Finite Element Method for Solids and Structures

Pankaj “Pankaj”
The University of Edinburgh
School of Engineering

Text books

- Cook, RD; Malkus, DS; Plesha, ME; Witt, RJ. *Concepts and Applications of Finite Element Analysis*, Wiley, 2002.
- Zienkiewicz, OC; Taylor, RL. *The Finite Element Method for Solid and Structural Mechanics*, Butterworth-Heinemann, 2005.
- Bathe, KJ. *Finite Element Procedures*, Prentice Hall, 1996.
- Smith, IM; Griffiths, DV. *Programming the Finite Element Method*, Wiley, 2004.

What is FEM used for?

FEM is used to obtain approximate numerical solutions to a variety of equations of calculus in several fields of science and engineering

Solid mechanics
- elasticity problems
- material nonlinear problems
- geometrically nonlinear problems
- fracture
- stability
- dynamics
- buckling

Fluid mechanics
- inviscid
- viscous
- compressible
- incompressible

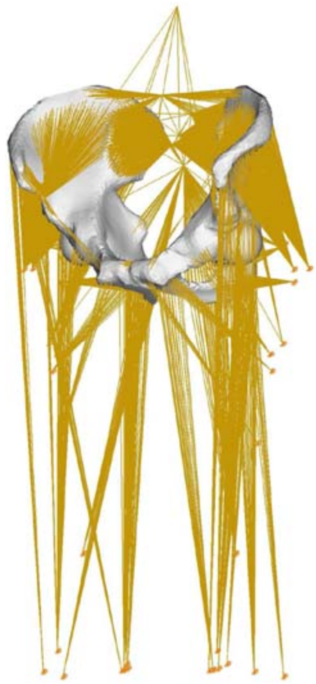
Heat transfer
- conduction
- convection
- radiation

Electromagnetism

Acoustics

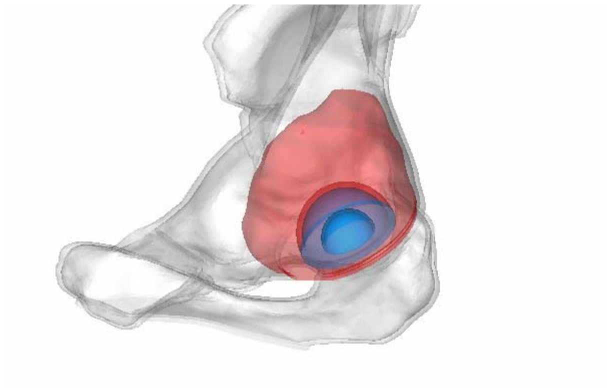
Coupled problems

Finite element model of the pelvis supported by muscles and ligaments



Phillips, Pankaj et al., Medical Eng. & Physics, 2007

Rotation of the acetabular cup after impaction grafting



5 walking cycles

- Phillips, Pankaj et al., Biomaterials, 2006
- Phillips, Pankaj et al., Comp Meth Biomech Biomed Eng, 2006

How does FEM work?

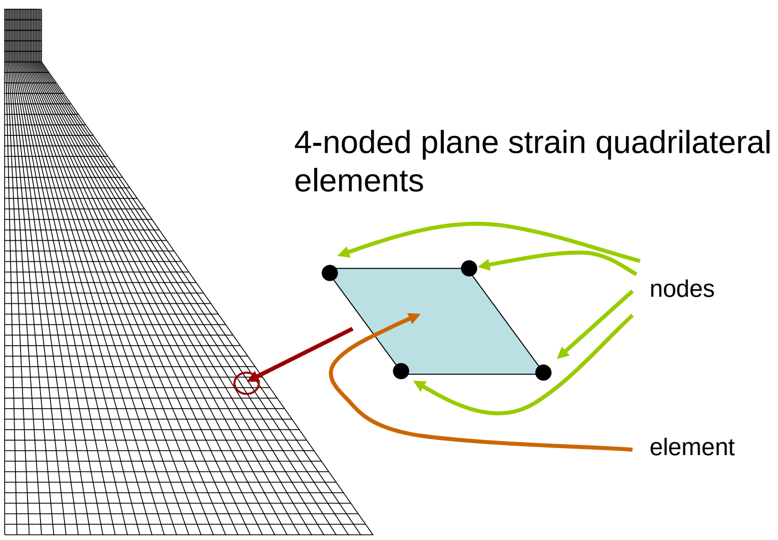
It uses a process of subdividing a complex system into small components or “elements” whose behaviour can be understood, and then rebuilding the original system from such components.

In structural analysis the method gives an approximate solution based on an *assumed displacement field,* stress field or a mixture. This course only discusses the method based on assumed displacement field, which most commonly used.

FE terminology and steps

1. Divide the continuum into sub-regions with simple geometry (e.g. triangles, rectangles, cuboids etc.) called *elements*
2. Select key points on the elements to serve as *nodes*, where conditions of equilibrium and compatibility are to be enforced; and where the displacement will be evaluated

FE *mesh* or *discretised continuum*– assemblage of nodes and elements

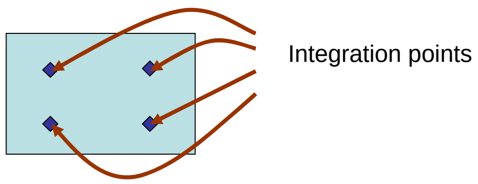


FE terminology and steps (cont.)

3. Assume interpolation functions (also called *shape functions*) within each element so that the displacements at each *generic point* are dependent on nodal values
4. Satisfy *strain-displacement* and *stress-strain relationships* within an element
5. Determine *stiffness* and *equivalent nodal loads* for each element
6. Develop *equilibrium equations* for the nodes of the discretised continuum in terms of element contributions

FE terminology and steps (cont.)

7. Solve the equilibrium equations for *nodal displacements*
8. Calculate *strains* and *stresses* at selected points within elements. These points are generally *integration points* (also called *Gauss points*)
9. Determine *support reactions* at *restrained nodes* if required



More FE vocabulary

Preprocessing: Includes mesh generation, assignment of material properties, boundary conditions

Postprocessing: Examination of results in numerical or graphical form after FE analysis

Boundary conditions: Typically assigning displacements to certain degrees of freedom at some nodes

Problem definition
- Geometry
- Loads
- Material properties
- Boundary conditions

↓

Preprocessing
- Mesh generation
- Assignment of material properties to elements
- Assigning boundary conditions to nodes
- Application of loads

↓

FE analysis

↓

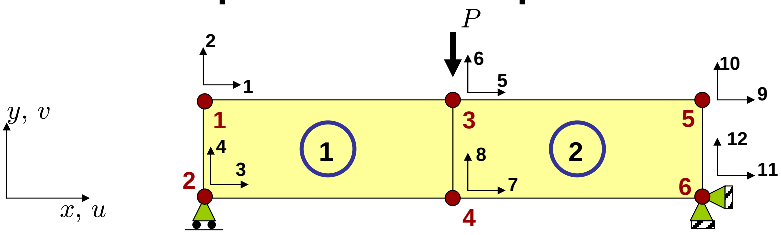
Postprocessing
- Examination of results for consistency and accuracy

More FE vocabulary

Degrees of freedom

- Nodal displacements necessary to specify the deformation of a finite element mesh are its degrees of freedom
- Degrees of freedom are associated with nodes
- Each node can have one or more degrees of freedom
- The number of equilibrium equations to be solved in a finite element analysis equals the number of degrees of freedom. Therefore, the size of the problem to be solved increases as degrees of freedom increase

Equilibrium equations



Consider a very crude mesh of a beam subjected to a concentrated load

How many elements are there in the mesh?

