Chapter 6 Inner Product Spaces

6.1 Inner Products and Norms

Definition. ***Inner Product*** *Let V be a vector space over F. An inner product on V is a function that assigns, to every ordered pair of vectors x and y in V, a scalar in F, denoted $\langle x, y\rangle$, such that for all $x, y, z \in VE$ and all $c \in F$,*

1. $\langle x + z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$
2. $\langle cx, y\rangle = c\langle x, y\rangle$
3. $\overline{\langle x, y\rangle} = \langle y, x\rangle$
4. $\langle x, x\rangle > 0$ *if* $x \neq 0$

First two condition requires inner product be linear in the first component. Also

$$\langle \sum_i a_i v_i, y\rangle = \sum_i a_i \langle v_i, y\rangle$$

Definition. ***Conjugate Transpose or Adjoint of a Matrix*** *Let $A \in M_{m\times n}(F)$, the conjugate transpose or adjoint of A is an $n \times m$ matrix A^* such that $(A^*)_{ij} = \overline{A_{ji}}$ for all i, j. For $F = \mathbb{R}$, $A^* = A^T$*

Definition. ***Inner Product Definition Example***

1. ***Standard Inner Product on F^n*** *For $x = (a_1, a_2, \cdots, a_n)$ and $y = (b_1, b_2, \cdots, b_n)$ in F^n, the standard inner product on F^n is given by*

$$\langle x, y\rangle = \sum_{i=1}^{n} a_i \bar{b}_i$$

2. ***Inner Product for Real-valued Continuous Functions on*** $[0, 1]$ *Let $V = C([0, 1])$, $f, g \in V$, define*

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt$$

3. ***Frobenius Inner Product for Matrices*** *Let $V = M_{n\times n}(F)$, $A, B \in V$, then*

$$\langle A, B\rangle = tr(B^*A) = \sum_{i=1}^{n} (B^*A)_{ii}$$

Definition. ***Inner Product Space*** *A vector space over F endowed with a specific inner product is called an inner product space. If $F = C$, V is a complex inner product space; if $F = \mathbb{R}$, then V is a real inner product space*

Theorem. ***6.1 Properties From Inner Product Conditions*** *Let V be an inner product space. Then for $x, y, z \in V$ and $c \in F$, the following statements are true*

1. $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$
2. $\langle x, cy \rangle = \bar{c}\langle x, y \rangle$
3. $\langle x, 0 \rangle = \langle 0, x \rangle = 0$
4. $\langle x, x \rangle = 0$ *if and only if* $x = 0$
5. *If* $\langle x, y \rangle = \langle x, z \rangle$ *for all* $x \in V$*, then* $y = z$

The inner product is conjugate linear in the second argument

Definition. ***Norm/Length*** *Let V be an inner product space. For $x \in V$, define norm or length of x by*

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Definition. ***6.2 Properties of Norm*** *Let V be an inner product space over F. Then for all $x, y \in V$ and $c \in F$, the following statements are true*

1. $\|cx\| = |c| \cdot \|x\|$
2. $\|x\| = 0$ *if and only if* $x = 0$*. In any case,* $\|x\| \geq 0$
3. ***Cauchy-Schwarz Inequality*** $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$
4. ***Triangular Inequality*** $\|x + y\| \leq \|x\| + \|y\|$

Definition. ***Angle*** *For $F = \mathbb{R}$, $x, y \neq 0$, and θ be angle between x and y*

$$\cos\theta = \frac{\langle x, y \rangle}{\|x\| \, \|y\|} \qquad\qquad \theta = \cos^{-1}\left(\frac{\langle x, y \rangle}{\|x\| \, \|y\|}\right)$$

Note

$$\left|\frac{\langle x, y \rangle}{\|x\| \, \|y\|}\right| \leq 1$$

So valid input to arccos function

Definition. ***Orthogonal Vectors*** *Let V be an inner product space. Vectors x and y in V are orthogonal (perpendicular) if $\langle x, y \rangle = 0$.*

Definition. ***Orthogonal Sets and Orthonormal Sets*** *A subset S of V is orthogonal if any two distinct vectors in S are orthogonal. A vector x in V is a unit vctor if $\|x\| = 1$. A subset S of V is orthonormal if S is orthogonal and consists entirely of unit vectors.*

1. $S = \{v_1, v_2, \cdots\}$*, then S is orthonormal if and only if* $\langle v_i, v_j \rangle = \delta_{ij}$

2. *We can **normalize** an orthogonal set S, by multiplying $1/\|x\|$ for each $x \in S$*

