST, i.e. $w(T) \leq w(T')$, then $w(T) = w(T')$. T' is also MST. Now we show (u, v) is safe for A . since $A \subseteq T$, $A \subseteq T'$ since $(x, y) \notin A$. hence $A \cup \{(u, v)\} \subseteq T'$. Since T' is MST, (u, v) is safe for A . \square

Corollary. Above theorem holds for cuts in form of connected component Let $G = (V, E)$ be a connected, undirected, weighted graph. Let $A \subseteq E$ such that A is included in some MST of G . Let $C = (V_C, E_C)$ be a connected component in the forest $G_A = (V, A)$. If (u, v) is a light edge connecting C to some other component in G_A , then (u, v) is safe for A

23.2 Kruskal and Prim's algorithms $O(E \lg V)$

Definition. Kruskal's algorithm Finds a safe edge to the growing forest by finding, of all edges that connect any two trees in the forest, an edge (u, v) of least weight.

1. **Implementation** Needs a fast way to determine if an edge crosses connected components. Tracks trees in disjoint sets. Initializes vertices to disjoint sets with MAKE-SET. Sort edges by weight in nondecreasing order. Loop over all edges and include edge (u, v) to $A \subseteq E$ if u and v are not in the same set (evaluate with FIND-SET). Update disjoint sets with UNION
2. **Complexity** Assume disjoint-set-forest impl with union-by-rank and path-compression. Sorting takes $O(E \lg E)$. $O(E)$ FIND-SET and UNION and $O(V)$ MAKE-SET takes a total of $O((V + E)\alpha(V))$. Since G connected, $|E| \geq |V| - 1$, so disjoint-set operation takes $O(E\alpha(V)) = O(E \lg V) = O(E \lg E)$. In total, algorithm takes $O(E \lg E)$. Note since $|E| < |V|^2$, $\lg |E| = O(\lg V)$, so running time is $O(E \lg V)$

Definition. Prim's algorithm Tree (A) starts from an arbitrary root vertex r and grows until the tree spans all vertices of V . Each step adds to the tree A a light edge that connects A to an isolated vertex, one on which no edge of A is incident. (so that the cut respects A)

1. **Implementation** Needs a fast way to select a new edge to add to tree. vertices not in the tree reside in a min-priority queue Q based on key attributes, where $v.\text{key}$ is the minimum weight of any edge connecting v to a vertex in the tree A . ($v.\text{key} = \infty$ if no such edge exists)
2. **Complexity** Assume Q impl with binary min-heap. building heap requires $O(\lg V)$ time. $O(V)$ EXTRACT-MIN each taking $O(\lg V)$ amounts to $O(V \lg V)$. While loop iterates $O(E)$ times. The test for membership is $O(1)$ by keeping a bit in G for each vertex and tells if its not in Q and updating the bit once vertex is removed from Q . DECREASE-KEY taking $O(\lg V)$ each. Hence total time is $O(V \lg V + E \lg V) = O(E \lg V)$. If use fibonacci heap, $O(E + V \lg V)$

24 Single-Source Shortest Path

Definition. The Single-Paths Problem Given a weighted, directed graph $G = (V, E)$, with $w : E \rightarrow \mathbb{R}$.

1. The **weight of path** $w(p)$ for $p = \langle v_0, \dots, v_k \rangle$ is given by

$$w(p) = \sum_{i=1}^k w(v_{i-1}, v_i)$$

2. A **Shortest-path weight** $\delta(u, v)$ from u to v is given by

$$\delta(u, v) = \begin{cases} \min\{w(p) : u \xrightarrow{p} v\} & \text{if there is a path from } u \text{ to } v \\ \infty & \text{otherwise} \end{cases}$$

3. A **Shortest path** from u to v is defined as any path p with weight

$$w(p) = \delta(u, v)$$

Definition. Variants

1. **Single-source shortest-path problem** Find the shortest path from a given **source** vertex $s \in V$ to each vertex $v \in V$
2. **Single-destination shortest-path problem** Find a shortest path to a given **destination** vertex t from each vertex $v \in V$. (By reversing direction of each edge, we can reduce this problem to a single-source problem)

3. **Single-Pair shortest-path problem** Find a shortest path from u to v for given vertices u and v . (If we solve single-source problem with source vertex u , we solve this problem also)
4. **All-pairs shortest-path problem** Find a shortest path from u to v for every pair of vertices u and v . (solving by single-source algo will be inefficient, there are better solutions)

Proposition. Subpaths of shortest paths are shortest path (Optimal substructure)

Given a weighted, directed graph $G = (V, E)$ with weight function $w : E \rightarrow \mathbb{R}$, let $p = \langle v_0, \dots, v_k \rangle$ be a shortest path from vertex v_0 to vertex v_k and, for any i and j such that $0 \leq i \leq j \leq k$, let $p_{ij} = \langle v_i, v_{i+1}, \dots, v_j \rangle$ be a subpath of p from vertex v_i to vertex v_j . Then p_{ij} is a shortest path from v_i to v_j

Proposition. Cycles

1. **Negative-weight cycle**

- (a) The shortest path cannot contain negative-weight cycles.
- (b) No path from s to a vertex on a cycle can be a shortest path, since we can always find a path with lower weight by following the proposed shortest path and then traversing the negative-weight cycle.
- (c) Hence we define for all $v \in C$ for some negative-weight cycle, $\delta(s, v) = \infty$

2. **Positive-weight cycle**

- (a) The shortest path cannot contain positive-weight cycle,
- (b) since removing the cycle from the path produces a path with the same source and destination vertices and a lower path weight
- (c) For 0-weight cycles, we can always remove the cycle and get a shortest path without a cycle.
- (d) Hence we assume shortest paths have no cycles (simple path). Since any acyclic path in G has at most $|V|$ distinct vertices, it contains at most $|V| - 1$ edges, hence we try to find shortest path of at most $|V| - 1$ edges

Definition. Representing shortest path Interested in predecessor subgraph $G_\pi = (V_\pi, E_\pi)$ where

$$V_\pi = \{v \in V : v.\pi \neq NIL\} \quad E_\pi = \{(v.\pi, v) : v \in V_\pi - \{s\}\}$$