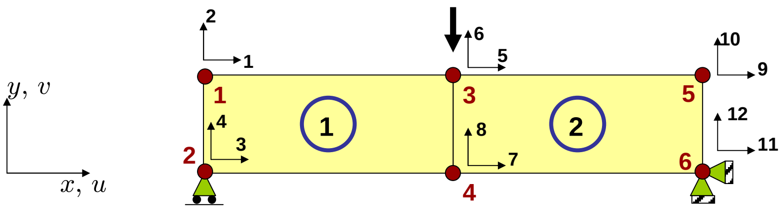
How many nodes are there in the mesh?

How many degrees of freedom does the discretisation have?

Which degrees of freedom are restrained against movement (or have zero displacement)?

What will the equilibrium equations look like and what are the sizes of different matrices?

Equilibrium equations



$$\mathbf{K}\mathbf{q} = \mathbf{p}$$

$$12 \times 12 \quad 12 \times 1 \qquad 12 \times 1$$

Global stiffness matrix

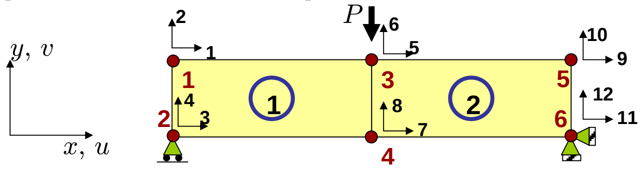
Global equivalent nodal loads matrix

$$\begin{bmatrix} k_{1,1} & \cdots & k_{1,12} \\ \vdots & & \vdots \\ k_{12,1} & \cdots & k_{12,12} \end{bmatrix} \begin{Bmatrix} q_1 \\ \vdots \\ q_{12} \end{Bmatrix} = \begin{Bmatrix} p_1 \\ \vdots \\ p_{12} \end{Bmatrix}$$

Nodal displacements – determined by solving equilibrium equations

Equilibrium equations

$$\mathbf{Kq} = \mathbf{p}$$

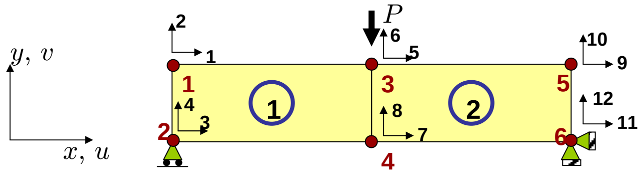


$$\mathbf{K} = \begin{bmatrix}
k^1_{1,1} & k^1_{1,2} & k^1_{1,3} & k^1_{1,4} & k^1_{1,5} & k^1_{1,6} & k^1_{1,7} & k^1_{1,8} & 0 & 0 & 0 & 0 \\
k^1_{2,1} & k^1_{2,2} & k^1_{2,3} & k^1_{2,4} & k^1_{2,5} & k^1_{2,6} & k^1_{2,7} & k^1_{2,8} & 0 & 0 & 0 & 0 \\
k^1_{3,1} & k^1_{3,2} & k^1_{3,3} & k^1_{3,4} & k^1_{3,5} & k^1_{3,6} & k^1_{3,7} & k^1_{3,8} & 0 & 0 & 0 & 0 \\
k^1_{4,1} & k^1_{4,2} & k^1_{4,3} & k^1_{4,4} & k^1_{4,5} & k^1_{4,6} & k^1_{4,7} & k^1_{4,8} & 0 & 0 & 0 & 0 \\
k^1_{5,1} & k^2_{5,2} & k^1_{5,3} & k^1_{5,4} & k^1_{5,5}+k^2_{5,5} & k^1_{5,6}+k^2_{5,6} & k^1_{5,7}+k^2_{5,7} & k^1_{5,8}+k^2_{5,8} & k^2_{5,9} & k^2_{5,10} & k^2_{5,11} & k^2_{5,12} \\
k^1_{6,1} & k^1_{6,2} & k^1_{6,3} & k^1_{6,4} & k^1_{6,5}+k^2_{6,5} & k^1_{6,6}+k^2_{6,6} & k^1_{6,7}+k^2_{6,7} & k^1_{6,8}+k^2_{6,8} & k^2_{6,9} & k^2_{6,10} & k^2_{6,11} & k^2_{6,12} \\
k^1_{7,1} & k^1_{7,2} & k^1_{7,3} & k^1_{7,4} & k^1_{7,5}+k^2_{7,5} & k^1_{7,6}+k^2_{7,6} & k^1_{7,7}+k^2_{7,7} & k^1_{7,8}+k^2_{7,8} & k^2_{7,9} & k^2_{7,10} & k^2_{7,11} & k^2_{7,12} \\
k^1_{8,1} & k^1_{8,2} & k^1_{8,3} & k^1_{8,4} & k^1_{8,5}+k^2_{8,5} & k^1_{8,6}+k^2_{8,6} & k^1_{8,7}+k^2_{8,7} & k^1_{8,8}+k^2_{8,8} & k^2_{8,9} & k^2_{8,10} & k^2_{8,11} & k^2_{8,12} \\
0 & 0 & 0 & 0 & k^2_{9,5} & k^2_{9,6} & k^2_{9,7} & k^2_{9,8} & k^2_{9,9} & k^2_{9,10} & k^2_{9,11} & k^2_{9,12} \\
0 & 0 & 0 & 0 & k^2_{10,5} & k^2_{10,6} & k^2_{10,7} & k^2_{10,8} & k^2_{10,9} & k^2_{10,10} & k^2_{10,11} & k^2_{10,12} \\
0 & 0 & 0 & 0 & k^2_{11,5} & k^2_{11,6} & k^2_{11,7} & k^2_{11,8} & k^2_{11,9} & k^2_{11,10} & k^2_{11,11} & k^2_{11,12} \\
0 & 0 & 0 & 0 & k^2_{12,5} & k^2_{12,6} & k^2_{12,7} & k^2_{12,8} & k^2_{12,9} & k^2_{12,10} & k^2_{12,11} & k^2_{12,12}
\end{bmatrix}$$

- ★ Superscripts indicate the element contributing
- ★ If we apply unit displacement along DOF j, holding all other displacements to zero; to maintain these displacements forces must be applied generally along all DOFs. k_{ij} is the force required along DOF i to maintain unit displacement at j

