Complex HW2

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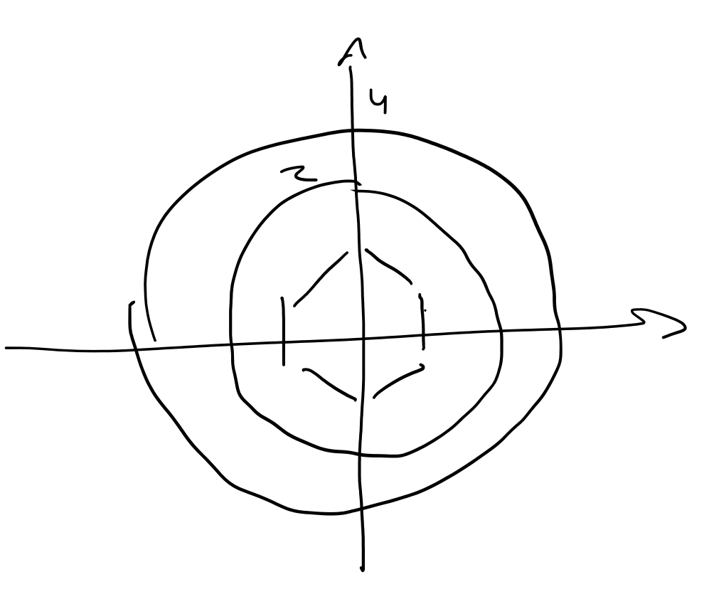
⑥ $$\oint_{C_2}\frac{z^5}{z^6+1}dz=\oint_{C_{R\to\infty}}\frac{z^5}{z^6+1}dz\simeq\oint_{C_{R\to\infty}}\frac{dz}{z}=(*)$$



$z=e^{i\varphi}$

$dz=ie^{i\varphi}d\varphi$ $\quad (*)=\int_0^{2\pi} i\,d\varphi=2\pi i$ //

⑦ $$\int_{-\infty}^{+\infty}\frac{\sin^2x\,dx}{x^2(x^2+1)}=\int_{-\infty}^{+\infty}\frac{e^{2ix}+e^{-2ix}-2}{(-4)x^2(x^2+1)}dx\ (*)=$$

1) $-\frac{1}{4}\int_{-\infty}^{+\infty}\frac{e^{2ix}dx}{x^2(x^2+1)}=2\pi i\sum\operatorname{Res}f(z)=(*)\ z_0=\{0;i\}\ (*)=$

$=-\frac{\pi}{4}\left[1-e^{-2}\right]=-\frac{\pi}{4}+\frac{\pi}{4e^2}$

2) $-\frac{1}{4}\int_{-\infty}^{+\infty}\frac{e^{-2ix}dx}{x^2(x^2+1)}=2\pi i\sum\operatorname{Res}f(z)=(*)\ z_0=\{0;-i\}\ (*)=$

$=-\frac{\pi}{4}\left[1-e^{-2}\right]=-\frac{\pi}{4}+\frac{\pi}{4e^2}$

3) $\frac{1}{2}\int_{-\infty}^{+\infty}\frac{dx}{x^2(x^2+1)}=2\pi i\sum\operatorname{Res}f(z)=(*)\ z_0=\{0;\pm i\}\ (*)=$

$=\pi\left[1-1-1\right]=-\pi,\ \Rightarrow$

$(*)=-\frac{\pi}{4}+\frac{\pi}{4e^2}-\frac{\pi}{4}+\frac{\pi}{4e^2}+\pi=\boxed{\frac{\pi}{2}+\frac{\pi}{2e^2}}$ //

⑧ $$\int_0^{\infty}\frac{x\sin ax}{x^2+k^2}dx=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{x\sin ax}{x^2+k^2}dx=(*)\ \sin ax=\operatorname{Im}\left[e^{iax}\right]\ (*)=$$

$=\frac{1}{2}\operatorname{Im}\int_{-\infty}^{+\infty}\frac{xe^{iax}}{x^2+k^2}dx=\frac{1}{2}\operatorname{Im}\left[2\pi i\operatorname{Res}f(z_0)\right]=(*)\ z_0=ik\ (*)=$

$=\pi\operatorname{Im}\left[i\lim_{z\to z_0}\frac{e^{iaz}(z-z_0)\cdot z}{(z+ik)(z-ik)}\right]=\pi\operatorname{Im}\left[i\lim_{z\to ik}\frac{z\,e^{iaz}}{z+ik}\right]=$

