STA 247
Probability with Computer Applications

Professor Karen H. Wong

Week 2

Tentative Schedule for This Week

Learning Goals

- Apply counting rules to "unusual" problems (cases, distinct combinations with non-distinct elements)
- Visualize and compute conditional probability
- Determine whether two (or more) events are independent
- Bayes' Theorem - Computation and tree diagram & Law of Total Probability

Tentative Schedule for This Week

Success Criteria You will have successfully learned the concepts if...

- You can choose and **justify** the use of fundamental principle of counting, permutations, combinations to solve problems
- You can explain the difference between conditional probability and probabilities involving union and intersection of sets
- You can identify given information and compute conditional probabilities (or any of $P(A \cap B), P(A), P(B)$)
- You can show through computation that two (or more events) are dependent/independent
- You can identify given information and correctly use Bayes' theorem (or an associated tree diagram) to solve problems

Independent Events

Recall that two events are independent if the occurrence of one (A) does not affect the occurrence of the other (B). Formally,

Independent Events

Two events A and B are **independent** if:

"A given B" - knowing B occurs does not affect the probability of A occurring → independent

If: $P(A|B) = P(A)$

$$P(A|B) = P(A)$$
$$P(B|A) = P(B)$$

$\frac{P(A \cap B)}{P(B)} = P(A)$

The result is equivalent to: $P(A \cap B) = P(A) \cdot P(B)$

$P(A \cap B) = P(A) \cdot P(B)$

Two events that are *not independent* are often called **dependent**.

Mutually Exclusive: Two events are mutually exclusive if the occurrence of one (A) excludes the occurrence of the other (B). Mathematically, the two sets are disjoint or $P(A \cap B) = \emptyset$. This implies that the events are dependent.

Independent Events

If events A and B are independent, then their complements A^c and B^c are also independent. As an exercise, check this for yourself! In the case of

multiple events $A_1, A_2, ...A_n$, if all n events are independent then

$$P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1) \cdot P(A_2) \cdot ... \cdot P(A_n)$$

and are **mutually independent** if any subset of events are also independent.

Example: Events A, B, C, D are independent. They are also mutually independent if each of the following are also independent.

- (A, B), (A, C), (A, D), (B, C), (B, D), (C, D)
- (A, B, C), (A, B, D), (A, C, D), (B, C D)

$C = (C \cap D^c) \cup (C \cap D)$

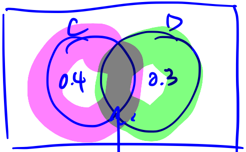
Example 4 Two events C and D have the following probabilities: $P(C) = 0.4, P(D) = 0.3, P(C \cup D) = 0.5$, determine

$\hookrightarrow$ C or D.

a. $P(C \cap D)$

$= P(C) + P(D) - P(C \cap D)$

$= 0.4 + 0.3 - 0.5 \quad = 0.2$



$P(C \cap D) = 0.2$

b. $P(C' \cap D')$

① $= P(C') + P(D') - P(C' \cup D')$

$P(C \cup D) = P(C) + P(D) - P(C \cap D)$

② $= P(C' \cap D') = P((C \cup D)^c)$

$= 1 - P(C \cup D) = 0.5$

c. $P(C|D') = \dfrac{P(C \cap D')}{P(D')} = \dfrac{P(C) - P(C \cap D)}{P(D')} = \dfrac{0.4 - 0.2}{1 - 0.3} = \dfrac{2}{7}$

Independent Events - Examples

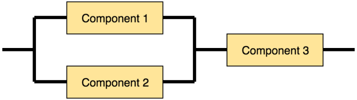
Example 5 Two events E and F have the following probabilities: $P(E) = 0.44, P(F) = 0.6, P(E \cap F) = 0.35$. Determine if E and F are independent. Check: $P(E \cap F) = P(E)\,P(F)$

Independent Events - Examples

Example 5 Two events E and F have the following probabilities: $P(E) = 0.44, P(F) = 0.6, P(E \cap F) = 0.35$. Determine if E and F are independent.