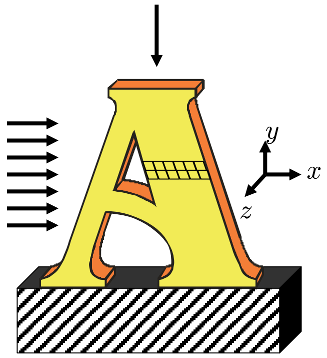
Equilibrium equations

$$\mathbf{Kq} = \mathbf{p}$$



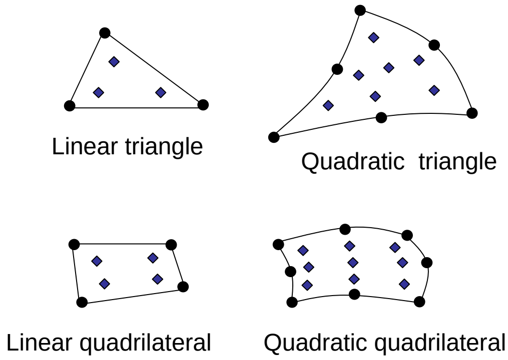
$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ 0 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{p} = \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \\ p_{10} \\ p_{11} \\ p_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ p_4 \\ 0 \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ p_{11} \\ p_{12} \end{Bmatrix}$$

FE modelling – plane stress



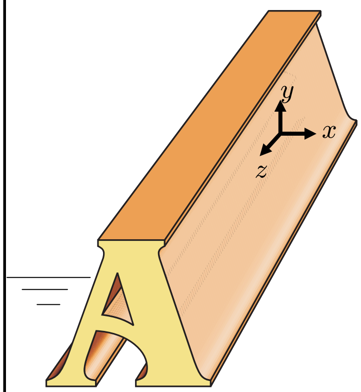
For thin plate-like structures with loading in plane of the plate

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$



All nodes have 2 degrees of freedom i.e. they can move in *x* and *y* directions

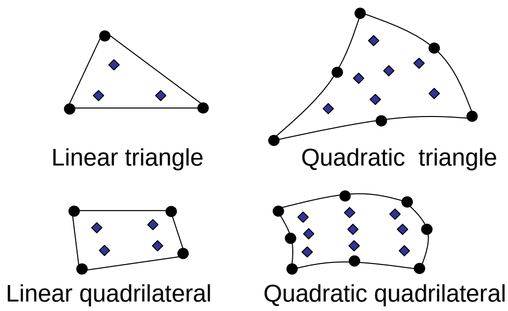
FE modelling – plane strain



For structures that are

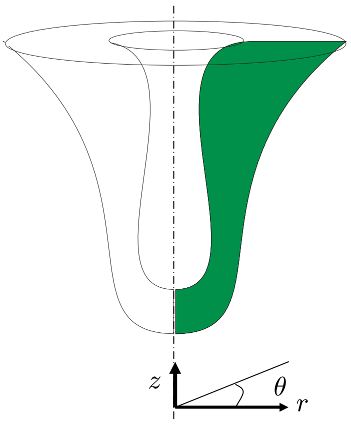
- very long in *z* direction
- have identical cross-section along *z*
- have identical loading along *z*

$$\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$



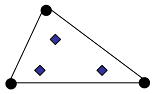
- a section can be analysed in 2D
- each node has 2 dof

FE modelling – axisymmetric

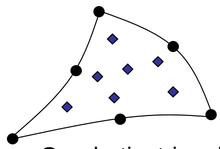


• solids created by rotating a planar section about an axis (say z)
• for simple analysis loading is also axisymmetric

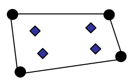
$$\epsilon_\theta = \gamma_{r\theta} = \gamma_{z\theta} = 0$$



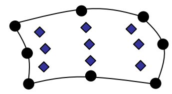
Linear triangle



Quadratic triangle



Linear quadrilateral

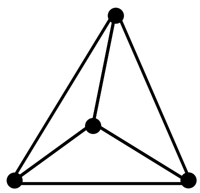


Quadratic quadrilateral

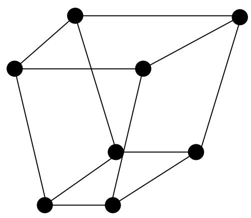
• a section can be analysed in 2D

• each node has 2 dof – they can move in r and z directions

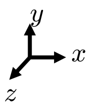
FE modelling – 3D solid

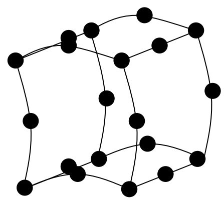


tetrahedron



Hexahedron or brick

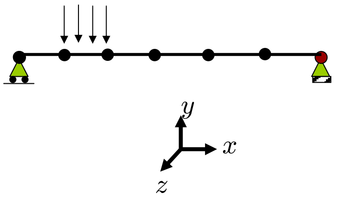


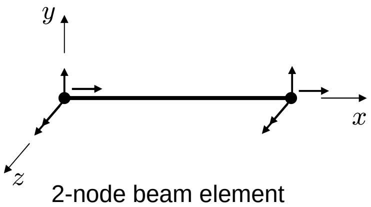


Quadratic hexahedron

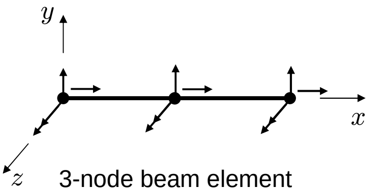
Each node has 3 dof – in x, y and z directions

FE modelling – beam





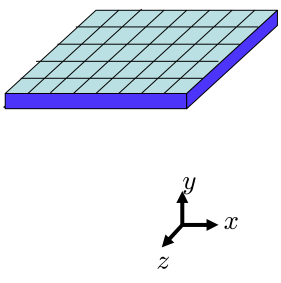
2-node beam element

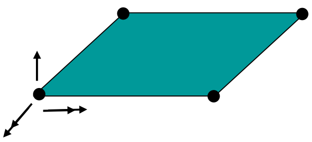


3-node beam element

For beams in a plane each node typically has either 3 dof – *x* & *y* translations and rotation about *z*

FE modelling – plates

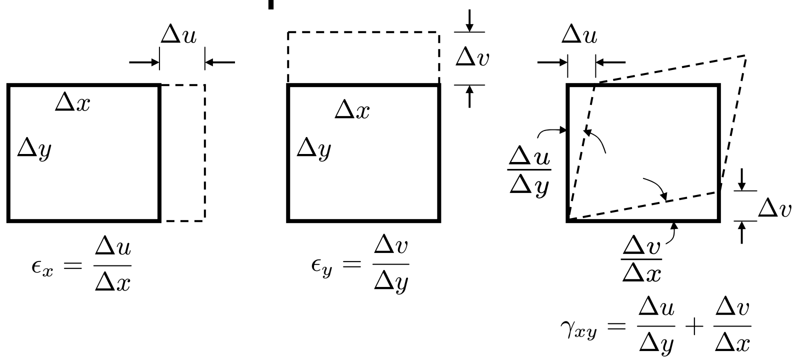




4-node plate element

Each node typically has either 3 dof – *y* translation and rotation about *x* and *z*

