Chapter 2 Linear Transformations and Matrices

2.1 Linear Transformations, Null Spaces, and Ranges

Definition. *Linear Transformation*
Let V and W be vector spaces (over F). We call a function $T : V \to W$ a linear transformation from V to W if, for all $x, y \in V$ and $c \in F$, we have

1. $T(x + y) = T(x) + T(y)$
2. $T(cx) = cT(x)$

T is called linear, with properties

1. *If T is linear $T(0) = 0$*
2. *T is linear if and only if $T(cx + y) = cT(x) + T(y)$ for all $x, y \in V$ and $c \in F$* ***(For proving a transformation is linear)***
3. *If T is linear, then $T(x - y) = T(x) - T(y)$ for all $x, y \in V$*
4. *T is linear if and only if, for $x_1, x_2, \cdots, x_n \in V$ and $a_1, a_2, \cdots, a_n \in F$, we have*
$$T(\sum_i a_i x_i) = \sum_i a_i T(x_i)$$

Some examples of linear transformations

1. ***Rotation*** *For any angle θ, define $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$. $T_\theta(a_1, a_2)$ is the vector obtained by rotating (a_1, a_2) counterclockwise by θ if $(a_1, a_2) \neq (0, 0)$, and $T_\theta = (0, 0)$. Then T_θ is a linear transformation called rotation by θ, Let α be angle that (a_1, a_2) makes with the positive axis. Note $a_1 = r\cos\alpha$ and $a_2 = r\sin\alpha$, and suppose $r = \sqrt{a_1^2 + a_2^2}$*
$$T_\theta(a_1, a_2) = (r\cos\alpha + \theta, r\sin\alpha + \theta) = (a_1\cos\theta - a2\sin\theta, a_2\sin\theta + a_2\cos\theta)$$
2. ***Reflection*** *Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1, a_2) = (a_1, -a_2)$. T is called the reflection about the x-axis*
3. ***Projection*** *Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1, a_2) = (a_1, 0)$. T is called the projection on the x-axis*
4. ***Taking transpose is linear*** *Define $T : M_{m\times n}(F) \to M_{n\times m}(F)$ by $T(A) = A^t$ (by $(A + B)^t = A^t + B^t$ and $(cA)^t = cA^t$)*
5. ***Taking derivative is linear*** *Define $T : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ by $T(f(x)) = f'(x)$, where $f'(x)$ denotes the derivative of $f(x)$. Let $g(x), h(x) \in P_n(\mathbb{R})$ and $a \in \mathbb{R}$,*
$$T(ag(x) + h(x)) = (ag(x) + h(x))' = ag'(x) + h'(x) = aT(g(x)) + T(h(x))$$
so T is linear.

6. ***Taking integral is linear*** *Let $V = C(\mathbb{R})$, the set of continuous real-valued functions on $\mathbb{R}$., Let $a, b \in \mathbb{R}$, $a < b$. Define $T : V \to \mathbb{R}$ by*

$$T(f) = \int_a^b f(t)dt$$

for all $f \in V$. THen T is linear because the definite integral of a linear combination of functions is same as combination of the detinite integrals of the functions.

Definition. ***Identity and Zero Transformation*** *For vector spaces V and W (over F), define identity transformation $I_V : V \to V$ by $I_V(x) = x$ for all $x \in V$ and the zero transformation $T_0 : V \to W$ by $T_0(x) = 0$ for all $x \in V$.*

Definition. ***Null Space and Range*** *Let V and W be vector spaces, and let $T : V \to W$ be linear. We define the null space (or kernel) $N(T)$ of T to be the set of all vectors $x \in V$ such that $T(x) = 0$; that is $N(T) = \{x \in V : T(x) = 0\}$. We define the range (or image) $R(T)$ of T to be the subset of W consisting all images (under T) of vectors in V; that is $R(T) = \{T(x) : x \in V\}$*

1. ***identity and zero transformation*** *$N(I) = \{0\}$ and $R(I) = V$, $R(T_0) = V$ and $R(T_0) = \{0\}$*

Theorem. ***2.1 Range and null space are subspaces***
Let V and W be vector spaces and $T : V \to W$ be linear. Then $N(T)$ and $R(T)$ are subspaces of V and W, respectively.

