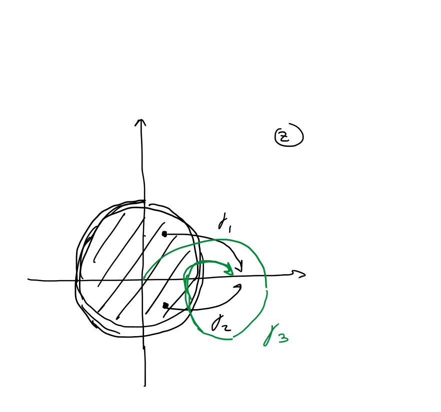
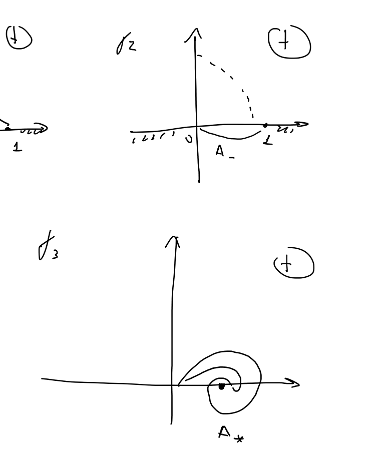
Complex HW4

Sunday, 14 March 2021 18:53

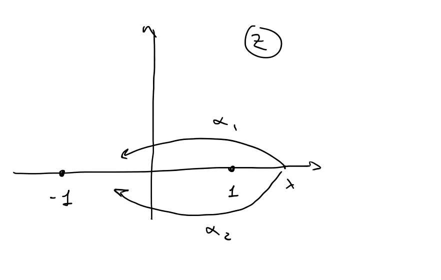
(1.) [illegible]

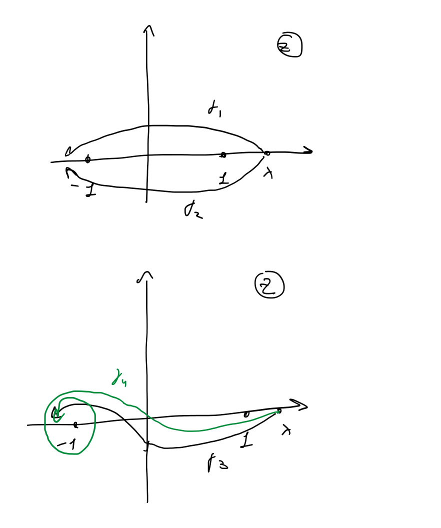
(2.) [illegible]





(3.) [illegible]

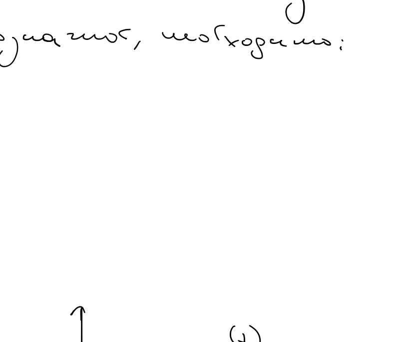




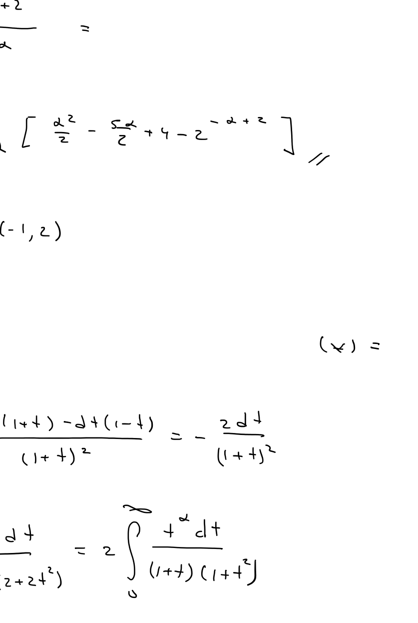
(4.) [illegible]

(5.) [illegible]

(6.) [illegible]



(7.) [illegible]