Strain-displacement relations



$$
\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} \qquad & \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\epsilon_y &= \frac{\partial v}{\partial y} & \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\
\epsilon_z &= \frac{\partial w}{\partial z} & \gamma_{zx} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}
\end{aligned}
$$

Strain-displacement relations

3D

$$
\underbrace{\left\{ \begin{array}{c} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{array} \right\}}_{6 \times 1} = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}}_{6 \times 3} \underbrace{\left\{ \begin{array}{c} u \\ v \\ w \end{array} \right\}}_{3 \times 1}
$$

2D

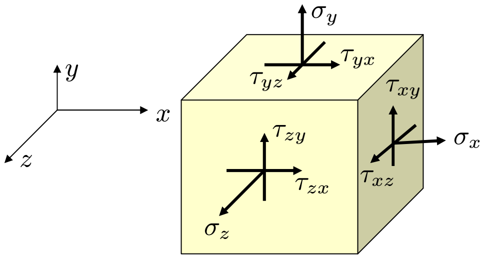
$$
\underbrace{\left\{ \begin{array}{c} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{array} \right\}}_{3 \times 1} = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}}_{3 \times 2} \underbrace{\left\{ \begin{array}{c} u \\ v \end{array} \right\}}_{2 \times 1}
$$

$$\boldsymbol{\epsilon} = \mathbf{L}\mathbf{u}$$

Where $\mathbf{L}$ is a linear differential operator

Stress-strain relations

Linear elasticity – generalised Hooke's law



$$
\begin{aligned}
\epsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \\
\epsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \\
\epsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \\
\gamma_{xy} &= \frac{\tau_{xy}}{G} \\
\gamma_{yz} &= \frac{\tau_{yz}}{G} \\
\gamma_{zx} &= \frac{\tau_{zx}}{G}
\end{aligned}
$$

$E =$ Young's modulus

$\nu =$ Poisson's ratio

$G = \dfrac{E}{2(1+\nu)} =$ shear modulus

Stress-strain relations

$$
\underset{6\times 1}{\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}} = \frac{1}{E} \underset{6\times 6}{\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}} \underset{6\times 1}{\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}}
$$

$$\boldsymbol{\epsilon} = \mathbf{C}\,\boldsymbol{\sigma}$$

We need stress expressed in terms of strain, therefore

$$\boldsymbol{\sigma} = \mathbf{C}^{-1}\,\boldsymbol{\epsilon} \qquad \mathbf{D} = \mathbf{C}^{-1}$$

$$\boldsymbol{\sigma} = \mathbf{D}\,\boldsymbol{\epsilon}$$

$\boxed{\mathbf{D} \text{ is called elasticity matrix}}$

Stress-strain relations

3D

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

Plane strain

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Plane stress

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

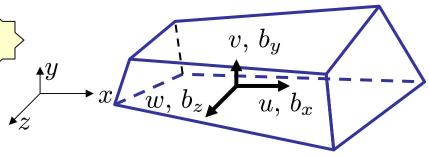
Virtual work basis of FEM

Some definitions first

Assume a 3D finite element exists in Cartesian coordinates x, y and z. Let *generic displacements* (displacements at any point within the element) be defined as

Translations in x, y and z directions

$$\underset{3\times 1}{\mathbf{u}} = \begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \quad (1)$$



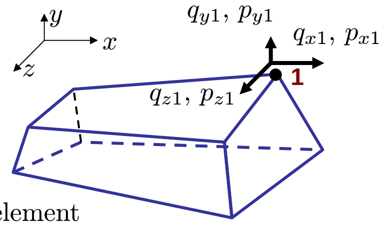
If the element is subjected to *body forces* (forces acting at a generic point):

$$\underset{3\times 1}{\mathbf{b}} = \begin{Bmatrix} b_x(x,y,z) \\ b_y(x,y,z) \\ b_z(x,y,z) \end{Bmatrix} = \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix} \quad (2)$$

Body forces can be forces per unit volume, forces per unit area, forces per unit length or concentrated forces

Virtual work basis of FEM – definitions (cont.)

Nodal displacements matrix: This matrix contains displacements of all nodes along all degrees of freedom



$$\mathbf{q} = \begin{Bmatrix} q_{x1} \\ q_{y1} \\ q_{z1} \\ \vdots \\ q_{xn} \\ q_{yn} \\ q_{zn} \end{Bmatrix}_{3n \times 1} \tag{3}$$

n = number of nodes in the element

Nodal loads matrix: This matrix contains loads acting directly on all nodes along all degrees of freedom

$$\mathbf{p} = \begin{Bmatrix} p_{x1} \\ p_{y1} \\ p_{z1} \\ \vdots \\ p_{xn} \\ p_{yn} \\ p_{zn} \end{Bmatrix}_{3n \times 1} \tag{4}$$

Virtual work basis of FEM – definitions (cont.)

For the kind of finite elements that we will be working with, assumed *displacement shape functions* relate generic displacements to nodal displacements as:

$$\underset{3 \times 1}{\mathbf{u}} = \underset{3 \times 3n}{\mathbf{N}}\, \underset{3n \times 1}{\mathbf{q}} \tag{5}$$

where $\mathbf{N}$ is a shape function matrix, typically of the form

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \cdots & N_n & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \cdots & 0 & N_n & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \cdots & 0 & 0 & N_n \end{bmatrix} \tag{5a}$$

Thus $\mathbf{N}$ is a rectangular matrix that makes $\mathbf{u}$ completely dependent on $\mathbf{q}$

Expanding the above equation you can see how u, v and w relate to nodal displacement

$$\begin{aligned} u &= N_1 q_{x1} + N_2 q_{x2} + \cdots + N_n q_{xn} \\ v &= N_1 q_{y1} + N_2 q_{y2} + \cdots + N_n q_{yn} \\ w &= N_1 q_{z1} + N_2 q_{z2} + \cdots + N_n q_{zn} \end{aligned}$$

Virtual work basis of FEM – definitions (cont.)

We have discussed *strain-displacement relations* that relate generic displacements to strains at a generic point i.e.

$$\underset{6\times 1}{\boldsymbol{\epsilon}} = \underset{6\times 3}{\mathbf{L}}\underset{3\times 1}{\mathbf{u}} \qquad (6)$$

or

$$\underset{6\times 1}{\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}} = \underset{6\times 3}{\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}} \underset{3\times 1}{\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}}$$

Virtual work basis of FEM – definitions (cont.)

Substituting (5) into (6)

$$\underset{6\times 1}{\boldsymbol{\epsilon}} = \underset{6\times 3}{\mathbf{L}}\underset{3\times 3n}{\mathbf{N}}\underset{3n\times 1}{\mathbf{q}}$$

or

$$\underset{6\times 1}{\boldsymbol{\epsilon}} = \underset{6\times 3n}{\mathbf{B}}\underset{3n\times 1}{\mathbf{q}} \qquad (7)$$

Thus we have strains related to nodal displacements using the $\mathbf{B}$ where

$$\mathbf{B} = \mathbf{LN} \qquad (8)$$

Virtual work basis of FEM – definitions (cont.)

