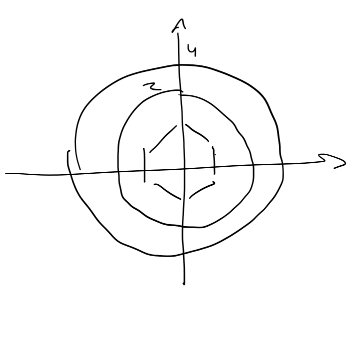
Complex HW2

Thursday, 4 February 2021 21:56

(6) $\oint_{C_R} \frac{z^2}{e^{z}+1}\,dz = \oint_{C_{R'}} \frac{z^2}{e^{z}+1}\,dz = \oint_{C_{R''}} \frac{dz}{z} = (*)$



$z = e^{i\varphi}$, $dz = ie^{i\varphi}d\varphi$, $(*) = \int_0^{2\pi} i\,d\varphi = 2\pi i$

(7) $\int_{-\infty}^{+\infty} \frac{x\sin x\,dx}{x^2(x^2+1)} = \int \frac{e^{iz}}{z(z^2+1)}\,dz = (*)$

1) $= 2\pi i \sum \mathrm{Res}\, f(z) = 2\pi i \lim_{z\to i} \frac{e^{iz}}{z(z+i)} = \frac{\pi}{e}$ ...

2) ...

3) ... $(*) = \frac{\pi}{e} + \dots = \boxed{\frac{\pi}{2}\ldots}$

(8) $\int_0^\infty \frac{x\sin ax}{x^2+b^2}\,dx = \frac12\int_{-\infty}^{\infty} \frac{x\sin ax}{x^2+b^2}\,dx = (*)\quad \sin ax = \mathrm{Im}[e^{iax}]$, $(*) = \frac12\mathrm{Im}\int \frac{xe^{iax}}{x^2+b^2}\,dx = \frac12\mathrm{Im}[2\pi i\,\mathrm{Res}\, f(z)] = (*)\quad z_0 = ib$

$= \pi\,\mathrm{Im}\left[i \lim_{z\to ib} \frac{ze^{iaz}}{z+ib}\right] = \pi\,\mathrm{Im}\left[i\,\frac{e^{-ab}}{2}\right] = \frac{\pi}{2}e^{-ab},\ a,b>0$

$I(a,b) = \int_0^\infty \frac{x\sin ax}{x^2+b^2}\,dx$

$I(a,-b) = I(a,b)$, $I(-a,b) = -I(a,b) \Rightarrow \boxed{I = \frac{\pi}{2}e^{-|ab|}\,\mathrm{sign}(a)}$

(9) $\int_0^\infty \frac{\cos(x-b)}{1+x^2}\,dx = \mathrm{Re}\int \frac{e^{i(x-b)}}{1+x^2}\,dx = \mathrm{Re}[2\pi i\,\mathrm{Res}\, f(z)] = (*)$

$z_0 = i$, $(*) = \mathrm{Re}\left[2\pi i \frac{e^{i(z-b)}}{2i}\right] = \mathrm{Re}[\pi e^{-1-ib}] = \boxed{\pi e^{-1}\cos b}$

(10) $\int_0^\infty \frac{x-\sin x}{x^3}\,dx = \frac12\int_{-\infty}^{\infty} \frac{x-\sin x}{x^3}\,dx = \frac12\mathrm{Im}\left[\int \frac{x - e^{ix}}{x^3}\right] =$

$= \frac12\mathrm{Im}[\pi i\,\mathrm{Res}\, f(0)] = \dots = \frac{\pi}{4}$

$\int_0^\infty \frac{dx}{\dots} = \dots = 2\pi i\,\mathrm{Res}\, f(z_0) = \dots = \boxed{\frac{\pi e^{-3}}{3}}$

(14) $\lim_{R\to\infty}\int_{C_R} e^{iz^2}dz = \lim_{R\to\infty}\int e^{iR^2(\cos 2\varphi + i\sin 2\varphi)}\cdot iRe^{i\varphi}\,d\varphi$

Оценим:

$\left|\int_{C_R} e^{iz^2}dz\right| \le R\int_0^{\pi/4} e^{-R^2\sin 2\varphi}\,d\varphi \le 2R\int_0^{\pi/4} e^{-4R^2\varphi/\pi}\,d\varphi = \dots = \frac{\pi}{2R}(1-e^{-R^2}) \to 0$

С другой стороны:

$\lim_{R\to\infty}\int_{C_R} e^{iz}dz = \lim_{R\to\infty}[-ie^{iz}]\Big|_{-R}^{R} = \lim_{R\to\infty} i[e^{-iR} - e^{iR}] = \lim_{R\to\infty} 2\sin R$, =>

предел не сущ.