$I_- = -I_+ e^{2\pi i\alpha}$, =>

$2\pi i \sum \operatorname{Res} f(t) = I_+ - I_+ e^{2\pi i\alpha} = I(1-e^{2\pi i\alpha})$

$$I = \frac{2\pi i \sum \operatorname{Res} f(t)}{1-e^{2\pi i\alpha}} = 2\pi i \frac{-(\frac{\alpha^2}{2} - \frac{5\alpha}{2} + 4) + 2^{-\alpha+2}}{e^{-i\pi\alpha} - e^{i\pi\alpha}} =$$

$$= -\frac{\pi}{\sin\pi\alpha}\left[-(\frac{\alpha^2}{2} - \frac{5\alpha}{2} + 4) + 2^{-\alpha+2}\right] = \frac{\pi}{\sin\pi\alpha}\left[\frac{\alpha^2}{2} - \frac{5\alpha}{2} + 4 - 2^{-\alpha+2}\right] //$$

(8.) $I(\alpha) = \int_{-1}^{1} \frac{(1-x)^\alpha(1+x)^{1-\alpha}}{x^2+1} dx$, $\operatorname{Re}\alpha \in (-1, 2)$

$I(\alpha) = \int_{-1}^{1} \left(\frac{1-x}{1+x}\right)^\alpha \frac{1+x}{x^2+1} dx = (*)$ $t = \frac{1-x}{1+x}$, $x = \frac{1-t}{1+t}$ $(*) =$

$dx = \frac{-dt(1+t) - dt(1-t)}{(1+t)^2} = -\frac{2dt}{(1+t)^2}$

$$= \int_\infty^0 t^\alpha \frac{1+\frac{1-t}{1+t}}{(\frac{1-t}{1+t})^2+1} \cdot \frac{-2dt}{(1+t)^2} = 2\int_0^\infty t^\alpha \frac{2dt}{(1+t)(2+2t^2)} = 2\int_0^\infty \frac{t^\alpha dt}{(1+t)(1+t^2)}$$

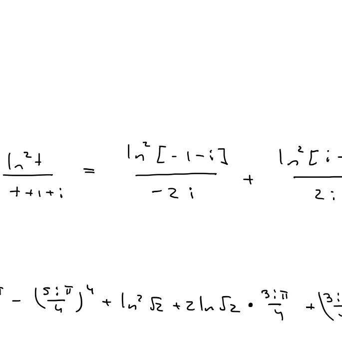
$$\frac{1}{2}\sum \operatorname{Res} f(t) = \operatorname{Res}_{t=-1} f(t) + \operatorname{Res}_{t=i} f(t) + \operatorname{Res}_{t=-i} f(t) = \lim_{t\to-1}\frac{t^\alpha}{1+t^2} + \lim_{t\to i}\frac{t^\alpha}{(1+t)(t+i)} +$$

$$+ \lim_{t\to -i}\frac{t^\alpha}{(1+t)(t-i)} = \frac{(-1)^\alpha}{2} + \frac{i^\alpha}{2i-2} + \frac{(-i)^\alpha}{-2i-2} = \frac{e^{i\pi\alpha}}{2} - \frac{1}{2}\left[\frac{-i^{\alpha+1} - i^\alpha + (-i)^{\alpha+1} - (-i)^\alpha}{(i-1)(i+1)}\right] =$$

$$= \frac{e^{i\pi\alpha}}{2} + \frac{1}{2}\left[\frac{-e^{i\pi\alpha/2} - e^{-i\pi\alpha/2} - e^{i\pi(\alpha+1)/2} + e^{-i\pi(\alpha+1)/2}}{2}\right] =$$

$$= \frac{e^{i\pi\alpha}}{2}\left[\frac{-e^{-i\pi\alpha/2} - e^{i\pi\alpha/2} - ie^{-i\pi\alpha/2} + ie^{i\pi\alpha/2}}{2} + 1\right]$$

$$= \frac{e^{i\pi\alpha}}{2}\left[1 - \cos\frac{\pi\alpha}{2} - \sin\frac{\pi\alpha}{2}\right]$$