Recall *stress-strain relations*

$$\underset{6\times 1}{\boldsymbol{\sigma}} = \underset{6\times 6}{\mathbf{D}}\,\underset{6\times 1}{\boldsymbol{\epsilon}} \qquad (9)$$

where $\mathbf{D}$ is the elasticity matrix

Substituting for strain matrix from (7) we obtain

$$\boldsymbol{\sigma} = \mathbf{DBq} \qquad (10)$$

Virtual work basis of FEM – derivation of equilibrium equations

Virtual Work Principal

If a general structure in equilibrium is subjected to a system of small virtual displacements within a compatible state of deformation, the virtual work of external forces is equal to the virtual strain energy of internal stresses

Applying this to a finite element

$$\delta U_e = \delta W_e \qquad (11)$$

δU_e: Virtual strain energy of internal stresses in the element

δW_e: Virtual work of external forces on the element

Virtual work basis of FEM – derivation of equilibrium equations

To develop from the virtual work principle assume a set of small virtual displacements at the nodes

$$\delta \mathbf{q} = \begin{Bmatrix} \delta q_{x1} \\ \delta q_{y1} \\ \delta q_{z1} \\ \vdots \\ \delta q_{xn} \\ \delta q_{yn} \\ \delta q_{zn} \end{Bmatrix} \qquad \text{a}$$

The virtual generic displacements from 5

$$\delta \mathbf{u} = \mathbf{N} \delta \mathbf{q} \qquad \text{b}$$

Using strain-displacement relations from 7

$$\delta \boldsymbol{\epsilon} = \mathbf{B} \delta \mathbf{q} \qquad \text{c}$$

Virtual work basis of FEM – derivation of equilibrium equations

Now virtual strain energy can be written as

$$\delta U_e = \int_V \delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma} dV \qquad \text{d}$$

The virtual work of external nodal and body forces is

$$\delta W_e = \delta \mathbf{q}^T \mathbf{p} + \int_V \delta \mathbf{u}^T \mathbf{b} dV \qquad \text{e}$$

Equating d and e

$$\int_V \boxed{\delta \boldsymbol{\epsilon}^T} \boldsymbol{\sigma} dV = \delta \mathbf{q}^T \mathbf{p} + \int_V \boxed{\delta \mathbf{u}^T} \mathbf{b} dV$$

c b

$$\int_V \boxed{\delta \mathbf{q}^T \mathbf{B}^T} \boldsymbol{\sigma} dV = \delta \mathbf{q}^T \mathbf{p} + \int_V \boxed{\delta \mathbf{q}^T \mathbf{N}^T} \mathbf{b} dV$$

Virtual work basis of FEM – derivation of equilibrium equations

$$\int_V \delta\mathbf{q}^T\mathbf{B}^T\boxed{\boldsymbol{\sigma}}\,dV = \delta\mathbf{q}^T\mathbf{p} + \int_V \delta\mathbf{q}^T\mathbf{N}^T\mathbf{b}dV$$

9

$$\int_V \delta\mathbf{q}^T\mathbf{B}^T\boxed{\mathbf{D}\,\boldsymbol{\epsilon}}\,dV = \delta\mathbf{q}^T\mathbf{p} + \int_V \delta\mathbf{q}^T\mathbf{N}^T\mathbf{b}dV$$

7

$$\int_V \delta\mathbf{q}^T\mathbf{B}^T\boxed{\mathbf{D}\ \mathbf{B}\mathbf{q}}\,dV = \delta\mathbf{q}^T\mathbf{p} + \int_V \delta\mathbf{q}^T\mathbf{N}^T\mathbf{b}dV$$

Since $\delta\mathbf{q}$ represents nodal displacements that are specific values and not integrable functions we can bring $\delta\mathbf{q}^T$ out of the integral and cancel them

$$\cancel{\delta\mathbf{q}^T}\int_V \mathbf{B}^T\mathbf{D}\ \mathbf{B}\mathbf{q}\,dV = \cancel{\delta\mathbf{q}^T}\mathbf{p} + \cancel{\delta\mathbf{q}^T}\int_V \mathbf{N}^T\mathbf{b}dV$$

Virtual work basis of FEM – derivation of equilibrium equations

$$\underbrace{\left(\int_V \overset{3n\times 6}{\mathbf{B}^T}\,\overset{6\times 6}{\mathbf{D}}\,\overset{6\times 3n}{\mathbf{B}}\,dV\right)}_{3n\times 3n}\underset{3n\times 1}{\mathbf{q}} = \underset{3n\times 1}{\mathbf{p}} + \underbrace{\int_V \overset{3n\times 3}{\mathbf{N}^T}\,\overset{3\times 1}{\mathbf{b}}dV}_{3n\times 1}$$

or

$$\mathbf{K}\mathbf{q} = \mathbf{p} + \mathbf{p}_b \qquad (12)$$

where $\mathbf{K}$ is the *element stiffness matrix* given by

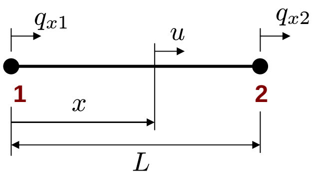
$$\mathbf{K} = \int_V \mathbf{B}^T\mathbf{D}\ \mathbf{B}\,dV \qquad (13)$$

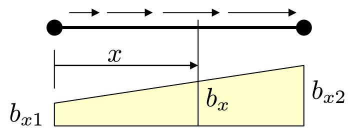
and $\mathbf{p}_b$ is the matrix of *equivalent nodal forces due to body loads*

$$\mathbf{p}_b = \int_V \mathbf{N}^T\mathbf{b}dV \qquad (14)$$

Axial finite element

Consider a 2-noded axial element (or a truss element) with only 1-dof per node subjected to linearly varying axial load (per unit length) as shown. We will develop equations that lead to the evaluation of the element stiffness matrix and equivalent nodal loads.





Generic displacement (see 1) $\mathbf{u} = u$

Body forces (see 2) $\mathbf{b} = b_x$

Nodal displacements (see 3) $\mathbf{q} = \begin{Bmatrix} q_{x1} \\ q_{x2} \end{Bmatrix}$

Nodal actions (see 4) $\mathbf{p} = \begin{Bmatrix} p_{x1} \\ p_{x2} \end{Bmatrix}$ (These will be zero if none applied)

Axial finite element (cont.)

