Inferences in Regression and Correlation Analysis

2.1 Inference Concerning β_1

Definition. ***Inference concerning*** β_1 *Test concerning* β_1 *is of the form*

$$\begin{cases} \mathcal{H}_0 : \beta_1 = 0 \\ \mathcal{H}_\alpha : \beta_1 \neq 0 \end{cases}$$

$\beta_1 = 0$ *implies that there is no linear association between* Y *and* X*. On assumption of gaussian noise, there is also no relation of any type between* Y *and* X*, since probability of* Y *are identical for all levels of* X

Definition. ***Linear estimators*** *Least square estimator* $\hat{\beta}_1$ *is a linear estimator*

$$\hat{\beta}_1 = \sum_i k_i y_i \qquad k_i = \frac{x_i - \overline{x}}{\sum_j (x_j - \overline{x})^2}$$

with properties

$$\sum k_i = 0 \quad \sum k_i x_i = 1 \quad \sum k_i^2 = \frac{1}{S_{XX}}$$

Proof. Note

$$\sum (x_i - \overline{x})(y_i - \overline{y}) = \sum (x_i - \overline{x}) y_i - \sum (x_i - \overline{x}) \overline{y} = \sum (x_i - \overline{x}) y_i \quad \text{by} \sum (x_i - \overline{x}) = 0$$

the result follows. Proof of properties are simple, i.e.

$$\sum k_i = \sum_i \left(\frac{x_i - \overline{x}}{\sum_j (x_j - \overline{x})^2} \right) = \frac{1}{\sum_j (x_j - \overline{x})^2} \sum_i (x_i - \overline{x}) = 0$$

□

Definition. ***Sampling distribution of*** $\hat{\beta}_1$ *The sampling distribution of* $\hat{\beta}_1$ *refers to different values of the estimator obtained with repeated sampling when the levels of the predictor variable* X *are held constant from sample to sample. Given* $\hat{\beta}_1$

$$\hat{\beta}_1 = \frac{\sum (x_i - \overline{x})(y_i - \overline{y})}{\sum (x_i - \overline{x})^2}$$

The sampling distribution of $\hat{\beta}_1$ *is* ***normal*** *with mean and variance,*

$$\mathbb{E}(\hat{\beta}_1 | X) = \beta_1 \quad Var(\hat{\beta}_1 | X) = \frac{\sigma^2}{\sum (x_i - \overline{x})^2} = \frac{\sigma^2}{S_{XX}}$$

$$\hat{\beta}_1 \sim \mathcal{N}(\beta_1, \frac{\sigma^2}{S_{XX}})$$

Proof.

1. **Mean and variance**

$$\mathbb{E}(\hat{\beta}_1) = \mathbb{E}\{\sum k_i y_i\} \overset{ind}{=} \sum(k_i \mathbb{E}\{y_i\}) = \sum k_i(\beta_0 + \beta_1 x_i) = \beta_0 \sum k_i + \beta_1 \sum k_i x_i = \beta_1$$

$$Var(\hat{\beta}_1) = Var\left(\sum k_i y_i\right) = \sum k_i^2 Var(y_i) = \sum k_i^2 \sigma^2 = \sigma^2 \sum k_i^2 = \frac{\sigma^2}{S_{XX}}$$

   with last step of both derivation given by properties of $\hat{\beta}_1$ as a linear estimator.

2. **Normality** of sampling distribution given by the fact that $\hat{\beta}_1$ is a linear combination of y_is. Since $y_i \overset{i.i.d}{\sim} \mathcal{N}(\beta_0 + \beta 1 x_i, \sigma^2)$, the estimator is normaly distributed because it is a linear combination of independent normal random variables

3. **Estimated variance** We can estimate the variance of $\hat{\beta}_1$ by substituting σ^2 (unknown) with its unbiased estimator MSE s^2

$$s^2(\hat{\beta}_1) = \frac{MSE}{S_{XX}} \quad \text{where} \quad MSE = \frac{\sum e_i^2}{n-2}$$

   Note s^2 carries a denominator of S_{XX}, this is from the variance of $\hat{\beta}_1$, we are simply substituting the unknown σ^2 with MSE

