1: Solving Recurrences

Stated Below is the Master Theorem which will be used exaustively in the following answers.

$$T(n) = \begin{cases} c & \text{if } n < d, \\ aT(n/b) + f(n) & \text{if } n \geq d, \end{cases}$$

where $a \geq 1, b > 1$, and d are integers and c is a positive constant. Let $\nu = \log_b a$.

Case (i) $f(n)$ is "definitely smaller" than n^ν: If there is a small contant $\epsilon > 0$, such that $f(n) \preceq n^{\nu-\epsilon}$, that is, $f(n) \prec n^\nu$, then $T(n) \sim n^\nu$.

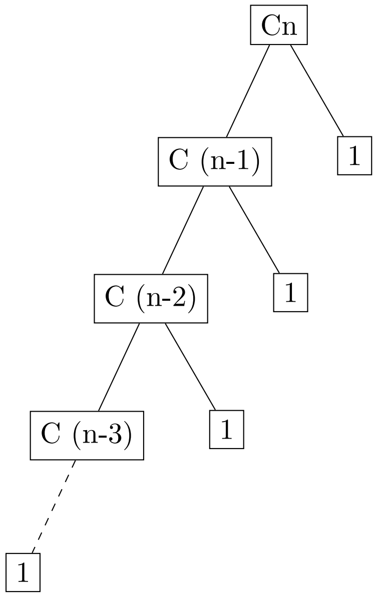
Case (ii) $f(n)$ is "similar in size" to n^ν: If there is a constant $k \geq 0$, such that $f(n) \sim n^\nu(\log n)^k$, then $T(n) \sim n^\nu(\log n)^{k+1}$.

Case (iii) $f(n)$ is "definitely larger" than n^ν: If there are small constants $\epsilon > 0$ and $\delta < 1$, such that $f(n) \succeq n^{\nu+\epsilon}$ and $af(n/b) \leq \delta f(n)$, for $n \geq d$, then $T(n) \sim f(n)$.

(a) $T(n) = 4T(n/4) + n$**.**
Here $a = 4, b = 4, f(n) = n$.
a, b both are greater than 1,
Hence applying master theorem:
Compare f(n) with $n^{\log_b a}$
$n^{log_b a}$ = n = $f(n)$.
Condition 2 of Master Method is valid.
Therefore, $T(n) = O(n^{\log_b a} \log n) = O(n \log n)$

(b) $T(n) = 4T(n/4) + 1$.
a, b both are greater than 1
Hence applying master theorem:
Compare f(n) with $n^{\log_b a}$
$n^{\log_b a}$ = n
We know that f(n)=1 which is in the form $O(n^{\log_4 4-\epsilon})$ where ϵ is a constant.
Condition 1 for master method holds
Hence, T(n)= $O(n^{\log_4 4}) = O(n)$

(c) $T(n) = T(n-1) + n$.
Solution using Recursion Tree
Cost of all subproblem is C_n, where n is subproblem.
The tree structure is



Size of sub prob decreases by 1 at each node
depth at which size of subproblem = 1 is 'n'.
depth of tree =n
Recurrence = Sum of costs at each level =
$Cn + C(n-1) + C(n-2) + ...1$
$\implies$ n*Cn - e, where e = sum of constant terms
$\implies Cn^2 - e$
$\implies O(n^2)$

(d) $T(n) = T(n/3) + T(n/2) + \sqrt{n}$.

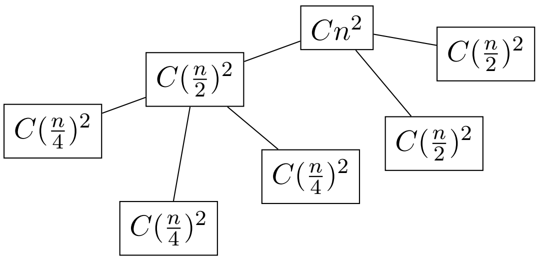
Guess: O(n)
$T(n) < d(n/3) + d(n/2) + \sqrt{n}$
$T(n) < d(5n/6) + \sqrt{n}$
$T(n) < Cn + \sqrt{n}$
$T(n) < Cn$
$T(n) = O(n)$