Specifically, let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \rightarrow \mathbb{R}$ and assume G contains no negative-weight cycles reachable from source vertex $s \in V$, so that shortest path are well-defined. A **Shortest-paths tree** rooted at s is a directed subgraph $G' = (V', E')$, where $V' \subseteq V$ and $E' \subseteq E$ such that

1. V' is the set of vertices reachable from s in G

2. G' forms a rooted tree with root s and
3. for all $v \in V'$, the unique simple path from s to v in G' is a shortest path from s to v in G

Note shortest path or shortest path are not necessarily unique

Proposition. Shortest-path estimate and Relaxation

1. **Shortest-path estimate** For each $v \in V$, the shortest-path estimate $v.d$ is an upper bound on the weight of a shortest path from source s to v .
2. **Relaxation** The process of relaxing an edge (u, v) consists of testing whether we can improve the shortest path to v found so far by going through u and, if so, updating (improving) $v.d$ and $v.\pi$
 - (a) Given $u.d$, $v.d$ and $w(u, v)$ for edge $u \rightarrow v$
 - (b) Update $v.d$ if $u.d + w(u, v) < v.d$. In essence, take path along u instead of some other path
 - (c) Only way to change $v.d$ and $v.\pi$
3. **Triangular Inequality (for weighted graphs)** For any edge $(u, v) \in E$ we have

$$\delta(s, v) \leq \delta(s, u) + w(u, v)$$

Proof. Suppose p where $s \xrightarrow{p} v$ is a shortest path, then claim holds by definition of shortest path. Otherwise, there is no shortest path from s to v . This implies that there is no shortest path to from s to u , since otherwise there exists shortest path p' such that $s \xrightarrow{p'} u \rightarrow v$ which is a shortest path. Hence $\delta(s, v) = \delta(s, u)$ are either ∞ or $-\infty$. Hence the claim holds \square

Proposition. Effect of relaxation on shortest-path estimates

1. **Upper-bound property** Let $G = (V, E)$ be weighted, directed graph with w . Let $s \in V$ be source vertex and graph initialized by INITIALIZE-SINGLE-SOURCE(G, s) then $v.d \geq \delta(s, v)$ for all $v \in V$ over any sequence of relaxation steps on edges of G . Moreover, once $v.d$ achieves value $\delta(s, v)$, it never changes.

Proof. Prove by induction the claim $v.d \geq \delta(s, v)$ holds for all $v \in V$ on the number of relaxation steps. After initialization, $\infty = v.d \geq \delta(s, v)$ holds for all $v \in V \setminus \{s\}$, and since $s.d = 0 \geq \delta(s, s)$. For inductive step, we have that $x.d \geq \delta(s, x)$ for all $x \in V$. Assume we relax an edge (u, v) , only $v.d$ will be changed

$$v.d = u.d + w(u, v) \geq \delta(s, u) + w(u, v) \geq \delta(s, v)$$

by I.H. and triangular inequality. In addition $v.d$ never change once $v.d = \delta(s, v)$. This is because $v.d$ never decreases as $v.d \geq \delta(s, v)$ holds for all $v \in V$ just proved and no operation increases $v.d$ \square

2. **No-path property** Given a weighted, directed graph $G = (V, E)$ with $w : E \rightarrow \mathbb{R}$ and there is no path from s to v . Then after graph is initialized by INITIALIZE-SINGLE-SOURCE(G, s), we have $v.d = \delta(s, v) = \infty$ and this equality is maintained as an invariant over any sequence of relaxation steps on edges of G

Proof. By definition of shortest path, $\delta(s, v) = \infty$ as there is no path from s to v . By upper-bound property $v.d \geq \delta(s, v) = \infty$, hence $v.d = \delta(s, v) = \infty$ \square

Lemma. Let $(u, v) \in E$, then immediately after relaxing edge (u, v) by executing RELAX(u, v, w), we have $v.d \leq u.d + w(u, v)$

Proof. If $v.d > u.d + w(u, v)$, then by RELAX, $v.d = u.d + w(u, v)$. If $v.d \leq u.d + w(u, v)$, $v.d$ not updated. Hence $v.d \leq u.d + w(u, v)$ afterwards \square

3. **Convergence property (Given $u.d = \delta(s, u)$, $v.d \xrightarrow{\text{Relax}(u, v)} \delta(s, v)$)** Let $G = (V, E)$ be weighted, directed graph with weight function $w : E \rightarrow \mathbb{R}$, let $s \in V$ be source vertex, and let $s \rightsquigarrow u \rightarrow v$ is a shortest path in G for some $u, v \in V$. Suppose G is initialized by INITIALIZE-SINGLE-SOURCE(G, s) and then execute a sequence of relaxation steps that includes the call RELAX(u, v, w) on edges of G . If $u.d = \delta(s, u)$ at any time prior to relaxing edge (u, v) , then $v.d = \delta(s, v)$ at all times afterwards

Proof. If $u.d = \delta(s, u)$ at any time prior to relaxing (u, v) , then by upper-bound property, $u.d = \delta(s, u)$ stays the same. After relaxation on (u, v) , by previous lemma

$$v.d \leq u.d + w(u, v) = \delta(s, u) + w(u, v) = \delta(s, v)$$

the last equality given by optimal substructure of shortest path. By upper-bound property, $v.d \geq \delta(s, v)$, hence $v.d = \delta(s, v)$ and this property is maintained afterwards \square

4. **Path-relaxation property** Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \rightarrow \mathbb{R}$ and let $s \in V$ be source vertex. Consider any shortest path $p = \langle v_0, \dots, v_k \rangle$ from $s = v_0$ to v_k . If G is initialized with INITIALIZE-SINGLE-SOURCE(G, s) and then a sequence of relaxation steps occurs that includes, in order, relaxing the edges $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$. This property holds regardless of any other relaxation steps that occur, even if they are intermixed with relaxations of the edges of p

Proof. Proof by induction $v_i.d = \delta(s, v_i)$ holds on i -th vertex in p relaxed. When $i = 0$, $v_0.d = s.d = 0 = \delta(s, s)$, by upper-bound property, the value never changes afterwards. In induction step, assume $v_{i-1}.d = \delta(s, v_{i-1})$. After relaxation of (v_{i-1}, v_i) , $v_i.d = \delta(s, v_i)$ by convergence property and the equality is maintained thereafter by upper-bound property \square

Proposition. Relaxation and Shortest-paths tree

1. Let $G = (V, E)$ be a weighted, directed graph with $w : E \rightarrow \mathbb{R}$, let $s \in V$ be a source vertex, and assume G contains no negative-weight cycles that are reachable from s . Then after the graph is initialized with $\text{INITIALIZE-SINGLE-SOURCE}(G, s)$, the predecessor subgraph G_π forms a **rooted tree** with root s , and any sequence of relaxation steps on edges of G maintains this property as an invariant.