For deriving the displacement shape functions let us assume generic displacement u varies linearly with x i.e.

$$u = c_1 + c_2 x$$

Constants c_1 and c_2 can be evaluated in terms of nodal displacements

$$\begin{aligned} \text{at} \quad x = 0; \qquad & u = q_{x1} = c_1 \\ \text{at} \quad x = L; \qquad & u = q_{x2} = c_1 + c_2 L \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} q_{x1} \\ q_{x2} \end{Bmatrix}$$

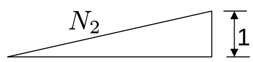
Therefore $c_1 = q_{x1}$ and $c_2 = (q_{x2} - q_{x1})/L$ or

$$\begin{aligned} u &= q_{x1} + \frac{q_{x2} - q_{x1}}{L} x \\ &= q_{x1} - q_{x1}\frac{x}{L} + q_{x2}\frac{x}{L} \\ &= \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{Bmatrix} q_{x1} \\ q_{x2} \end{Bmatrix} \\ &= \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} q_{x1} \\ q_{x2} \end{Bmatrix} \end{aligned}$$ 5

$$\mathbf{u} = \mathbf{N}\mathbf{q}$$

$N_1 = 1 - \dfrac{x}{L}$

$N_2 = \dfrac{x}{L}$



Shape functions have a value of unity at the nodes they represent and zero at other nodes.

Axial finite element (cont.)

Strain-displacement relationship

$$\boldsymbol{\epsilon} = \epsilon_x = \mathbf{L}\mathbf{u} = \frac{\partial u}{\partial x} = \left[\frac{\partial}{\partial x}\right](u) = \frac{\partial \mathbf{N}}{\partial x}\mathbf{q} = \mathbf{B}\mathbf{q}$$

6 & 7

$$\mathbf{B} = \frac{\partial \mathbf{N}}{\partial x} = \left[\begin{array}{cc} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} \end{array}\right] = \frac{1}{L}\left[\begin{array}{cc} -1 & 1 \end{array}\right]$$

8

Stress-strain relationship

$$\boldsymbol{\sigma} = \sigma_x = \mathbf{D}\,\boldsymbol{\epsilon} = E\epsilon_x = E\mathbf{B}\mathbf{q}$$

9

Axial finite element (cont.)

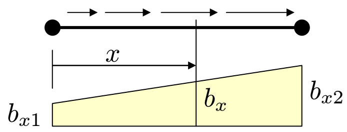
Element stiffness matrix

$$\begin{aligned}
\mathbf{K} &= \int_V \mathbf{B}^T \mathbf{D}\mathbf{B}\,dV \\
&= \int_L \frac{1}{L}\left\{\begin{array}{c} -1 \\ 1 \end{array}\right\} E\frac{1}{L}\left\{\begin{array}{cc} -1 & 1 \end{array}\right\} A\,dx \\
&= \frac{E}{L^2}\left\{\begin{array}{c} -1 \\ 1 \end{array}\right\}\left\{\begin{array}{cc} -1 & 1 \end{array}\right\}\int_L A\,dx \\
&= \frac{EA}{L}\left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right]
\end{aligned}$$

13

AL
if we assume
A is constant along
the length.
If it varies with x then
appropriate function needs to be
included.

Axial finite element (cont.)



Equivalent nodal loads matrix

$$b_x = b_{x1} + \frac{b_{x2} - b_{x1}}{L}x$$

14

$$\begin{aligned}
\mathbf{p}_b &= \int_V \mathbf{N}^T \mathbf{b} dV \\
&= \int_L \left\{ \begin{array}{c} 1 - \frac{x}{L} \\ \frac{x}{L} \end{array} \right\} \left[b_{x1} + \frac{b_{x2} - b_{x1}}{L}x \right] dx \\
&= \frac{L}{6} \left\{ \begin{array}{c} 2b_{x1} + b_{x2} \\ b_{x1} + 2b_{x2} \end{array} \right\}
\end{aligned}$$

- ★ Note b_x was body force per unit length
- ★ If in the above $b_{x1} = b_{x2} = b_x$ then

$$\mathbf{p}_b = \frac{L}{2} \left\{ \begin{array}{c} b_x \\ b_x \end{array} \right\}$$

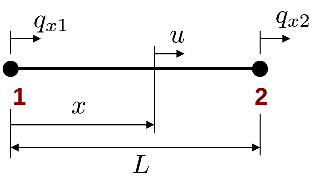
On shape functions

To obtain the shape functions of the 2-noded axial element we assumed a linear variation of displacement i.e.

$u = c_1 + c_2 x$

and then solved for c_1 and c_2 in the equations:

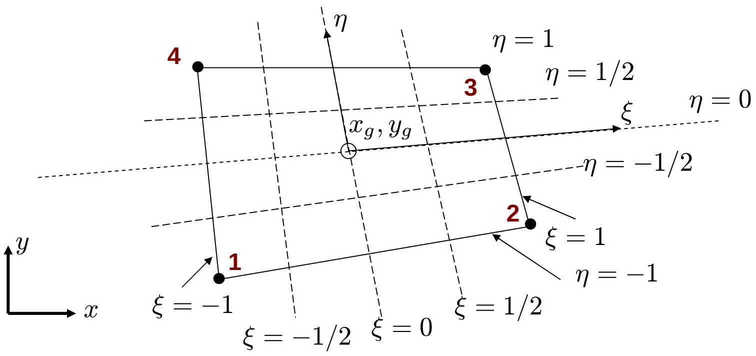
$$\left[\begin{array}{cc} 1 & 0 \\ 1 & L \end{array} \right] \left\{ \begin{array}{c} c_1 \\ c_2 \end{array} \right\} = \left\{ \begin{array}{c} q_{x1} \\ q_{x2} \end{array} \right\}$$



This was not difficult as there were only two equations in two unknowns and we could find c_1 and c_2 in general terms.
This process becomes complicated for more complex elements:

- ★ The number of coefficients (c_1, c_2 etc) to be determined depends on the dofs.
- ★ It is not always easy to find the coefficiants in general terms.
- ★ We should have a technique using which we can find $\mathbf{N}$ directly. We can use the fact that shape functions have a value of unity for the nodes they represent and zero at other nodes.
- ★ We should use a coordinate system that works identically for all elements of a similar shape.

Q4 element – natural coordinates & shape functions



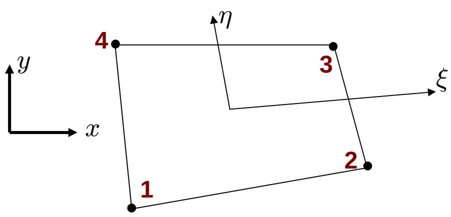
$$
\begin{aligned}
N_1 &= \frac{1}{4}(1-\xi)(1-\eta) \\
N_2 &= \frac{1}{4}(1+\xi)(1-\eta) \\
N_3 &= \frac{1}{4}(1+\xi)(1+\eta) \\
N_4 &= \frac{1}{4}(1-\xi)(1+\eta)
\end{aligned}
$$

Q4-N

$$
\begin{aligned}
x_g &= \frac{1}{4}(x_1+x_2+x_3+x_4) \\
y_g &= \frac{1}{4}(y_1+y_2+y_3+y_4)
\end{aligned}
$$

Q4 element – natural coordinates & shape functions

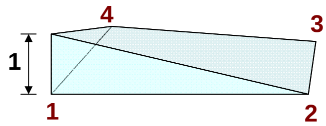
- ★ We can use these functions for all shapes of straight-edged quadrilaterals
- ★ All shape functions have a value of unity at the node they represent and zero at all other nodes.



