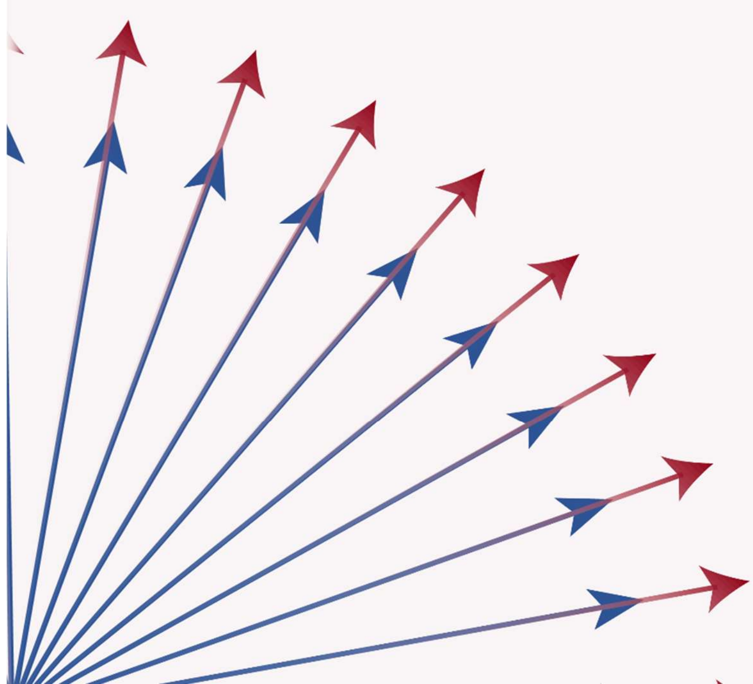
Eigen Values & Eigen Vectors

E-JUST



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I. Introduction to Eigenvalues and Eigenvectors:

In matrix linear transformation (or matrix multiplication), similar to functions in elementary algebra, an operator $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ transforms or moves vector $\mathbf{x}$ via matrix multiplication to a new vector in the same or in different space with a new magnitude. A special case of the subspace of the domain of $T(\mathbf{x})$ is the one-dimensional invariant subspaces (invariant lines); a non-zero vector $\mathbf{x}$ that belongs to invariant lines, also known as invariant directions or eigenrays, is mapped by the operator $\mathbf{A}$ to a new vector on the invariant line. Such vector $\mathbf{x}$ is called an eigenvector of $\mathbf{A}$. As this vector $\mathbf{x}$, the eigenvector, is carried into a collinear vector by matrix multiplication by operator **A,** then.