Proof. Proof consists of proving G_π is an acyclic graph by contradiction (i.e. assume there is a cycle and prove the cycle is in fact a negative weight cycle, which contradicts assumption of the problem). Then proving the graph is rooted at s , i.e. proving there is unique simple path from s to v in G_π \square

2. **Predecessor-subgraph property** Let $G = (V, E)$ be a weighted, directed graph with $w : E \rightarrow \mathbb{R}$, let $s \in V$ be a source vertex, and assume G contains no negative-weight cycles that are reachable from s . Then after the graph is initialized with $\text{INITIALIZE-SINGLE-SOURCE}(G, s)$ and execute any sequence of relaxation steps on edges of G that produces $v.d = \delta(s, v)$ for all $v \in V$, then the predecessor subgraph G_π is a shortest-path tree rooted at s .

Proof. Prove 3 properties of shortest-path trees given.

- (a) Prove V_π is the set of vertices reachable from s . Let $v \in V$ be not reachable from s , hence $\delta(s, v) = \infty$. Since $v.d$ and $v.\pi$ updated together in RELAX, implying $v.\pi = \text{NIL}$ and hence $v \notin V_\pi$
- (b) Prove G_π forms a rooted tree with root s , follows from previous proposition
- (c) Prove for all $v \in V_\pi$, the unique simple path $s \stackrel{p}{\rightsquigarrow} v$ in G_π is a shortest path from s to v in G . Let $p = \langle v_0, \dots, v_k \rangle$ where $v_0 = s$ and $v_k = v$. For $i = 1, \dots, k$, we have $v_i.d = \delta(s, v_i)$ (Path-Relaxation property) and $v_i.d \geq v_{i-1}.d + w(v_{i-1}, v_i)$, hence $w(v_{i-1}, v_i) \leq \delta(s, v_i) - \delta(s, v_{i-1})$. Summing weights along p

$$w(p) = \sum_{i=1}^k w(v_{i-1}, v_i) = \sum_{i=1}^k (\delta(s, v_i) - \delta(s, v_{i-1})) = \delta(s, v_k) - \delta(s, v_0) = \delta(s, v_k)$$

hence $w(p) = \delta(s, v_k)$ and thus p is a shortest path from s to $v = v_k$ \square

24.2 Bellman-Ford algorithm $O(VE)$

Definition. Bellman-Ford algorithm

1. **Goal** solves single-source shortest-paths problem in which edges may be negative

- (a) returns a boolean indicating whether or not there is a negative-weight cycle that is reachable from source
 - (b) and the shortest path and their weight if no such cycle exists
2. **Implementation** works by progressively decreasing estimate $v.d$ until it achieves $\delta(s, v)$ by making $|V| - 1$ passes, where in each pass, every $v \in V$ is relaxed once.
 3. **Runtime** $O(VE)$, initialization $\Theta(V)$, each $|V| - 1$ passes takes $\Theta(E)$ times (since relax every $e \in E$ requires traversing the entire adjacency list).

Lemma. Let $G = (V, E)$ be weighted, directed graph with source s and weight function $w : E \rightarrow \mathbb{R}$ assume G contains no negative-weight cycles that are reachable from s . Then after $|V| - 1$ iterations of for loop in the algorithm, we have $v.d = \delta(s, v)$ for all vertices v that are reachable from s

Proof. Let $v \in V$ be arbitrary vertices reachable from s , let $p = \langle v_0 = s, \dots, v_k = v \rangle$ be any shortest path from s to v . Since shortest path are simple, there are at most $|V| - 1$ edges. So $k \leq |V| - 1$. Since each of $|V| - 1$ iterations relax all $|E|$ edges, amongst them is the edge (v_{i-1}, v_i) . By path-relaxation property, $v.d = v_k.d = \delta(s, v_k) = \delta(s, v)$ \square

Corollary. Let $G = (V, E)$ be weighted, directed graph with source s and weight function $w : E \rightarrow \mathbb{R}$ assume G contains no negative-weight cycles that are **reachable from s** . For each $v \in V$, there is a path from s to v if and only if BELLMAN-FORD terminates with $v.d < \infty$ when it is run on G

Theorem. Correctness of Bellman-Ford Algorithm Let BELLMAN-FORD be run on a weighted, directed graph $G = (V, E)$ with source s and weight $w : E \rightarrow \mathbb{R}$. If G contains no negative-weight cycles that are reachable from S , then the algorithm returns TRUE, we have $v.d = \delta(s, v)$ for all vertices $v \in V$, and the predecessor subgraph G_π is a shortest-paths tree rooted at s . If G does contain a negative-weight cycle reachable from s , then the algorithm returns FALSE

Proof. Suppose G contains **no negative-weight cycles**. Prove $v.d = \delta(s, v)$ for all vertices $v \in V$. If v is reachable from s , previous lemma proves the claim. Otherwise v not reachable from s , then claim follows from no-path property, i.e. $v.d = \delta(s, v) = \infty$. The predecessor subgraph property, along with the claim, implies G_π is shortest path tree. Now prove the algorithm returns TRUE. At termination, for all $v \in V$

$$v.d = \delta(s, v) \leq \delta(s, u) + w(u, v) = u.d + w(u, v)$$

so none of test for negative cycle in the algorithm returns FALSE hence will return TRUE. Suppose G has negative cycles reachable from s , let $c = \langle v_0, \dots, v_k \rangle$, where $v_0 = v_k$, then

$$\sum_{i=1}^k w(v_{i-1}, v_i) < 0$$

Prove by contradiction, if the algorithm returns TRUE, then we have

$$v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$$

for $i = 1, \dots, k$. Summing equalities around the cycle

$$\sum_{i=1}^k v_i.d \leq \sum_{i=1}^k (v_{i-1}.d + w(v_{i-1}, v_i)) = \sum_{i=1}^k v_{i-1}.d + \sum_{i=1}^k w(v_{i-1}, v_i)$$

