CSC236 Assignment #1

Zian Qin, Peiqi Wang

October 7, 2016

Problem 1

Consider the Fibonacci-esque function g:

$$g(n) = \begin{cases} 1 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ g(n-2) + g(n-1) & \text{if } n > 1 \end{cases}$$

Use complete induction to prove that if n is a natural number greater than 1, then $2^{n/2} \leq g(n) \leq 2^n$. You may not derive or use a closed-form $g(n)$ in your proof.

Proof.

Let predicate P be

$$P(n) : 2^{n/2} \leq g(n) \leq 2^n \text{ with the given } g(n)$$

Basis:
Show $P(n)$ holds for $n = 2$,

$$g(2) = g(1) + g(0) = 3 + 1 = 4$$
$$2^{2/2} = 2 \leq g(2) = 4 \leq 2^2 = 4$$
$$\therefore P(2) \text{ is true}$$

Show $P(n)$ holds for $n = 3$,

$$g(3) = g(2) + g(1) = 4 + 3 = 7$$
$$2^{3/2} = 2\sqrt{2} < 2(3) \leq g(3) = 7 \leq 2^3 = 8$$
$$\therefore P(3) \text{ is true}$$

Inductive Step:
Let n ≥ 4 be any
Assume $2^{m/2} \leq g(m) \leq 2^m \ \forall m \in [2, n]$

$$\text{WTS } 2^{(n+1)/2} \leq g(n+1) \leq 2^{n+1}$$

$$g(n+1) = g(n) + g(n-1)$$

By the inductive hypothesis, we know the following:

$$2^{n/2} \leq g(n) \leq 2^n$$

and

$$2^{(n-1)/2} \leq g(n-1) \leq 2^{n-1}$$

Combining the two inequalities we get:

$$2^{n/2} + 2^{(n-1)/2} \leq g(n) + g(n-1) \leq 2^n + 2^{n-1}$$
$$2^{n/2} + 2^{n/2-1/2} \leq g(n+1) \leq 2^n + 2^n 2^{-1}$$
$$2^{n/2} + 2^{n/2} 2^{-1/2} \leq g(n+1) \leq 2^n(1 + 1/2)$$
$$2^{n/2}(1 + 2^{-1/2}) \leq g(n+1) \leq 2^n(1.5) \leq 2^n(2)$$
$$2^{n/2}(1.5) \leq 2^{n/2}(1 + 2^{-1/2}) \leq g(n+1) \leq 2^n 2 = 2^{n+1}$$
$$2^{n/2}(\sqrt{2}) \leq 2^{n/2}(1.5) \leq g(n+1) \leq 2^{n+1}$$
$$2^{(n+1)/2} = 2^{n/2+1/2} = 2^{n/2}(\sqrt{2}) \leq g(n+1) \leq 2^{n+1}$$
$$2^{(n+1)/2} \leq g(n+1) \leq 2^{n+1}$$
$$\therefore P(n+1) \text{ holds}$$

Therefore Given any $n \geq 4$, we proved $\forall 2 \leq m \leq n, P(m) \Rightarrow P(n+1)$.
By Complete Induction, $\forall n \in \mathbb{N}, n > 1 \Rightarrow 2^{n/2} \leq g(n) \leq 2^n$

□

Problem 2

Suppose B is a set of binary strings of length n, where n is positive (greater than 0), and no two strings in B differ in fewer than 2 positions. Use simple induction to prove that B has no more than 2^{n-1} elements

Proof.
Before formally proving the problem, here is a claim that is derived in both lecture notes and lecture sessions, and therefore I will be using it directly:

$$\forall l \in \mathbb{N}, \text{ there are } 2^l \text{ binary strings of length } l$$