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

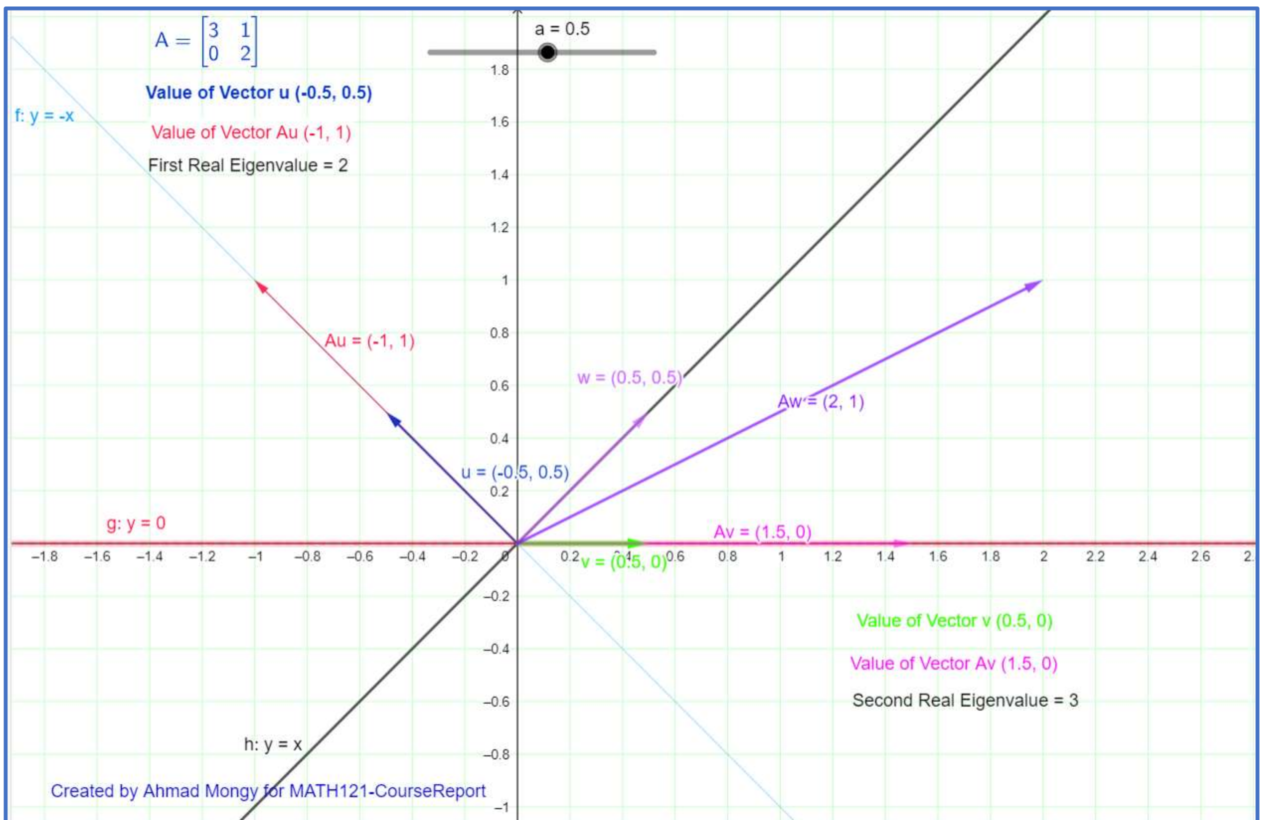

Figure 1: Illustration of eigenvectors of the transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$, where vectors **u** and **v** are eigenvectors and **w** not an eigenvector but rather demonstrated to show the difference. vector **u** and its transformation along line $y = -x$, vector v and its transformation along line $y = 0$.
https://www.geogebra.org/m/wnataapv

Where **λ,** a scalar, is called the eigenvalue (or characteristic value) of the operator **A**, corresponding to the eigenvector **x.** The eigenvalue **λ** tells whether the eigenvector **x** is stretched, shrunk, or reversed when it is multiplied by **A.** While the eigenvector **x** cannot be the zero vector the eigenvalue, λ, could equal zero. That is **Ax** =**0x** meaning that this eigenvector is in the null space of **A**, Nul(A).

II. Definition of eigenvalues and eigenvectors:

Based on our introduction of eigenvalues and eigenvectors, we can come to a rigorous definition of both eigenvalues and eigenvectors, but first note the following: from our discussion in the introduction, we stated that **x** is an eigenvector of operator **A** if **Ax** is in the one-dimensional invariant subspace of **x**; that means the applied operator **A** should not transform the vector **x** to neither higher nor lower dimension. With this we can have our rigorous definition of eigenvalues and eigenvectors

Definition:

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation of $\mathbb{R}^n$, then a nonzero vector $\mathbf{x} \in \mathbb{R}^n - \{\mathbf{0}\}$, is called an eigenvector of T if there exists some number, scalar, $\boldsymbol{\lambda} \in \mathbb{R}$ such that

$$\mathrm{T}(\mathbf{x}) = \boldsymbol{\lambda}\mathbf{x}$$

The real number $\boldsymbol{\lambda}$ is called a real eigenvalue of the real linear transformation T. Let A be an $n \times n$ matrix representing the linear transformation T; then, $\mathbf{x}$ is an eigenvector of the matrix A if and only it is an eigenvector of T, if and only if

$$\mathbf{Ax} = \boldsymbol{\lambda}\mathbf{x}$$

for an eigenvalue $\boldsymbol{\lambda}$

In a more simple words; for an $n \times n$ matrix (operator) A, an eigenvector is a nonzero vector **x** such that **Ax=λx** for some scalar λ, where λ is called an eigenvalue of A if there is a nontrivial solution **x** of **Ax** = **λx.**

Another important thing to note is that for any $n \times n$ matrix (operator) A, there are **at most n real eigenvalues of such matrix**. However, we will later see from the characteristic equation that there are exactly n eigenvalues (n roots), counting multiplicities and provided that complex roots are included.

