Contents

1. **(Taylor Expansion)** Given $f \in C^\infty(\mathbb{R})$, the Taylor expansion of f at a is given by

$$\begin{aligned} T(a) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \end{aligned}$$

2. **(Taylor's Theorem)** Let $k \geq 1$ and function $f : \mathbb{R} \to \mathbb{R}$ be k times differentiable at a point $a \in \mathbb{R}$, then exists $h_k : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n + h_k(x)(x-a)^k$$

and $\lim_{x\to a} h_k(x) = 0$, i.e. the reminder term $R_k(x) = f(x) - P_k(x)$ is asymptotically trivial. If f is $k+1$ times differentiable on the open interval and f^k continous on the closed interval $[a, x]$, then the Lagrange remind is given by

$$R_k(x) = \frac{f^{k+1}(\zeta)}{(k+1)!}(x-a)^{k+1}$$

for soem $\zeta \in [a, x]$ by the mean value theorem

3. **(Taylor's Theorem for multivariable function)** If $f : \mathbb{R}^n \to \mathbb{R}$ are k times differentiable function at point $\mathbf{a} \in \mathbb{R}^n$ then exists $h_\alpha : \mathbb{R}^n \to \mathbb{R}$ such that

$$f(\boldsymbol{x}) = \sum_{|\alpha|\leq k} \frac{D^\alpha f(\mathbf{a})}{\alpha!}(\boldsymbol{x}-\mathbf{a})^\alpha + \sum_{|\alpha|=k} h_\alpha(\boldsymbol{x})(\boldsymbol{x}-\mathbf{a})^\alpha \qquad \lim_{\boldsymbol{x}\to\mathbf{a}} h_\alpha(\boldsymbol{x}) = 0$$

where

$$|\alpha| = \alpha_1 + \cdots + \alpha_n \qquad \alpha! = \alpha_1! \cdots \alpha_n! \qquad \boldsymbol{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \qquad D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots x_n^{\alpha_n}}$$

4. **($\mathcal{O}$ notation)** $f(x) = \mathcal{O}(g(x))$ describes asymptotic behavior of function f
   (a) (as $x \to \infty$) if there exists $M \geq 0$ and $x_0 \in \mathbb{R}$ such that $|f(x)| \leq Mg(x)$ for all $x > x_0$.
   (b) (as $x \to a$) if there exists $M \geq 0$ and $\delta \in \mathbb{R}$ such that $|f(x)| \leq Mg(x)$ when $0 < |x-a| < \delta$. Alternatively we can say

$$\lim_{x\to a} \sup \left| \frac{f(x)}{g(x)} \right| < \infty$$

5. **(binomial theorem)**

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

6. **(power series expansions)**

$$
\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \\
(1+x)^\alpha &= \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \\
\ln(1+x) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} && \text{(convergent if } |x| < 1) \\
\ln(1-x) &= \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} && \text{(convergent if } |x| < 1) \\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots \\
\sin^2 x &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!} = \frac{2x^2}{2!} - \frac{8x^4}{4!} + \frac{32x^6}{6!} +
\end{aligned}
$$

7. **(trigonometric identities)**

$$
\begin{aligned}
\sin(\theta+\phi) &= \sin(\theta)\cos(\phi) + \sin(\phi)\sin(\theta) \\
\sin(\theta+\phi) + \sin(\theta-\phi) &= 2\sin(\theta)\cos(\phi) \\
1 - \cos(\theta) &= 2\sin^2\left(\frac{\theta}{2}\right)
\end{aligned}
$$

1 Euler's Method and Beyond

1.1 Ordinary differential equations and Lipschitz condition

1. **(Goal)** Approximate solution to

$$
\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y}) \quad \text{with initial condition} \quad \boldsymbol{y}(t_0) = \boldsymbol{y}_0
$$

   where $t > t_0$ and $\boldsymbol{f} : [t_0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is a sufficiently well behaved function

2. **(Lipschitz condition)** Given $\boldsymbol{f}$ and norm $\|\cdot\|$, the Lipschitz condition is defined by

$$
\|\boldsymbol{f}(t, \boldsymbol{x}) - \boldsymbol{f}(t, \boldsymbol{y})\| \leq \lambda \|\boldsymbol{x} - \boldsymbol{y}\| \qquad \text{for all} \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d \,,\ t > t_0
$$

   where $\lambda \in \mathbb{R}$ is called Lipschitz constant.

3. **(Picard Lindelof theorem)** Consider initial value problem

$$
y'(t) = f(t, y(t)) \quad y(t_0) = y_0
$$

   If f is uniformly Lipschitz continous in y and continous in t, then for some $\epsilon > 0$, there exists unique solution $y(t)$ to the initial value problem on the interval $[t_0 - \epsilon, t_0 + \epsilon]$

4. **(Analytic function)** A function $\boldsymbol{f}$ is an analytic function if it is a function that is locally given by a convergent power series, i.e. an infinitely differentiable function such that at any point $(t, \boldsymbol{y}_0) \in [0, \infty) \times \mathbb{R}^d$ in its domain, the Taylor series converges to $\boldsymbol{f}(\boldsymbol{x})$ for $\boldsymbol{x}$ in a neighborhood of $(t, \boldsymbol{y}_0)$.