(e) $T(n) = T(\sqrt{n}) + 4$.
let m = log n $\implies$ n= 2^m
Replacing n with m
$T(2^m)$ = $T(2^m)$ + 1
let $T(2^m)$ = S(m)
$S(m) = S(m/2) + 1$
Applying Master,
$a = 1, b = 2, f(n) = 1$
$n^{\log_b a}$ = $n^{\log_2 1}$ = 1 = f(n)
Condition 2 holds = $S(m) = O(\log m) = O(\log \log n)$

(f) Suppose we have $T(n) = 3T(n/2) + g(n)$.
Analyze the behavior when (a) $g(n) = n^2$ (b) $g(n) = n$, and (c) $g(n) = n^{\log_2 3}$.
For 3 cases:

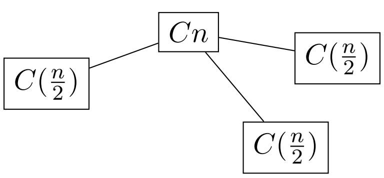
Size of subproblem at every level $= n/2^i$, where 'i' denotes level.
When $\frac{n}{2^i} = 1$, lets find leaf node
$\implies i = \log_2 n$
levels $= \log_2 n + 1$
nodes at depth i $= 3^i$

(a) when g(n) $=n^2 \implies T(n) = 3T(n/2) + n^2$



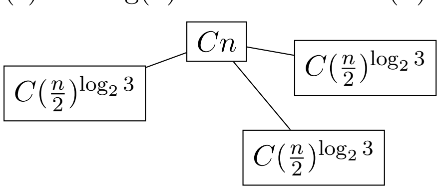
Cost of a subproblem at each level $= (\frac{n}{2^i})^2$
Cost = each level$= 3^i(\frac{n}{2^i})^2 = (\frac{3}{4})^i * n^2)$
$Cost$ = leaf node $= 3^i = 3^{\log n} = n^{log_2 3}$, where leaf = T(1) to the cost. So cost at level $\log n$ is $\theta(n^{log_2 3})$.
T(n) $= cn^2 + \frac{3}{4}cn^2 + \frac{9}{16}cn^2 + ... + (\frac{3}{4})^{log_2(n-1)}cn^2 + \theta(n^{log_2 3})$
Infinite GP case:
$T(n) < \frac{cn^2}{1-\frac{3}{4}} + \theta(n^{\log_2 3})$
$\implies \mathcal{O}(n^2)$.
Condition 3 of the Master Theorem:

(b) when g(n) $=n \implies T(n) = 3T(n/2) + n$



Cost of a subproblem $= (\frac{n}{2^i})$
Cost = each level$= 3^{i(\frac{n}{2^i})} = (\frac{3}{2})^{i*n})$
Cost = leaf node $= 3^i = 3^{logn} = n^{log_2 3}$, where leaf = T(1) to the cost. So cost at level log n is $\theta(n^{log_2 3})$.
$\implies T(n) = \mathcal{O}(n^{\log_2 3})$
Condition 1 of the Master Theorem:

(c) when g(n) $=n^{\log_2 3} \implies T(n) = 3T(n/2) + n^{\log_2 3}$



Cost = leaf node $=3^i = 3^{logn} = n^{log_2 3}$, where leaf = T(1) to the cost. So cost at level $logn$ is $\theta(n^{log_2 3})$.
T(n) $= cn^{log_2 3} + \frac{3}{2}cn^{log_2 3} + \frac{9}{4}cn^{log_2 3} + ... + (\frac{3}{2})^{log_2 n-1}cn^{log_2 3} + \theta(n^{log_2 3})$

T(n) = $(n^{log_2 3}) \sum_{j=0}^{logn-1} (\frac{3}{2})^{log_2 3}$
T(n) = $\mathcal{O}(n^{2(\log_2 3)} \log n)$
Condition 2 of the Master Theorem.

2: Sorting "nearby" numbers

Algorithm: The given case can be solved using the concept of Counting Sort. The first step is to scan through the array to determine the minimum and the maximum element. Let B be a new array of size M created that has each value as a constant 0. The array is scanned again and for each number in A[i], we increment $B[min_i\{A[i]\} + A[i]]$. A new array C[0..n-1] is created. In the final step we scan through the array B[] and for each number B[i], we can place B[i] values of i into the coming B[i] null slots of C[].
Thus we count the exact time value i occurs in the array and store the data in B[i]. We get the sorted array when we scan through B[i] and get the values in order.

