MACRO III

Exercise 4

- The Solow Model with Endogenous Growth -

Major in Economics

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Exercise 4.1

This is an exercise related to the Solow model with productive externalities and endogenous growth, introduced in chapter 8 of the textbook. Therefore, productivity A_t is now not exogenously given any longer, but depends on the aggregate capital stock K_t because of learning-by-doing externalities:

$A = K_t^{\phi}$.

Therefore, we can summarize the model with the following equations:

$$Y_t = {K_t}^{\alpha}(A_t L_t)^{1-\alpha} \tag{1}$$

$$A_t = {K_t}^{\phi} \tag{2}$$

$$K_{t+1} = sY_t + (1-\delta)K_t \tag{3}$$

$$L_{t+1} = (1+n)L_t \tag{4}$$

As usual, we denote per efficiency units with a tilde, e.g. $\tilde{y}_t \equiv \frac{Y_t}{A_t L_t} = \tilde{k}_t^{\alpha}$.

For parts a) to g), assume that $\phi < 1$.

a) From the definitions and equations above, we know that

$$\begin{aligned}\frac{\tilde{k}t+1}{\tilde{k}_t} = \frac{\frac{Kt+1}{A_{t+1}L_{t+1}}}{\frac{K_t}{A_tL_t}} = \frac{K_{t+1}/K_t}{(A_{t+1}/A_t)(L_{t+1}/L_t)} \\ = \frac{K_{t+1}/K_t}{(K_{t+1}/K_t)^{\phi}(L_{t+1}/L_t)} = \frac{1}{1+n}\left(\frac{K_{t+1}}{k_t}\right)^{1-\phi}\end{aligned} \tag{5}$$

where the third equality uses (2) and the fourth equality uses (4). Using equation (3) and the definition of $\tilde{y}_t$, show that we can write the law of motion of capita (per efficiency unit) as:

$$\tilde{k}_{t+1} = \frac{1}{1+n}\tilde{k_t}^{\phi}\left(s\tilde{k_t}^{\alpha} + (1-\delta)\tilde{k}_t\right)^{1-\phi} \tag{6}$$

Solution:

$$\frac{\tilde{k}_{t+1}}{\tilde{k}_t} = \frac{1}{1+n}\cdot\left(\frac{k}{t+1k_t}\right)^{1-\phi} \tag{7}$$

$$\Rightarrow \tilde{k}_{t+1} = \frac{1}{1+n}\cdot\tilde{k}_t\left(\frac{k}{t+1k_t}\right)^{1-\phi} \tag{8}$$

$$= \frac{1}{1+n}\cdot\tilde{k}_t\left(s\frac{Y_t}{K_t} + (1-\delta)\right)^{1-\phi} \tag{9}$$

$$= \frac{1}{1+n}\cdot\tilde{k}_t\left(s\frac{\tilde{y}_t}{\tilde{k}_t} + (1-\delta)\right)^{1-\phi} \tag{10}$$

$$= \frac{1}{1+n}\cdot\tilde{k}_t\left(s\tilde{k}_t^{\alpha-1} + (1-\delta)\right)^{1-\phi} \tag{11}$$

Extracting $\tilde{k}_t^{-1}$ from the parenthesis, we get the **Law of Motion**:

$$\tilde{k}_{t+1} = \frac{1}{1+n}\cdot\tilde{k}_t^{\phi}\left(s\tilde{k}_t^{\alpha} + (1-\delta)\tilde{k}_t\right)^{1-\phi} \tag{12}$$

b) Find the steady-states $\tilde{k}_t^*$ and $\tilde{y}_t^*$ as functions of model parameters.