Algebraic Manipulation of the Equation: $\mathbf{Ax} = \boldsymbol{\lambda}\mathbf{x}$

We have defined an eigenvector x and eigenvalue (a scalar) $\boldsymbol{\lambda}$ as $\mathbf{Ax}=\boldsymbol{\lambda}\mathbf{x}$, where $\mathbf{A}$ is an $n \times n$ matrix hence;

$$\boldsymbol{Ax} = \boldsymbol{\lambda x}$$
$$(\boldsymbol{A} - \boldsymbol{\lambda I_n})\boldsymbol{x} = \mathbf{0}$$

The set of all solutions of this equation is the null space of the matrix $(\boldsymbol{A} - \boldsymbol{\lambda I_n})$, and is called the eigenspace of $\boldsymbol{A}$ corresponding to $\boldsymbol{\lambda}$. The eigenspace consists of the zero vector and all the eigenvectors corresponding to $\boldsymbol{\lambda}$**.**

Definition:

Given a matrix $\boldsymbol{A} \in \mathbb{R}^{\boldsymbol{n}\times\boldsymbol{n}}$**,** the characteristic equation for $\mathbf{A}$ is

$$\det(\boldsymbol{A} - \boldsymbol{\lambda I_n}) = \mathbf{0}.$$

III. Characteristic Equations:

Let $\boldsymbol{A} \in \mathbb{R}^{\boldsymbol{n}\times\boldsymbol{n}}$, and $\boldsymbol{\lambda}$ is an eigenvalue of $\boldsymbol{A}$ with eigenvector $\mathbf{x}$ then:

$$\boldsymbol{Ax} = \boldsymbol{\lambda x} \Rightarrow (\boldsymbol{A} - \boldsymbol{\lambda I_n})\boldsymbol{x} = \mathbf{0} \Rightarrow \boldsymbol{x} \in \boldsymbol{Nul}((\boldsymbol{A} - \boldsymbol{\lambda I_n})$$

Since $\mathbf{x} \neq \mathbf{0}$ (being an eigenvector), we deduce that Nul($(\boldsymbol{A} - \boldsymbol{\lambda I_n})$ is nontrivial, if it is noninvertible, that is $\det(\boldsymbol{A} - \boldsymbol{\lambda I_n}) = \mathbf{0}$

The left-hand expression $\det(\boldsymbol{A} - \boldsymbol{\lambda I_n})$ determines a polynomial in $\boldsymbol{\lambda}$ called the characteristic polynomial, whose real roots are precisely the real eigenvalues of $\mathbf{A}$.

For a matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $(\boldsymbol{A} - \boldsymbol{\lambda I_n}) = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$

Hence the characteristic equation for $\mathbf{A}$ is

$$det(\boldsymbol{A}-\boldsymbol{\lambda I_n}) = (a-\boldsymbol{\lambda})\times(d-\boldsymbol{\lambda})-(b\times c)$$

Where the real roots of this polynomial are the real eigenvalues. This can be applied to any $n\times n$ matrix; however as mentioned before and as expected from a polynomial equation, the roots of the characteristic equation (the eigenvalues) could be a complex number which indicates the rotation of the eigenvectors rather than scaling over the invariant line and such complex roots are called complex eigenvectors.

After computing the eigenvalues, we can solve for vector **x** the eigenvector since $(\boldsymbol{A}-\boldsymbol{\lambda I_n})\boldsymbol{x}=\boldsymbol{0}$ by row operation.

This similar to finding the null space of $(\boldsymbol{A}-\boldsymbol{\lambda I_n})$, actully the eigenvector of a certain eigenvalue is the null space of $(\boldsymbol{A}-\boldsymbol{\lambda I_n})$.