Note $\sum_{i=1}^k v_i.d = \sum_{i=1}^k v_{i-1}.d$ since cycles hold the same vertices despite difference in the way they are indexed. Note $v_i.d$ is finite, so

$$\sum_{i=1}^k w(v_{i-1}, v_i) \geq 0$$

which contradicts the negative cycle assumption. \square

24.3 Single-Source shortest paths in directed acyclic graphs $O(V + E)$

Definition. DAG Shortest-Path

1. **Motivation** Increase runtime by relaxing edges according to a topological sort of its vertices (so that we can use path-relaxation property and only relax every edge once). The drawback is that the algorithm assumes a acyclic graph
2. **Implementation**
 - (a) Topologically sort the dag, i.e. if there is p such that $u \xrightarrow{p} v$, then u precedes v
 - (b) Then make one pass over vertices in topologically sorted order. Relax each edge that leaves the vertex
3. **Runtime** $O(V + E)$. Topological sort takes $\Theta(V + E)$ time, INITIALIZE-SINGLE-SOURCE takes $\Theta(V)$. There is 1 pass where each edge is relaxed exactly once, each taking $O(1)$, hence amounts to $\Theta(V + E)$

Theorem. Correctness of DAG Shortest-Path algorithm If a weighted, directed graph $G = (V, E)$ has source vertex s and no cycles, then at termination of DAG-SHORTEST-PATHS procedure, $v.d = \delta(s, v)$ for all vertices $v \in V$, and the predecessor subgraph G_π is a shortest-path tree

Proof. Show $v.d = \delta(s, v)$ for all $v \in V$ at termination. If v not reachable from s , $v.d = \delta(s, v) = \infty$ by no-path property. If v is reachable from s , then there is a shortest path $p = \langle v_0 = s, \dots, v_k \rangle = v$. Because we process vertices in topological sorted order, the edges are relaxed in order. The path-relaxation property implies $v_i = \delta(s, v_i)$ at termination. The predecessor subgraph property implies G_π is a shortest path tree \square

24.4 Dijkstra's Algorithm $O(V^2)$ or $O(E \lg V)$

Definition. *Dijkstra's algorithm*

1. **Use case** Solves single-source shortest-paths problem on a weighted, directed graph $G = (V, E)$ for the case in which all edges weights are **nonnegative**, i.e. $w(u, v) \geq 0$ for all $(u, v) \in E$
2. **Implementation**
 - (a) Maintains set S of vertices whose final shortest-path weights from s have already been determined.
 - (b) Repeatedly selects a vertex $u \in V \setminus S = Q$, implemented with min-priority queue, with minimum shortest-path estimate.
 - (c) Adds u to S
 - (d) Relax all edges leaving u , i.e. $\text{Adj}[u]$
3. **Greedy** Since it chooses the lightest/closest vertex in $V \setminus S$ to add to set S
4. **Analysis Min-priority queue** INSERT EXTRACT-MIN called once per vertex, since each $u \in V$ added to S exactly once. The loop iterates $|E|$, size of adjacency list, and DECREASE-KEY is called at most once per loop (in RELAX). The runtime depends on how min-priority queue is implemented
 - (a) **Array** INSERT and DECREASE-KEY $O(1)$, EXTRACT-MIN $O(V)$ (have to go through entire array) total time $O(V^2 + E) = O(V^2)$
 - (b) **binary min-heap** INSERT, DECREASE-KEY and EXTRACT-MIN take $O(\lg n)$. Total runtime $O((V + E) \lg V)$, which is $O(E \lg V)$ if all vertices are reachable from source. Good if graph is sparse
 - (c) **Fibonacci heap** $O(V \lg V + E)$
5. **Comparison** Both Dijkstra's and Prim's algorithm uses a min-priority queue and grow the tree from source s , while updating other vertices

Theorem. Correctness of Dijkstra's algorithm Dijkstra's algorithm run on a weighted, directed graph $G = (V, E)$ with nonnegative weight w and source s , terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$.

Proposition. *The Loop invariant*

At the start of each iteration, $v.d = \delta(s, v)$ for all vertex $v \in S$

It's enough to show for each vertex $u \in V$, $u.d = \delta(s, u)$ at time when u is added to the set, The upper-bound guarantees $u.d = \delta(s, u)$ holds afterwards

Proof. Prove algo correct by proving invariant holds

1. **Initialization** Initially, $S = \emptyset$, hence invariant trivially true.
2. **Maintenance** Now we prove $u.d = \delta(s, u)$ for u added to S . Prove by contradiction, let u be first vertex added for which $u.d \neq \delta(s, u)$ when it was added to the set. Note $u \neq s$ since s is first added with $s.d = \delta(s, s) = 0$, hence $S \neq \emptyset$. Also there must be some path connecting s to u , otherwise $u.d \neq \delta(s, u) = \infty$ which violates assumption that $u.d \neq \delta(s, u)$. If there is a path, there is a shortest path, let p be such path that connects $s \in S$ to $u \in V \setminus S$. Then at some point p crosses the cut $(S, V \setminus S)$. Let $y \in V \setminus S$ be the first vertices and x be y 's parent, i.e. $y.\pi = x$. Now we decompose p

$$s \xrightarrow{p_1} x \rightarrow y \xrightarrow{p_2} u$$

Now we claim $y.d = \delta(s, y)$ when u is added to S . This is true because u is the first vertex added to S such that $u.d \neq \delta(s, u)$. Since $x \in S$, by I.H. we have $x.d = \delta(s, x)$ when x was added to S . Then (x, y) is relaxed at that time, hence the claim follows by convergence property. Now we obtain a contradiction, since y comes before u on a shortest path from s to u and all other edge in p_2 weights are non-negative, we have $\delta(s, y) \leq \delta(s, u)$, hence

$$y.d = \delta(s, y) \leq \delta(s, u) \leq u.d$$

But since both u and y is in $V \setminus S$ when u was chosen and we picked u instead of y hence $u.d \leq y.d$. The two inequalities yield a equality

$$y.d = \delta(s, y) = \delta(s, u) = u.d$$

Hence $\delta(s, u) = u.d$ contradicts the choice of u . Hence $u.d = \delta(s, u)$ when it was first added to S .