Solution:

$$\tilde{k}_{t+1} = \frac{1}{1+n} \cdot \tilde{k}_t \left(s\tilde{k}_t^{\alpha-1} + (1-\delta)\right)^{1-\phi} \tag{13}$$

Since in equilibrium $\tilde{k}_{t+1} = \tilde{k}_t$:

$$\tilde{k}_t = \frac{1}{1+n} \cdot \tilde{k}_t \left(s\tilde{k}_t^{\alpha-1} + (1-\delta)\right)^{1-\phi} \tag{14}$$

$$1 = \frac{1}{1+n} \left(s\tilde{k}_t^{\alpha-1} + (1-\delta)\right)^{1-\phi} \tag{15}$$

$$(1+n)^{\frac{1}{1-\phi}} = s\tilde{k}_t^{\alpha-1} + (1-\delta) \tag{16}$$

$$\tilde{k}_t^* = \left(\frac{s}{(1+n)^{1/(1-\phi)} - (1-\delta)}\right)^{\frac{1}{1-\alpha}} \tag{17}$$

Using the equation $\tilde{y}_t = \tilde{k}_t^{\alpha}$ gives us:

$$\tilde{y}_t^* = \left(\frac{s}{(1+n)^{1/(1-\phi)} - (1-\delta)}\right)^{\frac{\alpha}{1-\alpha}} \tag{18}$$

c) By definition, $y_t^* = A_t\tilde{y}^*$ on the balanced growth path (BGP), and therefore, as $\tilde{y}^*$ is constant, ${y_t}^*$ will grow at the same rate as A_t. Unlike in the models discussed so far, however, A_t is endogenous and therefore does not just grow at rate g. Using equations (2) and (5) along with the fact that $\tilde{k}_{t+1} = \tilde{k}_t$ on the BGP, show that the growth rate on the BGP is

$$g_t^{y^*} = \frac{A_{t+1}}{A_t} - 1 = (1+n)^{\frac{\phi}{1-\phi}} - 1 \tag{19}$$

Solution:

$$\frac{\tilde{k}_{t+1}}{\tilde{k}_t} = \frac{1}{1+n}\left(\frac{K_{t+1}}{K_t}\right)^{1-\phi} \tag{20}$$

Since, on the BGP $\tilde{k}_{t+1} = \tilde{k}_t$:

$$1 = \frac{1}{1+n}\left(\frac{K_{t+1}}{K_t}\right)^{1-\phi} \tag{21}$$

$$\left(\frac{K_{t+1}}{K_t}\right)^{\frac{1}{1-\phi}} = \frac{1}{1+n} \tag{22}$$

$$\frac{K_{t+1}}{K_t} = \left(\frac{1}{1+n}\right)^{1-\phi} \tag{23}$$

using $A_t = K_t^{\alpha}$

$$\left(\frac{A_{t+1}}{A_t}\right)^{-\phi} = (1+n)^{\frac{1}{1-\phi}} \tag{24}$$

$$\frac{A_{t+1}}{A_t} = (1+n)^{\frac{\phi}{1-\phi}} \tag{25}$$

since $g_t = \frac{A_{t+1}-A_t}{A_t}$

$$g_t^{y^*} = \frac{A_{t+1}}{A_t} - 1 = (1+n)^{\phi/(1-\phi)} - 1 \tag{26}$$

d) How does the growth rate depend on the model parameters n and ϕ? Explain!

The growth rate depends positively on n and ϕ and if, and only if, n and ϕ are > 0, then is ${g_t}^y$ also > 0. This is controversial: On the one hand, it is quite logical that augmenting parameter ϕ will yield a higher ${g_t}^y$, since we are rising the return of the "learning-by-doing" effect that capital has over technology ($A_t = {K_t}^\phi$). Since A_t is "labour-augmenting" and therefore the input of an extra unit of labour is higher with A_t, in the model it makes sense that n grows with ${g_t}^{y^*}$. On the other hand, this goes in the opposite direction of what the empirics demonstrates (growth of GDP per capita is negatively correlated with population growth). Unfortunately, the model only yields a positive growth, if the country's population grows ($n > 0$).

e) What is the impact of the increase in the savings rate on the steady- state growth rate $g_t^{y^*}$? What is the impact on the steady-state levels? Intuitively, is the impact on the steady-state levels larger or smaller in this model compared to the Solow model with exogenous growth?

Since $g_t^{y^*} = \frac{A_{t+1}}{A_t} - 1$ and we know that $A_t = {K_t}^{\phi}$, we can rewrite the growth rate formula as

$$ {g_t}^{y^*} = \left(\frac{K_{t+1}}{K_t}\right)^{\phi} - 1 $$

Since increasing the savings rate increases the quantity of future capital (K_{t+1}), we see that ${g_t}^{y^*}$ must also increase.