Theorem. ***2.2 Transformation on basis yields a spanning set for the range***
Let V and W be vector spaces, and let $T : V \to W$ be linear. If $\beta = \{v_1, \cdots, v_n\}$ is a basis for V, then

$$R(T) = span(T(\beta)) = span(\{T(v_1), \cdots, T(v_n)\})$$

So we simply transform the original basis to find the generating set for the range of a transformation, then reduce the generating set to a linearly independent set to find the basis.

Definition. ***Nullity and Rank*** *Let V and W be vector spaces, and let $T : V \to W$ be linear. If $N(T)$ and $R(T)$ are finite-dimensional, then we define the nullity of T, denoted by $nullity(T)$, and the rank of T, denoted $rank(T)$, to be the dimensions of $N(T)$ and $R(T)$, respectively.*

Theorem. ***2.3 Rank-Nullity (Dimension) Theorem***
Let V and W be vector spaces, and let $T : V \to W$ be linear. If V is finite-dimensional, then

$$nullity(T) + rank(T) = dim(V)$$

In the context of matrcies, the rank and the nullity of a matrix add up to the number of columns of the matrix.

Theorem. ***2.4 One-to-One Transformation***
Let V and W be vector spaces, and let $T : V \to W$ be linear. Then T is one-to-one if and only if $N(T) = \{0\}$, or $nullity(T) = 0$

Theorem. ***2.5 One-to-One and Onto Equivalence***
Let V and W be vector spaces of equal (finite) dimension, and let $T : V \to W$ be linear. THen the following are equivalent

1. *T is one-to-one*
2. *T is onto*
3. *$rank(T) = dim(V)$*

If not a special case to see if a transformation is onto we verify that $R(T) = W$

Theorem. ***2.6 Uniqueness Linear Transformation***
Let V and W be vector spaces over F, and suppose that $\{v_1, v_2, \cdots, v_n\}$ is a basis for V. For $w_1, w_2, \cdots, w_n \in W$, there exists exactly one linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for $i = 1, 2, \cdots, n$

Corollary. ***Transformation is determined completely by action on a basis***
Let V and W be vector spaces, and suppose that V has a finite basis $\{v_1, v_2, \cdots, v_n\}$. If $U, T : V \to W$ are linear and $U(v_i) = T(v_i)$ for $i = 1, 2, \cdots, n$, then $U = T$.

2.2 The Matrix Representatino of Linear Transformation

Definition. ***Ordered Basis*** *Let V be a finite-dimensional vector space. An ordered basis for V is a basis for V endowed with a specific order; that is an ordered basis for V is a finite sequence of linearly independent vectors in V that generates V.*

1. ***Standard ordered basis*** *$\{e_1, e_2, \cdots, e_n\}$ is the standard ordered basis for F^n and $\{1, x, \cdots, x^n\}$ is the standard ordered basis for $P_n(F)$*
2. *In F^3, $\beta = \{e_1, e_2, e_3\}$ and $\gamma = \{e_2, e_1, e_3\}$ are 2 different ordered basis, i.e. $\beta \neq \gamma$*

Definition. ***Coordinate Vector*** *Let $\beta = \{u_1, u_2, \cdots, u_n\}$ be an ordered basis for a finite-dimensional vector space V. For $x \in V$, let $a_1, a_2, \cdots, a_n$ be the unique scalars such that*

$$x = \sum_{i=1}^{n} a_i u_i$$

We define the coordinate vector of x relative to β, denoted $[x]_\beta$, by

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Note $[u_i]_\beta = e_i$ and that a linear tranformation maps $x \to [x]_\beta$ with $V \to F^n$