□

Definition. ***standardization of*** $\hat{\beta}_1$
Standardization of sampling distribution of $\hat{\beta}_1$ gives

$$Z = \frac{\hat{\beta}_1 - \beta_1}{\sigma(\hat{\beta}_1)} = \mathcal{N}(0,1) \quad \textit{wher} \quad \sigma(\hat{\beta}_1) = \frac{\sigma}{\sqrt{S_{XX}}}$$

Usually, have to estimate standard error $\sigma(\hat{\beta}_1)$ with $s(\hat{\beta}_1)$. Standardization where the denominator is an estimated standard error is called ***studentized statistic****, given by*

$$T = \frac{\hat{\beta}_1 - \beta_1}{s(\hat{\beta}_1)} \sim t_{n-2} \quad \textit{where} \quad s^2(\hat{\beta}_1) = \frac{MSE}{S_{XX}}$$

Proof. Assume proposition

$$\frac{\sum(\hat{e}_i)^2}{\sigma^2} \sim \chi^2_{n-2}$$

Then we have

$$\frac{\hat{\beta}_1 - \beta_1}{s(\hat{\beta}_1)} = \frac{\hat{\beta}_1 - \beta_1}{\sigma(\hat{\beta}_1)} \Bigg/ \frac{s(\hat{\beta}_1)}{\sigma(\hat{\beta}_1)} = \frac{\hat{\beta}_1 - \beta_1}{\sigma(\hat{\beta}_1)} \Bigg/ \sqrt{\frac{\sum e_i^2 S_{XX}}{(n-2) S_{XX} \sigma^2}} \sim \frac{Z}{\sqrt{\chi^2_{n-2} \Big/ (n-2)}} = t_{n-2}$$

□

Definition. ***Confidence Interval and test for*** β_1
We use the previously derived distribution as a pivot

$$\Pr\left(t_{\alpha/2,n-2} \leq \frac{\hat{\beta}_1 - \beta_1}{s(\hat{\beta}_1)} \leq t_{1-\alpha/2,n-2}\right) = 1 - \alpha$$

$$\Pr\left(\hat{\beta}_1 - s(\hat{\beta}_1)t_{1-\alpha/2,n-2} \leq \beta_1 \leq \hat{\beta}_1 + s(\hat{\beta}_1)t_{1-\alpha/2,n-2}\right) = 1 - \alpha$$

Hence the confidence interval is given by

$$\hat{\beta}_1 \pm s(\hat{\beta}_1)t_{1-\alpha/2,n-2} \qquad \textit{where} \qquad s^2(\hat{\beta}_1) = \frac{MSE}{S_{XX}}$$

For 2-sided tests

$$\begin{cases} \mathcal{H}_0 : \beta_1 = b \\ \mathcal{H}_\alpha : \beta_1 \neq b \end{cases}$$

We compute test statistics

$$t^* = \frac{\hat{\beta}_1 - b}{s(\hat{\beta}_1)} \qquad \textit{and reject } \mathcal{H}_0 \textit{ if } |t^*| > t_{1-\alpha/2,n-2}$$

2.2 Inference Concerning of $\hat{\beta}_0$

Definition. ***Sampling Distribution of*** $\hat{\beta}_0$
Given point estimator

$$\hat{\beta}_0 = \overline{y} - \beta_1\overline{x}$$

$\hat{\beta}_0$ *refers to different values of* β_0 *that would be obtained with repeated sampling when levels of predictor variable* x *are held constant from sample to sample. The* ***sampling distribution of*** $\hat{\beta}_0$ *is* ***normal*** *with mean and variance*

$$\mathbb{E}(\hat{\beta}_0) = \beta_0 \qquad Var(\hat{\beta}_0) = \sigma^2\left[\frac{1}{n} + \frac{\overline{x}^2}{\sum(x_i - \overline{x})^2}\right] = \sigma^2\left[\frac{1}{n} + \frac{\overline{x}^2}{S_{XX}}\right]$$