The other steady state levels must increase as well, since we have positive returns on capital.

The impact on the steady state level is larger than in the Solow Model with exogenous growth, because now capital has also a positive return on technology, which augments the productivity of labour, making the return on capital in this model bigger than the one in the exogenous growth model.

f) In order to be able to simulate this economy, we need to recover A_t. Using $\tilde{k}_t = K_t/(A_t L_t)$ and $A_t = K_t^{\phi}$, we get

$$\tilde{k}_t = \frac{K_t}{K_t^{\phi} L_t} = \frac{K_t^{1-\phi}}{L_t} \tag{27}$$

$$\leftrightarrow K_t = (\tilde{k}_t L_t)^{\frac{1}{1-\phi}} \tag{28}$$

$$\leftrightarrow A_t = (\tilde{k}_t L_t)^{\frac{\phi}{1-\phi}} \tag{29}$$

Simulate this economy, using the parameters $\alpha = 0.33$, $\phi = 0.6$, $\delta = 0.1$, $s = 0.25$, $n = 0.02$, assuming it is on its steady-state in period zero and using $L_0 = 1$. Assume that in period 10, the savings rate increases to $s = 0.35$. Produce time series for $\tilde{k}_t$, $\tilde{y}_t$, $\tilde{c}_t$, $s\tilde{y}_t$, L_t, K_t, A_t, $\ln(y_t)$, $\ln(k_t)$, $\ln(c_t)$ and $g_t^y = \ln(y_t) - \ln(y_{t-1})$. Simulate 200 periods.

See file "semi-endogenous.csv" attached in the solution folder.

g) Produce diagrams showing the evolution of $\ln(y_t)$, $\ln(c_t)$ *and* g_t^y.