Summary: To solve the eigenvalue problem for an $n\times n$ matrix:

1. Compute $det(\boldsymbol{A}-\boldsymbol{\lambda I_n})$, and you will have a polynomial in $\boldsymbol{\lambda}$ of degree n
2. Find the roots of this polynomial, by solving the polynomial you will get n eigenvalues of A (real and complex values)
3. For each eigenvalue $\boldsymbol{\lambda}$, solve$(\boldsymbol{A}-\boldsymbol{\lambda I_n})\boldsymbol{x}$=**0** to find an eigenvector **x.**

Trick for computing eigenvalues for a 2x2 matrix:

For a matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$**,** it can be noted that the mean of the two eigenvalues (m) is the mean of the two diagonal entries, and the product of the two eigenvalues (p) is the det(**A**). Hence $\lambda 1, \lambda 2 \;=\; m \pm \sqrt{m^2-p}$

Numerical Example:

Find the Eigenvalues and eigenvectors of matrix **A** = $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$.

Solution:

Step 1:$det(\boldsymbol{A}-\boldsymbol{\lambda I_n}) = det\begin{bmatrix} 3-\boldsymbol{\lambda} & 1 \\ 0 & 2-\boldsymbol{\lambda} \end{bmatrix} = (3-\boldsymbol{\lambda})(2-\boldsymbol{\lambda}) = \boldsymbol{0}$

Step 2: $\boldsymbol{\lambda = 2, or\ \lambda = 3}$

Step3: For $\boldsymbol{\lambda = 2,}$

$$(\boldsymbol{A}-\boldsymbol{2I_n})\boldsymbol{x}=\begin{bmatrix}3-2 & 1\\ 0 & 2-2\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_1\\ \boldsymbol{x}_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\text{Aug}(\boldsymbol{A}-\boldsymbol{2I\_n}) = \begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}, \boldsymbol{x} = \begin{bmatrix}\boldsymbol{x}_1\\ \boldsymbol{x}_2\end{bmatrix} = \boldsymbol{x}_2\begin{bmatrix}-1\\ 1\end{bmatrix}$$

Definition:

Given two $n \times n$ matrices $\boldsymbol{A}$ ***and*** $\boldsymbol{B}$ **,** the matrix **A** is said to be similar to **B** if there exists an invertible matrix P such that:

$$\boldsymbol{A} = \boldsymbol{PBP^{-1}}$$

If A is similar to B via some matrix P, them taking $\boldsymbol{Q} = \boldsymbol{P^{-1}}$, we get:

$$\boldsymbol{QAQ^{-1}} = \boldsymbol{B}$$

And the same can be done for $\boldsymbol{\lambda} = \mathbf{3}$

Hence the eigenvalues are $\boldsymbol{\lambda} = \mathbf{2}$, with $\boldsymbol{x}$ ***as span*** $\begin{bmatrix}-1\\ 1\end{bmatrix}$

and $\boldsymbol{\lambda} = \mathbf{3}$ with $\boldsymbol{x}$ ***as span*** $\begin{bmatrix}1\\ 0\end{bmatrix}$

IV. Similarity of Matrices

Similar matrices are not necessarily row equivalent, but there is a relationship between their characteristic polynomials, and correspondingly, their eigenvalues:

Assume $\boldsymbol{A} = \boldsymbol{PBP^{-1}}$ ***for some invertible matrix*** $\boldsymbol{P} \in \mathbb{R}^{\boldsymbol{n\times n}}$

$$\boldsymbol{A}-\boldsymbol{\lambda I_n} = \boldsymbol{PBP^{-1}} - \boldsymbol{\lambda PP^{-1}} = \boldsymbol{P}(\boldsymbol{B}-\boldsymbol{\lambda I_n})\boldsymbol{P^{-1}}$$

Hence;