1. *Let* $V = P_2(\mathbb{R})$*, let* $\beta = \{1, x, x^2\}$ *be standard ordered basis for* V*. If* $f(x) = 4 + 6x - 7x^2$*, then*

$$[f]_\beta = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}$$

Definition. ***Matrix*** *Let* V *and* W *be finite-dimensional vector spaces with ordered basis* $\beta = \{v_1, v_2, \cdots, v_n\}$ *and* $\gamma = \{w_1, w_2, \cdots, w_m\}$*, respectively. Let* $T : V \to W$ *be linear. Then for each* $j, 1 \le j \le n$*, there exist unique scalar* $a_{ij} \in F$*,* $1 \le i \le m$*, such that*

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \qquad \text{for } 1 \le j \le n$$

The $m \times n$ *matrix* A *defined by* $A_{ij} = a_{ij}$ *the matrix representation of* T *in the ordered basis* β *and* γ *and write* $[T]_\beta^\gamma$*. If* $V = W$ *and* $\beta = \gamma$*, then write* $A = [T]_\beta$

1. j*th column of* A *is simply* $[T(v_j)]_\gamma$

2. *Observe if* $U : V \to W$ *is a linear transformation such that* $[U]_\beta^\gamma = [T]_\beta^\gamma$*, then* $U = T$ *by previous corollary*

3. *Let* $T : \mathbb{R}^2 \to \mathbb{R}^3$ *with* $T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$*. Let* β *and* γ *be standard ordered bases for* $\mathbb{R}^2$ *and* $\mathbb{R}^3$*. Now*

$$T(1,0) = (1,0,2) = 1e_1 + 0e_2 + 2e_3 \qquad T(0,1) = (3,0,-4) = 3e_1 + 0e_2 - 4e_3$$

hence

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}$$

Definition. ***Addition and Scalar Multipliation Operations for Function*** *Let* $T, U : V \to W$ *be arbitrary functions, where* V *and* W *are vector spaces over* F*, and let* $a \in F$*. We define* $T + U : V \to W$ *by* $(T + U)(x) = T(x) + U(x)$ *for all* $x \in V$*, and* $aT : V \to W$ *by* $(aT)(x) = aT(x)$ *for all* $x \in V$*.*

Theorem. ***2.7 Sums and Scalar Multiples of Linear Transformation are also Linear*** *Let* V *and* W *be vector spaces over a field* F*, and let* $T, U : V \to W$ *be linear.*

1. *For all* $a \in F$*,* $aT + U$ *is linear (Prove* $(aT+U)(cx+y) = c(aT+U)(x) + (aT+U)(y)$*)*

2. *Using the operations of addition and scalar multiplication in the preceding definition, the collection of all linear transformations from* V *to* W *is a vector space over* F*. (With* T_0 *the zero transformation as the zero vector)*

Definition. ***Vector space of Linear Transformations*** *Let* V *and* W *be vector spaces over* F*. We denote the vector space of all linear transformations from* V *into* W *by* $\mathcal{L}(V, W)$*. In the case that* $V = W$*, we write* $\mathcal{L}(V)$ *instead of* $\mathcal{L}(V, W)$

Theorem. ***2.8 Linearity of Matrix Representations*** *Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively, and let $T, U : V \to W$ be linear transformations. Then*

1. $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$
2. $[aT]_\beta^\gamma = a\,[T]_\beta^\gamma$

2.3 Compositino of Linear Transformations and Matrix Multiplication

Theorem. ***2.9 Sum and Scalar Multiple of Linear Transformation is Linear*** *Let V, W, and Z be vector spaces over the same field F, and let $T : V \to W$ and $U : W \to Z$ be linear. Then $UT : V \to Z$ is linear. (Prove $UT(ax+y) = a(UT)(x) + UT(y)$)*

Theorem. ***2.10 Properties of Sum and Scalar Multiple of Linear Transformations*** *Let $T, U_1, U_2 \in \mathcal{L}(V)$. Then*

1. $T(U_1+U_2) = TU_1 + TU_2$ *and* $(U_1+U_2)T = U_1T + U_2T$
2. $T(U_1U_2) = (TU_1)U_2$
3. $TI = IT = T$
4. $a(U_1U_2) = (aU_1)U_2 + U_1(aU_2)$

Definition. ***Matrix Product*** *Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. We define the product of A and B, denoted AB, to be the $m \times p$ matrix such that*