- If independent, then $P(E \cap F) = P(E) \cdot P(F)$
  $P(E) \cdot P(F) = 0.44 \cdot 0.6 = 0.264 \neq 0.35$
- Events E and F are dependent.

Example 6 A system below is made of independent components. The probability that the first component works is 0.9, 0.95 for the second component, and 0.99 for the third component. The signal can travel from left to right if there is a circuit made of working components. Find the probability that the signal is blocked.



• Let B = event that signal is blocked. Let C_1, C_2, C_3 be the events that components 1, 2, and 3 are operational.

↳ Ways this can work:

union of these is the event C_3^c

$$\begin{cases} C_1^c \cap C_2^c \cap C_3^c \\ C_1 \cap C_2 \cap C_3^c \\ C_1 \cap C_2^c \cap C_3^c \\ C_1^c \cap C_2 \cap C_3^c \end{cases}$$

, $C_1^c \cap C_2^c \cap C_3$

Independent Events - Examples

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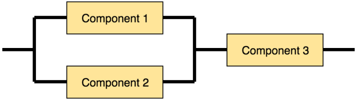

- Let B be the event that the signal is blocked. Let C_1, C_2, C_3 be the events that components 1, 2, 3 are working.

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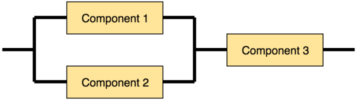

- Let B be the event that the signal is blocked. Let C_1, C_2, C_3 be the events that components 1, 2, 3 are working.
- $P(B) = P(C_3$ doesn't work or C_1 and C_2 don't work while C_3 does)

$= P(C_3' \cup (C_1' \cap C_2' \cap C_3))$ -> two disjoint events

$= P(C_3') + P(C_1' \cap C_2 \cap C_3) = 1 - P(C_3) + P(C_1')P(C_2')P(C_3)$

$= (1 - P(C_3)) + (1 - 0.9) \cdot (1 - 0.95) \cdot 0.99 = 0.01495$

Example 6

Alternatively, you can take the indirect approach.

Example 6 Signal works if C_3 always works <u>and</u> C_1 or C_2 or both work

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$P(B) = 1 - P(B')$

$$P(B') = P(C_3 \cap (C_1 \cup C_2)) = P(C_1 \cup C_2) \cdot P(C_3)$$

$$= [P(C_1) + P(C_2) - P(C_1 \cap C_2)] \cdot P(C_3)$$

$$= [0.9 + 0.95 - 0.9 \cdot 0.95](0.99)$$

$$= 0.98505 \rightarrow \text{signal not blocked}$$

$$P(B) = 1 - 0.98505 = 0.01495$$

$P(C|M) = P(C) \mid P(C|F) = P(C)$

Example 7 A study of colourblindness among males and females was performed. The results are tabulated below: $(\sim 5 \text{mins})$ $P(C) = \frac{0.042}{1}$

	Men (M)	Women (M')	Total
Colourblind (C)	0.04	0.002	0.042
Not Colourblind (C')	0.47	0.488	0.958
Total	0.51	0.49	1.00

$P(C|M) = \frac{0.04}{0.51} = 7.8\%$

a. What is the probability of being colourblind if you are male?

b. What is the probability of being colourblind if you are female?

$P(C|M') = \frac{0.02}{0.49} = 0.4\%$

or just verify 0.51 * 0.042 ?= 0.04

c. Is colourblindness independent of sex?

↳ No. $P(C|M) \neq P(C|M') \neq P(C)$

Work with a partner to answer this problem. In a test situation, you might only be asked to determine independence (c.). You need to (independently) connect independence to the appropriate conditional probabilities.

Law of Total Probability

Given two events, we've learned that their conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

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Suppose now we have a sample space consisting **only** of events $A, B_1, B_2, ...B_k$ where the $B_i's$ form a **partition of the sample space**.