$\lim_{R\to\infty}\int_{C_R} e^{iz^2}dz = (*)\quad s = z^2,\ ds = 2z\,dz \quad (*) = \lim_{R\to\infty}\int_{C_R'} \frac{e^{is}}{\sqrt{s}}\,ds$ => по $\frac{1}{\sqrt{s}} \to 0$ при $s\to\infty$

лемме Жордана $\lim_{R\to\infty}\int_{C_R} e^{iz^2}dz = 0$

(13.) $f(z) = \frac{\sin\frac1z}{1-z} = \sin\frac1z\cdot\frac{1}{1-z} = \sum_{n=0}^{\infty}\left[(-1)^n\cdot\frac{x^{-2n-1}}{(2n+1)!}\right]\cdot\sum_{n=0}^{\infty} z^n$, =>

Коэфор. перед $\frac1z$: $\sum_{n=0}^{\infty}(-1)^n\cdot\frac{1}{(2n+1)!} = \sin 1$

$g(z) = \exp(-\exp(\frac1z)) = \exp\left(-\sum_{n=0}^{\infty}\frac{z^{-n}}{n!}\right) = \prod_{n=0}^{\infty} e^{-z^{-n}/n!} =$

$= e^{-1}\cdot\sum_{n=0}^{\infty}(-1)^n\frac{z^{-n}}{n!}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{2!}\frac{z^{-2n}}{n!}\cdot\ \dots$ =>

Коэф. перед $\frac1z$: $-\frac1e$

(1.) $f(z) = \frac{1}{\sin z} + \frac{2z}{z^2-\pi^2} = \frac{1}{\sin z} + \frac{2z}{(z-\pi)(z+\pi)} = (*)\quad A(z+\pi) + B(z-\pi) = 2z$

$\begin{cases} A+B=2 \\ A-B=0 \end{cases} \Rightarrow A=B=1 \qquad (*) =$

$= \frac{1}{\sin z} + \frac{1}{z-\pi} + \frac{1}{z+\pi}$

Пусть $z\pm\pi = \varepsilon \to 0$, тогда

$\sin z = \sin(\varepsilon\mp\pi) = \sin\varepsilon\cos\pi \mp \sin\pi\cos\varepsilon = -\sin\varepsilon \simeq -\varepsilon + \frac{\varepsilon^3}{6}$, =>

$\frac{1}{\sin z} = \frac{6}{\varepsilon^3 - 6\varepsilon} = \frac{1}{\varepsilon}\left(\frac{6}{\varepsilon^2-6}\right) = \frac{1}{\varepsilon}\frac{1}{\varepsilon^2/6 - 1} \approx -\frac{1}{\varepsilon}\left(1+\frac{\varepsilon^2}{6}\right) = -\frac1\varepsilon - \frac\varepsilon6 =$

$= -\frac{1}{z\pm\pi} - \frac{z\pm\pi}{6}$, тогда

$\lim_{z\to\pm\pi}\left[\frac{1}{\sin z} + \frac{1}{z-\pi} + \frac{1}{z+\pi}\right] = -\frac{1}{z\pm\pi} - \frac{z\pm\pi}{6} + \frac{1}{z\pm\pi} + \frac{1}{z\mp\pi} = \frac{1}{z\mp\pi} - \frac{z\pm\pi}{6}$, =>

$\lim_{z\to\pm\pi} f(z) = \frac{1}{z\mp\pi} - \frac{z\pm\pi}{6}$, => $z=\pm\pi$ — точки устрани. разрыва

(2.) 1) $f(z) = \frac{\sin z}{1-\operatorname{tg} z}$; $\operatorname{tg} z = 1$ при $z_0 = \frac{\pi}{4} + \pi n$

$\lim_{z\to z_0} f(z_0) = \infty$, => z_0 — полюс

2) $f(z) = \frac{e^{c/z-a}}{e^{z/a}-1} = \frac{\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{c}{z-a}\right)^n}{\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{z}{a}\right)^n - 1} = \frac{\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{c}{z-a}\right)^n}{\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{z}{a}\right)^n}$ =>

$z_0 = a$ — существен. уединр. особен.