Figure 1: $\ln y$

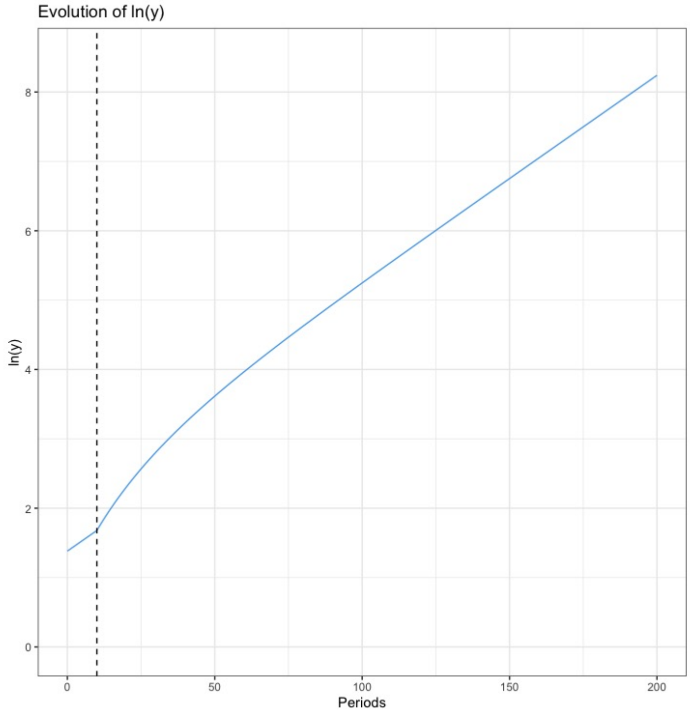


Figure 2: $\ln c$

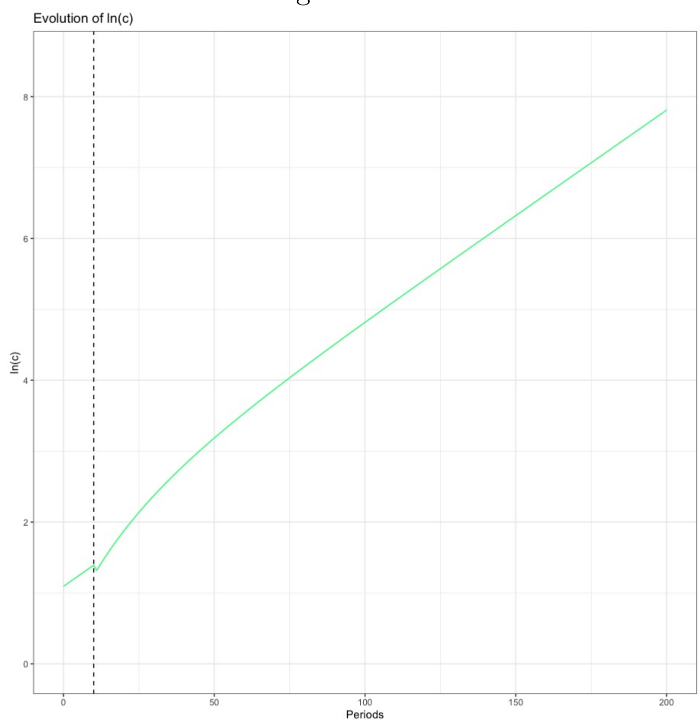


Figure 3: Growth Rate

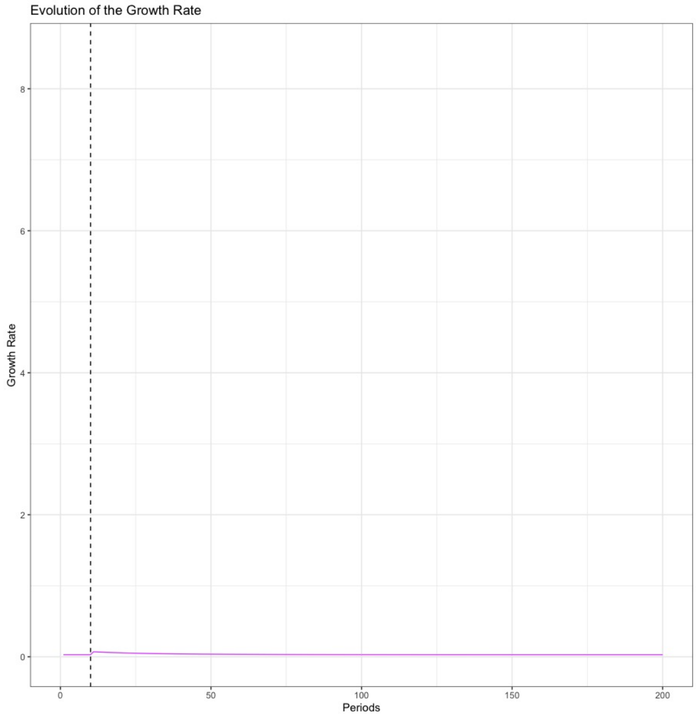
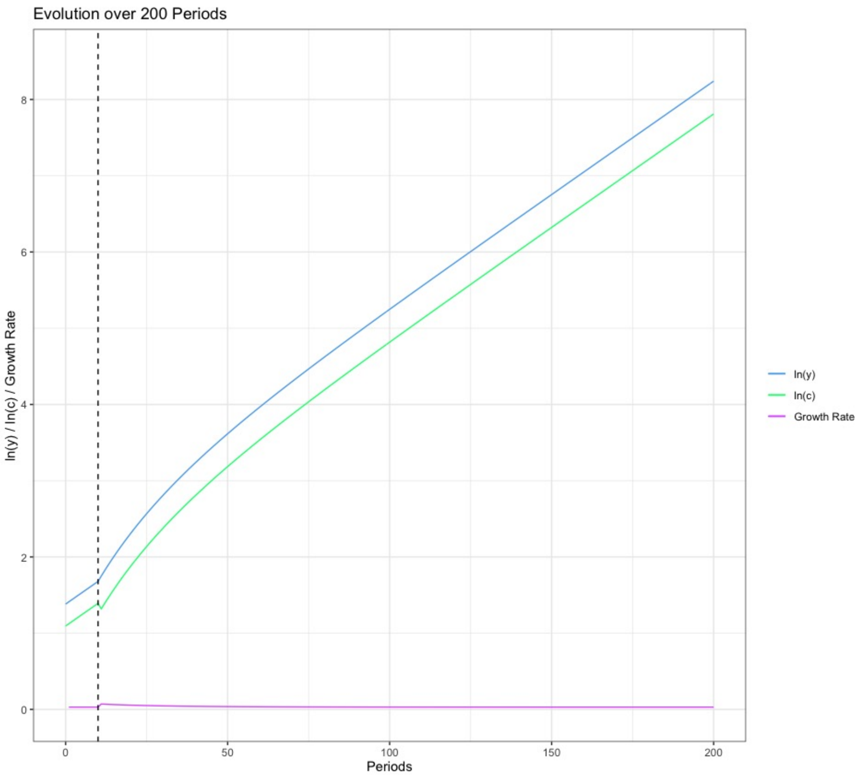


