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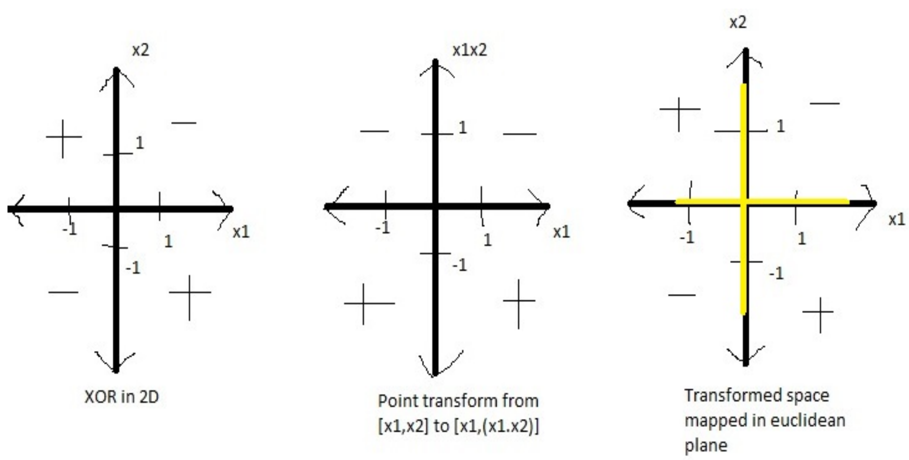
Referrences: Lecture Notes UC Berkeley

1: Warm up: Margins

(1) We need to map the input into a space consisting of two features: x_1 and x_1x_2.
We can say that the coordinate values of the examples will map from $[x_1, x_2]$ to $[x_1, x_1x_2]$ as follows:

From point	Type of point	will map to
$[-1, -1]$	*Negative*	[-1,+1]
$[-1, +1]$	*Positive*	[-1,-1]
$[+1, -1]$	*Positive*	[+1,-1]
$[+1, +1]$	*Positive*	[+1,+1]

Therefore, as we observe the positive and negative examples, we know that positive examples are having x_1x_2 = -1 and negative examples are having x_1x_2 = +1.
As asked in the question, the max margin separator, can be written as the line equation $x_1x_2 = 0$, having a margin of 1. The separator can be observed to correspond the x_1=0 and x_2=0 axes.
This statement can be represented as limiting the hyperbolic separator with 2 separating paths.
Please refer to diagram below:



(2A) For the first required explaination i.e. Dataset1=$\{x_1, x_2, x_3, x_5, x_7\}$.
In this case, the linear classifier line having the max margin will run parallely (never meeting) with the line connecting the points x_1 and x_3, and also the distance of this classifier line equation will be same from both the points x_1 and x_3 (on 1 part) and having x_5 on opposite part.
Therefore we can say that, for D_1, the max possible margin will be exactly 1/2 length of point x_5 from the line which is connecting x_1 and x_3.
The line connecting points x_1 and x_3 can be observed as Q - P = 0.
Here, max margin D_1 can be written as:
$D_1 = (1)/(2 * \sqrt{2})$
For the second required explaination i.e. Dataset2=$\{x_1, x_5, x_6, x_8\}$. In this case, the linear classifier line having the max margin will run parallely (never meeting) with the line connecting

the points x_5 and x_6, and also the distance of classifier line equation will be same from both the points x_5 and x_6 (on 1 part) and having x_1 opposite part. Therefore we can say that, for D_2, the max possible margin will be exactly 1/2 length of point x_1 from the line which is connecting x_5 and x_6.
The line connecting points x_5 and x_6 can be observed as $\sqrt{3}P + Q - \sqrt{3} = 0$.
Here, max margin D_2 can be written as:
$D_2 = (\sqrt{3})/(4)$
For the third required explaination i.e. Dataset3=$\{x_3, x_4, x_5, x_7\}$. In this case, the linear classifier line having the max margin will run parallely (never meeting) with the line connecting the points x_4 and x_3, and also the distance of classifier line equation will be same from both the points x_4 and x_3 (on 1 part) and having x_5 on the opposite part. Therefore we can say that, for D_3, the max possible margin will be exactly 1/2 length of point x_5 from the line which is connecting x_3 and x_4.
The line connecting points x_3 and x_4 can be observed as $2P - Q - 1 = 0$.
Here, max margin D_3 can be written as:
$D_3 = (1)/(2 * \sqrt{5})$

(2B) We know according to perceptron algorithm, mistake bound is given as $<= (R/\gamma)^2$.

