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Sheet 2 - Convergence & Bisection

CSE 213 - Numerical Analysis

120200033 CSE Section 01
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Question 1: Rate of Convergence and Order of Convergence

(a) Determine the rate of convergence of the following sequences:

(i) $x_n = 1 + (\frac{1}{2})^n$

Solution: $x_{n\to\infty} = 1$, $\dfrac{\|x_{n+1} - x_\infty\|}{\|x_n - x_\infty\|} = \dfrac{\left(\frac{1}{2}\right)^{n+1}}{\left(\frac{1}{2}\right)^{n}} = \dfrac{1}{2}$, $O(\frac{1}{2}^n)$

(converges linearly, p=1, $0 < k = 0.5 < 1$)

(ii) $x_n = 1 + (\frac{1}{n})^n$

Solution: $x_{n\to\infty} = 1$, $\dfrac{\|x_{n+1} - x_\infty\|}{\|x_n - x_\infty\|} = \dfrac{\left(\frac{1}{n}\right)^{n+1}}{\left(\frac{1}{n}\right)^{n}} = \dfrac{1}{n} \to \lim_{n\to\infty} \dfrac{1}{n} = 0$, $O(\frac{1}{n}^n)$

(converges super-linearly, p=1, $k = 0$)

(b) Suppose you apply an iterative method and obtain the following errors from the first four steps: $10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}$, ... How would you characterize the order of convergence of this method?

(A) Linear $\Rightarrow \|e_{n+1}\| \approx 10^{-2}\|e_n\| \Rightarrow$

e_n	k= $\|e_{n+1}\|/\|e_n\|$
10^{-2}	—
10^{-4}	10^{-2}
10^{-6}	10^{-2}
10^{-8}	10^{-2}

(B) Super-linear

(C) Quadratic

(D) Faster than Quadratic

(c) Suppose you apply an iterative method and obtain the following errors from the first four steps: $10^{-2}, 10^{-4}, 10^{-8}, 10^{-16}$, ... How would you characterize the order of convergence of this method?

(A) Linear

(B) Super-linear

(C) Quadratic $\Rightarrow \|e_{n+1}\| \approx 1\|e_n\|^2 \Rightarrow$

e_n	k= $\|e_{n+1}\|/\|e_n^2\|$
10^{-2}	—
10^{-4}	1
10^{-8}	1
10^{-16}	1

(D) Faster than Quadratic

(d) Limits Involving Continuous Functions Defined on Convergent Sequences: Determine whether the sequence $\lim_{n\to\infty} \cos(\frac{3}{n^2})$ converges. If it converges, find its limit.
Solution: $\lim_{n\to\infty} \cos(\frac{3}{n^2}) \to \lim_{n\to\infty} \cos(0) = 1$, The sequence converges with limit =1

(e) Rate of convergence of functions. Definition. Let f be a function defined on the interval (a,b) that contains x=0, and suppose $\lim_{x\to 0} f(x) = L$. If there exists a function g for which $\lim_{x\to 0} g(x) = 0$ and a positive constant K such that

$$|f(x) - L| \leq K|g(x)|$$

for all sufficiently small values of x, then f(x) is said to converge to L with rate of convergence O(g(x)).
Find the Rate of convergence of the following function as $h \to 0 : \lim_{h\to 0} \frac{sin(h)}{h} = 1$

Solution: $\lim_{h\to 0} \frac{sin(h)}{h} = 1 \to \sin(h) = h - \dfrac{h^3}{3!} \cdots$,

$\to \lim_{h\to 0} \dfrac{sin(h)}{h} = \lim_{h\to 0} \dfrac{h - \frac{h^3}{3!} \cdots}{h} = \lim_{h\to 0} 1 - \dfrac{h^2}{6} \cdots$

$\therefore \| \lim_{h\to 0} \dfrac{sin(h)}{h} - 1\| \leq \dfrac{1}{6} h^2$

$k = \dfrac{1}{6}$, $O(h^2)$

(f) Determine the order of convergence of the sequence generated by the iterative algorithm used to find $\sqrt{a}$, where a is a positive real number using the following recursive formula:

$$x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$$

To accomplish this we must be able to compare the error in the $(n+1)^{\text{th}}$ term in the sequence $x_{n+1} - \sqrt{a}$ with error in the n$^{\text{th}}$, $x_n - \sqrt{a}$

Solution:

$$
\begin{aligned}
x_{n+1} - \sqrt{a} &= \frac{1}{2}(x_n + \frac{a}{x_n}) \\
&= \frac{1}{2}\left(\frac{x_n^2 + a}{x_n}\right) - \sqrt{a} \\
&= \frac{1}{2x_n}\left(x_n^2 + a - 2x_n\sqrt{a}\right) \\
&= \frac{(x_n - \sqrt{a})^2}{2x_n}
\end{aligned}
$$

$$
\therefore \lim_{n\to\infty} \frac{\|x_{n+1} - \sqrt{a}\|}{\|x_n - \sqrt{a}\|^p} = \frac{(x_n - \sqrt{a})^2}{2x_n(\|x_n - \sqrt{a}\|^p)}, \quad \textbf{with } p = 2
$$

$$
\lim_{n\to\infty} \frac{1}{2x_n} = \frac{1}{2\sqrt{a}} \therefore \quad \textbf{The order of convergence is quadratic}
$$

Question 2: Bisection Method for root finding– Textbook (9th edition) – Problem Set 2.1 (Page 54). Answers should be given in a tabular form (as in the lecture slides).