   (a) (example) polynomial, exponential, trigonometric, logarithm, power function

   (b) (note) if $\boldsymbol{f}$ is analytic, solution $\boldsymbol{y}$ to the initial value problem is also analytic

1.2 Euler's method

Definition. ***(Euler's Method)*** *Given initial value problem $\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y})$ for $t \geq t_0$ and initial value $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$. If we assume $\boldsymbol{y}'(t) = \boldsymbol{f}(t, \boldsymbol{y}(t)) \approx \boldsymbol{f}(t_0, \boldsymbol{y}(t_0))$ for $t \in [t_0, t_0 + h)$ (i.e. deriviative in $[t_n, t_{n+1}]$ is approximated by value of derivative at t_n) for some sufficiently small time step $h > 0$, we can approximate the value of $\boldsymbol{y}(t)$ by*

$$
\begin{aligned}
\boldsymbol{y}(t) &= \boldsymbol{y}(t_0) + \int_{t_0}^{t} \boldsymbol{f}(\tau, \boldsymbol{y}(\tau)) d\tau \\
&\approx \boldsymbol{y}_0 + (t - t_0)\boldsymbol{f}(t_0, \boldsymbol{y}_0)
\end{aligned}
$$

Given a sequence of times $(t_n)_{n\in\mathbb{N}} = (t_0, t_0 + h, \cdots)$ we have numerical approximation $(\boldsymbol{y}_n)_{n\in\mathbb{N}}$ by

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{f}(t_n, \boldsymbol{y}_n)$$

1. *(intuition) euler's method is a time-stepping numerical method that covers interval by an equidistant grid and produe numerical solution at the grid points. we can show that euler's method is convergent, i.e. as $h \to 0$, grid is refined, the numerical solution tends to exact solution*

Definition. ***(convergent method)*** *Given a time-stepping numerical method on a compact inteval $[t_0, t_0 + t^*]$, we can compute numerical solutions dependent upon h*

$$\boldsymbol{y}_n = \boldsymbol{y}_{n,h} \quad \text{for} \quad n = 0, 1, \cdots, \lfloor t^*/h \rfloor$$

A method is said to be convergent if for every ODE with Lipschitz function $\boldsymbol{f}$, the numerical solution tends to the true solution as the grid becomes increasingly fine. More rigorously, if every ODE with Lipschitz function $\boldsymbol{y}$ and for every $t^* > 0$, then following holds

$$\lim_{h\to 0^+} \max_{n=0,1,\cdots,\lfloor t*/h \rfloor} \|\boldsymbol{y}_{n,h} - \boldsymbol{y}(t_n)\| = 0$$

Theorem. ***(Euler's method is convergent)***

Proof. Assume $\boldsymbol{f}$ and therefore also $\boldsymbol{y}$ is analytic, i.e. convergent Taylor expansion. Let $\boldsymbol{e}_{n,h} = \boldsymbol{y}_{n,h} - \boldsymbol{y}(t_n)$ be the numerical error. Show $\lim_{h\to 0^+} \max_n \|\boldsymbol{e}_{n,h}\| = 0$. By Taylor's theorem

$$\boldsymbol{y}(t_{n+1}) = \boldsymbol{y}(t_n) + h\boldsymbol{y}'(t_n) + \mathcal{O}(h^2) = \boldsymbol{y}(t_n) + h\boldsymbol{f}(t_n, \boldsymbol{y}(t_n)) + \mathcal{O}(h^2)$$

given $\boldsymbol{y}$ continously differentiable, $\mathcal{O}(h^2)$ can be bounded uniformly for all $h > 0$ by a term ch^2 for some $c > 0$. Subtract previous from iterative formula of euler's method

$$\boldsymbol{e}_{n+1,h} = \boldsymbol{e}_{n,h} + h\left(\boldsymbol{f}(t_n, \boldsymbol{y}(t_n) + \boldsymbol{e}_{n,h}) - \boldsymbol{f}(t_m, \boldsymbol{y}(t_n))\right) + \mathcal{O}(h^2)$$

By triangle inequality and Lipschitz condition

$$
\begin{aligned}
\|\boldsymbol{e}_{n+1,h}\| &\leq \|\boldsymbol{e}_{n,h}\| + h\left\|\boldsymbol{f}(t_n, \boldsymbol{y}(t_n) + \boldsymbol{e}_{n,h}) - \boldsymbol{f}(t_n, \boldsymbol{y}(t_n))\right\| + ch^2 \\
&\leq (1 + h\lambda)\|\boldsymbol{e}_{n,h}\| + ch^2
\end{aligned}
$$

for $n = 0, 1, \cdots, \lfloor t^*/h \rfloor - 1$. By induction on n, we can show $\|\boldsymbol{e}_{n,h}\| \leq \frac{c}{\lambda} h\left((1 + h\lambda)^n - 1\right)$. Since $1 + h\lambda < e^{h\lambda}$ we have $(1 + h\lambda)^n < e^{nh\lambda} < e^{\lfloor t*/h \rfloor h\lambda} \leq e^{t^*\lambda}$. Therefore

$$\|\boldsymbol{e}_{n,h}\| \leq \frac{c}{\lambda}(e^{t^*\lambda} - 1)h$$

for $n = 0, 1, \cdots, \lfloor t * /h \rfloor$. This is an upper bound on the error that is independent of h, hence $\lim_{h\to 0} \|\boldsymbol{e}_{n,h}\| = 0$. from which we can infer that error decays globally as $\mathcal{O}(h)$ □

Definition. ***(order p method)*** *Given arbitrary time-stepping method*

$$\boldsymbol{y}_{n+1} = \boldsymbol{\mathcal{Y}}_n(\boldsymbol{f}, h.\boldsymbol{y}_0, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_n) \qquad n = 0, 1, \cdots$$

for initial value problem, it is of order p if

$$\boldsymbol{y}(t_{n+1}) - \boldsymbol{\mathcal{Y}}_n(\boldsymbol{f}, h, \boldsymbol{y}(t_0), \boldsymbol{y}(t_1), \cdots, \boldsymbol{y}(t_n)) = \mathcal{O}(h^{p+1})$$

for every analytic $\boldsymbol{f}$ and $n = 0, 1, \cdots$. Intuitively, a method is of order p if it recovers exactly every polynomial oslution of degrees p or less.

