APM462: Homework 1

Due: Sat, May 25 before 9pm.[1]

Recall that a function of the form $f(x) = \frac{1}{2}x \cdot Qx - b \cdot x$, where Q is positive definite $n \times n$ symmetric matrix and $x \in \mathbb{R}^n$, can be written in the form $f(x) = \frac{1}{2}(x-x^*)\cdot Q(x-x^*) - \frac{1}{2}x^* \cdot Qx^*$, where $x^* = Q^{-1}b$. This is basically what is called "completing the square". The first 2 questions explore different functions which essentially have this form.

(1) Let $\alpha \in \mathbb{R}$ and let $f_\alpha : \mathbb{R}^2 \to \mathbb{R}$ be given as $f_\alpha(x,y) = 2x^2 + \frac{\alpha}{2}y^2 + xy - y$.
   (a) For each α find the point satisfying the first order condition for a local minimum of f_α.
   (b) For which α do the points you found in (a) satisfy the second order condition for a local minimum of f_α?
   (c) Prove that the candidate you found in (b) for alocal minimun of f_α is in fact a global minimum. (Prove this in two different ways: by completing the square and by showing that f_α is a convex function.)

(2) Find the local minimum point(s) for the function f below on the set $\{(x,y,z) \in \mathbb{R}^3 \mid x, y \geq 0\}$ by finding the point(s) which satisfy the first and second order conditions for a local minimum:

$$f(x,y,z) = (x - \frac{y}{2})^2 + \frac{3}{4}(y-2)^2 + z^2 - 3.$$

Prove that your solution is actually a global minimum.

(3) Show that any $n \times n$ matrix of the form xx^T, where $x^T = (x_1, \ldots, x_n)$ is a row vector, is positive semidefinite.

(4) (a) Let $f(x) = b \cdot Ax$, where A is an $n \times m$ matrix, $x \in \mathbb{R}^m$, and $b \in \mathbb{R}^n$. Show that $\nabla f(x) = A^T b$, where A^T is the transpose of A.
   (b) Let $f(x) = x \cdot Ax$, where A is an $n \times n$ matrix and $x \in \mathbb{R}^n$. Show that $\nabla f(x) = (A + A^T)x$.

(5) Let $f : \mathbb{R}^{2n} \to \mathbb{R}$ be defined as $f(x,y) = \frac{1}{2}|Ax - By|^2$, where A, B are $m \times n$ matrices, $x, y \in \mathbb{R}^n$.
   (a) Find $\nabla f(x,y)$ and $\nabla^2 f(x,y)$.
   (b) Let (x_0, y_0) be such that $Ax_0 = By_0$. Show that (x_0, y_0) satisfies the necessary first and second order conditions for a minimizer of $f(x)$ on $\Omega = \mathbb{R}^n$.

(6) Assume that g is a convex function on $\mathbb{R}^n$, that f is a linear function of a single variable, and in addition that f is a nondecreasing function (which means that $f(r) \geq f(s)$ whenever $r \geq s$).

(a) Show that $F := f \circ g$ is convex by directly verifying the convexity inequality

$$F(\theta x + (1-\theta)y) \leq \theta F(x) + (1-\theta)F(y).$$

Explain where each hypothesis (convexity of g, convexity of f, and the fact that f is nondecreasing) is used in your reasoning. (The notation $F = f \circ g$ means that $F(x) = f(g(x))$.)

(b) Now assume that f and g are both C^2. Express the matrix of second derivatives $\nabla^2 F(x)$ in terms of f and g. Prove directly (without using part (a)) that $\nabla^2 F(x)$ is positive semidefinite at every x. **Hint**: see problem (3) above.

Discussion: Expressing $\nabla^2 F$ in terms of f and g is basically an exercise in using the chain rule for functions of several variables. If you find it at all difficult, then **review the chain rule** until you have completely mastered it!

When showing that $\nabla^2 F$ is positive semidefinite, please explain again, as you did in part (a), where each hypothesis is used in your reasoning.

(7) Prove that if $f_1(x), \ldots, f_k(x)$ are k convex functions on $\mathbb{R}^n$, then

$$g(x) := \max\{f_1(x), \ldots, f_k(x)\}$$

is also convex.

(8) Let $f : \Omega \to \mathbb{R}$ be a convex function, where $\Omega = (a, b)$ is an open (nonempty) subset of $\mathbb{R}^1$. Prove that such a function f is continuous. Hint: pick $x_0 \in \Omega$. Let c, d be any two numbers in Ω on either side of x_0: $a < c < x_0 < d < b$. Use convexity to show that for any x such that $x_0 < x < d$, $f(x)$ is above the line connecting $(c, f(c))$ and $(x_0, f(x_0))$ and below the line connecting $(x_0, f(x_0))$ and $(d, f(d))$. Why is this enough to conclude that f is continuous from the right at x_0? Now you can do a similar thing on the left of x_0.

(9) Let $f : \Omega \to \mathbb{R}$ be a continuous convex function, where $\Omega \subseteq \mathbb{R}^n$ is a convex set. Suppose that the maximum of f on Ω occurs at an interior point x_0 of Ω. Prove that f must be a constant function.

(10) Let $A \subset \mathbb{R}^n \times \mathbb{R}$ be a compact and convex set and denote its projection onto $\mathbb{R}^n$ by $\pi(A) := \{x \in \mathbb{R}^n \mid (x, \lambda) \in A\}$.

a) Prove that the set $\pi(A)$ is convex.

b) Define $f : \pi(A) \to \mathbb{R}$ by $f(x) = \min\{\lambda \mid (x, \lambda) \in A)\}$. Prove that $f(x)$ is a convex function. Hint: draw a pictutre.

(11) Given a C^1-function $f : \mathbb{R}^n \to \mathbb{R}$, let $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be given by $g(x, z) = f(x) - z$. Prove that if f is convex, then so is g.

(12) Let $g : \mathbb{R}^n \to \mathbb{R}$ be a C^1 convex function. Consider the following optimization problem of minimizing the distance squared from a point $x_* \in \mathbb{R}^n$ to the convex set $\Omega \subset \mathbb{R}^n$:

$$\text{minimize: } f(x) = |x - x_*|^2$$
$$\text{subject to: } x \in \Omega,$$

Prove that the minimizer x_0 is unique. Hint: triangle inequality. Recall that minimizing $|x - x_*|^2$ is equivalent to minimizing $|x - x_*|$.

(13) Consider the following unconstrained optimization problem where A is an $m \times n$ matrix (not assumed to be invertible) and b is a vector in $\mathbb{R}^m$:

$$\text{minimize: } |Ax - b|^2$$
$$\text{subject to: } x \in \mathbb{R}^n.$$

Let x_0 be a minimizer.

(a) Show that the function $|Ax - b|^2$ is convex. Hint: recall that $|v|^2 = v^T v$.

(b) What are the 1^{st} and 2^{nd} order necessary conditions at a minimizer x_0?

(c) Suppose that the first column of A is all zeros. Explain why the minimizer x_0 is not unique.