Аналогично № 7

$2\pi i \sum \operatorname{Res} f(t) = I(1-e^{2\pi i\alpha})$

$$\pi i \frac{2}{e^{-i\pi\alpha} - e^{i\pi\alpha}}\left[1 - \cos\frac{\pi\alpha}{2} - \sin\frac{\pi\alpha}{2}\right] =$$

$$= -\frac{\pi}{\sin\pi\alpha}\left[1 - \cos\frac{\pi\alpha}{2} - \sin\frac{\pi\alpha}{2}\right] //$$

(10.) $I = \int_0^1 \ln\frac{1-x}{x}\frac{dx}{x^2+1} = (*)$ $t = \frac{1-x}{x}$, $x = \frac{1}{t+1}$, $dx = -\frac{dt}{(t+1)^2}$ $(*) = \int_\infty^0 -\frac{dt}{(t+1)^2}\cdot\frac{1}{\frac{1}{(t+1)^2}+1}\ln t =$

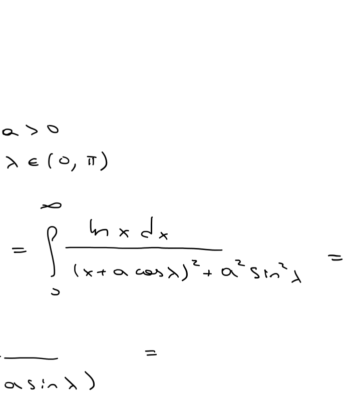
$$= \int_0^\infty \frac{\ln t\, dt}{1+(t+1)^2} = \int_0^\infty \frac{\ln t\, dt}{(t+1+i)(t+1-i)}$$

$$\sum \operatorname{Res}(f(t)\cdot \ln t) = \lim_{t\to-1-i}\frac{\ln^2 t}{t+1-i} + \lim_{t\to i-1}\frac{\ln^2 t}{t+1+i} = \frac{\ln^2[-1-i]}{-2i} + \frac{\ln^2[i-1]}{2i} =$$

$$= \frac{\ln^2 \sqrt{2}\cdot e^{5i\pi/4}}{-2i} + \frac{\ln^2 \sqrt{2} e^{3i\pi/4}}{2i} = \left[-\ln^2\sqrt{2} - 2\ln\sqrt{2}\cdot\frac{5i\pi}{4} - (\frac{5i\pi}{4})^2 + \ln^2\sqrt{2} + 2\ln\sqrt{2}\cdot\frac{3i\pi}{4} + (\frac{3i\pi}{4})^2\right]\cdot\frac{1}{2i} =$$

$$= \frac{1}{2i}\left[\ln 2\left(-\frac{i\pi}{2}\right) + \pi^2\right]$$

$I_+ = \int_0^\infty \frac{\ln^2 t}{1+(t+1)^2} dt$



$$I_- = \int_\infty^0 \frac{\ln^2(te^{2\pi i})\, dt}{1+(t+1)^2} =$$

$$= -\int_0^\infty \frac{\ln^2 t\, dt}{1+(t+1)^2} - 2\int_0^\infty \frac{2\pi i \ln t\, dt}{1+(t+1)^2} +$$

$$+ 4\pi^2\int_0^\infty \frac{dt}{1+(t+1)^2} = -I_+ - 4\pi i I + 4\pi^2\cdot \operatorname{arctg}(t+1)\Big|_0^\infty =$$

$= -I_+ - 4\pi i I + 4\pi^2\cdot\frac{\pi}{4}$, =>

$2\pi i\sum\operatorname{Res}(f(t)\cdot\ln t) = \pi^3 - 4\pi i I$

$-\frac{i\pi}{2}\ln 2 + \pi^2 - \pi^2 = -4iI$

$I = \frac{\pi}{8}\ln 2$ //

(11.) $I(a,\lambda) = \int_0^\infty \frac{\ln x\, dx}{x^2 + 2ax\cos\lambda + a^2}$, $a > 0$, $\lambda\in(0,\pi)$

$$I(a,\lambda) = \int_0^\infty \frac{\ln x\, dx}{(x+a\cos\lambda)^2 + a^2(1-\cos^2\lambda)} = \int_0^\infty \frac{\ln x\, dx}{(x+a\cos\lambda)^2 + a^2\sin^2\lambda} =$$

$$= \int_0^\infty \frac{\ln x\, dx}{(x+a\cos\lambda + ia\sin\lambda)(x+a\cos\lambda - ia\sin\lambda)} =$$

$$= \int_0^\infty \frac{\ln x\, dx}{(x+\frac{a}{2}\cdot 2e^{i\lambda})(x+\frac{a}{2}\cdot 2e^{-i\lambda})} = \int_0^\infty \frac{\ln x\, dx}{(x+ae^{i\lambda})(x+ae^{-i\lambda})}$$