1. *(intuition) order of a method gives information about local behavior, i.e. advancing from t_n to t_{n+1} where $h > 0$ is sufficiently small, we are incurring an error of $\mathcal{O}(h^{p+1})$. Generally want the the global (convergence) behavior of the method instead.*

2. ***(fact)*** *euler's method is order 1*

   Proof. Euler's method can be written as $\boldsymbol{y}_{n+1} - (\boldsymbol{y}_n + h\boldsymbol{f}(t_n, \boldsymbol{y}_n)) = 0$. Replace $\boldsymbol{y}_k$ by $\boldsymbol{y}(t_k)$ and expand terms of Taylor series about t_n we have

   $$\boldsymbol{y}(t_{n+1}) - (\boldsymbol{y}(t_n) + h\boldsymbol{f}(t_n, \boldsymbol{y}(t_n))) = \big(\boldsymbol{y}(t_n) + h\boldsymbol{y}'(t_n) + \mathcal{O}(h^2)\big) - \big(\boldsymbol{y}(t_n) + h\boldsymbol{y}'(t_n)\big) = \mathcal{O}(h^2)$$

   □

1.3 The trapezoidal rule

Definition. ***(Trapezoidal Rule)*** *Instead of approximating derivative by a constant in $[t_n, t_{n+1}]$, namely by its value at t_n, the trapezoidal rule approximates the value of the derivate by average of values at the endpoints. We can approximate solution $\boldsymbol{y}(t)$ by*

$$\begin{aligned} \boldsymbol{y}(t) &= \boldsymbol{y}(t_n) + \int_{t_n}^{t} \boldsymbol{f}(\tau, \boldsymbol{f}(\tau))d\tau \\ &\approx \boldsymbol{y}(t_n) + \frac{1}{2}(t - t_n)\left(\boldsymbol{f}(t_n, \boldsymbol{y}(t_n)) + \boldsymbol{f}(t, \boldsymbol{y}(t))\right) \end{aligned}$$

Given a sequence of times $(t_n)_{n\in\mathbb{N}} = (t_0, t_0 + h, \cdots)$ we have numerical approximation $(\boldsymbol{y}_n)_{n\in\mathbb{N}}$ by

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \frac{1}{2}h\left(\boldsymbol{f}(t_n, \boldsymbol{y}_n) + \boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})\right)$$

1. ***(theorem)*** *order of trapezoidal rule is 2*

   Proof. Compute by performing Taylor expansion on $\boldsymbol{y}(t_{n+1})$ and $\boldsymbol{y}'(t_{n+1})$ about t_n

   $$\boldsymbol{y}(t_{n+1}) - \left\{\boldsymbol{y}(t_n) + \frac{1}{2}h\left\{\boldsymbol{f}(t_n, \boldsymbol{y}(t_n)) + \boldsymbol{f}(t_{n+1}, \boldsymbol{y}(t_{n+1}))\right\}\right\} = \mathcal{O}(h^3)$$

   □

2. ***(theorem)*** *trapezoidal rule is convergent*

   Proof. Detail of proof here . We can show error is bounded by

   $$\|\boldsymbol{e}_{n,h}\| \leq \frac{ch^2}{\lambda} exp\left(\frac{t^*\lambda}{1 - \frac{1}{2}h\lambda}\right)$$

   from which we can infer that error decays globally as $\mathcal{O}(h^2)$ □

3. *(note) euler's method is explicit, since we can compute $\boldsymbol{y}_{n+1}$ with a few arithmetic operations by computing $\boldsymbol{f}$, a function of a known $\boldsymbol{y}_n$. Trapezoidal rule is implicit, i.e. finding $\boldsymbol{y}_{n+1}$ is not trivial and $\boldsymbol{f}$ is a function of both $\boldsymbol{y}_n$ and $\boldsymbol{y}_{n+1}$. We might need to solve a nonlinear equation of $\boldsymbol{y}_{n+1}$*

   $$\boldsymbol{y}_{n+1} - \frac{1}{2}h\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1}) = \boldsymbol{v}$$

   where $\boldsymbol{v} = \boldsymbol{y}_n + \frac{1}{2}h\boldsymbol{f}(t_n, \boldsymbol{y}_n)$ can be evaluated easily from assumptions.

1.4 The theta method

Definition. ***(theta method)*** *is a generalization of Euler's method ($\theta = 1$) and the trapezoidal rule ($\theta = 1/2$), whereby the derivates are assumed to be piecewise constant and provided by a linear combination of derivatives at the endpoints of each interval. The numerical approximates are,*

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\left(\theta \boldsymbol{f}(t_n, \boldsymbol{y}_n) + (1-\theta)\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})\right) \qquad n = 0, 1, \cdots$$

for some fixed $\theta \in [0,1]$

1. *(fact) theta method is explicit for $\theta = 1$ and implicit otherwise*
2. ***(theorem)*** *theta method is of order 2 for $\theta = 1/2$ and order 1 otherwise.*
3. ***(theorem)*** *theta method is convergent for every $\theta \in [0,1]$*

2 Multistep Method

3 8 Finite Differences Schemes

3.1 8.1 Finite differences

1. **(Finite difference operators)** Given real sequences $\boldsymbol{z} = \{z_k\}_{k\in\mathbb{Z}} = z(kh)$ for $k \in \mathbb{Z}$ as discrete sampling of a function z for some $h > 0$. Let $x_k = kh$. We can define finite difference operators mapping the space $\mathbb{R}^{\mathbb{Z}}$ of all such sequences to itself.

$$\begin{aligned}
(\boldsymbol{\mathcal{E}}\boldsymbol{z})_k &= z_{k+1} && \text{(shift)}\\
(\boldsymbol{\Delta}_+\boldsymbol{z})_k &= z_{k+1} - z_k && \text{(forward difference)}\\
(\boldsymbol{\Delta}_-\boldsymbol{z})_k &= z_k - z_{k-1} && \text{(backward difference)}\\
(\boldsymbol{\Delta}_0\boldsymbol{z})_k &= z_{k+\frac{1}{2}} - z_{k-\frac{1}{2}} && \text{(central difference)}\\
(\boldsymbol{\Upsilon}_0\boldsymbol{z})_k &= \tfrac{1}{2}(z_{k-\frac{1}{2}} + z_{k+\frac{1}{2}}) && \text{(averaging)}
\end{aligned}$$

   Finite difference operators are composed under function composition.

