Integration Beyond 2-dimensions

Definition. ***Generalization of Integrals***

1. *A* ***Rectangle*** *in* $\mathbb{R}^n$ *is any set of the form*

$$R = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

with ***volume*** $V(R) = (b_1 - a_1) \times \cdots \times (b_n - a_n)$

2. *A* ***Partition*** *of* R *is an* n *partition of* $\mathbb{R}$ *each decomposing* $[a_i, b_i]$

3. *Let* $R_{i_1, \cdots, i_n}$ *be* ***sub-rectangle*** *corresponding to* $(i_1, \cdots, i_n)$ *element, then* ***Riemann Sum*** *over* R *is any of the form*

$$S(f, P) = \sum_{(i_1, \cdots, -In)} f(t_{(i_1, \cdots, i_n)}) V(R_{(i_1, \cdots, i_n)}) \qquad t \in R_{(i_1, \cdots, i_n)}$$

4. $f : \mathbb{R}^n \to \mathbb{R}$ *is* ***Riemann Integrable*** *if and only if for every* $\epsilon > 0$ *there exists a partition* P *such that*

$$U(f, P) - u(f, P) < \epsilon$$

5. *The* ***Jordan measure*** *of a set* S *is defined as the infimum of the volumes of all covering rectangles, and* S *is* ***Jordan measurable*** *if its boundary has measure zero.*

6. *If* $k < n$ *then the image of a* C^1 *map* $f : \mathbb{R}^k \to \mathbb{R}^n$ *has Jordan measure zero.*

7. *A function* $f : S \to \mathbb{R}$ *is* ***integrable*** *if* S *is Jordan measurable and if the set of discontinuities of* f *on* S *has Jordan measure zero. We denote such integral to be*

$$\int \cdots \int_S f dV = \int \cdots \int f(x) d^n x = \int \cdots \int f(x_1, \cdots, x_n) dx_1 \cdots dx_n$$

Iterated Integrals

Theorem. ***Fubini's Theorem*** *Let* $= [a, b] \times [c, d]$ *be a rectangle and* $f : R \to \mathbb{R}$ *an integrable function on* R. *If for each* $y_0 \in [c, d]$ *the function* $f_{y_0} : [a, b] \to \mathbb{R}$ *given by* $x \mapsto f(x, y_0)$ *is integrable on* $[a, b]$, *and* $g(y) = \int_a^b f(x, y) dx$ *is integrable on* $[c, d]$, *then*

$$\int_R f dA = \int_c^d \int_a^b f(x, y) dx dy$$

Definition. ***Double Integral over non-rectangles*** *Integration over Jordan measurable sets* $S \subseteq \mathbb{R}^2$ *can be done in a similar manner. Suppose* S *has its boundary defined by* *piecewise C^1 function (hence S Jordan measurable, and if f continuous over a measure zero set of discontinuities, f integrable)*

$$S = \{(x,y) : a \le x \le b, \alpha(x) \le y \le \beta(x)\}$$

Then integration becomes

$$\int_S f dA = \int_a^b \int_{\alpha(x)}^{\beta(x)} f(x,y) dy dx$$

Sometimes we need to change the boundary of S so that integration becomes easier.

Definition. ***Triple Integral over non-rectangles*** *Suppose S has its boundary defined by piecewise C^1 function.*

$$S = \{(x,y,z) : a \le x \le b, \alpha(x) \le y \le \beta(x), \varphi(x,y) \le z \le \psi(x,y)\}$$

and integration becomes

$$\iiint_S f(x,y,z) dA = \int_a^b \int_{\alpha(x)}^{\beta x} \int_{\varphi(x,y)}^{\phi(x,y)} f(x,y,z) dz dy dx$$

Integral rules

1. $$\int \ln(x) dx = x \ln(x) - x + C$$

Definition. ***Trig substitition***

integrand	$x =$	*identity*
$a^2 - x^2$	$a \sin\theta$	$1 - sin^2\theta = \cos^2\theta$
$a^2 + x^2$	$a \tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$x^2 - a^2$	$a \sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

Definition. ***Some Trig identities***

1. $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
2. $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$
3. $\cos(u)\cos(v) = \frac{1}{2}(\cos(u - v) + \cos(u + v))$

4.4 Change of Variables

Definition. ***Diffeomorphism*** *If $U, V \in \mathbb{R}^n$ and $f : U \to V$ is a C^1 bijection with C^1 inverse $f^{-1} : V \to U$, then we say that f is a diffeomorphism.*

Remark. Space U and V are identical with respect to differentiation.