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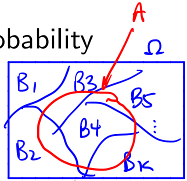

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$A = (A \cap B_3) \cup (A \cap B_2) \cup (A \cap B_4) \cup (A \cap B_5) \cup \ldots (A \cap B_k)$

Suppose now we have a sample space consisting **only** of events $A, B_1, B_2, ...B_k$ where the $B_i's$ form a **partition of the sample space**. This means that each B_i is disjoint and $\bigcup_{i=1}^{k} B_i = \Omega$. From last week, we have:

$A = (A \cap B) \cup (A \cap B^c)$

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) + ... + P(A \cap B_k)$$

Combining this with our knowledge of conditional probability, we get the **Law of Total Probability**

Law of Total Probability

Law of Total Probability

If $B_1, B_2, ...B_k$ is a collection of mutually exclusive (**disjoint**) and exhaustive events, then for any event A,

$P(A \cap B_1) + \cdots P(A \cap B_2) + \ldots + P(A \cap B_k)$

$$P(A) = P(A|B_1) \cdot P(B_1) + P(A|B_2) \cdot P(B_2) + ... + P(A|B_k) \cdot P(B_k)$$

$$P(A) = \sum_{i=1}^{k} P(A|B_i) \cdot P(B_i)$$

Bayes' Rule

We can now combine Law of Total Probability with Conditional Probability to get **Bayes' Rule**

Bayes' Rule

Let $B_1, B_2, ...B_k$ form a partition of the sample space and let A be an event in Ω. Then

$$P(B_i|A) = \frac{P(A \cap B_i)}{P(A)}$$

$$P(B_i|A) = \frac{P(A|B_i) \cdot P(B_i)}{\sum_{i=1}^{k} P(A|B_i) \cdot P(B_i)}$$

Bayes' Rule Examples

Example 8 Returning to HIV testing. An HIV test will correctly test positive 95% of the time, and incorrectly test positive 1% of the time. Suppose we know that 99.99% of the population is HIV-free. Under these conditions, what is the probability that a patient who tested positive is actually HIV positive?

$P(+ \mid HIV) = 0.95$

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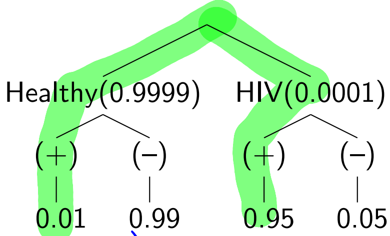
- Let's represent the situation with a tree diagram: $P(+ \mid Healthy) = 0.01$



$P(+ \mid Healthy)$ $P(+ \mid HIV)$

Bayes' Rule Examples

Example 8

Note that the end probabilities are **conditional** probabilities. i.e. the first probability of testing positive (0.01) is only on the condition that the patient is healthy.

Let H denote the event that the patient is healthy. Let $(+)$ denote the event that the patient tests positive. Then we have:

Example 8

The probability that a person actually has HIV given that they test positive is:

$$P(H'|(+)) = \frac{P(H' \cap (+))}{P((+))}$$

$P(+ \mid H) = 0.01$ $\quad$ $P(- \mid H) = 0.99$

$P(+ \mid H^c) = 0.95$ $\quad$ $P(- \mid H^c) = 0.05$

Probabilities within a condition sum to 1

$P(H) = 0.9999$ $\quad$ $P(H^c) = 0.0001$

$$P(+) = P(+ \cap H) + P(+ \cap H^c)$$
$$= P(+|H) \cdot P(H) + P(+|H^c) \cdot P(H^c)$$
$$= 0.01 \times 0.9999 + 0.95 \times 0.0001 = 0.010094$$

Bayes' Rule Examples

Example 8

The probability that a person actually has HIV given that they test positive is:

$$P(H'|(+)) = \frac{P(H' \cap (+))}{P((+))}$$

We have computed the denominator. We need to find the numerator, and there are two ways of doing so. We can use either $P(H'|(+)) \cdot P((+))$ or $P((+)|H') \cdot P(H')$. The latter form is the most useful to us. $P(H) = 0.010094$

$P(H' \cap +) = P(+|H') \cdot P(H') = 0.95 \times 0.0001$

$P(H'|+) = \frac{0.95 \times 0.0001}{0.010094} = 0.00941 \rightarrow 0.941\%$

Thus if a patient tests positive (for what one would consider a highly accurate test), there is only a 0.941% chance that the patient actually has HIV.