   (a) **(fact)** $\mathcal{T} \in \{\boldsymbol{\mathcal{E}}, \boldsymbol{\Delta}_+, \boldsymbol{\Delta}_-, \boldsymbol{\Delta}_0, \boldsymbol{\Upsilon}_0, \boldsymbol{\mathcal{D}}\}$ are linear operators

$$\mathcal{T}(a\mathbf{w} + b\boldsymbol{z}) = a\mathcal{T}\mathbf{w} + b\mathcal{T}\boldsymbol{z} \quad \text{for} \quad \mathbf{w}, \boldsymbol{z} \in \mathbb{R}^{\mathbb{Z}}, \quad a, b \in \mathbb{R}$$

   (b) **(convention)** $\mathcal{T}z_k$ stands for $(\mathcal{T}\boldsymbol{z})_k$

2. **(Differential operator)** The goal is to approximate derivatives $\boldsymbol{\mathcal{D}}$ by expresing it with a linear combination of values along the grid.

$$(\boldsymbol{\mathcal{D}}\boldsymbol{z})_k = z'(kh) \qquad \text{(differential)}$$

3. **(Functions of operators)** Finite difference operators are functions of h. Given an analytic function as Taylor series, $g(x) = \sum_{j=0}^{\infty} a_j x^j$, we can expand g about $\{\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{I}}, \boldsymbol{\Upsilon}_0 - \boldsymbol{\mathcal{I}}, \boldsymbol{\Delta}_+, \boldsymbol{\Delta}_-, \boldsymbol{\Delta}_0, h\boldsymbol{\mathcal{D}}\}$,

$$g(\boldsymbol{\Delta}_+)\boldsymbol{z} = \left(\sum_{j=0}^{\infty} a_j \boldsymbol{\Delta}_+^j\right)\boldsymbol{z} = \sum_{j=0}^{\infty} a_j(\boldsymbol{\Delta}_+^j\boldsymbol{z})$$

4. **(Asymptotics of operators)**

$$\{\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{I}}, \boldsymbol{\Upsilon}_0 - \boldsymbol{\mathcal{I}}, \boldsymbol{\Delta}_+, \boldsymbol{\Delta}_-, \boldsymbol{\Delta}_0, h\boldsymbol{\mathcal{D}}\} \overset{h\to 0+}{\longrightarrow} O$$

   (a) **(example)**

$$\boldsymbol{\Delta}_+ z_k = z_{k+1} - z_k = z(x_k + h) - z(x_k) = hz'(\eta_k) = \mathcal{O}(h)$$

   by some $\eta_k \in [x_k, x_{k+1}]$ by mean value theorem

5. **(Operator $\boldsymbol{\mathcal{E}}^{1/2}$)**

$$(\boldsymbol{\mathcal{E}}^{1/2}\boldsymbol{z})_k = z((k+\frac{1}{2})h)$$

by defining it with a power series expansion of $g(x) = \sqrt{1+x}$

$$\boldsymbol{\mathcal{E}}^{1/2} = (\boldsymbol{\mathcal{I}} + (\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{I}}))^{1/2} = \boldsymbol{\mathcal{I}} + \sum_{j=0}^{\infty} \frac{(-1)^{j-1}}{2^{2j-1}} \frac{(2j-2)!}{(j-1)!j!} (\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{I}})^j$$

6. **(Operator commutativity)** Idea is all operator can be expressed w.r.t. $\boldsymbol{\mathcal{E}}$

$$\begin{aligned}
\boldsymbol{\Delta}_+ &= \boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{I}} \\
\boldsymbol{\Delta}_- &= \boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{E}}^{-1} \\
\boldsymbol{\Delta}_0 &= \boldsymbol{\mathcal{E}}^{1/2} - \boldsymbol{\mathcal{E}}^{-1/2} \\
\boldsymbol{\Upsilon}_0 &= \frac{1}{2}(\boldsymbol{\mathcal{E}}^{1/2} + \boldsymbol{\mathcal{E}}^{-1/2}) \\
\boldsymbol{\mathcal{I}} &= \boldsymbol{\mathcal{E}}^0 \\
h\boldsymbol{\mathcal{D}} &= \ln \boldsymbol{\mathcal{E}}
\end{aligned}$$

Proof. Rest are trivial. To show $h\boldsymbol{\mathcal{D}} = \ln \boldsymbol{\mathcal{E}}$, note

$$\boldsymbol{\mathcal{E}} z(x) = z(x+h) = \sum_{i=0}^{\infty} \frac{h^i}{i!} \frac{d^i z(x)}{dx^i} = \left[\sum_{i=0}^{\infty} \frac{1}{i!}(h\boldsymbol{\mathcal{D}})^i\right] z(x) = e^{h\boldsymbol{\mathcal{D}}} z(x)$$

□

7. **(rewrite $\boldsymbol{\mathcal{D}}$ in terms of $\boldsymbol{\Delta}_+, \boldsymbol{\Delta}_-, \boldsymbol{\Delta}_0$)**

$$\begin{aligned}
h\boldsymbol{\mathcal{D}} &= \ln(\boldsymbol{\mathcal{I}} + \boldsymbol{\Delta}_+) \\
h\boldsymbol{\mathcal{D}} &= -\ln(\boldsymbol{\mathcal{I}} - \boldsymbol{\Delta}_-) \\
h\boldsymbol{\mathcal{D}} &= 2\ln\left(\frac{1}{2}\boldsymbol{\Delta}_0 + \sqrt{\boldsymbol{\mathcal{I}} + \frac{1}{4}\boldsymbol{\Delta}_0^2}\right)
\end{aligned}$$

Proof. From previous, $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{I}} + \boldsymbol{\Delta}_+ = (\boldsymbol{\mathcal{I}} - \boldsymbol{\Delta}_-)^{-1}$. For the last expression, consider

$$\begin{aligned}
\boldsymbol{\Delta}_0 &= \boldsymbol{\mathcal{E}}^{1/2} - \boldsymbol{\mathcal{E}}^{-1/2} \\
\boldsymbol{\mathcal{E}}^{1/2}\boldsymbol{\Delta}_0 &= \boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{I}} \\
(\boldsymbol{\mathcal{E}}^{1/2})^2 - \boldsymbol{\mathcal{E}}^{1/2}\boldsymbol{\Delta}_0 - \boldsymbol{\mathcal{I}} &= 0 \\
\boldsymbol{\mathcal{E}}^{1/2} &= \frac{1}{2}\boldsymbol{\Delta}_0 \pm \sqrt{\boldsymbol{\mathcal{I}} + \frac{1}{4}\boldsymbol{\Delta}_0^2} \qquad \text{(+ is correct)} \\
\boldsymbol{\mathcal{E}} &= \left(\frac{1}{2}\boldsymbol{\Delta}_0 + \sqrt{\boldsymbol{\mathcal{I}} + \frac{1}{4}\boldsymbol{\Delta}_0^2}\right)^2
\end{aligned}$$