Definition. ***Orthonormal Set Property*** *Let V be inner product space and $S = \{s_1, s_2, \cdots\} \subseteq V$ be an orthonormal set. Let $v \in span(S)$, then $v = a_1 s_1 + \cdots + a_k s_k$. Then*

$$\langle v, s_j \rangle = a_j$$

by

$$\langle v, s_j \rangle = \langle \sum_i a_i s_i, s_j \rangle = \sum_i a_i \langle s_i, s_j \rangle = \sum_i a_i \delta_{ij} = a_j$$

Gram-Schmidt Orthogonalization Process and Orthogonal Complements

Definition. ***Orthonormal Basis*** *Let V be an inner product space. A subset of V is an orthonormal basis for V if it is an ordered basis that is orthonormal*

Definition. ***Every Inner Product Space has n Orthogonal Basis*** *Let V be an inner product space and $S = \{v_1, v_2, \cdots, v_k\}$ be an orthogonal subset of V consisting of nonzero vectors. If $y \in span(S)$, then*

$$y = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i$$

Corollary. ***Special case for Orthonormal Set*** *If, in addition to hypotheses of previous theorem, S is orthonormal and $y \in S$, then*

$$y = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$$

Corollary. ***Nonzero Orthonormal Set is Linearly Independent*** *Let V be an inner product space, andf let S be an orthogonal subset of V consisting of nonzero vectors. Then S is linearly independent*

Theorem. ***6.4 Gram-Schmidt Process*** *Let V be an inner product space and $S = \{w_1, w_2, \cdots, w_n\}$ be a linearly independent subset of V. Define $S' = \{v_1, v_2, \cdots, v_n\}$, where $v_1 = w_1$ and*

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j \qquad 2 \leq k \leq n$$

Then S' is an orthogonal set of nonzero vectors such that $span(S') = span(S)$

Theorem. ***6.5 Every Finite Dimensional I.P.S has an Orthonormal Basis*** *Let V be a nonzero finite-dimensional inner product space. Then V has an orthonormal basis β. Furthermore, if $\beta = \{v_1, v_2, \cdots, v_n\}$ and $x \in V$, then*

$$x = \sum_{x=1}^{n} \langle x, v_i \rangle v_i$$

Corollary. ***Expression for Matrix Representation of Transformation on Orthonormal Basis*** *Let V be a finite-dimensional inner product space with an orthonormal basis $\beta = \{v_1, v_2, \cdots, v_n\}$. Let T be a linear operator on V, and let $A = [T]_\beta$. Then for any i and j, $A_{ij} = \langle T(v_j), v_i \rangle$, i.e.*

$$T(v_j) = \sum_{i=1}^{n} \langle T(v_j), v_i \rangle v_i$$

Definition. ***Fourier Coefficients*** *Let β be an orthonormal subset (possibly infinite) of an inner product space V, and let $x \in V$. We define the Fourier coefficients of x relative to β to be the scalars $\langle x, y \rangle$, where $y \in \beta$*

Orthogonal Complements

Definition. ***Orthogonal Complements*** *Let S be a nonempty subset of an inner product space V. We define $S^\perp = \{x \in V : \langle x, y \rangle = 0 \text{ for all } y \in S\}$. The set $S^\perp$ is called the orthogonal complement of S*

1. $\{0\}^\perp = V$ *and* $V^\perp = \{0\}$

Theorem. ***6.6 Finding Projection of a Vector onto a Subspace*** *Let W be a finite-dimensional subspace of an inner product space V, and let $y \in V$. Then there exist unique vectors $u \in W$ and $z \in W^\perp$ such that $y = u + z$. Furthermore, if $\{v_1, v_2, \cdots, v_k\}$ is an orthonormal basis for W, then*

$$u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$$

where u is the orthogonal projection of y on W.

Corollary. ***Orthogonal Projection is Unique and Closest to Projected Vector*** *In the notation of previous theorem, the vector u the unique vector in W that is closest to y; that is, for any $x \in W$, $\|y - x\| \geq \|y - u\|$, and this inequality is an equality if and only if $x = u$*

Theorem. ***6.7 Orthonormal Basis and Subspaces*** *Suppose that $S = \{v_1, v_2, \cdots, v_k\}$ is an orthonormal set in an n-dimensional inner product space V. Then*

1. *S can be extended to an orthonormal basis $\{v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_n\}$ for V.*
2. *If $W = span(S)$, then $S_1 = \{v_{k+1}, \cdots, v_n\}$ is an orthonormal basis for $W^\perp$*
3. *If W is any subspace of V, then $dim(V) = dim(W) + dim(W^\perp)$*

6.3 The Adjoint of a Linear Operator

Definition. ***Dual Space*** *is a space of all linear transformations from a vector space V to its field F.*