<u>**For D Value**</u>	<u>**Most Distant Point**</u>	<u>**R**</u>	<u>γ</u>	<u>**Perceptron Mistake Bound**</u>
D_1	x_7	3/2	$(1)/(2 * \sqrt{2})$	18
D_2	x_8	$\sqrt{5}/2$	$\sqrt{3}/4$	6
D_3	x_7	3/2	$1/(2 * \sqrt{5})$	45

Therefore, D_3 has largest mistake bound.

(2C) In terms of ease of learning, the ranking of the given Datasets will be: D_2 D_1 D_3, where D_2 is the easiest to learn.
Explaination: As we know that Perceptron Algorithm makes mistakes during learning process, so if lower number of mistakes occur we can tell that the learning of the algorithm did not take place properly, and thus the algorithm's prediction for labels will not be good on the test set. To be more precise, any classifier will learn a bit lower, if the number of mistakes occured during training is lower. Therefore, we can tell that a particular classifier with high mistake bound can behave in a way that is more easy to learn and inversely, smaller mistake bound can behave such that it is harder to learn. This explains the ranking mentioned above.

2: Kernels

(1a) For the requirements above, lets assume some respective gram matrices for a pre-defined space x1,x2,...,xd:

$$R = \{r_{mn}\}, in - which\{r_{mn}\} equals K_1(x_m, x_n)$$

$$S = \{s_{mn}\}, in - which\{s_{mn}\} equals K_2(x_m, x_n)$$

The kernel K (product of 2 kernel) can be denoted as:

$$K = K_1 X K_2$$

Gram Matrix of K will look like:

$$T = \{t_{mn}\} = \{r_{mn}\}\{s_{mn}\}, in - which\{t_{mn}\}equalsK(x_m, x_n)$$

As we see that the K_1 and K_2 are symmetric positive definite, so implying this we can say that respective components must be symmetric, that can further expand to:
$R = \{r_{mn}\} = \{r_{nm}\}$, therefore
$K_1(x_m, x_n) = K_1(x_n, x_m)$
$S = \{s_{mn}\} = \{s_{nm}\}$, therefore
$K_2(x_m, x_n) = K_2(x_n, x_m)$
Thus observing the above two equation, we can say that the particular components in the gram matrix for the resultant product kernel will give values as:
$\{t_{nm}\} = \{r_{nm}\}\{s_{nm}\} = \{r_{mn}\}\{s_{mn}\} = \{t_{mn}\}$
Hence, the above product kernel is symmetric, as we can see above.
So, as the further step we just need to show that new product kernel is also symmetric positive definite.
Lets assume $h \in R^n$ thus we have to represent that $h^T Th >= Zero$.
$h^T Th = \sum_{mn} h_m h_n t_{mn} = \sum_{mn} h_m h_n r_{mn} s_{mn}$
We know by property that any non-singular matrix A can be converted to SPD matrix if it can be multiplied by it's own transpose.
Any matrix can be segmented into any matrix and it's transpose.
Thus, we will try to segment R and S below:
$R = A^T A =$
$\{r_{mn}\} = a_m^T a_m = \sum_k a_{mk} a_{nk}$
$S = B^T B =$
$\{s_{mn}\} = b_m^T b_m = \sum_l b_{ml} b_{nl}$
Using these segmented values in above equation, the computation can be extended as:
$h^T Th = \sum_{mn} h_m h_n t_{mn} = \sum_{mn} h_m h_n \sum_k a_{mk} a_{nk} \sum_l b_{ml} b_{nl} = \sum_{kl} \sum_{mn} u_m u_n a_{mk} a_{nk} b_{ml} b_{nl}$
$h^T Th = \sum_{kl} \sum_{mn} h_m h_n a_{mk} a_{nk} b_{ml} b_{nl}$
As we observe that m and n are independent of each other, so we can segment these as:
$h^T Th = \sum_{kl} (\sum_m h_m a_{mk} b_{ml})(\sum_n h_n a_{nk} b_{nl})$
As we observe above, n values are not depending on m, but the values are same, so we can replace any one value:
$h^T Th = \sum_{kl} (\sum_m h_m a_{mk} b_{ml})^2 >= 0$
Thus we can say that $Gram\ Matrix$ of T is a SPD.
Therefore, it is extablished, K is valid Kernel.