Correctness: All the elements of A[] is stored in C[] in sorted order. So accuracy is 100%

Running time: During the scan process, we scan the elements of array with sizes 'n' and 'M' regularly, so it takes $O(n + M)$ times.

3: Selecting in an Union

Algorithm: Check the sum of middle index of array A and B.
$if\ sum < k$ and $middle\ element\ of\ k > middle$ element of B
discard the first half of B and new value of k =$K - (index$ of $mid - elem$ of $B) - 1$
$else\ if$
mid-elem of B is greater
discard the first half of A and new value of $k = K - (index$ of $mid - elem$ of $A) - 1$
repeat till length of A or B is zero.

Correctness: We get the desired kth smallest element using the abve steps, thus we have the 100% correctness.

Running time: In every repetition, we are performing computations on middle half of list, and based on if-else we remove half list in one of the array. Thus this dividing process takes $O(logn)$ time complexity.

4: Closest Pair

(a) Let the instances be 1,3,8 separated by an imaginary axis with instances 9,15,19. Now if we try to find the smallest distance, it comes as 4 between 15 and 19 and 2 between 1 and 3. So the closest distance is considered as 2. But actually points 8 and 9 are separated by just a point distance. So the consideration goes wrong in this case.

(b) Steps for proof:
We will consider a square.

We need to show that any 2 points are at a minimum distance d.
Let us consider a square area.
Let a point be placed on any corner.
The next point cannot be placed any where other than the other remaining vertices of the square.
Thus this way we get 4 different points placed on the vertices of the square area.
If we consider any 5th point, the distance of 5th point from other 4 points will be strictly smaller than d.
Thus proved.

(c) Steps involved and time computations:
finding median x = O(n)
recurse the two sub probs of size $n/2$ = 2T(n/2)
discard points = O(n)
$sort$ y values = $O(n \log n)$
iterate thorugh a list of y variables and computation for each = O(n)
Thus total =

$$T(n) = 2T(n/2) + O(n \log n).$$

For bound -

$$T(n) = 2T(n/2) + O(n \log n).$$

$$T(n) = \Sigma_{i=1}^{m} + \Sigma_{i=1}^{m} \log(n/2^{i-1}) + where(m) = logn$$

$$T(n) <= O(2^{\log n}) + O(n(logn + log(n/2) + log(n/4) + ... + 1)$$

$$T(n) <= O(n) + O(nlog(logn))$$

$$T(n) <= O(nlog^2 n)$$

5: Linear Time Median

(a) Using the recursive formula we know that $T(n) = O(n) + T(n/5) + T(7n/10)$
We assume that $T(n) = (a * n) + T(n/5) + T(7n/10)$
$C * n >= T(n/5) + T(7n/10) + a * n$
$C * n >= C * n/5 + C * 7n/10 + a * n$
$C > 9 * C/10 + a$
$C/10 >= a$
$C >= 10 * a$
so T(n)= O(n)
Here if we divide the group into 3 then:
$T(n) = O(n) + T(n/3) + T(2n/3) soT(n) > O(n)...$
if groups are divided into more than 5, Value of constant 5 is more,so that is the most optimal solution.
In which case running time is $T_{median}(n/5) + O(n)$

(b) Recurrence for finding the near median is $T(N) \leq T(N/S) + T(B/N) + O(N)$ as we see,
$T(N/S)$ = cost of finding median of $A_{medians}$ and N is the cost of partitioning algo.
$T(B/N)$ = cost of final recursive call
B = fraction of (BN) i.e. worst case length of either A_l A_r

M = median of $A_{medians}$
M must $= 1/2(N/5)$ except the median itself, not including 5 elements from last median cluster.
$G - 2 = 1/2(N/5)$
now at least 3 elements are smaller or equal to M
$3(G - 2) = 1/2(N/5)$ i.e $(3N/10) - 9 \geqslant N/4$
This proves the algo.