Complex HW1

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(14.) $\frac{z}{z^2+1} = \frac{z}{(z-i)(z+i)} = (*)\ \frac{A}{z-i} + \frac{B}{z+i} = \frac{z}{(z-i)(z+i)}\ (*) =$

$Az + iA + Bz - iB = z$

$\begin{cases} A = B \\ A+B=1 \end{cases}$

$A = B = 1/2$

$= \frac{1}{2(z-i)} - \frac{1}{2(z+i)}$

Разложение в окрестности точки $z = i$, =>

$\frac{z}{z^2+1} = \frac{1}{2(z-i)} \quad \frac{1}{2(z+i)} = \frac{1}{2(z-i)} - \frac{1}{2((z-i)+2i)} =$

$= \frac{1}{2(z-i)} - \frac{1}{4i} \cdot \frac{1}{\frac{z-i}{2i}+1} = \frac{1}{2(z-i)} - \frac{i}{4} \sum_{n=0}^{\infty} \left(\frac{1}{2i}\right)^n (z-i)^n$ //

$\left|\frac{z-i}{2i}\right| < 1$

$|z-i| < 2$ //

(13.) (i) $\frac{1}{z(z-1)} = (*)\ \frac{A}{z} + \frac{B}{z-1} = \frac{1}{z(z-1)}$ $\quad (*) = \frac{1}{z-1} - \frac{1}{z} =$

$Az - A + Bz = 1$

$A = -1$

$A = -B$

$= -\frac{1}{z} - \sum_{n=0}^{\infty} z^n$

(ii) $\frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z} = -\frac{1}{z} + \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = -\frac{1}{z} + \sum_{n=1}^{\infty} z^{-n} = \sum_{n=2}^{\infty} z^{-n}$ //

(11.) $\frac{1+2z^2}{z^3+z^5} = \frac{1}{z^3}\left(\frac{1+2z^2}{1+z^2}\right) = \frac{1}{z^3}\left(1+z^2\left(\frac{1}{1+z^2}\right)\right) = \frac{1}{z^3}\left(1+\sum_{n=1}^{\infty} z^2\right)$, =>

$\frac{1+2z^2}{z^3+z^5} \approx \frac{1}{z^3}\left(1+z^2+z^4+\dots\right) = \frac{1}{z^3} + \frac{1}{z} + z + \dots$ //

(7.) $\int_C z\,dz = (*)$ $\quad z = \cos t + i\sin t$, $dz = (-\sin t + i\cos t)\,dt$ $\quad (*) = \int_0^{2\pi} (-\cos t\sin t - i\sin^2 t - \sin t\cos t + i\cos^2 t)\,dt =$

$= \int_0^{2\pi} (i\cos 2t - 2\cos t\sin t)\,dt = 0$

$\int_C z^* dz = \int_0^{2\pi} (\cos t - i\sin t)(i\cos t - \sin t)\,dt = \int_0^{2\pi} i(\sin^2 t + \cos^2 t)\,dt = 2\pi i$

(8.) 1) $\int_C \frac{y\,dx - x\,dy}{x^2+y^2} = (*)$ $\quad x = r\cos\varphi$, $y = r\sin\varphi$, $r = 1$ $\quad (*) = \int_0^{2\pi} (\sin\varphi \cdot (-\sin\varphi) - \cos\varphi \cdot \cos\varphi)\,d\varphi =$

$= -2\pi$

2) $\int_C \frac{y\,dx - x\,dy}{x^2+y^2} = (*)$ $\quad x = r\cos\varphi + 2$, $y = r\sin\varphi$ $\quad (*) = \int_0^{2\pi} \frac{-\sin^2\varphi - \cos^2\varphi - 2\cos\varphi}{1+4\cos\varphi+4}\,d\varphi =$

$= -\int_0^{2\pi} \frac{1+2\cos\varphi}{5+4\cos\varphi}\,d\varphi = (*)$ $\quad u = \operatorname{tg}\frac{\varphi}{2}$, $\cos\varphi = \frac{1-u^2}{1+u^2}$, $dx = \frac{2\,du}{u^2+1}$ $\quad (*) = -2\int_0^{+\infty} \frac{1+u^2+2-2u^2}{(1+u^2)(5+5u^2+4-4u^2)}\,du =$

$= -2\int_0^{+\infty} \frac{3-u^2}{(1+u^2)(9+u^2)} = (*)$ $\quad \frac{A}{1+u^2} + \frac{B}{9+u^2} = \frac{u^2-3}{(1+u^2)(9+u^2)}$ $\quad (*) =$