Proof. The normality follows because $\hat{\beta}_0$ is also a linear estimator of y_is. We can estimate $Var(\hat{\beta}_0)$ by replacing σ^2 with MSE, as before

$$s^2(\hat{\beta}_0) = MSE\left[\frac{1}{n} + \frac{\overline{x}^2}{S_{XX}}\right]$$

□

Definition. ***Standardization of*** $\hat{\beta}_0$

$$Z = \frac{\hat{\beta}_0 - \beta_0}{\sigma(\hat{\beta}_0)} \sim t_{n-2} \quad \textit{where} \quad \sigma^2(\hat{\beta}_0) = \sigma^2 \left[\frac{1}{n} + \frac{\overline{x}^2}{S_{XX}}\right]$$

$$T = \frac{\hat{\beta}_0 - \beta_0}{s(\hat{\beta}_0)} \sim t_{n-2} \quad \textit{where} \quad s^2(\hat{\beta}_0) = MSE \left[\frac{1}{n} + \frac{\overline{x}^2}{S_{XX}}\right]$$

Definition. ***Confidence Interval for*** $\hat{\beta}_0$

$$\hat{\beta}_0 \pm s(\hat{\beta}_0) t_{1-\alpha/2,n-2} \quad \textit{where} \quad s(\hat{\beta}_0) = \sqrt{\frac{\sum e_i^2}{n-2}} \sqrt{\frac{1}{n} + \frac{\overline{x}^2}{S_{XX}}}$$

Definition. ***Consideration about inference***

1. ***Spacing of*** x ***levels*** *Looking at variance of* $\hat{\beta}_0$ *and* $\hat{\beta}_1$*, the larger the spread in* x *levels, the larger* S_{XX} *and smaller the variance.*

2.4 Interval Estimation of $\mathbb{E}(Y|X = x^*)$, the population regression line

Definition. ***Mean estimation*** *Often times want to estimate mean for one or more probability distribution of* Y*. (i.e. mean* Y *for low and high* X *levels). Let* x^* *be level of* X *for which we want to estimate the mean response* $\hat{y}^*$*. The mean response is given by*

$$\mathbb{E}(Y|X = x^*) = E(y^*) = \beta_0 + \beta_1 x^*$$

The idea is that the expectation is a random variable because of the estimated correlation coefficients. We would want to do inference on the mean response. We have a point estimator of the the mean response

$$\hat{y}^* = \hat{\beta}_0 + \hat{\beta}_1 x^*$$

which is simply the estimated response from estimated correlation coefficients and regression function. Note $X = x^*$ *is a known constant*

Definition. ***Sampling Distribution of mean response estimator*** $\hat{y}^*$
The sampling distribution of $\hat{y}^*$ *is* ***normal*** *with mean and variance*

$$\mathbb{E}(\hat{y}^*) = \mathbb{E}(\hat{y}|X = x^*) = \beta_0 + \beta_1 x^* \quad (= \mathbb{E}(y^*) \textit{ so unbiased})$$

$$Var(\hat{y}^*) = Var(\hat{y}|X = x^*) = \sigma^2 \left[\frac{1}{n} + \frac{(x^* - \overline{x})^2}{\sum(x_i - \overline{x})^2}\right] = \sigma^2 \left[\frac{1}{n} + \frac{(x^* - \overline{x})^2}{S_{XX}}\right]$$

$$\hat{y}^* = (\hat{y}|X = x^*) \sim \mathcal{N}(\beta_0 + \beta_1 x^*, \sigma^2 \left[\frac{1}{n} + \frac{(x^* - \overline{x})^2}{S_{XX}}\right])$$

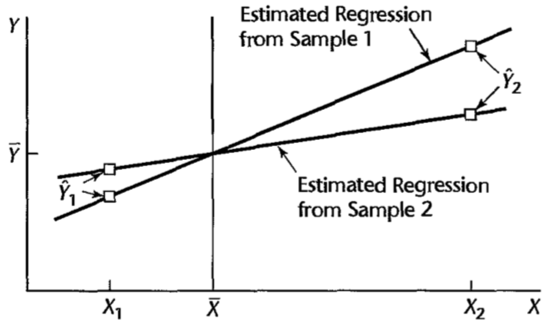
Note when $x^* = 0$, $Var(\hat{y}^*)$ *reduces to variance of* $\hat{\beta}_0$*.*