$\lim_{z\to z_0} f(z) = \infty$ при $e^{z_0/a} - 1 = 0$, =>

$e^{z_0/a} = 1$

$\frac{z_0}{a} = 2\pi i n$

$z_0 = 2\pi i n a$ — полюса

(3.) $f(z) = ze^{1/z}e^{-\frac{1}{z^2}} = z\cdot\sum_{n=0}^{\infty}\frac{z^{-n}}{n!}\cdot\sum_{n=0}^{\infty}(-1)^n\frac{z^{-2n}}{n!}$, => $z_0 = 0$ — существ. уединр. особенность

(4.) (1) $\oint_C \frac{ze^z}{\operatorname{tg} z^2}\,dz = 2\pi i\sum\mathrm{Res}\, f(z) = (*)\quad \operatorname{tg} z_0^2 = 0$, $z_0^2 = \frac{\pi n}{2}$, => в контур входит только $z_0 = 0$

$(*) = 2\pi i\lim_{z\to 0}\frac{z^2e^z}{\operatorname{tg} z^2} =$

$= 2\pi i\lim_{z\to 0}\frac{z^2e^z}{z^2} = 2\pi i$

(2) $\oint_C e^{-1/z}\sin\frac1z\,dz = \oint_C \sum_{n=0}^{\infty}(-1)^n z^{-n}\frac{1}{n!}\cdot\sum_{n=0}^{\infty} z^{-2n-1}\cdot(-1)^n\cdot\frac{1}{n!}\,dz =$

$= 2\pi i\,\mathrm{Res}\, f(z) = (*)\ z_0 = 0$ - сущ. уедин. особенность $(*) = 2\pi i\cdot 1 = 2\pi i$

(3) $\oint_C \frac{e^z}{z^n}\,dz = 2\pi i\,\mathrm{Res}\, f(z) = (*)\ z_0 = 0$ - полюс n-ного порядка $(*) = \frac{2\pi i}{(n-1)!}\lim_{z\to 0}\frac{d^{n-1}}{dz^{n-1}} f(z)\cdot z^n =$

$= \frac{2\pi i}{(n-1)!}\lim_{z\to 0}\frac{d^{n-1}}{dz^{n-1}} e^z = \boxed{\frac{2\pi i}{(n-1)!}}$

(12.) $f(z) = \frac{1}{z^3 - z^5} = \frac{1}{g(z)}$, => $g'(z) = 3z^2 - 5z^4$

- $\mathrm{Res}_{z=-1} f(z) = \frac{1}{g'(-1)} = -\frac12$
- $\mathrm{Res}_{z=0} f(z) = \mathrm{Res}_{z=0}\left[\frac{1}{z^3}\sum_{n=0}^{\infty} z^{2n}\right] = 1$
- $\mathrm{Res}_{z=1} f(z) = \frac{1}{g'(1)} = -\frac12$
- $\mathrm{Res}_{z=\infty} f(z) = \mathrm{Res}_{s=0}\frac{1}{\frac{1}{s^3} - \frac{1}{s^5}} = \mathrm{Res}_{s=0}\frac{s^5}{s^2-1} = \mathrm{Res}_{s=0} -s^5\sum_{n=0}^{\infty} s^{2n} = 0$

(11.) $f(z) = z^3\cos\frac{1}{z-2}$

$f(s=\frac1z) = \frac{1}{s^3}\cos\frac{1}{\frac1s - 2} = \frac{1}{s^3}\cdot\cos\frac{s}{1-2s} = \frac{1}{s^3}\cos\left[s\cdot\sum_{n=0}^{\infty}(2s)^n\right] = \frac{1}{s^3}\cdot\left(1 - \frac{s^2}{2} - 2s^3 - \frac{143}{24}s^4 + \dots\right) =$

$= \frac{1}{s^3} - \frac{1}{2s} - 2 - \frac{143}{24}s + \dots = z^3 - 2z - 2 - \frac{143}{24\cdot z} + \dots$, =>

$\mathrm{Res}_{z=\infty} f(z) = \frac{143}{24}$

(5.) 1) $\int_{-\infty}^{+\infty}\frac{x^4}{1+x^6}\,dx = 2\pi i\sum\mathrm{Res}\, f(z) = (*)\quad z_0^6 = -1 = e^{i\pi}$