Consider N_1

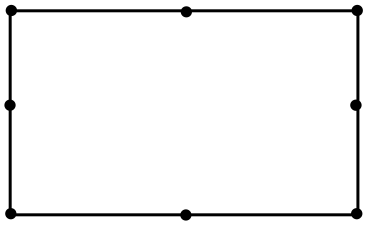
$$
\begin{aligned}
N_1\,(\xi=-1,\eta=-1) &= \frac{1}{4}(1-(-1))(1-(-1)) = 1 \\
N_1\,(\xi=1,\eta=-1) &= 0 \\
N_1\,(\xi=1,\eta=1) &= 0 \\
N_1\,(\xi=-1,\eta=1) &= 0
\end{aligned}
$$

Variation of N_1



Check that $N_1 = 0.5$ at the centre of line 1-2
And zero on lines 2-3 and 3-4.

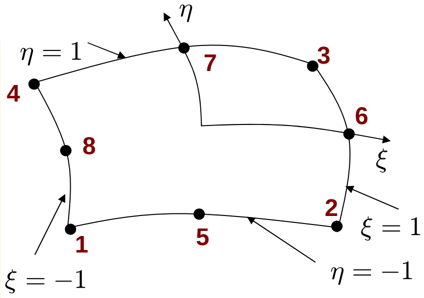
Serendipity element



Bruce Irons

Q8 element – natural coordinates & shape functions

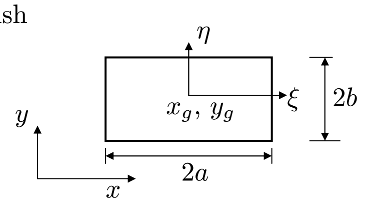
$$
\begin{aligned}
N_1 &= \frac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta-1)\\
N_2 &= \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1)\\
N_3 &= \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1)\\
N_4 &= \frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta-1)\\
N_5 &= \frac{1}{2}(1-\xi^2)(1-\eta)\\
N_6 &= \frac{1}{2}(1+\xi)(1-\eta^2)\\
N_7 &= \frac{1}{2}(1-\xi^2)(1+\eta)\\
N_8 &= \frac{1}{2}(1-\xi)(1-\eta^2)
\end{aligned}
$$



Q8-N

Concept of isoparametric elements

- We need to establish a one-to-one correspondence between (x, y) and (ξ, η)
- In terms of ξ and η all quadrilaterals (even those with curved sides) are rectangles as both coordinates (ξ and η) vary from -1 to $+1$
- For a real rectangle in the x-y system, this correspondence is easy to establish



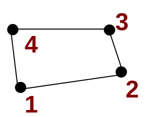
$$\xi = (x - x_g)/a$$
$$\eta = (y - y_g)/b$$

- We need to establish similar one-to-one correspondence for distorted shapes as well

Concept of isoparametric elements

A finite element is said to be isoparametric if the same interpolation functions define both geometry and displacement

Recall $\quad \mathbf{u} = \mathbf{N}\mathbf{q}$ 5



$$u = \sum_{i=1}^{4} N_i q_{xi} = N_1 q_{x1} + N_2 q_{x2} + N_3 q_{x3} + N_4 q_{x4}$$
$$v = \sum_{i=1}^{4} N_i q_{yi} = N_1 q_{y1} + N_2 q_{y2} + N_3 q_{y3} + N_4 q_{y4}$$
5

Now consider defining the geometry using the same shape functions i.e.

$$x = \sum_{i=1}^{4} N_i x_i = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$
$$y = \sum_{i=1}^{4} N_i y_i = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$
iso

Concept of isoparametric elements

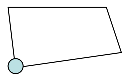
$$x = \frac{1}{4}[(1-\xi)(1-\eta)x_1 + (1+\xi)(1-\eta)x_2 + (1+\xi)(1+\eta)x_3 + (1-\xi)(1+\eta)x_4]$$

$$y = \frac{1}{4}[(1-\xi)(1-\eta)y_1 + (1+\xi)(1-\eta)y_2 + (1+\xi)(1+\eta)y_3 + (1-\xi)(1+\eta)y_4]$$

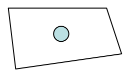
iso

Examples Q4-1

1. x, y coordinates of node 1
   Substituting $\xi = -1$ & $\eta = -1$
   $x = x_1$
   $y = y_1$
   ok – as expected



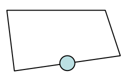
2. x, y coordinates of the geometric centre
   Substituting $\xi = 0$ & $\eta = 0$
   $x = (x_1 + x_2 + x_3 + x_4)/4$
   $y = (y_1 + y_2 + y_3 + y_4)/4$
   ok – as expected



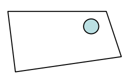
Concept of isoparametric elements

Examples Q4-1 (cont.)

3. x, y at $\xi = 0$ & $\eta = -1$
   $x = (x_1 + x_2)/2$
   $y = (y_1 + y_2)/2$
   ok – as expected



4. x, y at $\xi = 1/2$ & $\eta = 1/2$
   $x = (x_1 + 3x_2 + 9x_3 + 3x_4)/16$
   $y = (y_1 + 3y_2 + 9y_3 + 3y_4)/16$



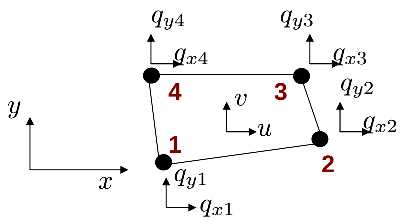
FE matrices for Q4 element

Generic displacements

$$\mathbf{u} = \left\{ \begin{array}{c} u \\ v \end{array} \right\} \tag{1}$$

Body forces

$$\mathbf{b} = \left\{ \begin{array}{c} b_x \\ b_y \end{array} \right\} \tag{2}$$



Nodal displacements matrix

$$\mathbf{q} = \left\{ \begin{array}{c} q_{x1} \\ q_{y1} \\ \vdots \\ q_{x4} \\ q_{y4} \end{array} \right\} \tag{3}$$

Nodal loads matrix

$$\mathbf{p} = \left\{ \begin{array}{c} p_{x1} \\ p_{y1} \\ \vdots \\ p_{x4} \\ p_{y4} \end{array} \right\} \tag{4}$$