Figure 4: Combined



h) Now assume that $\phi = 1$ and $n = 0$. What is the effect of the change in the savings rate on the growth-rate here? Simulate this economy, initializing $k_0 = 1$, and produce the same diagrams as in g).

Figure 5: $\ln y$

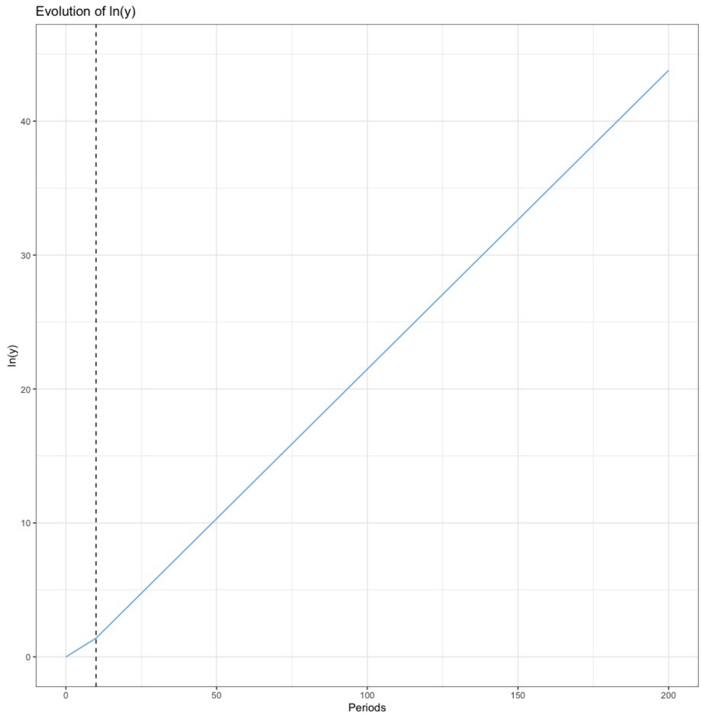