Аналогично № 10;

$$\sum\operatorname{Res}[f(x)\cdot\ln x] = \lim_{x\to -ae^{-i\lambda}}\frac{\ln^2 x}{x+ae^{i\lambda}} + \lim_{x\to -ae^{i\lambda}}\frac{\ln^2 x}{x+ae^{-i\lambda}} =$$

$$= \frac{1}{a\sin\lambda}\left[(\ln a + i(\pi-\lambda))^2 - (\ln a + i(\pi+\lambda))^2\right] = \frac{1}{2ia\sin\lambda}\left[2i\ln a(\pi-\lambda-\pi-\lambda) -\right.$$

$$\left. - (\pi-\lambda)^2 + (\pi+\lambda)^2\right] = \frac{1}{2ia\sin\lambda}\left[-4i\lambda\ln a + 4\pi\lambda\right] = \frac{2\lambda}{a\sin\lambda}\left[-4\pi\lambda i - \ln a\right]$$

$$I_+ = \int_0^\infty \frac{\ln^2 x\, dx}{\underbrace{x^2+2ax\cos\lambda + a^2}_{A}}$$

$$I_- = \int_\infty^0 \frac{(\ln x + 2\pi i)^2}{A}dx = -I_+ - 4\pi i I + 4\pi^2\int_0^\infty \frac{dx}{(x+a\cos\lambda)^2 + a^2\sin^2\lambda} =$$

$$= -I_+ - 4\pi i I + \frac{4\pi^2}{a\sin\lambda}\int_0^\infty \frac{dx}{(\frac{x}{a\sin\lambda} + \operatorname{ctg}\lambda)^2 + 1} =$$

$$= -I_+ - 4\pi i I + \frac{4\pi^2}{a^2\sin^2\lambda}\operatorname{arctg}(\operatorname{ctg}\lambda) = -I_+ - 4\pi i I - \frac{4\pi^2}{a\sin\lambda}\lambda$$

$$2\pi i\sum\operatorname{Res}[f(x)\ln x] = -4\pi i I - \frac{4\pi^2}{a\sin\lambda}\lambda$$

$$\frac{4\pi i\lambda}{a\sin\lambda}\cdot(-\ln a) = -4\pi i I$$

$$I = \frac{\lambda\ln a}{a\sin\lambda} //$$

(12.) $I(a,b) = \int_0^\infty \frac{x^{\alpha-1}}{1+x^b}dx = (*)$ $x^b = t$, $x = t^{1/b}$, $dx = \frac{1}{b}t^{-\frac{b-1}{b}}dt$ $(*) =$

$$= \frac{1}{b}\int_0^\infty \frac{t^{\frac{\alpha-1}{b}}}{1+t} t^{\frac{1-b}{b}}dt = \frac{1}{b}\int_0^\infty \frac{t^{\frac{\alpha}{b}-1}}{1+t}dt$$

$$\operatorname{Res}_{t=-1} f(t) = \frac{1}{b}\lim_{t\to-1} t^{\frac{\alpha}{b}-1} = \frac{1}{b}e^{i\pi(\frac{\alpha}{b}-1)} = -\frac{1}{b}e^{i\pi\frac{\alpha}{b}}$$

Аналогично № 7

$I_+ = I$

$I_- = -I_+ e^{2\pi i(\frac{\alpha}{b}-1)}$ => $2\pi i \operatorname{Res} f(t) = I(1-e^{2\pi i(\frac{\alpha}{b}-1)})$

$$-\frac{2\pi i}{b}e^{i\pi\frac{\alpha}{b}}\cdot\frac{1}{1-e^{2\pi i\frac{\alpha}{b}}} = I$$

$$-\frac{2\pi i}{b}\frac{1}{e^{-i\pi\frac{\alpha}{b}} - e^{i\pi\frac{\alpha}{b}}} = \boxed{\frac{\pi}{b\sin\frac{\alpha}{b}\pi} = I} //$$

