Problem 1

1. Let X be a set. The finite complement topology, denoted $\tau_{fc} \subset P(X)$, is ~~the set~~ topology that has for open sets subsets of X, U, so that $X \setminus U$ is finite, or $U = X$. or $\emptyset$.

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(To solve 2 I used the equivalence f is cont $\Leftrightarrow$ for every closed set B in the range, $f^{-1}(B)$ is closed)

2. Let $f : X, \tau_{fc}^X \to Y, \tau_{fc}^Y$. Suppose $Y \neq \emptyset$.

$\Rightarrow$): Suppose f is continuous:

Suppose f is not constant.

Let $y \in Y$. Then $Y \setminus \{y\} \in \tau_{fc}^Y$ because $Y \setminus (Y \setminus \{y\}) = \{y\}$ is finite. Than $\{y\}$ is a closed set.

f is continuous then for every $B \subset Y$ closed, $f^{-1}(B) \subset X$ is closed. so $f^{-1}(\{y\})$ is closed.

But that means $X \setminus f^{-1}(\{y\})$ is open, therefore its complement, $f^{-1}(y)$, is finite from the definition of τ_{fc}^X. (Because $f^{-1}(\{y\}) \neq X$ as it is not constant)

So f is finite to one. (otherwise, f must be constant by assumption), Therefore f is ~~either~~ constant or finite to one.

$\Leftarrow$: Suppose $f \equiv y$ is constant, then every $V \subset Y$ satisfies $f^{-1}(V) \in \{\emptyset, X\}$ Because $f^{-1}(V) = \begin{cases} X & y \in V \\ \emptyset & \text{otherwise} \end{cases}$, so f is continuous.

Suppose f is finite to one.

Let $B \subset Y$ be closed. Then B is finite. (like earlier)

Assume $B = \{y_i\}_{i=1}^N$ Then $f^{-1}(B) = \bigcup_{i=1}^N f^{-1}(\{y_i\})$ which is a finite union of finite sets, therefore finite.

So $f^{-1}(B)$ is closed in the fc topology, so f is continuous.

Problem 2

1. a topological space X is Hausdorff (T_2) if for every $x, y \in X$, if $x \neq y$ then there exist $x \in U$, $y \in V$ nbds of x and y, resp, s.t. $U \cap V = \emptyset$.

(For 2 I used the thm; $z \in \bar{A} \iff \forall U$ nbd of z, $U \cap A \neq \emptyset$)

2. $\Rightarrow$: Suppose X is T_2, and let $X \ni x \neq y \in X$, we will show that $(x,y) \notin \bar{\Delta} = \overline{\{(x,x)\}} \supset \{x,x\}$ and therefore it is closed $(\bar{\Delta} = \Delta)$. (because if Δ isn't closed, $\bar{\Delta} \not\subset \Delta \Rightarrow \exists x \neq y$ s.t (x,y)
There exist U, V such as in the def, $x \in U$, $y \in V$, U, V are open and $U \cap V = \emptyset$. Then $U \times V$ is open w.r. to the prod. top..
Let A be a set, $z \in \bar{A}$. ~~But every open set~~ We recall that for $z \in \bar{A}$, if $z \in U$ open, then $U \cap A \neq \emptyset$.
But $(x,y) \in U \times V$ while $U \times V \cap \Delta = \emptyset$ because if $(z,z) \in U \times V$, then $z \in U \cap V$, contradiction.
thus $(x,y) \notin \bar{\Delta} \Rightarrow \bar{\Delta} \subset \Delta \subset \bar{\Delta}$ so they are equal and Δ is closed.

$\Leftarrow$: Suppose Δ is closed, and let $x, y \in X$, $x \neq y$.
Then $(x,y) \notin \bar{\Delta}$ and from the other side of the equivalence we recall that $z \notin \bar{A} \Rightarrow \exists U \ni z$ open s.t. $U \cap A = \emptyset$ because $z \in \bar{A} \iff \forall U \ni z$, U open, $U \cap A \neq \emptyset$. (this is negation)
So $\exists U$ open s.t. $(x,y) \in U$ and $U \cap \Delta = \emptyset$. Let $D \subset U$ be a basis element of the prod. top. containing (x,y), then $D = B \times C$ for $X \supset B$ open, $X \supset C$ open, and ~~$D \cap \Delta \subset U \cap \Delta$~~
$D \cap \Delta \subset U \cap \Delta = \emptyset \Rightarrow B \cap C = \emptyset$ and we have B, C nbds of x, y s.t. $B \cap C = \emptyset$.
(because otherwise $(B \times C) \cap \Delta \neq \emptyset$)

So X is Hausdorff (T_2).