(1b) We will be using the above proved conclusions, to prove the required statement.
We need to prove that the addition of kernels and any positive coefficient will be a kernel.
According to the requirements, we can establish:
$K(x, z) = \alpha K_1(x, z) + \beta K_2(x, z)$
where it is given that $K_1(x, z)$, $K_2(x, z)$ = valid kernels.
Let us assume ϕ_1 as a function that maps data vector to feature space of K_1, so it can be computed as $K_1(x, z) = \phi_1^T(x)\phi_1(z)$, which can be further written as $< \phi_1(x), \phi_1(z) >$ denoting the dot product of two vectors
Let us assume ϕ_2 as a function that maps data vector to feature space of K_2, so it can be computed as $K_2(x, z) = \phi_2^T(x)\phi_2(z)$, which can be further written as $< \phi_2(x), \phi_2(z) >$ denoting the dot product of two vectors
So using the above equations we can expand as:

$K(x,z) = \alpha K_1(x,z) + \beta K_2(x,z) =< \sqrt[2]{\alpha}\phi_1(x), \sqrt[2]{\alpha}\phi_1(z) > + < \sqrt[2]{\beta}\phi_1(x), \sqrt[2]{\beta}\phi_1(z) >$
$K(x,z) =< [\sqrt[2]{\alpha}\phi_1(x), \sqrt[2]{\beta}\phi_2(x)], [\sqrt[2]{\alpha}\phi_1(z), \sqrt[2]{\beta}\phi_2(z)] >$
The above expression proves $K(x,z)$ can be represented as a inner product. Therefore, K is a valid kernel.
Now, we see that a polynomial is established by using some positive coefficients. As we saw above sum of two kernels is a kernel and product of a kernel with positive coefficient is a kernel, Therefore K is a valid kernel.

(2) From the above conclusions we have seen:
$K(x,z) = K_1(x,z)K_2(x,z)$
i.e product of two kernel is a kernel
Let's start by considering the 2 kernels to be $K_a(x,z) = 15(x^Tz)^2$ and $K_b = \exp(-||x-z||^2)$
This implies that if we show that K_a and K_b are indivisual kernels then it can be shown that K is also a valid kernel.
Firstly for K_a, We will show that K_a is a valid Kernel:
We will state the Mercer theorem here:
As we know K_a and K_b are kernels, according to mercer's theorem, K_1 and K_2 should show inner product representation.
Lets suppose that $A = p^Tq$ be the required *inner product* for p and q. We have also concluded above that multiplication of 2 kernel will give a kernel. Thus we can conclude that
$A^2 = (p^Tq)(p^Tq) = (p^Tq)^2$ is a valid kernel. Resultant kernel after multiplication with a positive coefficient is also a valid kernel. Here we know, 15 is positive number & $(p^Tq)^2 =$ valid kernel. therefore, we can say that $15(p^Tq)^2 =$ valid kernel.
Now, as a second case, we need to prove valid kernel condition for K_b
K_b can further be segmented into:
$K_b(x,z) = \exp(-||x-z||^2)$
$= \exp(-(x-z)^T(x-z))$
$= \exp(-(< x, x-z > - < z, x-z >))$

$K_b(x,z) = \exp(-(< x,x > - < x,z > - < z,x > + < z,z >))$
$= \exp(-(||x||^2 + ||z||^2 - 2 < x,z >))$

$K_b(x,z) = \exp(-(||x||^2 + ||z||^2))\exp(2 < x,z >)$
$= W\exp(2 < x,z >)$
Here we see that W (any constant) = $\exp(||x||^2 + ||z||^2)$
Then expanding the exponential we get:
$K_b(x,z) = W\sum_{n=0}^{\infty} \dfrac{< x,z >^n}{n!}$
Therefore, we can observe from the above equation that K_b is created by an ∞ addition over polynomial kernels. Also we can observe that these these infinite sum over polynomial kernels, are created using the multiplication of linear kernel x^Tz.
As we know that sum of valid kernels give valid kernel and product of kernels also give valid kernels, Thus K_b is a valid kernel.
As discussed in beginning of proof, K will be a valid kernel as K_a and K_b are valid kernels, as shown above.

(3) As required in the question, we need to prove that Gaussian kernel can be represented as, inner product of feature space with infinite diamension, which can be represented as:

$$K(x, z) = \exp(\frac{-||x - z||^2}{2\sigma^2})$$

As we have already seen in the above part 1 and part 2 that, the addition of two different kernels can be represented as a kernel:
$K(x, z) = K_1(x, z) + K_2(x, z)$
We can call the formations for the above equation to be $K(x, z) = \phi$, $K_1(x, z) = \phi_1$ and $K_2(x, z) = \phi_2$.
Thus we observe that ϕ is defined in such a way that it can form vector as defined below:
$\phi(x) = (\phi_1(x), \phi_2(x))$
Using this we may represent the same as we did for poof for addition of kernels:
$< \phi(x), \phi(z) >=< \phi_1(x), \phi_1(z) > + < \phi_2(x), \phi_2(z) >$
If we extend our discussion to the *Euclidean space*, We may conclude that the vector $= \phi(x)$ is made using segments of ϕ_2 and ϕ_1, which further implies:

$< \phi(x), \phi(z) >= \sum_{i=1}^{dim(K_1)} \phi_{1,i}(x)\phi_{1,i}(z) + \sum_{j=1}^{dim(K_2)} \phi_{1,j}(x)\phi_{1,j}(z)$

$\implies = \sum_{i=1}^{dim(K_2)+dim(K_2)} \phi_i(x)\phi_j(z)$

We can conclude using above statements that, K_1 and K_2, can be established as infinite sum over kernel polynomials. Therefore K can be mentioned as inner product of a feature space, having infinite diamension as we can observe.
In the above proof, K depicts the Radial Basis Function Kernel
K_1 and K_2 depicts the infinite sum Radial Basis Function.