(13.) $I = PV\int_0^\infty \frac{\sqrt{x}\,dx}{x^2-1}$

$I_+ + I_- + I_{\varepsilon_+} + I_{\varepsilon_-} =$

$= 2\pi i \operatorname{Res}_{z=-1} f(z)$, где $I_+ = I_- = I$

$$\operatorname{Res}_{z=-1} f(z) = \lim_{z\to-1}\frac{\sqrt{z}}{z-1} = -\frac{i}{2}$$

$$I_{\varepsilon_+} = (*)\quad x-1 = re^{i\varphi}\quad dx = ire^{i\varphi}d\varphi\quad (*) = \int_\pi^0 \frac{\sqrt{re^{i\varphi}+1}}{(re^{i\varphi}+1)^2-1} ire^{i\varphi}d\varphi = \int_\pi^0 \frac{(re^{i\varphi}+1)^{\frac{1}{2}}}{(re^{i\varphi}+1)^2-1} ire^{i\varphi}d\varphi =$$

$$\int_\pi^0 \frac{\frac{1}{2}re^{i\varphi}+1}{r^2e^{2i\varphi}+2re^{i\varphi}} ire^{i\varphi}d\varphi = \frac{i}{2}\int_\pi^0 \frac{r^2e^{2i\varphi}+2re^{i\varphi}}{r^2e^{2i\varphi}+2re^{i\varphi}}d\varphi = -\frac{\pi i}{2}$$

$$I_{\varepsilon_-} = e^{i\pi}\int_0^{-\pi}\frac{\sqrt{re^{i\varphi}+1}\, ire^{i\varphi}}{(re^{i\varphi}+1)^2-1}d\varphi = -\frac{i}{2}\int_0^{-\pi}d\varphi = \frac{\pi i}{2}, \Rightarrow I_{\varepsilon_+} + I_{\varepsilon_-} = 0,\ \text{тогда}$$

$2I = 2\pi i \operatorname{Res} f(x)$

$I = \frac{\pi}{2}$ //

(9.) $I = \int_0^\infty \frac{\ln x}{\sqrt[3]{x}(x+1)^2}$

$$\sum\operatorname{Res} f(z) = \operatorname{Res}_{z=-1} f(z) = \lim_{z\to-1}\frac{d}{dz}\frac{\ln z}{\sqrt[3]{z}} = \lim_{z\to-1}\frac{\frac{\sqrt[3]{z}}{z} - \ln z\cdot\frac{1}{3}z^{-\frac{2}{3}}}{z^{2/3}} =$$

$$= \lim_{z\to-1} z^{-4/3} - \frac{1}{3}z^{-4/3}\ln z = e^{-i\pi\cdot\frac{4}{3}}(1-\frac{1}{3}i\pi)$$

$$I_- = \int_\infty^0 \frac{\ln z + 2\pi i}{z^{1/3}e^{\frac{2\pi i}{3}}(z+1)^2} = -I_+ e^{-\frac{2\pi i}{3}} - 2\pi i e^{-2\pi i/3}\int_0^\infty \frac{dz}{z^{1/3}(z+1)^2} =$$

$$e^{-2\pi i/3}\left(-I - 2\pi i\cdot\frac{2\pi\sqrt{3}}{9}\right), \Rightarrow$$

$$2\pi i \operatorname{Res} f(z) = I(1-e^{-2\pi i/3}) - e^{2\pi i/3}\cdot\frac{4\pi^2 i\sqrt{3}}{9}$$

$$2\pi i e^{-i\pi\frac{4}{3}}(1-\frac{1}{3}i\pi) + e^{2\pi i/3}\cdot\frac{4\pi^2 i\sqrt{3}}{9} = I(1-e^{-2\pi i/3})$$

$$I = \frac{2\pi^2}{9} - \frac{2\pi\sqrt{3}}{3} //$$