Problem 3

1. It is enough to show that for every $x \in B$, ~~and~~ $y \in B^c$ $\exists U, V$ nbd of x, y s.t. $U \subset B$, $V \subset B^c$ resp.

Suppose $(x_n) \in B$, and let $\prod_{n \in \mathbb{N}} (x_n - 1, x_n + 1) =: U$, an open subset containing (x_n). $U \subset B$ because $\sup_{n \in \mathbb{N}} |y_n| \leq \sup_{n \in \mathbb{N}} (|x_n| + 1) \leq (\sup_{n \in \mathbb{N}} |x_n|) + 1 < \infty$ as $\sup_{n \in \mathbb{N}} |x_n| < \infty$. Then $(y_n) \in B$. $\boxed{y_n \in U}$

So $U \subset B$.

If $x_n \in B^c$, we take the subset $U = \prod_{n \in \mathbb{N}} (x_n - 1, x_n + 1)$,

but now as $\limsup |x_n| = \infty$, it follows that ~~$\sup |x_n + 1|$~~

~~$\inf |x_n + 1|$ $\inf |x_n - 1|$~~

~~$\inf |y_n| \geq \inf_{n \in \mathbb{N}} (|x_n| - 1) \geq \inf_{n \in \mathbb{N}} |x_n| - 1$~~

for $y_n \in U$: $|y_n| \geq |x_n| - 1$

$$\limsup_n |y_n| \geq \limsup_n |x_n| - 1 \geq (\limsup |x_n|) - 1 = \infty.$$

Then $y_n \in B^c \Rightarrow U \subset B^c$. So B, B^c are open $\Rightarrow B$ is open and closed.

2. ~~Let~~ We have $\bar{0} \in B$. Let U be a nbd of $\bar{0}$, we will show $U \not\subset B$. $U = \bigcap_{i=1}^{N} \pi_i^{-1}(U_i)$ for $0 \in U_i \subset \mathbb{R}$ open $\forall i$, $N \in \mathbb{N}$. We can take $y_n \in U$ s.t. $y_n = \begin{cases} 0 & n \leq N \\ n & n > N \end{cases}$ and thus get $(y_n) \in U$, $y_n \notin B$ as (y_n) is unbounded. ($\lim y_n = \infty$)

So B can't be open ~~closed~~ as there exists no nbd of $\bar{0}$ that is open $\subset B$. (because $U \not\subset B$)

Now let $(n)_{n \in \mathbb{N}} \in B^c$, we will show that there is no nbd of $(n)_{n \in \mathbb{N}}$ that is contained in B^c, therefore B^c can't be open. Let $(n)_{n \in \mathbb{N}} \in U$ such nbd, then $U = \bigcap_{i=1}^{N} \pi_i^{-1}(U_i)$ for $N \in \mathbb{N}$, $i \in U_i \subset \mathbb{R}$ open for every i.

We have that $y_n = \begin{cases} n & n \leq N \\ 0 & n > N \end{cases}$ is bounded by N,

solution continues on next page

Therefore $(y_n) \in U$, $(y_n) \notin B^c \Rightarrow U \not\subset B^c$ and thus B^c is not open, so B is not closed.
Thus B is not open nor closed.

Problem 5

We will show that the metric $d(x,y)=\sup \frac{1}{n} d_n(x(n),y(n))$ is a metric and it induces the same topology as the product topology.

First, it's a metric: Symmetry: $d(x,y)=\sup \frac{1}{n} d_n(x(n),y(n)) \overset{\circledast}{=} \sup \frac{1}{n} d_n(y(n),x(n))=d(y,x)$

where $\circledast$ is from the symmetry of each d_n.