□

8. **(approximate $\boldsymbol{\mathcal{D}}$ and its powers)** To approximate $\boldsymbol{\mathcal{D}}$ with $\boldsymbol{\Delta}_+$, we can expand $\ln(\boldsymbol{\mathcal{I}} + \boldsymbol{\Delta}_+)$ by power series expansion of $\ln(1+x) = \sum_{i=1}^{\infty}(-1)^{n+1}\frac{x^i}{i!}$

$$\begin{aligned}
\boldsymbol{\mathcal{D}} &= \frac{1}{h}\ln(\boldsymbol{\mathcal{I}} + \boldsymbol{\Delta}_+) = \frac{1}{h}\left[\boldsymbol{\Delta}_+ - \frac{1}{2}\boldsymbol{\Delta}_+^2 + \frac{1}{3}\boldsymbol{\Delta}_+^3 + \mathcal{O}(\boldsymbol{\Delta}_+^4)\right] \\
&= \frac{1}{h}\left(\boldsymbol{\Delta}_+ - \frac{1}{2}\boldsymbol{\Delta}_+^2 + \frac{1}{3}\boldsymbol{\Delta}_+^3\right) + \mathcal{O}(h^3) \qquad h \to 0
\end{aligned}$$

where $\boldsymbol{\Delta}_+ = \mathcal{O}(h)$ as $h \to 0$ shown perviously. Use binomial theorem on $\boldsymbol{\mathcal{D}}$ repeatedly and collect terms to $\mathcal{O}(h^3)$,

$$\boldsymbol{\mathcal{D}}^s = \frac{1}{h^s}\left[\boldsymbol{\Delta}_+^s - \frac{1}{2}s\boldsymbol{\Delta}_+^{s+1} + \frac{1}{24}s(3s+5)\boldsymbol{\Delta}_+^{s+2}\right] + \mathcal{O}(h^3) \qquad h \to 0$$

Inuititively, we can approximate $\boldsymbol{\mathcal{D}}^s z_k = d^s z(kh)/dx^s$ up to $\mathcal{O}(h^3)$ with $s+3$ grid points in the positive direction, i.e. $z_k, z_{k+1}, \cdots, z_{k+s+2}$. Similarly we can express $\boldsymbol{\mathcal{D}}$ in terms of grid points to the left with $\boldsymbol{\Delta}_-$.

$$\boldsymbol{\mathcal{D}}^s = \frac{(-1)^s}{h^s}\left(\ln\left(\boldsymbol{\mathcal{I}} - \boldsymbol{\Delta}_-\right)\right)^s = \frac{1}{h^s}\left[\boldsymbol{\Delta}_-^s + \frac{1}{2}s\boldsymbol{\Delta}_-^{s+1} + \frac{1}{24}s(3s+5)\boldsymbol{\Delta}_-^{s+2}\right] + \mathcal{O}(h^3) \qquad h \to 0$$

Similarly we can express $\boldsymbol{\mathcal{D}}$ in terms of grid points on the left and right with $\boldsymbol{\Delta}_0$ operator. Note, only even powers of $\boldsymbol{\Delta}_0$ maps $\mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$, i.e. onto grid points. ($\boldsymbol{\Delta}_0^2 z_k = z_{k+1} - 2z_k + z_{k-1}$ and for any power to $2s$, $\boldsymbol{\Delta}_0^{2s} = \left(\boldsymbol{\Delta}_0^2\right)^s$). We consider Maclaurin expansion of function $g(\xi) = \ln(\xi + \sqrt{1+\xi^2})$

$$g(\xi) = 2\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1}\binom{2j}{j}\left(\frac{1}{2}\xi\right)^{2j+1}$$

Let $\xi = \frac{1}{2}\boldsymbol{\Delta}_0$, we have power series expansion of $\boldsymbol{\mathcal{D}}$ in terms of $\boldsymbol{\Delta}_0$

$$\boldsymbol{\mathcal{D}} = \frac{2}{h}g(\frac{1}{2}\boldsymbol{\Delta}_0) = \frac{4}{h}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\binom{2j}{j}\left(\frac{1}{4}\boldsymbol{\Delta}_0\right)^{2j+1}$$

However powers of $\boldsymbol{\Delta}_0$ are all odd, we raise power to $2s$ to keep output of operator on the grid

$$\boldsymbol{\mathcal{D}}^{2s} = \frac{1}{h^{2s}}\left[\left(\boldsymbol{\Delta}_0^2\right)^s - \frac{s}{12}\left(\boldsymbol{\Delta}_0^2\right)^{s+1} + \frac{s(11+5s)}{1440}\left(\boldsymbol{\Delta}_0^2\right)^{s+2}\right] + \mathcal{O}(h^6) \qquad h \to 0$$

approximates $\boldsymbol{\mathcal{D}}$ up to $\mathcal{O}(h^6)$

9. **(comparing $\boldsymbol{\Delta}_+$ and $\boldsymbol{\Delta}_0$ for approximating $\boldsymbol{\mathcal{D}}$)** To attain $\mathcal{O}(h^{2p})$ error, $\boldsymbol{\Delta}_+$ requires $2s+2p$ grid points and $\boldsymbol{\Delta}_0$ requires $2s+2p-1$ grid points. However $\boldsymbol{\Delta}_+$ would have a smaller error constant. (exercise 8.3)

10. **(express $\boldsymbol{\Upsilon}_0$ in terms of $\boldsymbol{\Delta}_0$)**

$$\boldsymbol{\Upsilon}_0 = \left(\boldsymbol{\mathcal{I}} + \frac{1}{4}\boldsymbol{\Delta}_0^2\right)^{1/2}$$

Proof.