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} \qquad \text{for } 1 \le i \le m, 1 \le j \le p$$

Note

1. $(AB)_{ij}$ *is sum of products of corresponding entries from ith row of A and jth column of B.*
2. $(AB)^t = B^tA^t$

Remark. The motivation is as follows. Let $T : V \to W$ and $U : W \to Z$ be linear transformations, and let $A = [U]_\beta^\gamma$ and $B = [T]_\alpha^\beta$ where $\alpha = \{v_1, \cdots, v_n\}$, $\beta = \{w_1, \cdots, w_n\}$, and $\gamma = \{z_1, \cdots, z_p\}$ are ordered bases for V, W, and Z, respectively. We would like to define the product AB of two matrices such so that $AB = [UT]_\alpha^\gamma$. Consider for $1 \le j \le n$, we have

$$(UT)(v_j) = U(T(v_j)) = U\left(\sum_k^m B_{kj}w_k\right) = \sum_k^m B_{kj}U(w_k)$$
$$= \sum_k^m \left(\sum_i^p A_{ik}z_i\right) = \sum_i^p \left(\sum_k^m A_{ik}B_{kj}\right) z_i = \sum_i^p C_{ij}z_p$$

Theorem. ***2.11 Composition of Linear Transformation*** *Let V, W, and Z be finite-dimensional vector spaces with ordered bases α, β, γ, respectively. Let $T : V \to W$ and $U : W \to Z$ be linear transformations. Then*

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$$

Proof direct as a result of definition of matrix product

Corollary. *Let V be finite-dimensional vector space with ordered basis β. Let $T, U \in \mathcal{L}(V)$. Then $[UT]_\beta = [U]_\beta [T]_\beta$*

Definition. ***Identity Matrix*** *We define the Kronecker delta δ_{ij} by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. The $n \times n$ identity matrix I_n is defined by $(I_n)_{ij} = \delta_{ij}$*

Theorem. ***2.12 Properties of Composition of Linear Transformation*** *Let A be $n \times n$ matrix, B and C be $n \times p$ matrices, D and E be $q \times m$ matrices. Then*

1. *$A(B + C) = AB + AC$ and $(D + E)A = DA + EA$*
2. *$a(AB) = (aA)B = A(aB)$ for $a \in F$*
3. *$I_m A = A = AI_n$*
4. *If V is an n-dimensional vector space with ordered basis β, then $[I_V]_\beta = I_n$ (identity transformation)*

Proved using definition of matrix product

Proof. Proving number 3

$$(I_m A)_{ij} = \sum_k^m (I_m)_{ik} A_{kj} = \sum_k^m \delta_{ik} A_{kj} = A_{ij}$$

□

Corollary. *Let A be an $m \times n$ matrix, $B_1, B_2, \cdots, B_k$ be $n \times p$ matrices, $C_1, C_2, \cdots, C_k$ be $q \times m$ matrices, and $a_1, a_2, \cdots, a_k$ be scalars. Then*

$$A\left(\sum_i^k a_i B_i\right) = \sum_i^k = a_i AB_i \qquad \textit{and} \qquad \left(\sum_i^k a_i C_i\right) A = \sum_i^k a_i C_i A$$

Definition. ***Matrix Exponentials*** *Define $A^0 = I_n$ and $A^k = A^{k-1}A$ for $k > 1$.*

Theorem. ***2.13 Regarding columns in matrix multiplication*** *Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. For each j $(1 \leq j \leq p)$ let u_j and v_j denote the jth columns of AB and B, respectively. Then*

1. *$u_j = Av_j$*

2. $v_j = Be_j$*, where* e_j *is the* j*th standard vector of* F^p

Proof. We have

$$u_j = \begin{pmatrix} (AB)_{1j} \\ \vdots \\ (AB)_{mj} \end{pmatrix} = \begin{pmatrix} \sum_k^n A_{1k}B_{kj} \\ \vdots \\ \sum_k^n A_{mk}B_{kj} \end{pmatrix} = A \begin{pmatrix} B_{1j} \\ \vdots \\ B_{nj} \end{pmatrix} = Av_j$$

