CSC446 A3

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Problem 1

Implement Ritz-Galerkin method on equidistant grid with piecewise linear hat function for

$$
\begin{aligned}
-y'' + y = (\pi^2+1)\sin(\pi x) \quad x \in (0,1) \\
y(0) = 1 \quad y'(1) = \frac{1}{2}(e - \frac{1}{e}) - \pi
\end{aligned}
$$

where the real solution to the problem is given by

$$
y(x) = \frac{1}{2}(e^x + e^{-x}) + \sin(\pi x)
$$

solution. Note this BVP is a special case of BVP that appeared in the textbook, so

$$
a_{k,l} = \langle \varphi_l', \varphi_k' \rangle + \langle \varphi_l, \varphi_k \rangle = \begin{cases} \frac{2(h^2+3)}{3h} & l = k \\ \frac{h^2-6}{6h} & l = k-1, k+1 \\ 0 & \text{otherwise} \end{cases}
$$

for $k = 1, 2, \cdots, m-1$. Let

$$
\varphi_0(x) = \left(\frac{1}{2}\left(e - \frac{1}{e}\right) - \pi \right) x + 1
$$

satisfies the boundary conditions. Now we compute the right hand side of the linear system

$$
\begin{aligned}
b_k &= \langle f, \varphi_k \rangle - a_{k,0} \\
&= \int_0^1 (\pi^2+1)\sin(\pi x)\varphi_k(x)dx - \int_0^1 \varphi_0'(x)\varphi_k'(x) + \varphi_0(x)\varphi_k(x)dx \\
&= \int_{(k-1)h}^{kh} (\pi^2+1)\sin(\pi x)\left(1 - k + \frac{x}{h}\right)dx + \int_{kh}^{(k+1)h} (\pi^2+1)\sin(\pi x)\left(1 + k - \frac{x}{h}\right)dx \\
&\quad - \int_{(k-1)h}^{kh} \left(\frac{1}{2}\left(e - \frac{1}{e}\right) - \pi\right)\frac{1}{h} + \left(\left(\frac{1}{2}\left(e - \frac{1}{e}\right) - \pi\right)x + 1\right)\left(1 - k + \frac{x}{h}\right)dx \\
&\quad - \int_{kh}^{(k+1)h} \left(\frac{1}{2}\left(e - \frac{1}{e}\right) - \pi\right)\left(-\frac{1}{h}\right) + \left(\left(\frac{1}{2}\left(e - \frac{1}{e}\right) - \pi\right)x + 1\right)\left(1 + k - \frac{x}{h}\right)dx
\end{aligned}
$$

for $k = 1, 2, \cdots, m-1$. To allow for arbitrary approximate $y_m(x)$ at $x = 1$, we have a basis function $\varphi_m(x)$ supported over $[x_{m-1}, x_m]$ only. The corresponding entries in A and b is as follows

$$
\begin{aligned}
a_{m,m-1} &= \frac{h^2-6}{6h} \\
a_{m,m} &= \frac{h^2+3}{3h} \\
b_m &= \int_{(m-1)h}^{mh} (\pi^2+1)\sin(\pi x)\left(1 - m + \frac{x}{h}\right)dx \\
&\quad - \int_{(m-1)h}^{mh} \left(\frac{1}{2}\left(e - \frac{1}{e}\right) - \pi\right)\frac{1}{h} + \left(\left(\frac{1}{2}\left(e - \frac{1}{e}\right) - \pi\right)x + 1\right)\left(1 - m + \frac{x}{h}\right)dx
\end{aligned}
$$

During matrix assembly, we use `integral` to numerically integrate b_k for $k = 1, 2, \cdots, m$, as exact value of the integral are not easy to obtain. We set `AbsTol` to `1e-20` such that error resulted from numerical integration does not contribute to error of Ritz-Galerkin method. $\square$

Problem 2

Repeat qestion 1, but use cubic B-spline basis instead of piecewise linear hat basis function.