$=\pi\operatorname{Im}\left[i\cdot\frac{1}{2}e^{-ak}\right]=\frac{\pi}{2}e^{-ak},\ a,k>0$

$I(a,k)=\int_0^{\infty}\frac{x\sin ax}{x^2+k^2}dx$

$I(a,-k)=I(a,k)$

$I(-a,k)=-I(a,k),\ \Rightarrow\boxed{I=\frac{\pi}{2}e^{-|a||k|}\operatorname{sign}(a)}$ //

⑨ $$\int_{-\infty}^{+\infty}\frac{\cos\left(x-\frac{1}{x}\right)}{1+x^2}dx=\operatorname{Re}\int_{-\infty}^{+\infty}\frac{e^{i\left(x-\frac{1}{x}\right)}}{1+x^2}dx=\operatorname{Re}\left[2\pi i\sum\operatorname{Res}f(z_0)\right]=(*)$$

$z_0=i\ (*)=\operatorname{Re}\left[2\pi i\lim_{z\to i}\frac{e^{i\left(z-\frac{1}{z}\right)}}{z+i}\right]=\operatorname{Re}\left[2\pi i\frac{e^{-2}}{2i}\right]=\operatorname{Re}\left[\pi e^{-2}\right]=\boxed{\pi e^{-2}}$ //

⑩ $$\int_0^{+\infty}\frac{x-\sin x}{x^3}dx=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{x-\sin x}{x^3}dx=\frac{1}{2}\operatorname{Im}\left[\int_{-\infty}^{+\infty}\frac{x-e^{ix}}{x^3}dx\right]=$$

$=\frac{1}{2}\cdot\operatorname{Im}\left[\pi i\operatorname{Res}f(0)\right]=\frac{\pi}{2}\operatorname{Im}\left[i\lim_{z\to0}\frac{z-e^{iz}}{z^2}\right]=\frac{\pi}{2}\operatorname{Im}\left[i\lim_{z\to0}\frac{1-ie^{iz}}{2z}\right]=$

$=\frac{\pi}{2}\operatorname{Im}\left[i\lim_{z\to0}\frac{e^{iz}}{2}\right]=\frac{\pi}{2}\operatorname{Im}\frac{i}{2}=\boxed{\frac{\pi}{4}}$ //

$$\int_{-\infty}^{+\infty}\frac{e^{-iz}}{z^2+9}dz=\int_{-\infty}^{+\infty}\frac{e^{-iz}dz}{(z-3i)(z+3i)}=2\pi i\operatorname{Res}f(z_0)=(*)\ z_0=3i\ (*)=$$

$=2\pi i\frac{e^{-3}}{6i}=\boxed{\frac{\pi e^{-3}}{3}}$ //

⑭ $$\lim_{R\to\infty}\int_{C_R}e^{iz}dz=\lim_{R\to\infty}\int_0^{\pi}e^{iR(\cos\varphi+i\sin\varphi)}\cdot iRe^{i\varphi}d\varphi$$

Оценим:

$\left|\int_{C_R}e^{iz}dz\right|\leq R\int_0^{\pi}\left|e^{iR(\cos\varphi+i\sin\varphi)+i\varphi}\cdot i\right|d\varphi=R\int_0^{\pi}e^{-R\sin\varphi}d\varphi=$

$=2R\int_0^{\pi/2}e^{-R\sin\varphi}d\varphi\leq(*)$ где $\varphi\in[0;\pi/2]$ $\sin\varphi\geq2\varphi/\pi$ $(*)\leq2R\int_0^{\pi/2}e^{-2R\varphi/\pi}d\varphi=$

$=-\pi e^{-2R\varphi/\pi}\Big|_0^{\pi/2}=\pi-e^{-R}\leq\pi$

С другой стороны:

$\lim_{R\to\infty}\int_{C_R}e^{iz}dz=\lim_{R\to\infty}\left[-ie^{iz}\right]\Big|_{-R}^{R}=\lim_{R\to\infty}i\left[e^{-iR}-e^{iR}\right]=\lim_{R\to\infty}2\sin R,\ \Rightarrow$