$e^{6i\varphi} = e^{i\pi + 2\pi i n}$

$\varphi = \frac{\pi}{6} + \frac{\pi n}{3}$, =>

т.к. $\varphi \in [0;\pi]$

$z_0 = \{e^{i\pi/6};\ e^{i\pi/2};\ e^{5i\pi/6}\}$ $\qquad (*) =$

$= (*)\ \mathrm{Res}\, f(z) = \frac{z^4}{(1+z^6)'} = \frac{1}{6z_0}\quad (*) = 2\pi i\sum\left[\frac{1}{6e^{i\pi/6}} + \frac{1}{6e^{i\pi/2}} + \frac{1}{6e^{i\pi/6}}\right] =$

$= \frac{\pi i}{3}\left(\cancel{\cos -\frac{\pi}{6}} + i\underbrace{\sin -\frac{\pi}{6}}_{=-\frac12} + \underbrace{\cos\frac{\pi}{2}}_{=0} + i\underbrace{\sin -\frac{\pi}{2}}_{=-1} + \cancel{\cos -\frac{5\pi}{6}} + i\underbrace{\sin -\frac{5\pi}{6}}_{=-\frac12}\right) = \frac{\pi i}{3}\cdot(-2i) = \frac{2\pi}{3}$

2) $\int_0^{2\pi}\frac{\cos 2\theta}{2+\cos\theta}\,d\theta = (*)\quad z = e^{i\theta},\ \cos\theta = \frac12\left(z+\frac1z\right),\ \cos 2\theta = \frac12\left(z^2+\frac{1}{z^2}\right)\quad (*) = \oint_{C_R}\frac{2(z^4+1)z}{2z^2(4z+z^2+1)}\cdot\left(-\frac{i}{z}\right)dz =$

$= -i\oint_{C_R}\frac{z^4+1}{z^2(z^2+4z+1)}\,dz = 2\pi i\sum\mathrm{Res}\, f(z) = (*)\quad z_0 = 0,\ z_0^2 + 4z_0 + 1 = 0$, => $z_0 = -2+\sqrt3$

$\mathrm{Res}_{z_0=0} f(z) = -i\lim_{z\to 0}\frac{4z^3(z^2+4z+1) - (z^4+1)(2z+4)}{(z^2+4z+1)^2} = -4$

$\mathrm{Res}_{z_0=-2+\sqrt3} f(z) = -i\lim_{z\to z_0}\frac{z^4+1}{z^2(z-2+\sqrt3)} = \frac{(7-4\sqrt3)^2+1}{(7-4\sqrt3)\cdot 2\sqrt3} = \frac{96-8\sqrt3}{14\sqrt3-24} =$

$= \frac{(96-8\sqrt3)(14\sqrt3+24)}{12} = -i\frac{7\sqrt3}{3}$

$(*) = 2\pi i\left[-4 + \frac{7\sqrt3}{3}\right] = -8\pi + \frac{14\pi\sqrt3}{3}$

3) $\int_{-\infty}^{+\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)^2} = 2\pi i\sum\mathrm{Res}\, f(z) = (*)\quad z_0 = i|a|,\ z_0 = i|b|$ (полюс 2-го порядка) $\quad (*) =$

$= 2\pi i\left[\frac{1}{2i|a|\cdot(b^2-a^2)^2} + \lim_{z\to ib}\frac{d}{dz}\frac{1}{(z^2+a^2)(z+ib)^2}\right] =$

$= \frac{\pi}{|a|(b^2-a^2)^2} + \lim_{z\to ib}\frac{-2z(z+ib)^2 + 2(z+ib)(z^2+a^2)}{(z^2+a^2)^2(z+ib)^4}\cdot 2\pi i =$

$= \frac{\pi}{|a|(b^2-a^2)^2} - \frac{i4|b|\cdot(-b^2) + 4i|b|(a^2-b^2)}{(a^2-b^2)^2\cdot 16b^4}\cdot 2\pi i =$

$= \frac{\pi}{|a|(b^2-a^2)^2} + \pi\frac{-2b^2 + (a^2-b^2)}{2|b|^3(a^2-b^2)^2} = \frac{\pi}{2|a||b^3|(a^2-b^2)^2}\left[2|b|^3 + |a|(a^2-3b^2)\right] =$

$= \frac{\pi}{2|a||b|^3}\cdot\frac{1}{(|a|-|b|)^2\cdot(|a|+|b|)^2}\cdot(|a|+2|b|)(|a|-|b|)^2 =$

$= \boxed{\frac{\pi(|a|+2|b|)}{2|a||b|^3(|a|+|b|)^2}}$