□

Corollary. *The* j*th column of* AB *is a linear combination of the columns of* A *with the coefficients in the linear combination being the entries of the column* j *of* B*. Analogously, row* i *of* AB *is a linear combination of the rows of* B *with the coefficients in the linear combination being the entries of the row* i *of* A*.*

Theorem. ***2.14 Transformation as Matrix Left Multiplication*** *Let* V *and* W *be finite-dimensional vector spaces having ordered bases* β *and* γ*, respectively, and let* $T : V \to W$ *be linear. Then, for each* $u \in V$*, we have*

$$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$$

Proof. Fix $u \in V$, define linear transformations $f : F \to V$ by $f(a) = au$ and $g : F \to W$ by $g(a) = aT(u)$ for all $a \in F$. Let $\alpha = \{1\}$ be standard ordered basis for F. Note $g = Tf$. Identify column vectors as matrices, i.e. column vector $[g(1)]_\gamma$ is simply the matrix representing transformation g, $[g]_\alpha^\gamma$, since the transformation is determined by operation on the basis, which is a set of size 1.

$$[T(u)]_\gamma = [g(1)]_\gamma = [g]_\alpha^\gamma = [Tf]_\alpha^\gamma = [T]_\beta^\gamma [f]_\alpha^\beta = [T]_\beta^\gamma [f(1)]_\beta = [T]_\beta^\gamma [u]_\beta$$

□

As an example, Let $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ *be linear transformation defined by* $T(f(x)) = f'(x)$*, and let* β *and* γ *be standard ordered bases for* $P_3(\mathbb{R})$ *and* $P_2(\mathbb{R})$*. If* $A = [T]_\beta \gamma$*, then, we have*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

We verify the theorem. Let $p(x) \in P_3(\mathbb{R})$ *be* $p(x) = 2 - 4x + x^2 + 3x^3$*, let* $q(x) = T(p(x))$*, then* $q(x) = p'(x) = -4 + 2x + 9x^2$*. So*

$$[T(p(x))]_\gamma = [q(x)]_\gamma = \begin{pmatrix} -4 \\ 2 \\ 9 \end{pmatrix} \qquad [T]_\beta^\gamma [p(x)]_\beta = A [p(x)]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -4 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 9 \end{pmatrix}$$

Definition. ***Left-multiplication Transformation*** *Let A be $m \times n$ matrix with entries from a field F. We denote by L_A by mapping $L_A : F^n \to F^m$ defined by $L_A(x) = Ax$ (the matrix product of A and x) for each column vector $x \in F^n$. We L_A a left-multiplication transformation*

Theorem. ***Properties of Left-multiplication Transformation*** *Lett A be $m \times n$ matrix with entries from F. Then the left-multiplication transformation $L_A : F^n \to F^m$ is linear. Furthermore, if B is any other $m \times n$ matrix (with entries from F) and β and γ are the standard ordered bases for F^n and F^m, respectively, then we have the following properties*

1. $[L_A]_\beta^\gamma = A$
2. $L_A = L_B$ *if and only if* $A = B$
3. $L_{A+B} = L_A + L_B$ *and* $L_{aA} = aL_A$ *for all* $a \in F$
4. *If* $T : F^n \to F^m$ *is linear, then there exists a unique* $m \times n$ *matrix* C *such that* $T = L_C$. *In fact,* $C = [T]_\beta \gamma$
5. *If* E *is an* $n \times p$ *matrix, then* $L_{AE} = L_A L_E$
6. *If* $m = n$, *then* $L_{I_n} = I_{F^n}$

Theorem. ***Matrix Multiplication is Associative*** *Let A, B, and C be matrices such that $A(BC)$ is defined. Then $(AB)C$ is also defined and $A(BC) = (AB)C$; that is, the matrix multiplication is associative.*

Proof.

$$L_{A(BC)} = L_A L_{BC} = L_A(L_B L_C) = (L_A L_B)L_C = L_{AB} L_C = L_{(AB)C}$$

implies $A(BC) = (AB)C$ by 5th point in previous theorem. □

Definition. ***Incident Matrices*** *An incident matrix is a square matrix in which all the entries are either zero or one, and for convenience, all diagonal entries are zero. $A_{ij} = 1$ if i is related to j, and $A_{ij} = 0$ otherwise.*