предел не сущ. //

$\lim_{R\to\infty}\int_{C_R}e^{iz^2}dz=(*)\ s=z^2,\ ds=z\,dz\quad(*)=\lim_{R\to\infty}\int_{C'_R}\frac{e^{is}}{\sqrt{s}}ds\Rightarrow$ по

$\frac{1}{\sqrt{s}}\to0$ при $s\to\infty$

лемме Жордана $\lim_{R\to\infty}\int_{C_R}e^{iz^2}dz=0$ //

⑬ $$f(z)=\frac{\sin\frac{1}{z}}{1-z}=\sin\frac{1}{z}\cdot\frac{1}{1-z}=\sum_{n=0}^{\infty}\left[(-1)^n\cdot\frac{x^{-2n-1}}{(2n+1)!}\right]\cdot\sum_{n=0}^{\infty}z^n,\ \Rightarrow$$

Коэфф. перед $\frac{1}{z}$: $\sum_{n=0}^{\infty}(-1)^n\cdot\frac{1}{(2n+1)!}=\sin1$ //

$g(z)=\exp(-\exp(\frac{1}{z}))=\exp\left(-\sum_{n=0}^{\infty}\frac{z^{-n}}{n!}\right)=\prod_{n=0}^{\infty}e^{-z^{-n}/n!}=$

$=e^{-1}\cdot\sum_{n=0}^{\infty}(-1)^n\frac{z^{-n}}{n!}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{2!}\frac{z^{-2n}}{n!}\cdot\ldots\Rightarrow$

Коэфф. перед $\frac{1}{z}$: $-\frac{1}{e}$ //

① $$f(z)=\frac{1}{\sin z}+\frac{2z}{z^2-\pi^2}=\frac{1}{\sin z}+\frac{2z}{(z-\pi)(z+\pi)}=(*)\ A(z+\pi)+B(z-\pi)=2z$$

$\begin{cases}A+B=2\\A-B=0\end{cases}\Rightarrow A=B=1\quad(*)=$

$=\frac{1}{\sin z}+\frac{1}{z-\pi}+\frac{1}{z+\pi}$

Пусть $z\pm\pi=\varepsilon\to0$, тогда

$\sin z=\sin(\varepsilon\mp\pi)=\sin\varepsilon\cos\pi\mp\sin\pi\cos\varepsilon=-\sin\varepsilon\simeq-\varepsilon+\frac{\varepsilon^3}{6},\ \Rightarrow$

$\frac{1}{\sin z}=\frac{6}{\varepsilon^3-6\varepsilon}=\frac{1}{\varepsilon}\left(\frac{6}{\varepsilon^2-6}\right)=\frac{1}{\varepsilon}\frac{1}{\varepsilon^2/6-1}\approx-\frac{1}{\varepsilon}\left(1+\frac{\varepsilon^2}{6}\right)=-\frac{1}{\varepsilon}-\frac{\varepsilon}{6}=$

$=-\frac{1}{z\pm\pi}-\frac{z\pm\pi}{6}$, тогда

$\lim_{z\to\pm\pi}\left[\frac{1}{\sin z}+\frac{1}{z-\pi}+\frac{1}{z+\pi}\right]=-\frac{1}{z\pm\pi}-\frac{z\pm\pi}{6}+\frac{1}{z+\pi}+\frac{1}{z-\pi}=\frac{1}{z\mp\pi}-\frac{z\pm\pi}{6},\ \Rightarrow$

$\lim_{z\to\pm\pi}f(z)=\frac{1}{z\mp\pi}-\frac{z\pm\pi}{6},\ \Rightarrow z=\pm\pi$ — точки устрани. разрыва

② 1) $f(z)=\frac{\sin z}{1-\operatorname{tg}z}$; $\operatorname{tg}z=1$ при $z_0=\frac{\pi}{4}+\pi n$

$\lim_{z\to z_0}f(z_0)=\infty,\ \Rightarrow z_0$ — полюс //

2) $f(z)=\frac{e^{c/z-a}}{e^{z/a}-1}=\frac{\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{c}{z-a}\right)^n}{\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{z}{a}\right)^n-1}=\frac{\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{c}{z-a}\right)^n}{\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{z}{a}\right)^n},\ \Rightarrow$

$z_0=a$ — существенно изолир. особен.