Let A_n be a set of binary strings of length n where $n > 0$. Because of the previously stated claim, $|A_n| = 2^n$ holds. Let B_n be a set of binary strings of length n where $n > 0$ and no two strings in B_n differ in fewer than 2 positions, let predicate P be,

$$P(n) : |B_n| \leq 2^{n-1}$$

Proof by Simple induction,
Basis:
When $n = 1$, $B_1 = \emptyset$ because any binary string of length 1 cannot differ in 2 positions. Therefore $|B_1| = 0 \leq 2^{1-1} = 1$. Therefore $P(1)$ holds.
Inductive Step: Show $P(n) \Rightarrow P(n+1)$ for any $n \in \mathbb{N}$
Let arbitrary $i \in \mathbb{I}$, P(i) holds, meaning $|B_i| \leq 2^{i-1}$. Here A_i is every possible binary string of length i. Here we define a map $M : A_i \to B_{i+1}$ whereby binary string in the domain appends either a 0 or a 1. We claim that M cannot append 1 and append 0 to the same element in the A_i. Proof by contradiction. Assume $a \in A_i$, and that $b_0, b_1 \in B_{i+1}$ be binary string appended with 0 and 1 to a respectively. Because b_0, b_1 differ by exactly one position, which is the last position, b_0, b_1 cannot be B_{i+1} at the same time. However we assumed $b_0, b_1 \in B_{i+1}$, contradiction arises. Therefore, M can append 1 or 0 only once to the same element in A_i. Hence, there are at most $|A_i| = 2^i$ elements in B_{i+1}, or that $|B_{i+1}| \leq 2^{(i+1)-1}$. We proved $P(n+1)$ holds. Since i is arbitrary, $\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$

By simple induction, $P(n)$ holds for all $n \in \mathbb{N}, n > 0$.

□

Problem 3

Define T as the smallest set of strings such that:

1. "b" $\in T$
2. if $t_1, t_2 \in T$, then $t_1 +$ "ene" $+ t_2 \in T$, where + operator is string concatenation.

Use structural induction to prove that if $t \in T$ has n "b" characters, then t has $2n-2$ "e" characters.

Proof.

Define

$$C(t_i, t_2) : t_1 + \text{"ene"} + t_2 \text{ for some } t_1, t_2 \in T$$
$$b(t) : \text{ the number of "b" characters in t}$$
$$e(t) : \text{ the number of "e" characters in t}$$

$$P(t) : e(t) = 2b(t) - 2$$

Basis:
let $t_1 \in T$ be character "b".

$$b(t) = 1$$
$$e(t) = 0 = 2 * 1 - 2$$

Therefore P(t) holds

Recursive step:
Show that $\forall t_1, t_2 \in T, P(t_1) \wedge P(t_2) \Rightarrow P(C(t_1, t_2))$
$P(t_1)$ and $P(t_2)$ implies that $e(t_1) = 2b(t_1) - 2$ and $e(t_2) = 2b(t_2) - 2$. Since by construction of set T,

$$b(C(t_1, t_2)) = b(t_1) + b(t_2)$$
$$e(C(t_1, t_2)) = e(t_1) + e(e_2) + 2$$

Then,

$$\begin{aligned} e(C(t_1, t_2)) &= e(t_1) + e(e_2) + 2 \\ &= 2b(t_1) - 2 + 2b(t_2) - 2 + 2 \\ &= 2b(C(t_1, t_2)) - 2 \end{aligned}$$

Therefore $P(C(t_1, t_2))$ holds
By structural induction, for any $t \in T$, if t has n "b" characters, then t has $2n - 2$ "e" characters. $\square$

Problem 4

Note the quantity $\phi = {}^{(1+\sqrt{5})}/_{2}$ is shown closely related to Fibonacci function. You may assume that $1.61803 < \phi < 1.61804$. Complete the steps below to show that ϕ is irrational.