2.4 Invertibility and Isomorphisms

Definition. ***Function Invertibility*** *Let V and W be vector spaces, and let $T : V \to W$ be linear. A function $U : W \to V$ is said to be an inverse of T if $TU = I_W$ and $UT = I_V$. If T has an inverse, then T is said to be invertible. If T is invertible, the inverse of T is unique and is denoted by T^{-1}. The following holds for invertible functions T and U*

1. $(TU)^{-1} = U^{-1}T^{-1}$
2. $(T^{-1})^{-1} = T$, *in particular* T^{-1} *is invertible*

3. *Let $T : V \to W$ be a linear transformation, where V and W are finite-dimensional spaces of equal dimension. Then T is invertible if and only if $rank(T) = dim(V)$, i.e. T is one-to-one and onto*

Theorem. ***2.17 Inverse of Transformation is Linear*** *Let V and W be vector spaces, and let $T : V \to W$ be linear and invertible. Then $T^{-1} : W \to V$ is linear.*
Then it follows from theorem 2.5 that if T is a linear transformation between vector spaces of equal (finite) dimension, then the conditions of being invertible, one-to-one, and onto are all equivalent.

Proof. Let $y_1, y_2 \in W$ and $c \in F$. Since T is onto and one-to-one, there exists unique vectors x_1 and x_2 such that $T(x_1) = y_1$ and $T(x_2) = y_2$. So $x_1 = T^{-1}(y_1)$ and $x_2 = T^{-1}(y_2)$

$$T^{-1}(cy_1+y_2) = T^{-1}(cT(x_1)+T(x_2)) = T^{-1}(T(cx_1+x_2)) = cx_1+x_2 = cT^{-1}(y_1)+T^{-1}(y_2)$$

□

Definition. ***Matrix Invertibility*** *Let A be $n \times n$ matrix. Then A is invertible if there exists an $n \times n$ matrix B such that $AB = BA = I$. Such matrix B is unique, called inverse of A and denoted by A^{-1}*

Lemma. *Let T be an invertible linear transformation from V to W., Then V is finite-dimensional if and only if W is finite-dimensional. In this case, $dim(V) = dim(W)$*

Theorem. ***2.18 Matrix and Transformation Invertibility are Equivalent*** *Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively. Let $T : V \to W$ be linear. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible. Furthermore, $\left[T^{-1}\right]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$*

Corollary. ***1 Special case where*** $W = V$ *Let V be a finite-dimensional vector space with an ordered basis β, and let $T : V \to V$ be linear. Then T is invertible if and only if $[T]_\beta$ is invertible. Furthermore, $\left[T^{-1}\right]_\beta = ([T]_\beta)^{-1}$*

Corollary. *2 Let A be an $n \times n$ matrix. Then A is invertible if and only if L_A is invertible. Futhermore, $(L_A)^{-1} = L_{A^{-1}}$*

Definition. ***(Vector Space) Isomorphism*** *Let V and W be vector spaces. We say V is isomorphic to W if there exists a linear transformation $T : V \to W$ that is invertible. Such a linear transformation is called an isomorphism from V onto W.*

Theorem. ***2.19 Isomorphic vector space have equal dimensions*** *Let V and W be finite-dimensional vector spaces (over the same field). Then V is isomorphic to W if and only if $dim(V) = dim(W)$*

Corollary. *Let V be a vecctor space over F. Then V is isomorphic to F^n if and only if $dim(V) = n$ (finite)*

Theorem. ***2.20 Collection of all linear transformation may be identified with appropriate vector space of*** $m \times n$ ***matrices***
Let V and W be finite-dimensional vector spaces over F of dimensions n and m, respectively, and let β and γ be ordered bases for V and W, respectively. Then the function $\Phi : \mathcal{L}(V, W) \to M_{m\times n}(F)$, defined by $\Phi(T) = [T]_\beta^\gamma$ for $T \in \mathcal{L}(V, W)$, is an isomorphism