3.1: Experiments - Support Vector Machines

(1) This has been run on Handwriting dataset.
C=1 and γ_0=0.01
Weight vector is random ranging from -1 and 1.
Bias is also random ranging from -1 and 1.
Below results are reported after shuffling the data.

TEST ON TRAIN SET

BIAS	**EPOCH**	**ACCURACY**
1	3	93.1
-1	5	92.7
1	8	92.9
1	3	94.0
1	5	93.8
0	8	93.7

TEST ON TEST SET

BIAS	**EPOCH**	**ACCURACY**
-1	3	91.23
1	5	91.06
-1	8	91.38
-1	3	90.64
0	5	90.38
0	8	90.05

As we see due to random initialization of weight vector and bias and due to shuffling of data, the results vary by a small margin everytime we run the program. The results are however similar for epoch 3 and 5 in every run.

(2) 5 - Fold Cross Validation.
This has been run on madelon dataset.
Weight vector is random ranging from -1 and 1.
Bias is also random ranging from -3 and 3.
C value used = 0.0312, 0.0625, 0.125, 0.5, 0.25, 1.0
γ_0 = 100, 0.1, 0.001, 1
Below results are reported after shuffling the data.

TEST ON TRAIN SET

C = Value	γ_0	**BIAS**	**EPOCH**	**AVG ACCURACY VALUE**
0.0312	1	3	3	49.75
0.0312	1	-1	3	50.08
0.0312	1	1	3	51.50
1	0.1	-1	3	48.09
1	0.1	-1	5	49.75
1	0.1	-1	5	49.98
0.5	0.01	-3	5	53.50
0.5	0.001	-2	8	48.00
0.025	0.001	2	8	51.50
0.025	0.001	2	8	48.09
0.025	100	1	8	49.25
0.0625	100	1	5	49.98

BEST VALUE after running the Cross Validation =
C = 0.01
γ_0= 0.001
Accuracy = 53.50

TEST ON TEST SET

C = Value	γ_0	BIAS	EPOCH	AVG ACCURACY VALUE
1	1	3	3	49.25
1	1	-1	3	50.80
1	1	1	3	51.50
0.025	0.1	-1	3	48.00
0.025	0.1	-1	8	49.75
0.025	0.1	-1	5	50.25
0.025	0.001	-3	8	53.00
0.025	0.001	-2	8	53.50
0.5	0.001	2	5	51.50
0.5	0.01	2	8	52.50
0.5	100	1	8	49.25
0.0625	100	1	5	50.50

BEST VALUE after running the Cross Validation =
C = 0.01
γ_0= 0.001
Accuracy =53.50

(3) This has been run on Handwriting dataset.
Weight vector is random ranging from -1 and 1.
Bias is also random ranging from -3 and 3.
Below results are reported after shuffling the data.

Type	PRECISION	RECALL	F1 - SCORE	EPOCH
TRAIN	88.94	98.35	93.41	3
TRAIN	89.40	98.54	93.75	5
TRAIN	90.08	97.81	93.78	8
TEST	88.94	98.35	93.41	3
TEST	89.40	98.54	93.75	5
TEST	90.08	97.81	93.78	8

This has been run on madelon dataset.
Weight vector is random ranging from -1 and 1.
Bias is also random ranging from -3 and 3.
Below results are reported after shuffling the data.

Type	PRECISION	RECALL	F1 - SCORE	EPOCH
TRAIN	59.18	66.40	62.58	3
TRAIN	59.29	67.00	62.91	5
TRAIN	59.92	65.20	62.45	8
TEST	64.06	54.66	58.99	3
TEST	70.77	34.66	47.81	5
TEST	53.30	75.33	62.43	8

3.2: Experiments - Ensemble of decision trees

(1) Data run on Handwriting dataset.
Accuracy of random forest on data

Type	Accuracy
TRAIN	93.80
TEST	89.50
TRAIN	93.0
TEST	91.00
TRAIN	92.00
TEST	91.21
TRAIN	89.23
TEST	88.22