$9A + Au^2 + B + u^2B = u^2 - 3$

$\begin{cases} 9A+B=-3 \\ A+B=1 \end{cases} \sim \begin{cases} A=-\frac{1}{2} \\ B=\frac{3}{2} \end{cases}$

$= -\int_0^{+\infty} \frac{du}{1+u^2} + 3\int_0^{+\infty} \frac{du}{9+u^2} = \left[-\operatorname{arctg} u + \operatorname{arctg}\frac{u}{3}\right]\Big|_0^{+\infty} = -\pi + \pi = 0$ //

(10.) $y(1) = 0$

$y'(z) = \frac{1}{2z}$, => $y(z) = \frac{1}{2}\int_D \frac{dz}{z} = (*)$ $\quad z = |z|e^{i\varphi}$, $dz = d|z| \cdot e^{i\varphi} + i|z|e^{i\varphi}d\varphi$ $\quad (*) =$

$= \frac{1}{2}\left[\int_D \frac{e^{i\varphi}d|z|}{|z|e^{i\varphi}} + \frac{i|z|e^{i\varphi}}{|z|e^{i\varphi}}d\varphi\right] = \frac{1}{2}\int_D \frac{d|z|}{|z|} + i\,d\varphi = \frac{1}{2}\left[\ln|z| + i\varphi\right]\Big|_D$

а) $z = -1$, $\varphi \in (0;\pi)$, => $y(-1) = \frac{1}{2}[\ln 1 + i\pi - 0] = \frac{i\pi}{2}$

б) $z = -1$, $\varphi \in (\pi;0)$, => $y(-1) = \frac{1}{2}[\ln 1 + 0 - i\pi] = -\frac{i\pi}{2}$

(2.) $\int[1+2\varepsilon+3\varepsilon^2+\dots+n\varepsilon^{n-1}]\,d\varepsilon = \varepsilon+\varepsilon^2+\varepsilon^3+\dots+\varepsilon^n$

$\sum_{i=1}^{n} \varepsilon^n = \frac{\varepsilon(\varepsilon^n-1)}{\varepsilon-1} = \frac{\varepsilon^{n+1}-\varepsilon}{\varepsilon-1} = S_n$

$\frac{dS_n}{d\varepsilon} = \frac{((n+1)\varepsilon^n-1)(\varepsilon-1) - \varepsilon^{n+1} + \varepsilon}{(\varepsilon-1)^2} = \frac{(n\varepsilon^n+\varepsilon^n-1)(\varepsilon-1) - \varepsilon^{n+1} + \varepsilon}{(\varepsilon-1)^2} =$

$= \frac{n\varepsilon^n(\varepsilon-1) + \varepsilon^{n+1} - \varepsilon^n - \varepsilon + 1 - \varepsilon^{n+1} + \varepsilon}{(\varepsilon-1)^2} = \boxed{\frac{\varepsilon^n(n(\varepsilon-1)-1)+1}{(\varepsilon-1)^2}}$ //

(4.) $w(z) = x^2+y^2$, $u = x^2+y^2$, $v = 0$, =>

$\frac{\partial u}{\partial x} = 2x$; $\frac{\partial u}{\partial y} = 2y$

$\frac{\partial v}{\partial x} = 0$; $\frac{\partial v}{\partial y} = 0$ $\quad$ => $\frac{\partial u}{\partial x} \ne \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} \ne \frac{\partial v}{\partial x}$ // ⊖

$w(z) = x^2 - y^2 + 2ixy$, $u = x^2-y^2$, $v = 2xy$

$\frac{\partial u}{\partial x} = 2x$; $\frac{\partial u}{\partial y} = -2y$

$\frac{\partial v}{\partial x} = 2y$; $\frac{\partial v}{\partial y} = 2x$ $\quad$ => $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ // ⊕

$w(z) = \frac{1}{x+iy} = \frac{x-iy}{x^2-y^2}$, $u = \frac{x}{x^2-y^2}$, $v = -\frac{y}{x^2-y^2}$

$\frac{\partial u}{\partial x} = \frac{x^2-y^2-2x^2}{(x^2-y^2)^2}$; $\frac{\partial u}{\partial y} = \frac{-x\cdot(-2y)}{(x^2-y^2)^2}$

$\frac{\partial v}{\partial x} = \frac{y\cdot 2x}{(x^2+y^2)^2}$; $\frac{\partial v}{\partial y} = \frac{-(x^2-y^2) - y\cdot(-2y)}{(x^2+y^2)^2}$ $\quad$ => $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ // ⊕