3. **Termination** At termination $Q = V \setminus S = \emptyset$, hence $S = V$, hence by previous invariant, $u.d = \delta(s, u)$ for all $u \in V$

□

25 All-Pairs Shortest Path

Definition. All-Pairs shortest path

1. **Goal** Given $G = (V, E)$ with weight $w : E \rightarrow \mathbb{R}$. Find, for every pair $u, v \in V$, a shortest (least weight) path from u to v . Want to output in tabular form: each entry in u 's row and v 's column should be weight of a shortest path from u to v
2. **Naive solution** Run single-source shortest path algorithm $|V|$ times, once for each vertex as the source. Non-negative weight, use Dijkstra's algorithm, the min-heap impl of min-priority queue yields a runtime of $O(VE \lg V)$, fibonacci heap impl yields runtime of $O(V^2 \lg V + VE) = O(V^3)$. If have non-negative weights have to use Bellman-Ford algorithm, with runtime of $O(V^2 E)$, which is $O(V^4)$ if graph is dense.

3. **Representation of Graph** Use matrix representation. Assume vertices numbered $1, 2, \dots, |V|$, an n -vertex directed $G = (V, E)$ is represented as a $n \times n$ matrix $W = (w_{ij})$ representing edge weights.

$$w_{ij} = \begin{cases} 0 & \text{if } i = j \\ \text{weight of directed edge } (i, j) & \text{if } i \neq j \wedge (i, j) \in E \\ \infty & \text{if } i \neq j \wedge (i, j) \notin E \end{cases}$$

The all-pairs shortest-path algorithm outputs $n \times n$ matrix $D = (d_{ij})$, where $d_{ij} = \delta(i, j)$. A **Predecessor Matrix** $\Pi = (\pi_{ij})$, such that π_{ij} is NIL if either $i = j$ or there is no path from i to j , otherwise π_{ij} is predecessor of j on some shortest path from i .

25.1 Shortest path and matrix multiplication with DP $O(V^3 \lg V)$

Definition. Shortest path and matrix multiplication

1. **Structure of shortest path** Given $W = (w_{ij})$, consider shortest path p from i to j , where p has m edges, assume no negative-weight cycles, and m is finite. If $i = j$, p has weight 0 and no edges. If $i \neq j$, then we can decompose p into

$$i \xrightarrow{p'} k \rightarrow j$$

where path p' now contains at most $m - 1$ edges, By optimal substructure of shortest path, p' is a shortest path from i to k , and so $\delta(i, j) = \delta(i, k) + w_{kj}$

2. **Recursive solution** Let $l_{ij}^{(m)}$ be the minimum weight of any path from vertex i to vertex j that contains at most m edges. When $m = 0$, there is a shortest path from i to j with no edges if and only if $i = j$, hence

$$l_{ij}^{(0)} = \begin{cases} 0 & \text{if } i = j \\ \infty & \text{if } i \neq j \end{cases} \quad l_{ij}^{(m)} = \text{Min} \left\{ l_{ij}^{(m-1)}, \text{Min}_{1 \leq k \leq n} \{ l_{ik}^{(m-1)} + w_{kj} \} \right\} = \text{Min}_{1 \leq k \leq n} \{ l_{ik}^{(m-1)} + w_{kj} \}$$

The first term is the weight of a shortest path from i to j in potentially $m - 1$ edges, the latter is the minimum weight of paths, where all possible predecessor k of j is explored. The latter simplification is because $w_{jj} = 0$. The actual shortest-path weights are given by

$$\delta(i, j) = l_{ij}^{(n-1)} = l_{ij}^{(n)} = \dots$$

since a path from i to j with more than $n - 1$ edges is not simple anymore and hence cannot have a lower weight than a shortest path from i to j in under $n - 1$ edges

3. **Bottom Up approach** The algorithm computes a series of matrices $W = L^{(1)}, L^{(2)}, \dots, L^{(n-1)}$ for $m = 1, \dots, n-1$ and $L^{(m)} = (l_{ij}^{(m)})$ and the final matrix $L^{(n-1)}$ contains the shortest path weights. Requires 3 nested for loop, hence runtime of $O(n^3)$. The procedure is very much similar to matrix multiplication, where

$$c_{ij} = \sum_k a_{ik} \cdot b_{kj}$$

We have $L^{(m)} = L^{(m-1)} \cdot W$ where \cdot represent taking mins instead... The procedure EXTEND-SHORTEST-PATHS is run $n-1$ times to yield $L^{(n-1)}$ hence the total runtime amounts to $\Theta(n^4)$.

4. **Improvement in runtime** To improve the runtime, we notice that the matrix operation is associative and hence we can compute $L^{(n-1)}$ in $\lceil \lg(n-1) \rceil$ by computing $L^{(m)}$ such that m is a power of 2. And once we loop to a point where $m \geq n-1$, we have $L^{(m)} = L^{(n-1)}$ as $\delta(i, j) = l_{ij}^{(n-1)} = l_{ij}^{(n)} = \dots$. The total runtime is improved to $O(n^3 \lg n) = O(V^3 \lg V)$. The improvement lies in the fact that since there is no elaborate data structure, constant hidden in Θ is therefore small

The FLloyd-Warshall algorithm $\Theta(V^3)$

Definition. Structure of a shortest path

1. **Concepts** Consider *intermediate vertices* of a shortest path $p = \langle v_1, \dots, v_l \rangle$ is the set $\{v_2, \dots, v_{l-1}\}$
2. **Observation** Assume $V = \{1, 2, \dots, n\}$ For some subset $\{1, 2, \dots, k\} \subseteq V$. Let $i, j \in V$ and p be a minimum-weight path from i to j with all intermediate vertices in $\{2, \dots, k-1\}$.
 - (a) If k is not an intermediate vertex of p , The shortest path p with all intermediate vertices in $\{1, \dots, k\}$ is also in $\{1, \dots, k-1\}$
 - (b) If k is an intermediate vertex of p , then decompose p

$$i \xrightarrow{p_1} k \xrightarrow{p_2} j$$

By optimal substructure of shortest path, p_1 is a shortest path from i to k with all intermediate vertices in $\{1, 2, \dots, k\}$. Since k is not an intermediate vertex, all intermediate vertices of p_1 are in $\{1, 2, \dots, k-1\}$. Hence

p_1 is a shortest path from i to k with all intermediate vertices in the set $\{1, 2, \dots, k-1\}$; Similarly, p_2 is a shortest path from k to j with all intermediate vertices in the set $\{1, 2, \dots, k-1\}$