Theorem. ***6.8 Every Linear Transformation from V to F Can Be Written as a Inner Product*** *Let V be a finite-dimensional inner product space over F, and let $g : V \to F$ be a linear transformation. Then there exists a unique vector $y \in V$ such that $g(x) = \langle x, y\rangle$ for all $x \in V$, where*

$$y = \sum_i \overline{g(v_i)} v_i \qquad \beta = \{v_1, \cdots, v_n\} \text{ is orthonormal basis}$$

Definition. ***Adjoint Linear Operator*** *Given inner product space V, let T be a linear operator on V. The adjoint of operator T, T^*, is the unique operator on V satisfying*

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle \qquad \text{for all } x, y \in V$$

Theorem. ***6.9 Adjoint of an Linear Operator Exist for f.d. Inner Product Space*** *Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Then there exists a unique function, called the adjoint of T, $T^* : V \to V$ such that*

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle$$

for all $x, y \in V$. Furthermore, T^* is linear. We can view the equation symbolically as adding an asterik $*$ to T when shifting position inside the inner product symbol

Theorem. ***6.10 Adjoint of a Linear Operator in Matrix Form is the Adjoint of Matrix Form of that Linear Operator*** *Let v be a finite-dimensional inner product space. Let β be an orthonormal basis for V. If T is a linear operator on V, thern*

$$[T^*]_\beta = [T]_\beta^*$$

Corollary. ***For Left-Matrix Transformation*** *Let A be $n \times n$ matrix, then $L_{A^*} = (L_A)^*$.* *(theorem 2.16)*

Theorem. ***6.11 Properties of Adjoint of Linear Operators***
Let V bewr an inner product space, and let T, U be linear operators on V, then

1. $(T + U)^* = T^* + U^*$
2. $(cT)^* = \bar{c}T^*$ *for any* $c \in F$
3. $(TU)^* = U^*T^*$
4. $T^{**} = T$
5. $I^* = I$

assuming adjoints always exists.

Corollary. ***For Matrix***
Let A and B be $n \times n$ matrix, then

1. $(A+B)^* = A^* + B^*$
2. $(cA)^* = \bar{c}A^*$ *for all* $c \in F$
3. $(AB)^* = B^*A^*$
4. $A^{**} = A$
5. $I^* = I$

Least Squares Approximation

Definition. ***Some notation*** *Fort $x, y \in F^n$*

1. $\langle x, y\rangle_n$ *is the standard inner product of x and y in F^n*
2. *If x and y are column vectors, then* $\langle x, y\rangle_n = y^*x$

Lemma. *Let $A \in M_{m\times n}(F)$, $x \in F^n$ and $y \in F^m$, then*

$$\langle Ax, y\rangle_m = \langle x, A^*y\rangle_n$$

Lemma. *Let $A \in M_{m\times n}(F)$. Then $rank(A^*A) = rank(A)$*

Corollary. *If A is $m \times n$ matrix such that $rank(A) = n$, then A^*A is invertible*

Theorem. ***6.12 Close Form Solution for Least Squared Problem*** *Let $A \in M_{m\times n}(F)$ and $y \in F^m$. Then there exists $x_0 \in F^n$ such that $(A^*A)x_0 = A^*y$ and $\|Ax_0 - y\| \leq \|Ax - y\|$ for all $x \in F^n$. Furthermore, if $rank(A) = n$, then $x_0 = (A^*A)^{-1}A^*y$*

6.4 Normal and Self-Adjoint Operators

Lemma. ***Condition on Existence of Eigenvector for Adjoint Linear Operators***
Let T be a linear operator on a finite-dimensional inner product space V. If T has an eigenvector, then so does T^*. If λ is an eigenvalue of T, then $\overline{\lambda}$ is an eigenvalue of T^*

Proof. Let v be eigenvector of T with corresponding eigenvalue λ, then for any $x \in V$,

$$0 = \langle 0, x\rangle = \langle (T - \lambda I)(v), x\rangle = \langle v, (T - \lambda I)^*(x)\rangle = \langle v, (T^* - \overline{\lambda} I)(x)\rangle$$

Let $W = span(\{v\})$, so $R(T^* - \overline{\lambda}I) \subseteq W^\perp$. Note $rank(T^* - \overline{\lambda}I) \leq dim(W^\perp) = n - 1$, then $N(T^* - \overline{\lambda}I) \neq \{0\}$. So exists $u \in N(T^* - \overline{\lambda}I)$ such that $T^*(u) = \overline{\lambda}u$ □

Theorem. ***6.14 (Schur's Theorem)***

$P_T(t)$ *Splits Implies Exists O.N. Basis st.* $[T]_\beta$ *is Upper Triangular*