The Characteristic polynomial of **A**$= \boldsymbol{det}(\boldsymbol{A}-\boldsymbol{\lambda I_n})$

$$= \boldsymbol{det}(\boldsymbol{P}(\boldsymbol{B}-\boldsymbol{\lambda I_n})\boldsymbol{P^{-1}})$$

$$= \boldsymbol{det}(\boldsymbol{P})\boldsymbol{det}(\boldsymbol{B}-\boldsymbol{\lambda I_n})\boldsymbol{det}(\boldsymbol{P^{-1}})$$

$$= \boldsymbol{det}(\boldsymbol{B}-\boldsymbol{\lambda I_n})$$

$=$ The Characteristic polynomial of **B**

V. Diagonalization

A square matrix D is called diagonal if all but its diagonal enteries are zero that is

$$D_{n\times n} = \begin{bmatrix}a_1 & 0 & \dots & 0\\ 0 & a_2 & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & a_n\end{bmatrix}$$

Definition:

Diagonalization is the process of finding a corresponding diagonal matrix for a diagonalizable matrix, where a square matrix **A** is called diagonalizable if it is similar to a diagonal matrix, that is saying for matrix **A** an invertible matrix P such that $\boldsymbol{PAP^{-1}}$ is a diagonal matrix

Obviously, for any diagonal matrix, eigenvalues are all the diagonal entries. Due to the simplicity of diagonal matrices, if for a matrix **A** there exist a similar diagonal matrix **D**, we can easily determine the eigenvalues and eigenvectors of **A**, that are, from our discussion of similar matrices, the same as the eigenvalues and eigenvectors of matrix **D.**

What is great about diagonal matrices is that it is easy to compute its high powers as a diagonal matrix D raised to power k where k≥1 is equal to raising its diagonal elements by the same power k, and this is obviously a great method to calculating high powers of a matrix than the inexpedient method of ordinary matrix multiplication.

Numerical Examples:

1) Let **A** = $\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$, find a formula for $\mathbf{A}^k$; given that $\boldsymbol{A = PDP^{-1}}$, where

P = $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ and **D** = $\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

Solution:

$$\mathbf{A}^2 = (\boldsymbol{PDP^{-1}})(\boldsymbol{PDP^{-1}}) = \mathbf{PD^2P^{-1}} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\mathbf{A}^3 = (\mathbf{PD^2P^{-1}})(\boldsymbol{PDP^{-1}}) = \mathbf{PD^3P^{-1}} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^3 & 0 \\ 0 & 3^3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\mathbf{A}^k = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2\times 5^k - 3^k & 5^k - 3^k \\ 2\times 3^k - 2\times 5^k & 2\times 3^k - 5^k \end{bmatrix}$$

Summary: To solve the Diagonalizing matrices problem for an $n \times n$ matrix:

1. Find the eigenvalues of A.
2. Find n linearly independent eigenvectors of A.
3. Construct P from the LI vectors in 2.
4. Construct D from the corresponding eigenvalues.

2) Diagonalize the following matrix, if possible:

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Solution:

$$1. det(A - \lambda I_n) \; = \; -(\lambda - 1)(\lambda + 2)^2, \lambda = 1, \lambda = -2$$

$$2. Basis\ for\ \lambda = 1: \quad V_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$Basis\ for\ \lambda = -2: \quad V_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \ and\ V_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$3.\ P = [V_1 \quad V_2 \quad V_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$4.\ D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

We can check our solution by computing AP and PD, and verify that ***AP=PD***

Application of diagonalization include Google's Page Ranking in web search, the stress matrix diagonalization, and many other applications in different fields.

Command Window

```
A =

     1     6
     5     2

[V, D]= eig(A)

V =

   -0.7682   -0.7071
    0.6402   -0.7071
D =

    -4     0
     0     7

first_eigv = V(:,1)

first_eigv =

   -0.7682
    0.6402

second_eigv = V

(:,2)second_eigv =

   -0.7071
   -0.7071

first_eigval= D

(1,1)first_eigval =

    -4
```

Command Window Cont'

```
second_eigval = D(2,2)

second_eigval =

     7

>> A*first_eigv

ans =

    3.0729
   -2.5607

>> first_eigval*first_eigv

ans =

    3.0729
   -2.5607

>> A*second_eigv

ans =

   -4.9497
   -4.9497

>> second_eigval*second_eigv

ans =

   -4.9497

   -4.9497
```

Link to Eigenvector and eigenvalue interactive graph on GeoGebra created by us:
https://www.geogebra.org/m/wnataapv

References

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