3. **Recursive solution** Let $d_{ij}^{(k)}$ be weight of a shortest path from i to j for which all intermediate vertices are in the set $\{1, 2, \dots, k\}$. Note when $k = 1$, the set $\{1, 0\}$ has no intermediate vertex and includes i and j respectively and has one edge (i, j)

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 \\ \text{Min} \{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\} & \text{if } k \geq 1 \end{cases}$$

Hence $D^{(n)} = (d_{ij}^n)$ gives the right answer since all intermediate sets are in $\{1, \dots, n\}$.
So

$$d_{ij}^{(n)} = \delta(i, j) \quad \text{for all } i, j \in V$$

4. **Bottom Up Approach** Runtime $O(V^3)$ because of the triple for loop, each taking $O(1)$ to look up previously computed values and calculate the minimum. Again, the code is tight, and so constant hidden in Θ notation is small

5. **Constructing shortest path Π**

- (a) from D of shortest path weights after computing D
(b) at the same time D is calculated

Definition. Transitive Closure of a directed graph $G^* = (V, E^*)$ where

$$E^* = \{(i, j) : \text{there is a path from vertex } i \text{ to } j \text{ in } G\}$$

Solutions

1. We can compute transitive closure by assign weight of 1 to each edge in E and run Floyd-Warshall algorithm. So if $d_{ij} < \infty$ there is a path from i to j otherwise $d_{ij} = \infty$
2. To save time and space we substitute logical operations and and lor with arithmetic operation in Floyd-Warshall algorithm Define t_{ij}^k be 1 if there is a path from i to j with all intermediate set in G and 0 otherwise.

$$t_{ij}^{(0)} = \begin{cases} 0 & \text{if } i \neq j \text{ and } (i, j) \notin E \\ 1 & \text{if } i = j \text{ or } (i, j) \in E \end{cases} \quad t_{ij} = t_{ij}^{(k-1)} \vee (t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)})$$

Then compute $T^{(k)} = (t_{ij}^{(k)})$ in order of increasing k (bottom up). The runtime is $\Theta(n^3)$, same as previous algorithm. But is quite faster and memory efficient since operates on bits (logical) instead of on integer words (arithmetic)

Maximum Flow

26.1 Flow Networks

Definition. Flow networks

1. **Flow network** A flow network $G = (V, E)$ is a directed graph in which
 - (a) each edge $(u, v) \in E$ has a nonnegative **capacity** $c(u, v) \geq 0$.
 - (b) if $(u, v) \in E$, then the edge in reverse direction $(v, u) \notin E$
 - (c) if $(u, v) \notin E$, then $c(u, v) = 0$
 - (d) No self-loops
 - (e) **source** s and a **sink** t
 - (f) Assume each vertex lies on some path from s to t , i.e. for all $v \in V$, we have $s \rightsquigarrow v \rightsquigarrow t$
 - (g) $|E| \geq |V| - 1$ since each vertex other than s has at least one entering edge
2. **Flow** Let $G = (V, E)$ be flow network with capacity function c . A flow in G is a real-valued function $f : V \times V \rightarrow \mathbb{R}$ satisfying
 - (a) **Capacity Constraint** For all $u, v \in V$, we have $0 \leq f(u, v) \leq c(u, v)$
 - (b) **Flow Conservation** For all $u \in V \setminus \{s, t\}$, we have

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v)$$

If $(u, v) \notin E$, then no flow from u to v and $f(u, v) = 0$. Denote $f(u, v)$ the flow from vertex u to v . The **value of $|f|$ of a flow f** is defined as difference between total flow out of source and total flow into sink

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

as a typical flow network does not have edges into source s we have

$$|f| = \sum_{v \in V} f(s, v)$$

3. **Maximum-Flow problem** Given flow network G with source s and sink t , find a flow f of maximum value $|f|$
4. **Transformation to flow network**
 - (a) **Antiparallel edges** An antiparallel edge is the pair (v_1, v_2) and (v_2, v_1) , which violates flow network. We can transform such graph into a flow network by taking one edge and decompose into 2 edges with an additional intermediate vertex, while set bot new edges' capacity constraint to the original edge. The two graphs are equivalent
 - (b) **Multiple sources and sinks** Add a **supersource** s and directed edge (s, s_i) with capacity $c(s, s_i) = \infty$ for each $i = 1, \dots, n$ and likewise add a **supersink** t with directed edge (t_i, t) with capacity $c(t_i, t) = \infty$. In other words, provided unlimited flow as desired for multiple sources s_i and sinks t_i . The two graphs are equivalent

26.2 The Ford-Fulkerson Method

Definition. General Steps

1. let $f(u, v) = 0$ for all $u, v \in V$
2. At each step, increase flow value in G by finding an **augmenting path** in an associated **residual network** G_f
3. Repeat until the residue network has no more augmenting paths

Definition. Residual Network

1. **General Idea** G_f consists of edges with capacities that represent how we can change the flow on edges of G .
 - (a) An edge (u, v) of G can admit $c(u, v) - f(u, v) = c_f(u, v)$ amount of additional flow (if edge has flow equal to capacity then $c_f(u, v) = 0$)
 - (b) An edge (u, v) of G can also reduce their flow by an amount up to $f(u, v) = c_f(v, u)$. The edge (v, u) placed in G_f is able to admit flow in opposite direction to (u, v) , at most cancelling out the flow on (u, v)
2. **Residual Capacity** Given flow network G and a flow f . Consider $u, v \in V$, the residual capacity $c_f(u, v)$ is defined by

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E \\ f(v, u) & \text{if } (v, u) \in E \\ 0 & \text{otherwise} \end{cases}$$

Note since flow network disallows antiparallel edges, exactly one of the cases applies