Non-negative: $d(x,y)=\sup \frac{1}{n} d_n(x(n),y(n)) \overset{\forall n}{\geq} \frac{1}{n} d_n(x(n),y(n)) \geq 0$ (#)

$d(x,y)=0 \iff \sup \frac{1}{n} d(x(n),y(n))=0 \iff \forall n,\ \frac{1}{n} d_n(x(n),y(n)) \leq 0 \underset{\substack{\uparrow \\ d_n \text{ is a metric}}}{\iff} \forall n\ d_n(x(n),y(n))=0$

$\overset{(\#)}{\iff} \forall n\ x(n)=y(n) \iff x=y$

$\triangle$-inequality: $d(x,y)+d(y,z)=\sup \frac{1}{n} d_n(x(n),y(n))+\sup \frac{1}{n} d(y(n),z(n)) \overset{(i)}{\geq}$

$\overset{\forall n}{\geq} \frac{1}{n} d_n(x(n),y(n))+\frac{1}{n} d(y(n),z(n)) \overset{(ii)}{\geq} \frac{1}{n} d_n(x(n),z(n))$

as these are $\geq$ supremums

d_n is a metric

And because this holds for any $n\in\mathbb{N}$,

$d(x,y)+d(y,z) \overset{(i)+(ii)}{\geq} \sup \frac{1}{n} d_n(x(n),z(n))=d(x,z)$.

Now let U be open in the prod top, $x\in U$.

$U=\bigcap_{i=1}^{N} \pi_i^{-1}(U_i)$ for U_i's open in their resp. X_i's, $N\in\mathbb{N}$.

So for every U_i, there exists an $\varepsilon_i>0$ s.t. $B_{\varepsilon_i}(x(i))\subset U_i$ (as $x(i)\in U_i$). Let us take $\varepsilon=\min_{1\leq i\leq N} \frac{\varepsilon_i}{i}$, then

$B_\varepsilon(x)=\{(y_n):\sup \frac{1}{n} d_n(x(n),y(n))<\varepsilon\} \subset \{(y_n): \max_{1\leq i\leq N} \frac{1}{i} d_i(x(i),y(i))<\varepsilon\} \subset$

$\subset \{(y_n): \forall_{1\leq i\leq N}\ \frac{1}{i} d_i(x(i),y(i))<\frac{\varepsilon_i}{i}\} \subset \bigcap_{i=1}^{N} \pi_i^{-1}(U_i)=U$, so U is open w.r. to the induced metric top.

to be continued...

Now we will show that for every basis element (open ball), B, in the metric topology, $x \in B$, $\exists$ W open in the prod topology s.t. $W \subset B$, $x \in W$.

Let $B_\delta(x)$ be such ~~open~~ basis element, and let $y \in B_\delta(x)$, then $B_\varepsilon(y) \subset B_\delta(x)$ for some $\varepsilon > 0$.

Now, for $N = \lceil \frac{2}{\varepsilon} \rceil$, we can take for every $i \le N$ $U_i = (y(i) - \frac{\varepsilon}{2}, y(i) + \frac{\varepsilon}{2})$, and for $i > N$, $U_i = X_i$. then $\prod_{i \in \mathbb{N}} U_i =: U$ is open w.r.t ~~to~~ the prod topology. $y \in U$ because $\forall i, y(i) \in U_i$.

$U \subset B_\varepsilon(y)$ because for $(z_n) \in U$, $\sup_n \frac{1}{n} d_n(z(n), y(n)) \le$

$\le \max\left(\max_{i \le N} \frac{1}{i} d_i(z(i), y(i)), \sup_{n > N} \frac{1}{n} d_n(z(n), y(n))\right) \le \frac{\varepsilon}{2} < \varepsilon$

As: $\forall i \le N$ $z(i) \in B_{\varepsilon/2}(y(i)) \Rightarrow \frac{1}{i} d_i(z(i), y(i)) < \frac{\varepsilon}{2i} \le \varepsilon/2$

And $\forall n > N$, $\frac{1}{n} d_n(z(n), y(n)) \underset{\substack{\uparrow \\ d_n \le 1}}{\le} \frac{1}{n} \cdot 1 = \frac{1}{n} < \frac{1}{N} \le \varepsilon/2$

as the diameter ≤ 1

Then $d(y, z) = \sup \frac{1}{n} d_n(z(n), y(n)) \le \frac{\varepsilon}{2} < \varepsilon \Rightarrow U \subset B_\varepsilon(y) \subset B_\delta(x)$ and we are done, as $B_\delta(x)$ must be open w.r.t. the ~~met~~ prod top. the topologies are each finer than the other, hence they are equal.