a show that $\phi(\phi - 1) = 1$

$$\begin{aligned}
\phi(\phi-1) &= \frac{1+\sqrt{5}}{2}\left(\frac{1+\sqrt{5}}{2} - 1\right) \\
&= \frac{\sqrt{5}+1}{2}\left(\frac{\sqrt{5}+1}{2} - \frac{2}{2}\right) \\
&= \frac{\sqrt{5}+1}{2}\left(\frac{\sqrt{5}+1-2}{2}\right) \\
&= \frac{\sqrt{5}+1}{2}\left(\frac{\sqrt{5}-1}{2}\right) \\
&= \frac{(\sqrt{5})^2 - (1)^2}{4} && \text{\#difference of squares} \\
&= \frac{5-1}{4} = 4/4 = 1 \\
\therefore \phi(\phi-1) &= 1
\end{aligned}$$

b Rewrite the equation in the previous step so that you have ϕ on the left-hand side, and on the right-hand side a fraction whose numerator and denominator are expressions that may only have integers, $+$ or $-$, and ϕ. There are two different fractions, corresponding to the two different factors in the original equation's left-hand side. Keep both fractions around for future consideration.

$$\phi = \frac{1}{\phi - 1} \text{ and } \frac{1+\phi}{\phi}$$

c Assume, for a moment, that there are natural numbers m and n such that $\phi = {}^{n}/_{m}$. Re-write the right-hand side of both equations in the previous step so that you end up with fractions whose numerators and denominators are expressions that may only have integers, $+$ or $-$, m and n.

$$\begin{aligned}\phi &= \frac{1}{\phi - 1}\\ &= \frac{1}{\frac{n}{m} - 1}\\ &= \frac{1}{\frac{n-m}{m}}\\ &= \frac{m}{n-m}\end{aligned}$$

$$\begin{aligned}\phi &= \frac{1+\phi}{\phi}\\ &= \frac{1+\frac{n}{m}}{\frac{n}{m}}\\ &= \frac{m+n}{m}\frac{m}{n}\\ &= \frac{m+n}{n}\end{aligned}$$

d Use your assumption from the previous part to construct a non-empty subset of the natural numbers that contains m. Use the Principle of Well-Ordering, plus one of the two expressions for ϕ from the previous step to derive a contradiction.

Let $\{M_k\}$ $\{N_k\}$ be sequences. Let arbitrary $m_0, n_0 \in \mathbb{N}$ such that $\phi = \dfrac{m_0}{n_0 - m_0}$ as previously assumed. We construct $\forall k \in \mathbb{N}, k > 0, m_{k+1} = n_k - m_k$ and $n_{k+1} = m_k$. Note that $\{M_k\}$,$\{N_k\}$ $\in \mathbb{N}$ due to construction. Given arbitrary i such that $\phi = \dfrac{m_i}{n_i - m_i}$. Note that m_{k+1} will always be in the natural numbers because the difference between 2 natural number remains a natural number. Then, We can find m_{i+1}, n_{i+1} such that

$$\begin{aligned}\phi &= \frac{m_i}{n_i - m_i}\\ &= \frac{n_{i+1}}{m_{i+1}} && \text{(by definition of sequence)}\\ &= \frac{m_{i+1}}{n_{i+1} - m_{i+1}} && (\phi = \frac{n}{m} = \frac{m}{n-m} \text{ for some } n, m \in \mathbb{N})\end{aligned}$$

The fraction representation of $\phi = \dfrac{m}{n-m}$ persisted and we can always get another m_i such that $m_i < m_{i-1}$. The sequence $\{M_k\}$ is a proper sequence in that it has infinitely many elements. However by the well ordering principle, $\{M_k\} \in \mathbb{N}$ always has a smallest element. Here contradiction arises.

e Combine your assumption and contradiction from the previous step into a proof that ϕ cannot be the ratio of two natural numbers. Extend this to a proof that ϕ is irrational.

Given $\phi = \dfrac{1+\sqrt{5}}{2}$, we proved $\phi(\phi-1) = 1$ by computation. By arrangements, we get $\phi = \dfrac{1}{\phi - 1}$. We prove that ϕ is irrational by contradiction. Assume ϕ is rational, then $\exists n, m \in \mathbb{N}, \phi = \dfrac{n}{m}$. By substituting in $\phi = \dfrac{1}{\phi-1}$, we can express ϕ using another fraction $\phi = \dfrac{m}{n-m}$.

Remark. I will just copy things from previous question...