3. **Residual Network** Given flow network G and flow f , the residual network of G induced by f is $G_f = (V, E_f)$ where

$$E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\}$$

4. **Residual Edge** Edges in residual network is called residual edge E_f , which can be either edges in E , their reversal, or both

$$|E_f| \leq 2|E|$$

5. **Augmentation** If f is a flow in G and f' is a flow in corresponding residual network G_f , then $f \uparrow f'$, the augmentation of flow f by f' , to be a function $(f \uparrow f') : V \times V \rightarrow \mathbb{R}$

$$(f \uparrow f')(u, v) = \begin{cases} f(u, v) + f'(u, v) - f'(v, u) & \text{if } (u, v) \in E \\ 0 & \text{otherwise} \end{cases}$$

The idea is we increase flow on $(u, v) \in V$ by $f'(u, v)$ but decrease it by $f'(v, u)$ because pushing flow on reverse edge in residual network signifies decreasing the flow in the original network, this is called **cancellation**

Lemma. Let $G = (V, E)$ be flow network with source s and sink t , let f be a flow in G . Let G_f be residual network of G induced by f , and f' be a flow in G_f . Then $f \uparrow f'$ is a flow in G with value $|f \uparrow f'| = |f| + |f'|$

Definition. Augmenting Paths (Improves value of flow)

1. **Augmenting Path** Given flow network G and a flow f , an augmenting path p is a simple path from s to t in the residual network G_f .
2. **Residual Capacity of an Augmenting Path** The maximum amount by which we can increase the flow on each edge in an augmenting path p the residual capacity of p (such that capacity constraint is satisfied in G)

$$c_f(p) = \text{Min}\{c_f(u, v) : (u, v) \text{ is on } p\}$$

3. **Flow of an Augmenting Path** We get a flow of an augmenting path p by assigning the residual capacity of p , i.e. $c_f(p)$, to every edge on the path p . Define function $f_p : V \times V \rightarrow \mathbb{R}$ by

$$f_p(u, v) = \begin{cases} c_f(p) & \text{if } (u, v) \text{ is on } p \\ 0 & \text{otherwise} \end{cases}$$

Lemma. f_p is a flow in G_f with $|f_p| = c_f(p) > 0$

4. If we augment f by f_p , i.e. $f \uparrow f_p$, we get another flow in G whose value is closer to the maximum

Corollary. Let $G = (V, E)$ be a flow network, let f be a flow in G , and let p be an augmenting path in G_f . Let f_p be defined as previously, then the function $f \uparrow f_p$ is a flow in G with value

$$|f \uparrow f_p| = |f| + |f_p| > |f|$$

Proof. Follows from $|f_p| = c_f(p) > 0$ and $|f \uparrow f'| = |f| + |f'|$ □

Definition. Cut of Flow Networks (Determines when max flow is found)

1. **Cut** A cut (S, T) of a flow network $G = (V, E)$ is a partition of V into S and $T = V \setminus S$ such that $s \in S$ and $t \in T$
2. **Net Flow across a cut** If f is a flow, then the net flow $f(S, T)$ across the cut (S, T) is defined to be

$$f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u)$$

Note **value of flow** $|f|$ is the net flow across cut $(\{s\}, V \setminus \{s\})$

3. **Capacity of a cut** The capacity of cut (S, T) is

$$c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v)$$

Note we only consider flow from S to T , ignoring edges in the reverse direction (different from flow which considers both directions)

4. **Minimum Cut** The minimum cut of a network is a cut whose capacity is minimum over all cuts of the network

Lemma. Conservation of net flow cross arbitrary cuts Let f be a flow in a flow network G with source s and sink t , and let (S, T) be any cut of G . Then the net flow across (S, T) is $f(S, T) = |f|$

Corollary. The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G . (Implies that optimal $|f|$ is minimum capacity of all cuts in G)

Proof. Let (S, T) be any cut of G and f be any flow. By previous lemma we have

$$|f| = f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u) \leq \sum_{u \in S} \sum_{v \in T} f(u, v) \leq \sum_{u \in S} \sum_{v \in T} c(u, v) = c(S, T)$$

□

Theorem. Max-flow Min-cut theorem If f is in a flow network $G = (V, E)$ with source s and sink t , then the following conditions are equivalent

1. f is a maximum flow in G
2. The residual network G_f contains no augmenting paths
3. $|f| = c(S, T)$ for some cut (S, T) of G

Proof. Prove 3 parts

- (1) \rightarrow (2) Prove by contradiction. Assume f is a maximum flow in G but G_f has an augmenting path p with flow f_p . If we augment f by f_p , we have $|f \uparrow f_p| = |f| + |f_p| > |f|$, implies there is a larger flow value, contradicting f is the maximum flow
- (2) \rightarrow (3) Idea is to identify cut (S, T) , infer value of $f(u, v)$ from the fact there exists no path from s to t in G_f , then calculate net flow $f(S, T)$ across an arbitrary cut, which is identical for any cut, including $|f|$. Assume G_f has no augmenting path, that is there is no path from s to t , Define

$$S = \{v \in V : \text{there exists a path from } s \text{ to } v \text{ in } G_f\}$$

and $T = V \setminus S$. Consider vertices $u \in S$ and $v \in T$. $(u, v) \notin E_f$

1. If $(u, v) \in E$, then $f(u, v) = c(u, v)$ since otherwise we have $c_f(u, v) = c(u, v) - f(u, v) > 0$, implying $(u, v) \in E_f$
2. If $(v, u) \in E$, then $f(u, v) = 0$ since otherwise we have $c_f(u, v) = f(v, u) > 0$, implying $(u, v) \in E_f$
3. If $(u, v) \notin E$ or $(v, u) \notin E$, then $f(u, v) = f(v, u) = 0$

Now we compute a net flow over the cut (S, T) in G

$$f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{v \in T} \sum_{u \in S} f(v, u) = \sum_{u \in S} \sum_{v \in T} c(u, v) - \sum_{v \in T} \sum_{u \in S} 0 = c(S, T)$$

By previous corollary, net flow is same for all arbitrary cuts, we have

$$|f| = f(S, T) = c(S, T)$$

- (3) \rightarrow (1) By previous corollary, the value of flow $|f|$ is bounded above by capacity of any cuts. $|f| \leq c(S, T)$. hence when $|f| = c(S, T)$ implies f is a maximum flow

□

Definition. Ford-Fulkerson algorithm $O(E|f^*|)$

1. **Steps**

- (a) Initialize $(u, v).f$ to 0
- (b) Loop if there exists an augmenting path p from s to t in residual network G_f
- (c) Find residual capacity of the path $c_f(p) = \text{Min}\{c_f(u, v) : (u, v) \text{ is in } p\}$ in G_f
- (d) We replace f with $f \uparrow f_p$ to obtain a new flow whose value is $|f| + |f_p|$
 - i. If $(u, v) \in E$, i.e. residual edge in p is an edge in the original network, $(u, v) \in G_f$ specifies how much flow $(u, v) \in G$ can increase by, so add $c_f(p)$ amount of flow to $(u, v) \in G$
 - ii. If $(v, u) \in E$, i.e. residual edge in p is a reverse edge in the original network, $(u, v) \in G_f$ specifies how much flow $(v, u) \in G$ can decrease by, so decrease $c_f(p)$ amount of flow to $(v, u) \in G$

2. **Analysis** Runtime depends on finding the augmenting path p .