Figure 6: $\ln c$

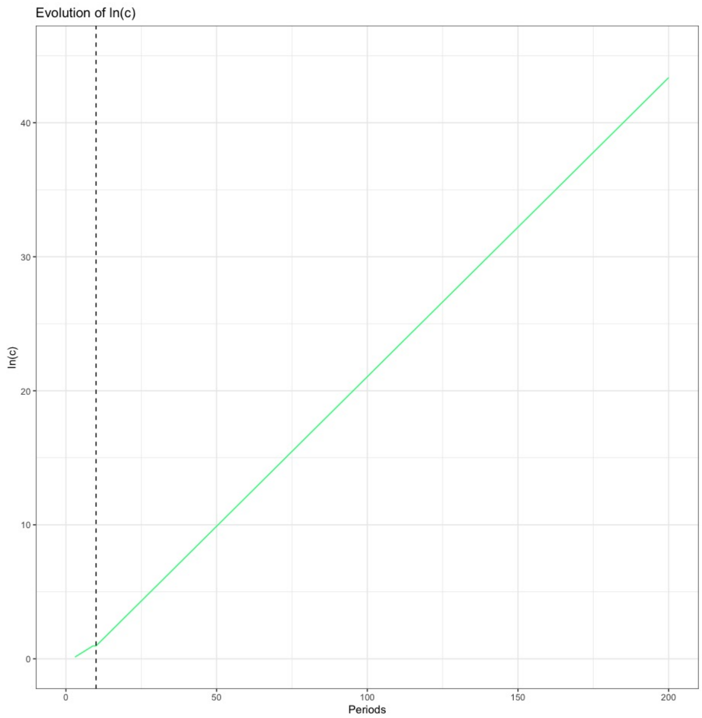


Figure 7: Growth Rate

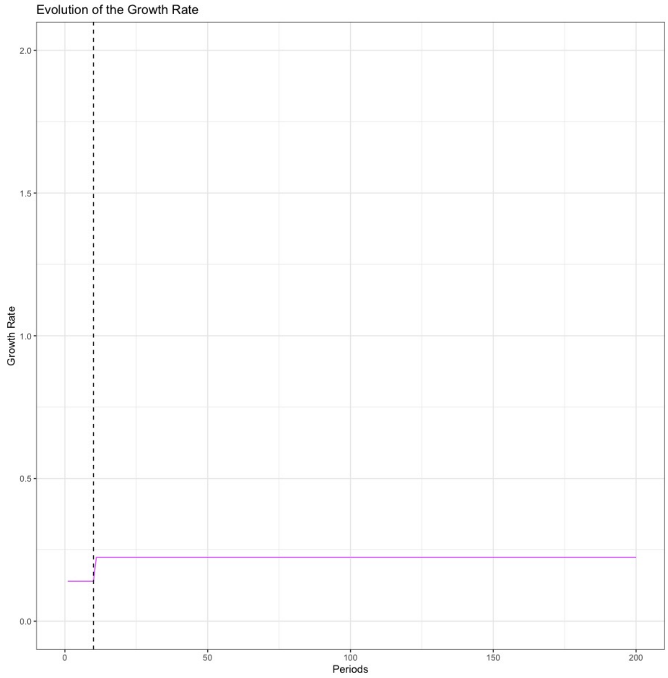
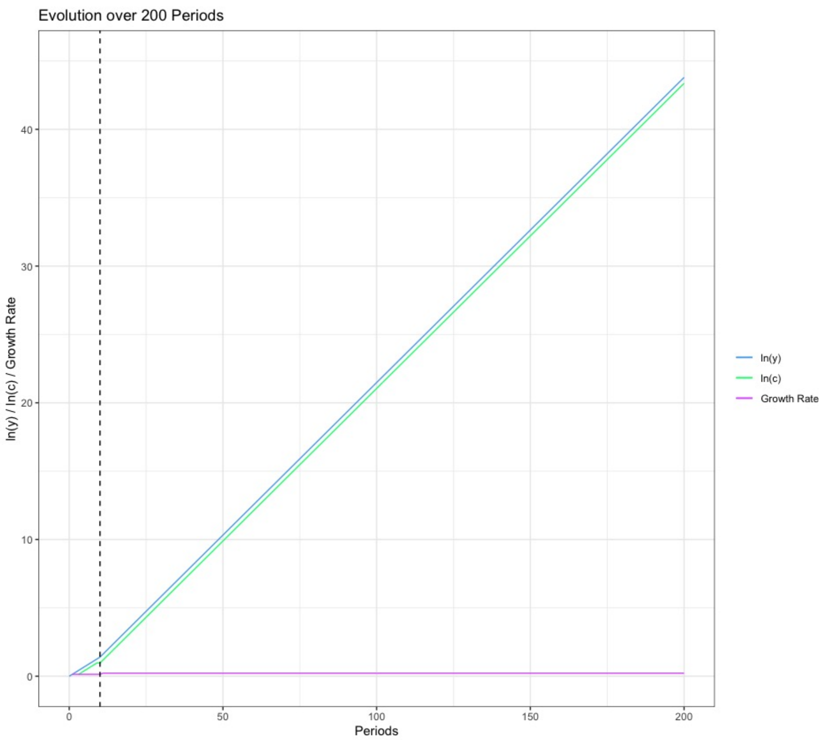


Figure 8: Combined



For the time series, see file "endogenous.csv" attached in the solution folder.