$\lim_{z\to z_0}f(z)=\infty$ при $e^{z_0/a}-1=0,\ \Rightarrow$

$e^{z_0/a}=1$

$\frac{z_0}{a}=2\pi in$

$z_0=2\pi ina$ — полюса //

③ $f(z)=ze^{1/z}e^{-\frac{1}{z^2}}=z\cdot\sum_{n=0}^{\infty}\frac{z^{-n}}{n!}\cdot\sum_{n=0}^{\infty}(-1)^n\frac{z^{-2n}}{n!},\ \Rightarrow z_0=0$ — существ. изолир. особенность //

④ (1) $\oint_C\frac{ze^z}{\operatorname{tg}z^2}dz=2\pi i\sum\operatorname{Res}f(z)=(*)\ \operatorname{tg}z_0^2=0$

$z_0^2=\frac{\pi n}{2},\ \Rightarrow$ в контур входит только $z_0=0$ $\quad(*)=2\pi i\lim_{z\to0}\frac{z^2e^z}{\operatorname{tg}z^2}=$

$=2\pi i\lim_{z\to0}\frac{z^2e^z}{z^2}=\underline{2\pi i}$ //

(2) $\oint_Ce^{-1/z}\sin\frac{1}{z}dz=\oint_C\sum_{n=0}^{\infty}(-1)^nz^{-n}\frac{1}{n!}\cdot\sum_{n=0}^{\infty}z^{-2n-1}\cdot(-1)^n\cdot\frac{1}{n!}dz=$

$=2\pi i\operatorname{Res}f(z)=(*)\ z_0=0$ - сущ. изол. особенность $(*)=2\pi i\cdot1=\underline{2\pi i}$ //

(3) $\oint_C\frac{e^z}{z^n}dz=2\pi i\operatorname{Res}f(z)=(*)\ z_0=0$ - полюс n-ного порядка $(*)=\frac{2\pi i}{(n-1)!}\cdot\lim_{z\to0}\frac{d^{n-1}}{dz^{n-1}}f(z)\cdot z^n=$

$=\frac{2\pi i}{(n-1)!}\lim_{z\to0}\frac{d^{n-1}}{dz^{n-1}}e^z=\boxed{\frac{2\pi i}{(n-1)!}}$ //

⑫ $f(z)=\frac{1}{z^3-z^5}=\frac{1}{g(z)},\ \Rightarrow g'(z)=3z^2-5z^4$

- $\operatorname{Res}_{z=-1}f(z)=\frac{1}{g'(-1)}=-\frac{1}{2}$ //
- $\operatorname{Res}_{z=0}f(z)=\operatorname{Res}_{z=0}\left[\frac{1}{z^3}\sum_{n=0}^{\infty}z^{2n}\right]=1$ //
- $\operatorname{Res}_{z=1}f(z)=\frac{1}{g'(1)}=-\frac{1}{2}$ //
- $\operatorname{Res}_{z=\infty}f(z)=\operatorname{Res}_{s=0}\frac{1}{\frac{1}{s^3}-\frac{1}{s^5}}=\operatorname{Res}_{s=0}\frac{s^5}{s^2-1}=\operatorname{Res}_{s=0}-s^5\sum_{n=0}^{\infty}s^{2n}=0$ //

⑪ $f(z)=z^3\cos\frac{1}{z-2}$

$f\left(s=\frac{1}{z}\right)=\frac{1}{s^3}\cos\frac{1}{\frac{1}{s}-2}=\frac{1}{s^3}\cdot\cos\frac{s}{1-2s}=\frac{1}{s^3}\cos\left[s\cdot\sum_{n=0}^{\infty}(2s)^n\right]=\frac{1}{s^3}\cdot\left(1-\frac{s^2}{2}-2s^3-\frac{143}{24}s^4+\ldots\right)=$

$=\frac{1}{s^3}-\frac{1}{2s}-2-\frac{143}{24}s+\ldots=z^3-2z-2-\frac{143}{24\cdot z}+\ldots,\ \Rightarrow$

$\operatorname{Res}_{z=\infty}f(z)=\frac{143}{24}$ //

⑤ 1) $\int_{-\infty}^{+\infty}\frac{x^4}{1+x^6}dx=2\pi i\sum\operatorname{Res}f(z)=(*)$