2. Let $f(x) = 3(x+1)(x-\frac{1}{2})(x-1)$. Use the bisection method on the following intervals to find p_3

(a) $[-2, 1.5]$ **Answers: a. -0.6875**

It#	a	b	p	f(a)	f(b)	abs(pn-pn_1)/pn
1	-2	1.5	-0.25	-22.5	3.75	-4.0
2	-2	-0.25	-1.125	-22.5	3.75	-0.8888888888888888
3	-1.125	-0.25	-0.6875	-22.5	3.75	-0.8181818181818182

(b) $[-1.25, 2.5]$ **Answers: b. 1.09375**

It#	a	b	p	f(a)	f(b)	abs(pn-pn_1)/pn
1	-1.25	2.5	0.625	-2.953125	31.5	1.0
2	0.625	2.5	1.5625	-2.953125	31.5	0.2
3	0.625	1.5625	1.09375	-2.953125	31.5	0.2857142857142857

5. use the bisection method to find solutions accurate to within 10^{-5} for the following problems.

(a) $x - 2^{-x} = 0$ for $0 \leq x \leq 1$ **Answers: a. p17= 0.641182**

It#	a	b	p	f(a)	f(b)	abs(pn-pn_1)/pn
1	0	1	0.5	-1	0.5	0.0
2	0.5	1	0.75	-1	0.5	0.3333333333333333
3	0.5	0.75	0.625	-1	0.5	0.4
4	0.625	0.75	0.6875	-1	0.5	0.45454545454545453
5	0.625	0.6875	0.65625	-1	0.5	0.47619047619047616
6	0.625	0.65625	0.640625	-1	0.5	0.4878048780487805
7	0.640625	0.65625	0.6484375	-1	0.5	0.4939759036144578
8	0.640625	0.6484375	0.64453125	-1	0.5	0.49696969696969695
9	0.640625	0.64453125	0.642578125	-1	0.5	0.49848024316109424
10	0.640625	0.642578125	0.6416015625	-1	0.5	0.4992389649923896
11	0.640625	0.6416015625	0.64111328125	-1	0.5	0.49961919268849964
12	0.64111328125	0.6416015625	0.641357421875	-1	0.5	0.49980966882375333
13	0.64111328125	0.641357421875	0.6412353515625	-1	0.5	0.4999048162954502
14	0.64111328125	0.6412353515625	0.64117431640625	-1	0.5	0.49995240361732507
15	0.64117431640625	0.6412353515625	0.641204833984375	-1	0.5	0.49997620294131645
16	0.64117431640625	0.641204833984375	0.6411895751953125	-1	0.5	0.4999881011875015

p17 = 0.6411819458007812

(b) $e^x - x^2 + 3x - 2 = 0$ for $0 \leq x \leq 1$ **Answers: b. p17= 0.257530**

It#	a	b	p	f(a)	f(b)	abs(pn-pn_1)/pn
1	0	1	0.5	-1.0	2.718281828459045	0.0
2	0	0.5	0.25	-1.0	2.718281828459045	0.0
3	0.25	0.5	0.375	-1.0	2.718281828459045	0.3333333333333333
4	0.25	0.375	0.3125	-1.0	2.718281828459045	0.4
5	0.25	0.3125	0.28125	-1.0	2.718281828459045	0.4444444444444444
6	0.25	0.28125	0.265625	-1.0	2.718281828459045	0.47058823529411764
7	0.25	0.265625	0.2578125	-1.0	2.718281828459045	0.48484848484848486
8	0.25	0.2578125	0.25390625	-1.0	2.718281828459045	0.4923076923076923
9	0.25390625	0.2578125	0.255859375	-1.0	2.718281828459045	0.4961832061068702
10	0.255859375	0.2578125	0.2568359375	-1.0	2.718281828459045	0.49809885931558934
11	0.2568359375	0.2578125	0.25732421875	-1.0	2.718281828459045	0.4990512333965844
12	0.25732421875	0.2578125	0.257568359375	-1.0	2.718281828459045	0.4995260663507109
13	0.25732421875	0.257568359375	0.2574462890625	-1.0	2.718281828459045	0.499762920815 5524
14	0.2574462890625	0.257568359375	0.25750732421875	-1.0	2.718281828459045	0.4998814885043849
15	0.25750732421875	0.257568359375	0.257537841796875	-1.0	2.718281828459045	0.49994075127384763
16	0.25750732421875	0.257537841796875	0.2575225830078125	-1.0	2.718281828459045	0.499970373881614

p17 = 0.25753021240234375

Tools used in creating this document:

- Texmaker 5.0.4
- Spyder IDE 5.1.5 with python 3.8.5

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