(12.) $f(z) = \frac{1}{z(e^z-1)} = \frac{1}{z} \cdot \frac{1}{e^z-1} = (*)$

Пусть $S(z) = \frac{e^z-1}{z} = \frac{1}{z} \cdot \sum_{n=1}^{\infty} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$, тогда

$(e^z-1)^{-1} = \frac{1}{zS(z)} = \sum_{-\infty}^{+\infty} a_n z^n$

$a_n = \frac{1}{2\pi i}\int_C \frac{dz}{z^{n+1}(e^z-1)} = \frac{1}{2\pi i}\int_C \frac{dz}{z^{n+2} \cdot S(z)}$

При $n \le -2$ $a_n = 0$, тогда

$\left(\frac{a_{n-1}}{z} + a_0 + a_1z + a_2z^2 + a_3z^3 + O(z^4)\right)\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + O(z^6)\right) = 1$

1) $\frac{a_{n-1}}{z} \cdot z = 1$, => $a_{n-1} = 1$

2) $\frac{a_{n-1}}{2}z + a_0z = 0$, => $a_0 = -\frac{1}{2}$

3) $\frac{a_{n-1}}{6}z^2 + a_0z^2 \cdot \frac{1}{2} + a_1 = 0$

$\frac{1}{6} - \frac{1}{4} + a_1 = 0$

$a_1 = \frac{1}{12}$, тогда

$f(z) \approx \frac{1}{z}\left(\frac{1}{z} - \frac{1}{2} + \frac{z}{12} + O(z^2)\right) = \frac{1}{z^2} - \frac{1}{2z} + \frac{1}{12} + \frac{1}{z}O(z^2)$ //

(5.) 1) $|f| = e^{r^2\cos 2\varphi} = e^{r^2(\cos^2\varphi - \sin^2\varphi)} = e^{x^2-y^2}$, => $u = x^2-y^2$

$w = \ln f = \ln|f|e^{i\arg f} = \ln|f| + i\arg f = x^2-y^2+i\arg f$, => $v = \arg f$

По условию Коши-Римана:

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, => $2x = \frac{\partial v}{\partial y}$, => $v = 2xy + \varphi(x)$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, => $2y = \frac{\partial v}{\partial x}$, => $v = 2xy + \gamma(y)$

$2xy + \varphi(x) = 2xy + \gamma(y)$

$\varphi(x) = \gamma(y)$, => $\varphi(x) = \gamma(y) = C$, => $v = 2xy + C$

$w = u + iv = x^2 - y^2 + 2ixy + iC = (x+iy)^2 + iC = z^2 + iC = \ln f$, =>

$f = e^w = e^{z^2+iC}$ //

2) $\operatorname{Arg} f = xy$

$w = \ln f = \ln|f| + i\arg f = \ln|f| + ixy$, => $\ln|f| = u$, $xy = v$

По условию Коши-Римана:

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, => $\frac{\partial u}{\partial x} = x$, => $u = \frac{x^2}{2} + \gamma(y)$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, => $\frac{\partial u}{\partial y} = -y$, => $u = -\frac{y^2}{2} + \varphi(x)$

$\frac{x^2}{2} + \gamma(y) = -\frac{y^2}{2} + \varphi(x)$

$\begin{cases} \gamma(y) = -\frac{y^2}{2} + C \\ \varphi(x) = \frac{x^2}{2} + C \end{cases}$, => $u = \frac{x^2-y^2}{2} + C$, =>

$w = \frac{x^2-y^2}{2} + ixy + C = \frac{z^2}{2} + C = \ln f$, =>

$f = e^w = e^{z^2/2 + C}$ //

(6.) $f(z) = u + iv$

1) $u = \varphi(x^2-y^2)$

Из ус-я Коши-Римана

$\begin{cases} \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\partial y} \\ \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x\partial y} \end{cases}$ => $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\frac{\partial^2 u}{\partial x^2} = (2x\varphi'(x^2-y^2))' = 2\varphi'(x^2-y^2) + 4x^2\varphi''(x^2-y^2)$

$\frac{\partial^2 u}{\partial y^2} = -2\varphi'(x^2-y^2) + 4y^2\varphi''(x^2-y^2)$ =>

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\varphi''(x^2-y^2)(x^2+y^2) = 0$, =>

$\varphi''(x^2-y^2) = 0$

$\varphi'(x^2-y^2) = C_1$

$\varphi(x^2-y^2) = C_1(x^2-y^2) + C_2 = u$

Из ус-я Коши-Римана:

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, => $\frac{\partial v}{\partial y} = 2xC_1$, => $v = 2xyC_1 + \gamma(y)$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, => $\frac{\partial v}{\partial x} = 2yC_1$, => $v = 2xyC_1 + \zeta(x)$