Let $\{M_k\}$ $\{N_k\}$ be sequences. Let arbitrary $m_0, n_0 \in \mathbb{N}$ such that $\phi = \dfrac{m_0}{n_0 - m_0}$ as previously assumed. We construct $\forall k \in \mathbb{N}, k > 0, m_{k+1} = n_k - m_k$ and $n_{k+1} = m_k$. Note that $\{M_k\}$,$\{N_k\}$ $\in \mathbb{N}$ due to construction. Given arbitrary i such that $\phi = \dfrac{m_i}{n_i - m_i}$. Note that m_{k+1} will always be in the natural numbers because the difference between 2 natural number remains a natural number. Then, We can find m_{i+1}, n_{i+1} such that

$$\begin{aligned}
\phi &= \frac{m_i}{n_i - m_i} \\
&= \frac{n_{i+1}}{m_{i+1}} && \text{(by definition of sequence)} \\
&= \frac{m_{i+1}}{n_{i+1} - m_{i+1}} && (\phi = \frac{n}{m} = \frac{m}{n-m} \text{ for some } n, m \in \mathbb{N})
\end{aligned}$$

The fraction representation of $\phi = \dfrac{m}{n-m}$ persisted and we can always get another m_i such that $m_i < m_{i-1}$. The sequence $\{M_k\}$ is a proper sequence in that it has infinitely many elements. However by the well ordering principle, $\{M_k\} \in \mathbb{N}$ always has a smallest element, hence contradiction.
Therefore, ϕ cannot be expressed as a ratio of two natural numbers and thus irrational.

Problem 5

Consider function f, where $3 \div 2 = 1$ (integer division)

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ f^2(n \div 3) + 3f(n \div 3) & \text{if } n > 0 \end{cases}$$

Use complete induction to prove that for every natural number n greater than 2, $f(n)$ is a multiple of 7. NB: Think carefully about which natural numbers you are justified in using the inductive hypothesis for.

Proof.
Let

$$P(n) : \exists k \in \mathbb{N}, f(n) = 7k$$

Basis:
Prove that $\forall x \in \{3, 4, 5, 6, 7, 8\} : P(x)$ holds. Given $f(n)$ defined previously, we can compute the $f(x)$,

x	3	4	5	6	7	8
$f(x)$	28	28	28	28	28	28

Just to show how the computation works,

$$\begin{aligned} f(1) &= f^2(1 \div 3) + 3f(1 \div 3) \\ &= f^2(0) + 3f(0) && (1 \div 3 = 0) \\ &= 1^2 + 3 * 1 && (f(0) = 1) \\ &= 4 \end{aligned}$$

$$\begin{aligned} f(3) &= f^2(3 \div 3) + 3f(3 \div 3) \\ &= f^2(1) + 3f(1) && (3 \div 3 = 1) \\ &= 4^2 + 3 * 4 && (f(1) = 4) \\ &= 28 \end{aligned}$$

Other $f(x)$ are computed in the same way and will not be listed here. In every case $28 = 4*7$. Therefore, $P(x)$ holds.

Inductive step:
For any $i \in \mathbb{N}, i \geq 9$, given that $\forall j \in \mathbb{N}, 3 \leq j < i$, P(j) holds. Need to prove that P(i) holds. Beause $i \geq 9$, $3 \leq i \div 3 < i$, and hence $P(i \div 3)$ holds. Therefore, $\exists k \in \mathbb{N}, f(i \div 3) = 7k$. Let $k_0 \in \mathbb{N}, f(i \div 3) = 7k_0$

$$\begin{aligned} f(i) &= f^2(i \div 3) + 3f(i \div 3) \\ &= (7k_0)^2 + 3(7k_0) \\ &= 7(7k_0^2 + 3k_0) \\ &= 7k_1 && (\text{Let } k_1 = 7k_0^2 + 3k_0\text{, then } k_1 \in \mathbb{N}) \end{aligned}$$

Therefore $\exists k \in \mathbb{N}, f(i) = 7k$. P(i) holds.
By complete induction, we proved that $f(n)$ is a multiple of 7 for all $n > 2$

□