Proof. 3 parts

1. **Normality** of $\hat{y}^*$ follows from the fact that it is composed of $\hat{\beta}_0$ and $\hat{\beta}_1$, both of which are linear estimators of y_i

2. **Mean**

$$\mathbb{E}(\hat{y}^*) = \mathbb{E}\left(\hat{\beta}_0 + \hat{\beta}_1 x^*\right) = \mathbb{E}(\hat{\beta}_0) + x^*\mathbb{E}(\hat{\beta}_1) = \beta_0 + \beta_1 x^* = \mathbb{E}(y^*)$$



3. **Variance** Idea is variability of $\hat{y}^*$ is affected by how far x^* is from $\overline{x}$, via

$$(x^* - \overline{x})^2 = S_{XX}$$

The further x^* is from $\overline{x}$, the greater the variability. Note in plot, x_1 near $\overline{x}$, the fitted value $\hat{y}_1$ for two sample regression line (from 2 experiments) are close to each other; the fitted values $\hat{y}_2$ differ substantially due to the fact that x_2 is far from $\overline{x}$. In summary,

variation in $\hat{y}^*$ value from sample to sample will be greater when x^* is far from mean than when x^* is near mean

We can substitute MSE for σ^2 to obtain $s^2(\hat{y}^*)$. The estimated variance is given by

$$s^2(\hat{y}^*) = MSE\left[\frac{1}{n} + \frac{(x^* - \overline{x})^2}{S_{XX}}\right]$$

□

Definition. ***Standardization of mean response estimator $\hat{y}^*$***

$$Z = \frac{\hat{y}^* - (\beta_0 + \beta_1 x^*)}{\sigma(\hat{y}^*)} \sim \mathcal{N}(0,1)$$

note $\mathbb{E}(y^*) = \beta_0 + \beta_1 x^*$

$$T = \frac{\hat{y}^* - (\beta_0 + \beta_1 x^*)}{s(\hat{y}^*)} \sim t_{n-2}$$

Definition. ***Confidence Interval for $\hat{y}^*$***
The $100(1-\alpha)\%$ confidence interval for $\mathbb{E}(Y|X=x^*) = \beta_0 + \beta_1 x^*$ is given by

$$\hat{y}^* \pm s(\hat{y}^*)t_{1-\alpha/2,n-2} = (\hat{\beta}_0 + \hat{\beta}_1 x^*) \pm t_{1-\alpha/2,n-2}\sqrt{\frac{\sum e_i^2}{n-2}}\sqrt{\frac{1}{n} + \frac{(x^* - \overline{x})^2}{S_{XX}}}$$

Definition. ***Obervations***

1. *variance of $\hat{y}^*$ is smallest when $x^* = \overline{x}$. So in an experiment to estimate mean response at a particular level x^* of predictor variable, the* ***precision of the estimate is greatest*** *if (everything else remain equal) the observations on X are spaced so that $\overline{x} = x^*$*

2. *confidence interval for $\hat{y}^*$ not sensitive to moderate departures from assumption of error being normally distributed. The robustness in estimating mean response is related to robustness of Confidence interval for β_0 and β_1*

2.5 Prediction of New Observations

Definition. ***Prediction of New Observations***

1. ***Motivation*** *Idea is that we have a model set up given a set of data, and we would want to extrapolate to new data points. The new observation Y is viewed as the result of a new trial, independent of trials on which the regression analysis is based. Let level of X be x^* and the new observation y^*_{new} (which is unknown, and which we want to characterize), assuming that the underlying regression model is still appliacble for basic sample data*

2. ***Estimate of mean response $\mathbb{E}(y^*)$ vs. Prediction of new response y^*_{new}***
In the former case, we estimate ***mean of distribution of Y****. In latter case, we predict an* ***individual outcome*** *drown from the distribution of Y. Idea is we have to take into account of the fact that the majority of individual outcomes deviate from the mean response*