Let T be a linear operator on a finite-dimensional inner product space V. Suppose that the characteristic polybnomial of T splits. Then there exists an orthonormal basis β for V such that the matrix $[T]_\beta$ is upper triangular

Proof. With induction, idea is to construct an orthonormal basis $\beta = \gamma \cup \{z\}$, where γ is an orthonormal basis for $W^\perp$ and $z \in W = span(z)$, where z is unit eigenvector for T^* whose existence ensured by previous lemma. The induction hypothesis mandates

1. $W^\perp$ is a T-invariant subspace as an assumption, i.e. if $y \in W^\perp, x \in W$, then $\langle T(y), x \rangle = 0$
2. $P_{T_{W^\perp}}(t) | P_T(t)$, so characteristic polyonmial of $T_{W^\perp}$ splits

to get the orthonormal basis γ, for which $[T_{W^\perp}]_\gamma$ is upper triangular. □

Definition. ***Normal Linear Operator*** *Let V be an inner product space, and let T be a linear operator on V. We say that T is normal if $TT^* = T^*T$. An $n \times n$ real or complex matrix A is normal if $AA^* = A^*A$ (Commutivity).*

1. *T is normal if and only if $[T]_\beta$ is normal, where β is an orthonormal basis*
2. *Skew-symmetric matrix ($A^t = -A$) is normal by $A^tA = -A^2 = AA^t$*
3. *Normality not sufficient to guarantee an orthonormal basis of eigenvectors. However, normality suffices if V is a complex inner product space*

Theorem. ***6.15 Properties of Normal Operator***
Let V be an inner product space, and let T be a normal operator on V. Then the following are true

1. *$\|T(x)\| = \|T^*(x)\|$ for all $x \in V$*
2. *$T - cI$ is normal for every $c \in F$*
3. *If x is an eigenvector of T, then x is also an eigenvector of T^*. In fact,if $T(x) = \lambda x$, then $T^*(x) = \overline{\lambda} x$*
4. *If λ_1 and λ_2 are distinct eigenvalues of T with corresponding eigenvectors x_1 and x_2, then x_1 and x_2 are orthogonal*

Theorem. ***6.16 Normal Operator iff Diagonalizable*** *($F = \mathbb{C}$)*
Let T be a linear operator on a finite-dimensional ***complex*** *inner product space V. Then T is normal if and only if there exists an orthonormal basis for V consisting of eigenvectors of T.*

Proof. Idea is the orthonormal basis that makes T an upper triangular matrix happens to be a set of eigenvectors. The proof consists of using fundamental theorem of algebra to show that $P_T(t)$ splits and by Schur's theorem there exists orthonrmal basis $\beta = \{v_1, v_2, \cdots, v_n\}$ for V such that $[T]_\beta = A$ is upper triangular. Then use induction on k to prove that $v_k \in \beta$ is in fact eigenvectors, given that $\{v_1, \cdots, v_{k-1}\}$ are eigenvectors, with base case $k = 1$, $T(v_1) = A_{11}v_1$ since A is upper triangular. The converse is true by $[T]_\beta$ diagonal and so $[T^*]_\beta$ also diagonal, diagonal matrix commute, so T is normal □

1. *example showing theorem does not work on infinite dimension vector spaces with problem definition here . Specifically an example where T is normal and that T has no eigenvectors*
2. *Normality not sufficient for existence of orthonormal basis of eigenvectors for real inner product spaces*

Definition. ***Self-Adjoint*** *Let T be a linear operator on an inner product space V. We say that T is self-adjoint (Hermitian) if $T = T^*$. An $n \times n$ real or complex matrix A is self-adjoint (Hermitian) if $A = A^*$*

1. *If β is orthonormal basis, then T is self-adjoint if and only if $[T]_\beta$ is self-adjoint (symmetric matrix for $F = \mathbb{R}$)*
2. *If T is self-adjoint, then T is normal*

Lemma. ***Properties of Self-Adjoints***
Let T be a self-adjoint operator on a finite-dimensional inner product space V. Then

1. *Every eigenvalue of T is real*
2. *Suppose that V is a real inner product space ($F = \mathbb{R}$). Then the characteristic polynomial of T splits*

Theorem. ***6.17 Self-Adjoints iff Diagonalizable*** *($F = \mathbb{R}$)*
Let T be a linear operator on a finite-dimensional real inner product space V. Then T is self-adjoint if and only if there exists an orthonormal basis β for v consisting of eigenvectors of T.

Definition. ***Computing Squared Root of Imaginary Number***
Relies on Euler's formula

$$e^{ix} = \cos x + i \sin x$$

Therefore we have

$$e^{i\pi} = -1 \qquad i = \sqrt{-1} = e^{i\pi/2} \qquad \sqrt{i} = e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$