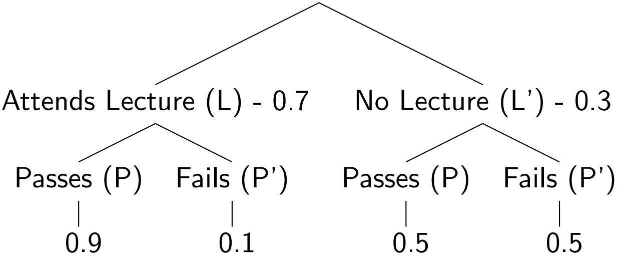
Bayes' Rule Examples

Example 9 A student will attend lecture 70% of the time. When she attends lecture, she has a 90% chance of passing. If she doesn't attend lecture, she has a 50% chance of passing.

a. Represent this situation with a tree diagram.
b. What is her probability of passing?
c. Hurray! She passes the course! What's the probability that she attended lecture?
d. What is the probability that she didn't attend lecture and she fails the course?
e. What is the probability she didn't attend lecture given that she fails the course?

Example 9

a. Represent this situation with a tree diagram.



b. What is her probability of passing?

Bayes' Rule Examples

Example 9

a. Represent this situation with a tree diagram.

Attends Lecture (L) - 0.7 | No Lecture (L') - 0.3

Passes (P) | Fails (P') | Passes (P) | Fails (P')

0.9 | 0.1 | 0.5 | 0.5

b. What is her probability of passing?

$$
\begin{aligned}
P(P) &= P(P|L) \cdot P(L) + P(P|L') \cdot P(L') \rightarrow \text{Law of Total Prob.} \\
&= (0.7)(0.9) + (0.3)(0.5) = 0.78
\end{aligned}
$$

Bayes' Rule Examples

Example 9

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Bayes' Rule Examples

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$$P(L|P) = \frac{P(L \cap P)}{P(P)} = \frac{P(P|L) \cdot P(L)}{P(P)}$$
$$= \frac{0.9 \cdot 0.7}{0.78} = 0.8077$$

d. What is the probability that she didn't attend lecture and she failed the course?

Bayes' Rule Examples

Example 9

c. Hurray! She passes the course! What's the probability that she attended lecture?

$$P(L|P) = \frac{P(L \cap P)}{P(P)} = \frac{P(P|L) \cdot P(L)}{P(P)}$$
$$= \frac{0.9 \cdot 0.7}{0.78} = 0.8077$$

d. What is the probability that she didn't attend lecture and she failed the course? Note that there's no condition stated here. Simply the event that both events occur.
$P(L' \cap P') = P(P'|L') \cdot P(L') = 0.5 \cdot 0.3 = 0.15$

Bayes' Rule Examples

Example 9

e. What is the probability she didn't attend lecture given that she fails the course?

Bayes' Rule Examples

Example 9

e. What is the probability she didn't attend lecture given that she fails the course? Here a condition is clearly stated: She has failed the course. Knowing this, what's the probability she didn't attend lecture?

$$
\begin{aligned}
P(L'|P') &= \frac{P(L' \cap P')}{P(P')} \\
&= \frac{0.15}{P(P'|L') \cdot P(L') + P(P'|L) \cdot P(L)} \\
&= \frac{0.15}{0.15 + (0.1)(0.7)} = 0.6818
\end{aligned}
$$

Practice 3!

Harmeet can take 3 different routes to school. 55% of the time, she takes the subway to school, but will be late 30% of the time. 30% of the time, she'll take the bus to school, but will be late 40% of the time. 15% of the time, she'll just walk to school, but will be late 10% of the time. Given that she showed up on time to her classes, what is the probability she walked to school?

Suggested Problems

p. 53: 2.68 - 2.70
p. 66: 3.2, 3.4, 3.7, 3.9, 3.14
p. 75: 3.23, 3.27-3.30, 3.33-3.35
p. 81: 3.43-3.45, 3.51, 3.54