$$\boldsymbol{\Upsilon}_0 = \frac{1}{2}\left(\boldsymbol{\mathcal{E}}^{1/2} + \boldsymbol{\mathcal{E}}^{-1/2}\right) \qquad \longrightarrow \qquad 4\boldsymbol{\Upsilon}_0^2 = \boldsymbol{\mathcal{E}} + 2\boldsymbol{\mathcal{I}} + \boldsymbol{\mathcal{E}}^{-1}$$
$$\boldsymbol{\Delta}_0 = \boldsymbol{\mathcal{E}}^{1/2} - \boldsymbol{\mathcal{E}}^{-1/2} \qquad \longrightarrow \qquad \boldsymbol{\Delta}_0^2 = \boldsymbol{\mathcal{E}} - 2\boldsymbol{\mathcal{I}} + \boldsymbol{\mathcal{E}}^{-1}$$

Therefore $4\boldsymbol{\Upsilon}_0 - \boldsymbol{\Delta}_0^2 = 4\boldsymbol{\mathcal{I}}$ and result follows □

11. **(approximate odd derivatives with $\boldsymbol{\Upsilon}_0$ with central difference)**

$$\boldsymbol{\mathcal{D}} = \frac{1}{h}\left(\boldsymbol{\Upsilon}_0\boldsymbol{\Delta}_0\right)\left[\sum_{j=0}^{\infty}(-1)^j\binom{2j}{j}\left(\frac{1}{16}\boldsymbol{\Delta}_0^2\right)^j\right]\left[\sum_{i=0}^{\infty}\frac{(-1)^i}{2i+1}\binom{2i}{i}\left(\frac{1}{16}\boldsymbol{\Delta}_0^2\right)^i\right]$$

which are constructed from even powers of $\boldsymbol{\Delta}_0$ and $\boldsymbol{\Upsilon}_0\boldsymbol{\Delta}_0$ which will make image of $\boldsymbol{\mathcal{D}}$ operator reside on the grid.

$$\boldsymbol{\Upsilon}_0\boldsymbol{\Delta}_0 z_k = \boldsymbol{\Upsilon}_0\left(z_{k+\frac{1}{2}} - z_{k-\frac{1}{2}}\right) = \frac{1}{2}\left(z_{k+1} - z_{k-1}\right)$$

Raising powers of $\boldsymbol{\mathcal{D}}$ yield

$$\boldsymbol{\mathcal{D}}^2 = \frac{1}{h^2}(\boldsymbol{\Upsilon}_0\boldsymbol{\Delta}_0)^2(\boldsymbol{\mathcal{I}} - \frac{1}{3}\boldsymbol{\Delta}_0^2) + \mathcal{O}(h^4)$$

12. **(practical use)** instead of mixing difference operators, opt for finite difference grids. For finite grids, one-sided finite differences can be employed to evaluate $\boldsymbol{\mathcal{D}}$ near boundaries.

3.2 8.2 The five-point formula for $\nabla^2 u = f$

1. **(consistent)** A method is consistent if the truncation error goes to 0 as step size goes to zero
2. **(order of consistency)** of $\mathcal{O}(\Delta x^p) + \mathcal{O}(\Delta y^q)$ is p in x and q in y.
3. **(theorem)** If a method is consistent and stable, then it is convergent and order of convergence will be same as the order of consistency

1. **(Poisson Equation)**

$$\nabla^2 u = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})u = f \qquad (x,y) \in \Omega$$

and $f = f(x,y)$ is continuous, domain $\Omega \subset \mathbb{R}^2$ is bounded, open, and connected and has a piecewise-smooth boundary. Assume *Dirichlet condition*, i.e.

$$u(x,y) = \phi(x,y) \qquad (x,y) \in \partial\Omega$$

2. **(Setup)** Inscribe a grid that is axis-aligned with equal spacing of Δx in both direction, i.e. pick $\Delta x > 0$, $(x_0, y_0) \in \Omega$ and let $\Omega_{\Delta x}$ be

$$\Omega_{\Delta x} = \{x_0 + k\Delta x, y_0 + l\Delta x\} \subset cl\,\Omega$$

Denote

$$\mathbf{I}_{\Delta x} = \left\{(k,l) \in \mathbb{Z}^2 \mid (x_0 + k\Delta x, y_0 + l\Delta x) \subset cl\,\Omega\right\}$$
$$\mathbf{I}^{\circ}_{\Delta x} = \left\{(k,l) \in \mathbb{Z}^2 \mid (x_0 + k\Delta x, y_0 + l\Delta x) \subset \Omega\right\}$$

and for every $(k,l) \in \mathbf{I}^{\circ}_{\Delta x}$, let $u_{k,l}$ be *approximation* to the solution $u(x_0 + k\Delta x, y_0 + l\Delta x)$ of the Poisson equation at the relevant grid point. Note there is no need to approximate points in $\mathbf{I}_{\Delta x} \setminus \mathbf{I}^{\circ}_{\Delta x}$ since they lie on $\partial\Omega$ and their exact values given by ϕ.

3. **(internal, near-boundary, boundary points)** A point on the grid $(k,l) \in \mathcal{I}_{\Delta x}$ whereby $(k\pm 1, l)$ and $(k, l \pm 1)$ are in $\mathbf{I}_{\Delta x}$ is called *internal point*. A point $(k,l) \in \mathcal{I}_{\Delta x}$ where we can no longer employ a finite difference scheme (and so requires a special approach) is called *near-boundary points*. $(k,l) \in \partial\Omega$ are called *boundary points*

4. **(Central difference approximation)** given u sufficiently smooth, we can approximate ∇^2

$$\nabla^2 = \frac{1}{(\Delta x)^2}\left(\boldsymbol{\Delta}^2_{0,x} + \boldsymbol{\Delta}^2_{0,y}\right) + \mathcal{O}((\Delta x)^2) \quad \text{where} \quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{\Delta x^2}\boldsymbol{\Delta}^2_{0,x} u_{k,l} + \mathcal{O}((\Delta x)^2)$$

with central difference operators, i.e. $\boldsymbol{\Delta}_{0,x}, \boldsymbol{\Delta}_{0,y}$ along the x,y-axis.We can rewrite Poisson equation by the *five point* finite difference scheme. For every internal grid point (k,l), we have

$$\frac{1}{(\Delta x)^2}(\boldsymbol{\Delta}^2_{0,x} + \boldsymbol{\Delta}^2_{0,y})u_{k,l} = f_{k,l}$$

where $f_{k,l} = f(x_0 + k\Delta x, y_0 + l\Delta x)$. Expanding expression, we have

$$u_{k-1,l} + u_{k+1,l} + u_{k,l-1} + u_{k,l+1} - 4u_{k,l} = (\Delta x)^2 f_{k,l}$$

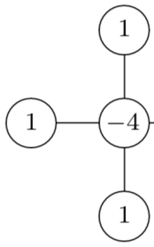
Intuitively, we have a linear combination of values of u at grid point and at immediate horizontal and vertical neighbors of this point.