$z_0^6=-1=e^{i\pi}$

$e^{6i\varphi}=e^{i\pi+2\pi in}$

$\varphi=\frac{\pi}{6}+\frac{\pi n}{3},\ \Rightarrow$

т.к. $\varphi\in[0;\pi]$

$z_0=\{e^{i\pi/6};e^{i\pi/2};e^{5i\pi/6}\}$ $\quad(*)=$

$=(*)\ \operatorname{Res}f(z)=\frac{z^4}{(1+z^6)'}=\frac{1}{6z_0}\ (*)=2\pi i\sum\left[\frac{1}{6e^{i\pi/6}}+\frac{1}{6e^{i\pi/2}}+\frac{1}{6e^{i\pi/6}}\right]=$

$=\frac{\pi i}{3}\left(\cancel{\cos}-\frac{\pi}{6}+i\overset{=-\frac{1}{2}}{\sin-\frac{\pi}{6}}+\overset{=0}{\cos\frac{\pi}{2}}+i\overset{=-1}{\sin-\frac{\pi}{2}}+\cancel{\cos}-\frac{5\pi}{6}+i\overset{=-\frac{1}{2}}{\sin-\frac{5\pi}{6}}\right]=\frac{\pi i}{3}\cdot(-2i)=\frac{2\pi}{3}$ //

2) $\int_0^{2\pi}\frac{\cos2\theta}{2+\cos\theta}d\theta=(*)$

$z=e^{i\theta}$

$\cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right)$

$\cos2\theta=\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)$

$(*)=\oint_{C_R}\frac{2(z^4+1)z}{2z^2(4z+z^2+1)}\cdot\left(-\frac{i}{z}\right)dz=$

$=-i\oint_{C_R}\frac{z^4+1}{z^2(z^2+4z+1)}dz=2\pi i\sum\operatorname{Res}f(z)=(*)$

$z_0=0$

$z_0^2+4z_0+1=0$

$z_0=-2+\sqrt{3}$, $\Rightarrow$

$\operatorname{Res}_{z_0=0}f(z)=-i\lim_{z\to0}\frac{4z^3(z^2+4z+1)-(z^4+1)(2z+4)}{(z^2+4z+1)^2}=-4$

$\operatorname{Res}_{z_0=-2+\sqrt{3}}f(z)=-i\lim_{z\to z_0}\frac{z^4+1}{z^2(z-2+\sqrt{3})}=\frac{(7-4\sqrt{3})^2+1}{(7-4\sqrt{3})\cdot2\sqrt{3}}=\frac{96-8\sqrt{3}}{14\sqrt{3}-24}=$

$=\frac{(96-8\sqrt{3})(14\sqrt{3}+24)}{12}=-i\frac{7\sqrt{3}}{3}$

$(*)=2\pi i\left[-4+\frac{7\sqrt{3}}{3}\right]=-8\pi+\frac{14\pi\sqrt{3}}{3}$

3) $\int_{-\infty}^{+\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)^2}=2\pi i\sum\operatorname{Res}f(z)=(*)$

$z_0=i|a|$

$z_0=i|b|$ (полюс 2-го порядка) $\quad(*)=$

$=2\pi i\left[\frac{1}{2i|a|\cdot(b^2-a^2)^2}+\lim_{z\to ib}\frac{d}{dz}\frac{1}{(z^2+a^2)(z+ib)^2}\right]=$

$=\frac{\pi}{|a|(b^2-a^2)^2}+\lim_{z\to ib}\frac{-2z(z+ib)^2+2(z+ib)(z^2+a^2)}{(z^2+a^2)^2(z+ib)^4}\cdot2\pi i=$

$=\frac{\pi}{|a|(b^2-a^2)^2}-\frac{i4|b|\cdot(-b^2)+4i|b|(a^2-b^2)}{(a^2-b^2)^2\cdot16b^4}\cdot2\pi i=$

$=\frac{\pi}{|a|(b^2-a^2)^2}+\pi\frac{-2b^2+(a^2-b^2)}{2|b|^3(a^2-b^2)^2}=\frac{\pi}{2|a||b|^3(a^2-b^2)^2}\left[2|b|^3+|a|(a^2-3b^2)\right]=$

$=\frac{\pi}{2|a||b|^3}\cdot\frac{1}{(|a|-|b|)^2\cdot(|a|+|b|)^2}\cdot(|a|+2|b|)(|a|-|b|)^2=$

$=\boxed{\frac{\pi(|a|+2|b|)}{2|a||b|^3(|a|+|b|)^2}}$ //