$2xyC_1 + \gamma(y) = 2xyC_1 + \zeta(x)$

$\gamma(y) = \zeta(x) = C_3$, =>

$f = C_1(x^2-y^2) + C_2 + i2xyC_1 + iC_3 = C_1z^2 + C_2 + iC_3 = C_1z^2 + C_4 =$

$= \boxed{bz^2 + a}$ //

2) $u = \varphi\left(\frac{y}{x}\right)$

Из усл-я Коши-Римана:

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\frac{\partial^2 u}{\partial x^2} = \left(y \cdot \left(-\frac{1}{x^2}\right)\varphi'\left(\frac{y}{x}\right)\right)' = 2y\frac{1}{x^3}\varphi'\left(\frac{y}{x}\right) + \frac{y}{x^2}\varphi''\left(\frac{y}{x}\right) \cdot \frac{y}{x^2}$

$\frac{\partial^2 u}{\partial y^2} = \left(\frac{1}{x}\varphi'\left(\frac{y}{x}\right)\right)' = \frac{1}{x^2}\varphi''\left(\frac{y}{x}\right)$, =>

$\frac{1}{x^2}\left(\frac{2y}{x}\varphi'\left(\frac{y}{x}\right) + \frac{y^2}{x^2}\varphi''\left(\frac{y}{x}\right) + \varphi''\left(\frac{y}{x}\right)\right) = 0$

Положим $\frac{y}{x} = t$, =>

$2t\varphi'(t) + \varphi''(t)(1+t^2) = 0$

$\varphi'(t) = \xi(t)$, тогда

$\xi'(t)(1+t^2) + 2t\xi(t) = 0$

$\frac{d\xi}{\xi(t)} = -\frac{2t}{t^2+1}dt$

$\frac{d\xi}{\xi(t)} = -\frac{1}{t^2+1}dt^2$

$\ln\xi(t) = -\ln C(t^2+1)$

$\xi(t) = \frac{C_1}{t^2+1}$

$\varphi'(t) = C_1(t^2-1)^{-1}$

$\varphi(t) = C_1\operatorname{arctg} t + C_2 = u$

Из условия Коши-Римана:

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, => $C_1\frac{1}{1+\frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right) = \frac{\partial v}{\partial y}$,

$-\frac{C_1y}{x^2+y^2} = \frac{\partial v}{\partial y}$, => $v = -\frac{C_1}{2}\int \frac{dy^2}{x^2+y^2} = -\frac{C_1}{2}\ln(x^2+y^2) + \zeta(x)$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, => $C_1\frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} = -\frac{\partial v}{\partial x}$

$C_1\frac{x}{x^2+y^2} = -\frac{\partial v}{\partial x}$, => $v = -\frac{C_1}{2}\ln(x^2+y^2) + \gamma(y)$, =>

$\zeta(x) = \gamma(y) = C_3$, =>

$f(z) = u + iv = C_1\operatorname{arctg}\frac{y}{x} + C_2 - \frac{iC_1}{2}\ln(x^2+y^2) + iC_3 =$

$= C_1\varphi + C_2 - iC_1\ln|z| + iC_3 = (*)$ $\quad z = |z|e^{i\varphi}$, $\ln z = \ln|z| + i\varphi$ $\quad (*) = i(C_1(i\varphi + \ln|z| + iC_2 + C_3)) =$

$= iC_1\ln z + C_4 = \boxed{bi\ln z + a}$ //

(9.) $p(n) = \frac{1}{2\pi i}\int_C dz\,z^{-1-n}\prod_{k=1}^{\infty}\frac{1}{1-z^k}$

$\prod_{k=1}^{\infty}\frac{1}{1-z^k} = \frac{1}{1-z} \cdot \frac{1}{1-z^2} \cdot \frac{1}{1-z^3} \cdot \dots = \sum_{i=0}^{\infty} z^i \cdot \sum_{i=0}^{\infty} z^{2i} \cdot \sum_{i=0}^{\infty} z^{3i} \cdot \dots$, =>

Интеграл $\int_C dz \cdot z^{-1-n}\prod_{k=1}^{\infty}\frac{1}{1-z^k} = \sum_k \int_C a_k z^{b_k}dz = \sum_k a_k \int_C z^{b_k}dz$

$\int_C z^k dz = 0$ при $k \ne -1$

$p(1) = \frac{1}{2\pi i}\int_C dz \cdot z^{-2} \cdot \left(\sum_{i=0}^{\infty} z^i \cdot \sum_{i=0}^{\infty} z^{2i} \cdot \sum_{i=0}^{\infty} z^{3i} \dots\right) =$