5. **(properties)**

(a) **(truncation error)** $\mathcal{O}(\Delta x^2)+\mathcal{O}(\Delta y^2)$ (computed by substituting exact solution $\tilde{u}_{i,j} = u(x_0 + \Delta x, y_0 + \Delta y)$ to finite difference formula in place of approximate values at grid points $u_{i,j}$ to compute the truncation error)

$$\begin{aligned}
&\frac{\tilde{u}_{i+1,j} - 2\tilde{u}_{i,j} + \tilde{u}_{i-1,j}}{\Delta x^2} + \frac{\tilde{u}_{i,j+1} - 2\tilde{u}_{i,j} + \tilde{u}_{i,j-1}}{\Delta y^2} - f(x_i, y_i) \\
=&\frac{\partial u(x_i, y_j)}{\partial x^2} + \mathcal{O}(\Delta x^2) + \frac{\partial u(x_i, y_j)}{\partial y^2} + \mathcal{O}(\Delta y^2) - f(x_i, y_i) \\
=&\mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta y^2)
\end{aligned}$$

6. **(computational stencil / molecule)**



$$u_{k,\ell} = (\Delta x)^2 f_{k,\ell}$$

7. **(solving linear equation)** Main idea of finite difference method is to associate every grid point having an index in $\mathbf{I}^{\circ}_{\Delta x}$ a single linear equation. Solving a system of linear equations whose solution is our approximation $\boldsymbol{u} = (u_{k,l})_{(k,l)\in \mathbf{I}^{\circ}_{\Delta x}}$. Performance of finite difference is evaluated with

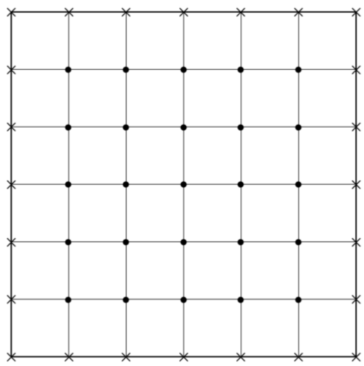
   (a) nonsingular linear system (such that $\boldsymbol{u}$ exists and is unique)

   (b) for $\Delta x \to 0$, the convergence of $\boldsymbol{u}$ to exact solution of Poisson equation and error

   (c) efficient and robust ways to solve sparse linear systems

8. **(A simplified grid)** over a square Ω. Let

$$\Omega = \{(x, y) \mid 0 < x, y < 1\} \qquad \Delta x = 1/(m+1) \qquad (x_0, y_0) = 0$$



9. **(numerical example on Laplace equation)** indicates that error decreases by a factor of 4 when number of steps m increase by a factor of 2.

10. **(discretization to a system of linear equations)** Rearrange $u_{k,l}$ to $\boldsymbol{u} \in \mathbb{R}^s$ where $s = m^2$ according to some permutation $\{(k_i, l_i))\}_{i=1,2,\cdots,m}$ and write

$$A\boldsymbol{u} = \boldsymbol{b}$$

where A is $s \times s$ matrix and $\boldsymbol{b} \in \mathbb{R}^s$ includes both $(\Delta x)^2 f_{k,l}$ and boundary values.

11. **(lemma (unique solution to linear system))** A from previous is symmetric and the set of its eigenvalues is

$$\sigma(A) = \{\lambda_{\alpha,\beta} \mid \alpha, \beta = 1, 2, \cdots, m\}$$

where

$$\lambda_{\alpha,\beta} = -4\left\{\sin^2\left[\frac{\alpha\pi}{2(m+1)}\right] + \sin^2\left[\frac{\beta\pi}{2(m+1)}\right]\right\}$$

Proof. Symmetry follows by examining matrix A. To find eigenvalues of A, find nonzero functions $(v_{k,l})_{k,l=0,1,\cdots,m+1}$ such that $v_{k,0} = v_{k,m+1} = v_{0,l} = v_{m+1,l} = 0$ where $k, l = 1, 2, \cdots, m$ such that

$$v_{k-1,l} + v_{k+1,l} + v_{k,l-1} + v_{k,l+1} - 4v_{k,l} = \lambda v_{k,l} \quad k, l = 1, 2, \cdots, m$$

is satisfied for some λ. It follows that $(v_{k,l})$ is an eigenvector and λ is the corresponding eignvalue of A. Given $\lambda_{\alpha,\beta}$ for some α, β, we show that

$$v_{k,l} = \sin\left(\frac{k\alpha\pi}{m+1}\right)\sin\left(\frac{l\beta\pi}{m+1}\right) \quad k, l = 0, 1, \cdots, m+1$$

is the corresponding eigenvector by verifying above formula. □

12. **(corollary)** The matrix A is negative definite and, therefore, nonsingular

Proof. A is symmetric and from previous lemma eigenvalues are negative, therefore it is negative definite and nonsingular □

13. **(eigenvalues of the Laplace operator)** The function v, not identically zero, is said to be an *eigenfunction* of ∇^2 in a domain Ω and λ is the corresponding eigenvalue if v vanishes along $\partial\Omega$ and satisfies within Ω the equation $\nabla^2 v = \lambda v$. Note eigenvalues and eigenfunctions of the Laplace operator ∇^2 over $(0,1)^2$ are *related to* eigenvalues and eigenvectors of the matrix A. Given α, β, eigenvalue of ∇^2 and the corresponding eigenfunction is given by

$$\begin{aligned}\lambda_{\alpha,\beta} &= -(\alpha^2 + \beta^2)\pi^2 \\ v(x, y) &= \sin(\alpha\pi x)\sin(\beta\pi y) \qquad x, y \in [0, 1]\end{aligned}$$