Theorem. *If $T : V \to W$ is a linear transformation between vector spaces of same dimension, and $S \subseteq V$ is measurable with measure $m(S)$, then*

$$m(TS) = |detT|m(S)$$

Remark. The absolute value of the Jacobian determinant at point $p \in V$ gives us the factor by which the function f expands or shrinks volumes near p;

Theorem. ***One-Var Change of Variable*** *Let $I \subseteq \mathbb{R}$ be an interval and $\varphi : [a, b] \to I$ be a differentiable function with integrable derivative. Suppose $f : I \to \mathbb{R}$ is a continuous function. Then*

$$\int_{\varphi(a)}^{\varphi(b)} f(x)dx = \int_a^b f(\varphi(t))\varphi'(t)dt$$

where $x = \varphi(t)$ and $dx = \varphi'(t)dt$. Equivalently

$$\int_I f(x)dx = \int_{\varphi^{-1}(I)} f(\varphi(t))\varphi'(t)dt$$

Proof. Let $f : I \to \mathbb{R}$ be a continuous; Let $\varphi : [a, b] \to I$ be a differentiable function such that φ' is integrable on $[a, b]$. Then function $f(\varphi(t))\varphi'(t)$ is also integrable on $[a, b]$. Hence,

$$\int_{\varphi(a)}^{\varphi(b)} f(x)dx \qquad \int_a^b f(\varphi(t))\varphi'(t)dt$$

exists. Since f is continuous, it has antiderivative F. Since F and φ are differentiable, we have

$$(F \circ \varphi)'(t) = F'(\varphi(t))\varphi'(t) = f(\varphi(t))\varphi'(t)$$

By the fundamental theorem of calculus twice

$$\int_a^b f(\varphi(t))\varphi'(t)dt = \int_a^b (F\circ\varphi)'(t)dt = (F\circ\varphi)(b)-(F\circ\varphi)(a) = F(\varphi(b))-F(\varphi(a)) = \int_{\varphi(a)}^{\varphi(b)} f(x)dx$$

□

Theorem. ***Multivariable Change of Variables*** *If $S, T \subseteq \mathbb{R}^n$ are measurable and $G : S \to T$ is a diffeomorphism, then for any integrable function $f : T \to \mathbb{R}$ we have*

$$\int_T f(u)du = \int_{G^{-1}(T)} f(G(x))|\det DG(x)|dx$$

Remark.

Polar Coordinate $f : (r, \theta) \to (x, y) = (r\cos(\theta), r\sin(\theta))$ we have

$$\det Df = r \qquad dxdy = rdrd\theta$$

As an example, the area of circle of radius $r = a$ is

$$A = \int_{x^2+y^2 \leq a^2} dxdy = \int_0^{2\pi}\int_0^a rdrd\theta = \pi a^2$$

Cylindrical Coordinate $f : (r, \theta, z) \to (x, y, z) = (r\cos(\theta), r\sin(\theta), z)$ we have

$$\det Df = r \qquad dxdydz = rdrd\theta dz$$

Spherical Coordinate $f : (\rho, \theta, \phi) = (x, y, z) = (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))$ where $0 \leq \rho \leq \mathbb{R}$, $0 \leq \phi \leq \pi$, and $0 \leq \theta \leq 2\pi$ so then

$$\det Df = -\rho^2\sin(\phi) \qquad dxdydz = \rho^2\sin(\phi)d\rho d\theta d\phi$$