- (a) **Initialization** $O(E)$
- (b) **While loop** If capacity is rational, scale to integer. If f^* is the max flow, FORD-FULKERSON executes while loop at most $|f^*|$ times, since flow value increases by at least one unit in each iteration.
- (c) **Finding path** Assume we have a data structure representing a directed graph $G' = (V, E')$ where $E' = \{(u, v) : (u, v) \in E \vee (v, u) \in E\}$. The edges in G_f consists off all edges $(u, v) \in E'$ such that $c_f(u, v) > 0$. If use DFS, or BFS, runtime $O(V + E') = O(E)$ (since $|E| \geq |V| - 1$) for finding a path from s to t .

In summary runtime of $O(E|f^*|)$. The algorithm is good if capacities are integral and the optimal flow value $|f^*|$ is small.

26.3 Maximum Bipartite Matching $O(VE)$

Definition. *Matching*

1. **Matching** Given undirected graph $G = (V, E)$, a matching is a subset of edges $M \subseteq E$ such that for all vertices $v \in V$, at most one edge of M is incident on v . (each edge symbolizes a pair)
2. **Matched and Unmatched** a vertex $v \in V$ is matched by the matching M if some edge in M is incident on v ; otherwise v is unmatched
3. **Maximum Matching** A maximum matching is a matching of maximum cardinality, that is, a matching M such that for any matching M' , we have $|M| \geq |M'|$
4. **Bipartite graphs** graphs in which V can be partitioned into 2 disjoint sets $V = L \cup R$, $L \cap R = \emptyset$ and all edges in E go between L and R . Assume every vertex in V has at least one incident edge

Definition. *Finding a Maximum Bipartite Matching*

1. **Corresponding Flow Network** $G' = (V', E')$ (directed) for a bipartite graph $G = (V, E)$ (undirected) with partition $V = L \cup R$ is defined as follows

(a) let source s and sink t be new vertices not in V

$$V' = V \cup \{s, t\}$$

(b) let directed edge of G' be edges of E , directed from L to R , along with $|V|$ new directed edges connecting s to L and R to t

$$E' = \{(s, u) : u \in L\} \cup \{(u, v) : (u, v) \in E\} \cup \{(v, t) : v \in R\}$$

Note $|E'| = \Theta(E)$, since $|E| \geq |V|/2$ (every vertex has an incident edge) implies

$$\Omega(E) = |E| \leq |E'| = |E| + |V| \leq 3|E| = O(E)$$

(c) assign unit capacity to each edge in E'

2. **Integer-valued flow** A flow f on a flow network G is integer valued if $f(u, v)$ is an integer for all $(u, v) \in V \times V$
3. A matching in G corresponds to a flow in G 's corresponding flow network G'

Lemma. Let $G = (V, E)$ be a bipartite graph with vertex partition $V = L \cup R$ and let $G' = (V', E')$ be corresponding flow network. If $M \subseteq E$ is a matching in G , then there is an integer-valued flow f in G' with value $|f| = |M|$. Conversely, if f is an integer-valued flow in G' , then there is a matching M in G with cardinality $|M| = |f|$

Proof. 2 steps

- (a) Find matching M in G corresponds to flow f in G' . Define f as follows. If $(u, v) \in M$, then $f(s, u) = f(u, v) = f(v, t) = 1$. For all other edges $(u, v) \in E'$, define $f(u, v) = 0$. Hence each $(u, v) \in M$ corresponds to one unit of flow in G' traversing path

$$s \rightarrow u \rightarrow v \rightarrow t$$

The cut $(L \cup \{s\}, R \cup \{t\})$ is equal to $|M|$ by the previous definition, and hence $|f| = |M|$ (net flow same for any cuts)

- (b) Prove converse. Let f be integer-valued flow in G' , prove there is a matching such that $|M| = |f|$. Let

$$M = \{(u, v) : u \in L, v \in R, f(u, v) > 0\}$$

Prove M is a matching (i.e. all edge $v \in V$ has at most 1 edge $e \in M$ incident on v). For $u \in L$, has one entering edge (s, u) of one unit of flow, by flow conservation, must have one unit of flow leaving it. Since f integer-valued, one unit of flow enter on at most 1 edge and leave on at most 1 edge. Hence there cannot be 2 edges leaving $u \in L$. Hence, one unit of flow entering u if and only if there is exactly one vertex $v \in R$ such that $f(u, v) = 1$. Similar argument to R .

Hence maximum matching M in bipartite graph G corresponds to a maximum flow in its corresponding flow network G' . \square

4. By previous lemma, we can compute maximum matching in G by running max-flow algorithm on G' , the following theorem guarantees the output from FORD-FULKERSON will be a integer-valued flow

Theorem. If the capacity function c takes on only integral values, then maximum flow f produced by FORD-FULKERSON Method has the property that $|f|$ is an integer. Moreover, for all vertices u and v , the value $f(u, v)$ is an integer

Corollary. The cardinality of a maximum matching M in a bipartite graph G equals the value of a maximum flow f in its corresponding flow network G'

5. Steps

- (a) Create corresponding flow network G'
- (b) Run FORD-FULKERSON
- (c) Obtain maximum matching M from integer-valued maximum flow f found

6. **Runtime** Note any matching in bipartite graph has cardinality of

$$|M| \leq \min(L, R) = O(V)$$

the value of maximum flow in G' is hence $O(V)$, therefore maximum matching in a bipartite graph takes $O(|f^*|E') = O(VE') = O(VE)$, since $|E'| = \Theta(E)$