We can easily verify that

$$\begin{aligned}\nabla^2 v &= -\alpha^2\pi\sin(\alpha\pi x)\sin(\beta\pi y) - \beta^2\pi\sin(\alpha\pi x)\sin(\beta\pi y) \\ &= -(\alpha^2 + \beta^2)\pi^2 v\end{aligned}$$

v obeys boundary conditions. Note eigenvectors $v_{k,l}$ for A can be obtained by sampling of the eigenfunction v at grid points

$$\left\{\left(\frac{k}{m+1}, \frac{l}{m+1}\right)\right\}_{k,l=0,1,\cdots,m+1}$$

Note $(\Delta x)^{-2}\lambda_{\alpha,\beta}$ is a good approximation to $-(\alpha^2 + \beta^2)\pi^2$ provided α, β are small in comparison with m. Note we can expand $\sin^2$ in a power series

$$\begin{aligned}\frac{\lambda_{\alpha,\beta}}{(\Delta x)^2} =& -4\left(\left\{\left(\frac{\alpha\pi}{2(m+1)}\right)^2 - \frac{1}{3}\left(\frac{\alpha\pi}{2(m+1)}\right)^4 + \cdots\right\} + \left\{\left(\frac{\beta\pi}{2(m+1)}\right)^2 - \frac{1}{3}\left(\frac{\beta\pi}{2(m+1)}\right)^4 + \cdots\right\}\right) \\ =& -(\alpha^2 + \beta^2)\pi^2 + \frac{1}{12}(\alpha^4 + \beta^4)\pi^4(\Delta x)^2 + \mathcal{O}((\Delta x)^4)\end{aligned}$$

14. **(theorem (convergence))** Subject to sufficient smoothness of the function f and the boundary conditions, there exists a number $c > 0$, independent of Δx, such that

$$\|\boldsymbol{e}\| \le c(\Delta x)^2 \qquad \Delta x \to 0$$

or equivalently

$$\lim_{\Delta x \to 0} \|\boldsymbol{e}\|_\infty = 0$$

where $\boldsymbol{e} \in \mathbb{R}^s$, $s = m^2$ in same order as that of $\boldsymbol{u}$. Denote $e_{k,l} = u_{k,l} - \tilde{u}_{k,l}$ as error of the five point formula at the (k,l)th grid point.

15. **(handle near boundary grid points)** approximate z'' at P in x direction as a linear combination of value of z at P, Q, T. The coefficient of terms can be determined via Taylor expansion of $z_{x_0 - \Delta x}, z_{x_0}, z_{x_0 + \tau\Delta x}$ at $a = x_0$ and solve a 3×3 linear system. The error of approximation is $\mathcal{O}(\Delta x)$

$$\begin{aligned}\frac{1}{(\Delta x)^2}&\left[\frac{2}{\tau+1}z(x_0-\Delta x) - \frac{2}{\tau}z(x_0) + \frac{2}{\tau(\tau+1)}z(x_0+\tau\Delta x)\right]\\ &= z''(x_0) + \frac{1}{3}(\tau-1)z'''(x_0)\Delta x + \mathcal{O}((\Delta x)^2)\end{aligned}$$

To achieve $\mathcal{O}((\Delta x)^2)$ order for the error, we use value of z at 4 grid points V, Q, P, T to approximate z''. Coefficient to linear term can be determined with Taylor expansion and solve a 4×4 linear system.

$$\begin{aligned}\frac{1}{(\Delta x)^2}&\left[\frac{\tau-1}{\tau+2}z(x_0-2\Delta x) + \frac{2(2-\tau)}{\tau+1}z(x_0-\Delta x) - \frac{3-\tau}{\tau}z(x_0) + \frac{6}{\tau(\tau+1)(\tau+2)}z(x_0+\tau\Delta x)\right]\\ &= z''(x_0) + \mathcal{O}((\Delta x)^2)\end{aligned}$$

In total, a good approximation to $\nabla^2 u$ at P requires 6 points. For P corresponding to grid (k,l), we obtain the following linear equation for constructing A and b for both first order and second order approximations

$$\frac{2}{\tau+1}u_{k-1,l} + \frac{2}{\tau(\tau+1)}u_{k+\tau,l} + u_{k,l-1} + u_{k,l+1} - \frac{2+2\tau}{\tau}u_{k,l} = (\Delta x)^2 f_{k,l}$$

$$\frac{\tau-1}{\tau+2}u_{k-2,l} + \frac{2(2-\tau)}{\tau+1}u_{k-1,l} + \frac{6}{\tau(\tau+1)(\tau+2)}u_{k+\tau,l} + u_{k,l-1} + u_{k,l+1} - \frac{3+\tau}{\tau}u_{k,l} = (\Delta x)^2 f_{k,l}$$

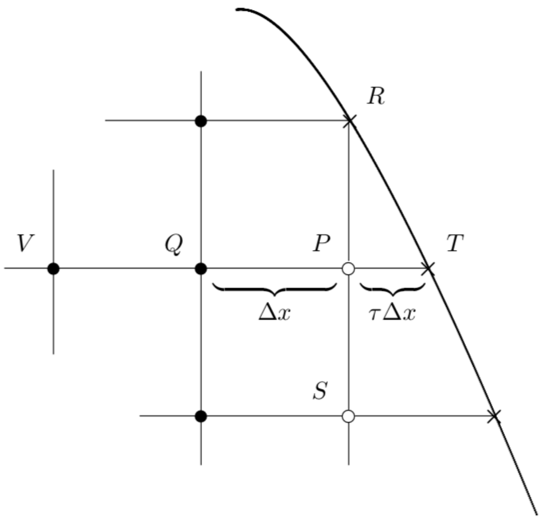


Figure 8.6 Computational grid near a curved boundary.