$= \frac{1}{2\pi i}\int_C dz \cdot z^{-2} \cdot (1 + z + 2z^2 + 3z^3 + 5z^4 + \dots) = \frac{1}{2\pi i}\int_C dz \cdot \frac{1}{z} =$

$= \frac{1}{2\pi i}\int_C |z|^{-1}e^{-i\varphi}(e^{i\varphi}d|z| + i|z|e^{i\varphi}d\varphi) = \frac{1}{2\pi i}\left[\ln|z| + i\varphi\right] = 1$ //

$p(4) = \frac{1}{2\pi i}\int_C dz \cdot z^{-5}(1 + z + 2z^2 + 3z^3 + 5z^4 + \dots) = 5$ //

(3.) (i) $\operatorname{Im} z = 1$

$z \to w(z) = z^3 + 3z - i = (x+iy)^3 + 3(x+iy) - i = (*)$ $\quad y = \operatorname{Im} z = 1$ $\quad (*) =$

$= x^3 + 3ix^2 - 3x - i + 3x + 3i - i = x^3 + 3ix^2 + i = x^3 + i(3x^2+1)$, =>

$\operatorname{Re}(w(z)) = x^3$, => $x = (\operatorname{Re} w)^{1/3}$

$\operatorname{Im}(w(z)) = 3x^2 + 1 = 3(\operatorname{Re} w)^{2/3} + 1$ //

(ii) $|z-i| = [x^2 + (y-1)^2]^{1/2} = [x^2 + y^2 - 2y + 1]^{1/2} = 1$, => $x^2 + y^2 - 2y = 0$

$z \to w(z) = \frac{1}{z-2i} = \frac{1}{x+iy-2i} = \frac{1}{x+i(y-2)} = \frac{x-i(y-2)}{x^2+(y-2)^2} = \frac{x-i(y-2)}{x^2+y^2-4y+4} =$

$= \frac{x-i(y-2)}{4-2y} = \frac{x}{4-2y} + \frac{i(y-2)}{2(y-2)} = \frac{x}{4-2y} + \frac{i}{2}$, =>

$\operatorname{Re} w = \frac{x}{4-2y}$

$\operatorname{Im} w = \frac{1}{2}$ //

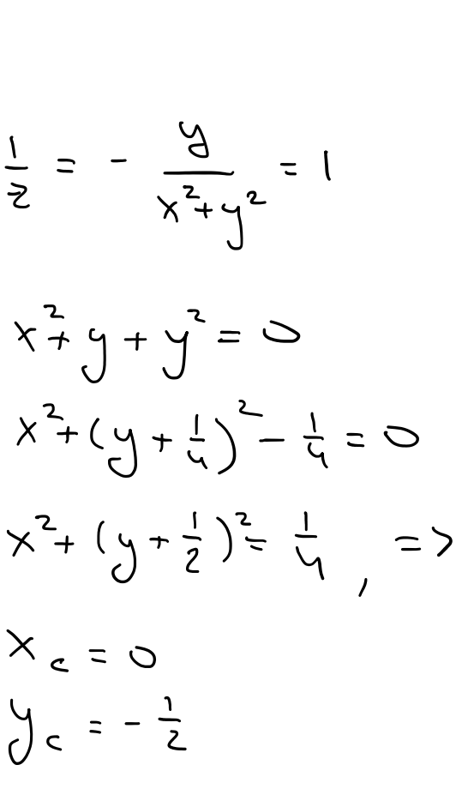
(1.)

1) $2 \le |z-i| \le 4$

$z - i = x + i(y-1) = 0$ при $x = 0$, $y = 1$, =>

$z_0 = i$

$S = \pi(4^2 - 2^2) = 12\pi$ //



2) $|z-4i| + |z+4i| = 10$

$|x + i(y-4)| + |x + i(y+4)| = 10$, =>

$z_1 = 4i$, $z_2 = -4i$ => $z_c = \frac{z_1 + z_2}{2} = 0$

$\sqrt{x^2 + (y-4)^2} + \sqrt{x^2 + (y+4)^2} = 2a$, => $a = 5$ //

3) $\operatorname{Im}\frac{1}{z} = 1$

$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$, => $\operatorname{Im}\frac{1}{z} = -\frac{y}{x^2+y^2} = 1$

$x^2 + y + y^2 = 0$

$x^2 + (y+\frac{1}{4})^2 - \frac{1}{4} = 0$

$x^2 + (y+\frac{1}{2})^2 = \frac{1}{4}$, =>

$x_c = 0$

$y_c = -\frac{1}{2}$

$R = \frac{1}{2}$ //