# UNIVERSITY OF TORONTO

## The Faculty of Arts and Science

**APRIL 2016 EXAMINATIONS**

**MAT247H1 S**
**Duration: 3 hours**

**NO AIDS ALLOWED**

**Last Name:** ______________________________

**First Name:** ______________________________

**ID:** ______________________________

**Section:** ______________________________

- Total marks for this paper: 300 marks
- This paper contains 2 pages, not including the cover page.

| Question | Possible Marks | Marks Earned |
|---|---|---|
| 1 | 40 | |
| 2 | 50 | |
| 3 | 40 | |
| 4 | 40 | |
| 5 | 30 | |
| 6 | 60 | |
| 7 | 40 | |
| | | |
| | | |
| | | |
| TOTAL | 300 | |

**University of Toronto**
**Faculty of Arts and Science**
**Final Examinations, April-May 2016**
**MAT247HS – Algebra II**
**Instructor: Stephen S. Kudla**
**Duration – 3 hours**
**No aids allowed or needed**

*Please write clearly and show all of your work.*
*The point value of each problem is indicated.*

**1.** (*40 points*) State and prove Schur's Theorem.

**2.** (*50 points*) Let $V$ be a finite dimensional vector space over a field $F$, with $\dim_F V = n$, and let $T : V \to V$ be a linear transformation.
(i) Suppose that the characteristic polynomial $P_T(t)$ of $T$ is irreducible[1]over $F$. Show that the only $T$-invariant subspaces of $V$ are $\{0\}$ and $V$.
(ii) Suppose that $P_T(t) = g(t)\,h(t)$ for polynomials $g(t)$ and $h(t)$ both of degree less than $n$. Show that $V$ has a $T$-invariant subspace $W$ with $0 < \dim_F W < n$.

**3.** (*40 points*) (i) Find the Jordan Canonical Form (JCF) of the matrix

$$A = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 1 & 3 & 0 & 1 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 5 \end{pmatrix}.$$

(ii) Find a matrix $Q$ such that $Q^{-1}AQ$ is in JCF.

**4.** (*40 points*) Let $V$, $\langle\,,\,\rangle$ be a finite dimensional complex inner product space with $\dim V = n$.
(i) Define what it means for a linear transformation $T : V \to V$ to be unitary.
(ii) Show that $T$ is unitary if and only if $||T(x)|| = ||x||$ for all $x$ in $V$.

**5.** (*30 points*) Let $V$, $\langle\,,\,\rangle$ be a finite dimensional inner product space with $\dim V = n$. Suppose that $S = \{v_1, \dots, v_k\}$ is an orthogonal subset of $V$.
(i) Show that $S$ can be extended to an orthogonal basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ for $V$.
(ii) Show that $\{v_{k+1}, \dots, v_n\}$ is an orthogonal basis for $W^\perp$.
(iii) Show that $\dim V = \dim W + \dim W^\perp$.

[1]Recall that this means that $P_T(t)$ cannot be written as a product of two polynomials $g(t)$ and $h(t)$ both of degree less than $n$.

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**6.** (*60 points*) Let $V$, $\langle\, , \rangle$ be a finite dimensional inner product space over $\mathbb{C}$, $\dim_{\mathbb{C}} V = n$, and let $T : V \to V$ be a normal linear transformation of $V$.
(i) State the spectral theorem.
(ii) Suppose that $T$ has $n$ distinct eigenvalues and that $S$ is a linear transformation of $V$ such that $ST = TS$. Show that $S$ is normal.
(iii) Show that there is a polynomial $g(t)$ such that $S = g(T)$.
(iv) Give an example of such a pair of operators $S$ and $T$ where $T$ does *not* have $n$ distinct eigenvalues and $S$ *cannot* be written as a polynomial in $T$.

**7.** (*40 points*) (i) Suppose that $A$ is an $11 \times 11$ real matrix with characteristic polynomial
$$P_A(t) = t(t-1)^5(t-2)^3(t-3)^2$$
and minimal polynomial
$$M_A(t) = t(t-1)^2(t-2)^2(t-3).$$
What are the possible Jordan canonical forms of $A$? Explain your answer.
(ii) Suppose that $B$ is a $8 \times 8$ real matrix with characteristic polynomial
$$P_B(t) = (t^2+7)^2(t^2+11)^2.$$
What are the possible minimal polynomials for $B$? For each of these, describe the rational canonical form of $B$.