Corollary. *Let V and W be finite-dimensional vector spaces of dimensions n and m, respectively. Then $\mathcal{L}(V, W)$ is finite-dimensional of dimension mn (From the fact that $dim(M_{m\times n}(F)) = mn$)*

Definition. ***Standard Representation of Vector Space*** *Let β be an ordered basis for an n-dimensional vector space V over the field F. The standard representation of V with respect to β is the function $\phi_\beta : V \to F^n$ defined by $\phi_\beta(x) = [x]_\beta$ for each $x \in V$*

Theorem. ***2.21*** *For any finite-dimensional vector space V with ordered basis β, ϕ_β is an isomorphism*

Definition. *Let V and W be vector spaces of dimensions n and m. let $T : V \to W$ be a linear transformation. Define $A = [T]_\beta^\gamma$, where β and γ are arbitrary ordered bases of V and W, respectively. We can use ϕ_β and ϕ_γ to study the relationship between linear transformations T and $L_A : F^n \to F^m$. We can use two composites of linear transformation to map V into F^m*

1. *Map V into F^n with ϕ_β and follow transformation with L_A, yielding $L_A\phi_\beta$*
2. *Map V into W with T and follow it by ϕ_γ to obtain the composite $\phi_\gamma T$*

Together, we can conclude that the two ways of composition commutes

$$L_A\phi_\beta = \phi_\gamma T$$

This allows us to transfer operations on abstract vector spaces to ones on F^n and F^m

2.5 The Change of Coordinate Matrix

Theorem. ***Coordinate Vector Change of Basis*** *Let β and β' be two ordered basis for a finite-dimensional vector space V, and let $Q = [I_V]_{\beta'}^\beta$. Then*

1. *Q is invertible*
2. *For any $v \in V$, $[v]_\beta = Q\,[v]_{\beta'}$*

*where Q is called a **change of coordinate matrix**. We say that Q changes β'-coordinates into β-coordinates. Observe that if $\beta = \{x_1, x_2, \cdots, x_n\}$ and $\beta' = \{x'_1, x'_2, \cdots, x'_n\}$, then*

$$x'_j = \sum_i^n Q_{ij}x_i$$

for $j = 1, 2, \cdots, n$ that is jth column of Q is $\left[x'_j\right]_\beta$

Proof. For any $v \in V$

$$[v]_\beta = [I_V(v)]_\beta = [I_V]_{\beta'}^{\beta} [v]_{\beta'} = Q [v]_{\beta'}$$

$\square$

Definition. ***Linear Operator*** *A linear transformation that map a vector space V into itself*

Theorem. ***2.23 Linear Operator Change of Basis*** *Let T be a linear operator on a finite-dimensional vector space V, and let β and β' be ordered bases for V. Suppose that Q is the change of the coordinate matrix that changes β'-coordinates into β-coordinates. Then*

$$[T]_{\beta'} = Q^{-1} [T]_\beta Q$$

Proof. Let I be identity transformation on V. Then $T = IT = TI$ hence, by multiplication of linear transformations

$$Q [T]_{\beta'} = [I]_{\beta'}^{\beta} [T]_{\beta'}^{\beta'} = [IT]_{\beta'}^{\beta} = [TI]_{\beta'}^{\beta} = [T]_{\beta}^{\beta} [I]_{\beta'}^{\beta} = [T]_\beta Q$$

Therefore $[T]_{\beta'} = Q^{-1} [T]_\beta Q$ $\square$

Corollary. *Let $A \in M_{n\times n}(F)$, and let γ be an ordered basis for F^n. Then $[L_A]_\gamma = Q^{-1}AQ$, where Q is the $n \times n$ matrix whose jth column is the jth vectort of γ,*

Remark. Note we make distinction between A and L_A. The former is a matrix, the latter is a function. They are not equivalent when represented as matrices since A is the same regardless but L_A is subject to a change of basis.

Definition. ***Similar Matrices*** *Let A and B be matrices in $M_{n\times n}(F)$. We say that B is similar to A if there exists an invertible matrix Q such that $B = Q^{-1}AQ$*