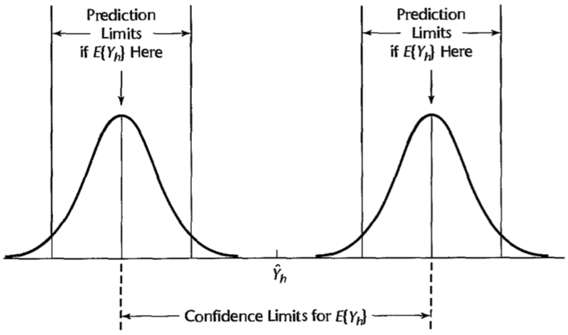
Definition. ***Prediction Interval for y^*_{new} when parameter is known***
Assume all parameters are known, we have y^* follow a normal distribution

$$y^*_{new} \sim \mathcal{N}(\beta_0 + \beta_1 x^*, \sigma^2)$$

So we have confidence interval,

$$(\beta_0 + \beta_1 x^*) \pm \sigma z_{1-\alpha/2}$$

Definition. ***Prediction Interval for y^*_{new} when parameter is unknown***
When parameter is unknown, we must estimate regression parameters. We might want to estimate mean distribution of Y with $\hat{y}^*$ and variance of distribution of Y with MSE.



However, we cannot substitute these estimate into the previous distribution because $\mathbb{E}(y^*)$ is a random variable. Since we do not know the mean $\mathbb{E}(y^*)$, and only estiamte it by a confidence interval (shown previously), we cannot be certain of the distribution of Y. It could be anywhere along within its confidence intervals (for $\mathbb{E}\{Y_h\}$). Hence ***prediction limit*** *for y^*_{new} must take into account two elements*

1. *variation in possible location of distribution of Y (i.e. sampling distributino of $\hat{y}^*$)*
2. *variation within the probability distribution of Y (namely σ^2, same as that of error terms')*

We can prove that

$$\mathbb{E}(y^*_{new} - \hat{y}^*) = \mathbb{E}(y_{new} - \hat{y}|X = x^*) = \mathbb{E}(\hat{\beta}_0 + \hat{\beta}_1 x^*) - \mathbb{E}(\hat{y}^*) = 0$$

$$Var(y^*_{new} - \hat{y}^*) = Var(\hat{\beta}_0 + \hat{\beta}_1 x^*) + Var(\hat{y}^*) = \sigma^2 + \sigma^2\left[\frac{1}{n} + \frac{(x^* - \overline{x})^2}{S_{XX}}\right] = \sigma^2\left[1 + \frac{1}{n} + \frac{(x^* - \overline{x})^2}{S_{XX}}\right]$$

Note $Cov(y^*_{new}, \hat{y}^*) = 0$ by independence

$$y^*_{new} - \hat{y}^* \sim \mathcal{N}(0, \sigma^2\left[1 + \frac{1}{n} + \frac{(x^* - \overline{x})^2}{S_{XX}}\right])$$

An unbiased estimtor of the variance is given by

$$s^2(y^*_{new} - \hat{y}^*) = MSE\left[1 + \frac{1}{n} + \frac{(x^* - \overline{x})^2}{S_{XX}}\right]$$

Standardization of *y^*_{new} gives*

$$T = \frac{y^*_{new} - \hat{y}^*}{s^2(y^*_{new} - \hat{y}^*)} \sim t_{n-2}$$

The $(100 - \alpha)\%$ ***Prediction limit*** *for y^*_{new} at $X = x^*$ is thus given by,*

$$(\hat{\beta}_0 + \hat{\beta}_1 x^*) \pm t_{1-\alpha/2, n-2} MSE\sqrt{1 + \frac{1}{n} + \frac{(x^* - \overline{x})^2}{S_{XX}}}$$

Definition. ***Comments***

1. *prediction limit is subject to departure from normality of error term distributions*
2. *A confidence interval represents an inference on a paramter and is an interval that is intended to cover the value of* ***parameter****. A prediction interval, is a statement about the value to be taken by a* ***random variable****, the new observation* y^*_{new}

Confidence-band for Regression line

Definition. ***Confidence-band*** *represents uncertainty in the estimate of regression line (i.e.* $\mathbb{E}(Y) = \beta_0 + \beta_1 X$*)*