FE matrices for Q4 element

Relating generic displacements to nodal displacements using shape functions

$$\mathbf{u} = \mathbf{N}\mathbf{q} \tag{5}$$

or

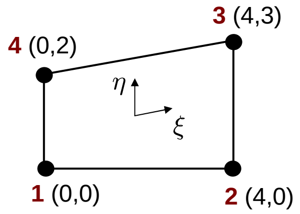
$$\begin{aligned} u &= N_1 q_{x1} + N_2 q_{x2} + N_3 q_{x3} + N_4 q_{x4} \\ v &= N_1 q_{y1} + N_2 q_{y2} + N_3 q_{y3} + N_4 q_{y4} \end{aligned} \tag{5}$$

where

$$\mathbf{N} = \left[\begin{array}{cccccccc} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{array} \right] \tag{5a}$$

$$\begin{aligned} N_1 &= \frac{1}{4}(1-\xi)(1-\eta) \\ N_2 &= \frac{1}{4}(1+\xi)(1-\eta) \\ N_3 &= \frac{1}{4}(1+\xi)(1+\eta) \\ N_4 &= \frac{1}{4}(1-\xi)(1+\eta) \end{aligned} \tag{Q4-N}$$

Example Q4-2



For the element shown the nodal displacements in the x and y directions are (0.1, 0.2), (0,0.2), (0.1, 0.2) and (-0.1,0) for nodes 1, 2, 3 and 4 respectively. Find the generic displacements at $(\xi,\eta)=(0,0)$ and $(\xi,\eta)=(-0.5,-0.5)$

$$\mathbf{u}=\mathbf{N}\mathbf{q}$$

$$\left\{\begin{array}{c} u \\ v \end{array}\right\} = \left[\begin{array}{cccccccc} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{array}\right] \left\{\begin{array}{c} 0.1 \\ 0.2 \\ 0 \\ 0.2 \\ 0.1 \\ 0.2 \\ -0.1 \\ 0 \end{array}\right\}$$

5

$$\begin{aligned} u &= 0.1N_1+0.1N_3-0.1N_4 \\ v &= 0.2N_1+0.2N_2+0.2N_3 \end{aligned}$$

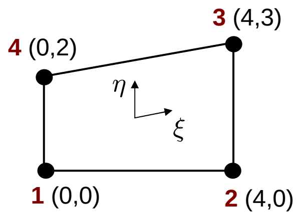
using Q4-N

at $(\xi,\eta)=(0,0)$:
$N_1=N_2=N_3=N_4=1/4.$
$u=0.025$
$v=0.15$

using Q4-N

at $(\xi,\eta)=(-0.5,-0.5)$:
$u=0.04375$
$v=0.1625$

Example Q4-2 (cont.)



If the nodal displacements are (0.3, 0.2) at all nodes, what are the generic displacements at any point within the element?
What are the stresses and strains in this case?

$$\begin{aligned} u &= 0.3N_1+0.3N_2+0.3N_3+0.3N_4 \\ v &= 0.2N_1+0.2N_2+0.2N_3+0.2N_4 \end{aligned}$$

5

at all points:
$u=0.3$
$v=0.2$

What is this called?

Stresses?
Strains?

FE matrices for Q4 element

Strain displacement relations

$$\boldsymbol{\epsilon} = \mathbf{Lu}$$ 6 or $$\left\{ \begin{array}{c} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{array} \right\} = \left[\begin{array}{cc} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{array} \right] \left\{ \begin{array}{c} u \\ v \end{array} \right\}$$ 6

or

$$\boldsymbol{\epsilon} = \mathbf{LNq}$$ or $$\boldsymbol{\epsilon} = \mathbf{Bq}$$ 7 where $$\mathbf{B} = \mathbf{LN}$$ 8

$$\begin{aligned}
\mathbf{B} &= \left[\begin{array}{cc} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{array} \right] \left[\begin{array}{cccccccc} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{array} \right] \\
&= \left[\begin{array}{cccccccc} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{array} \right]
\end{aligned}$$ 8

We need derivatives of N_1, N_2 etc in terms of global coordinates x and y.
We have N_1, N_2 etc in terms of natural coordinates ξ and η.

FE matrices for Q4 element

To obtain partial derivatives of shape functions w.r.t. the the natural coordinates we use the following chain rule of differenciation

$$\begin{aligned}
\frac{\partial N_i}{\partial \xi} &= \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \xi} \\
\frac{\partial N_i}{\partial \eta} &= \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \eta}
\end{aligned}$$

$$\left\{ \begin{array}{c} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{array} \right\} = \left[\begin{array}{cc} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{array} \right] \left\{ \begin{array}{c} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{array} \right\}$$ Q-J1

No problem evaluating these as shape functions are in terms of natural coordinates

These can be evaluated using the isoparametric concept
This matrix is called the Jacobian matrix and denoted by $\mathbf{J}$
$\mathbf{J}$ contains derivatives of global coordinates w.r.t. natural coordinates

We need these terms for matrix $\mathbf{B}$

FE matrices for Q4 element

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$ Q-J2

$$\begin{aligned} J_{11} &= \frac{\partial x}{\partial \xi} \qquad \text{using} \\ &= \frac{\partial}{\partial \xi}\sum_1^4 N_i x_i = \frac{\partial N_1}{\partial \xi}x_1 + \frac{\partial N_2}{\partial \xi}x_2 + \frac{\partial N_3}{\partial \xi}x_3 + \frac{\partial N_4}{\partial \xi}x_4 \\ &= \sum_1^4 \frac{\partial N_i}{\partial \xi}x_i \end{aligned}$$

iso

Thus

$$J_{11} = \sum_1^4 \frac{\partial N_i}{\partial \xi}x_i \qquad J_{12} = \sum_1^4 \frac{\partial N_i}{\partial \xi}y_i$$

$$J_{21} = \sum_1^4 \frac{\partial N_i}{\partial \eta}x_i \qquad J_{22} = \sum_1^4 \frac{\partial N_i}{\partial \eta}y_i$$

Q-J2

FE matrices for Q4 element

$$\begin{aligned} N_1 &= \frac{1}{4}(1-\xi)(1-\eta) \\ N_2 &= \frac{1}{4}(1+\xi)(1-\eta) \\ N_3 &= \frac{1}{4}(1+\xi)(1+\eta) \\ N_4 &= \frac{1}{4}(1-\xi)(1+\eta) \end{aligned}$$

$$\begin{aligned} \frac{\partial N_1}{\partial \xi} &= -\frac{1}{4}(1-\eta) \\ \frac{\partial N_2}{\partial \xi} &= \frac{1}{4}(1-\eta) \\ \frac{\partial N_3}{\partial \xi} &= \frac{1}{4}(1+\eta) \\ \frac{\partial N_4}{\partial \xi} &= -\frac{1}{4}(1+\eta) \end{aligned}$$

$$\begin{aligned} \frac{\partial N_1}{\partial \eta} &= -\frac{1}{4}(1-\xi) \\ \frac{\partial N_2}{\partial \eta} &= -\frac{1}{4}(1+\xi) \\ \frac{\partial N_3}{\partial \eta} &= \frac{1}{4}(1+\xi) \\ \frac{\partial N_4}{\partial \eta} &= \frac{1}{4}(1-\xi) \end{aligned}$$

Deriv-N

$$\begin{aligned} J_{11} &= \frac{1}{4}[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4] \\ J_{12} &= \frac{1}{4}[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4] \\ J_{21} &= \frac{1}{4}[-(1-\xi)x_1 - (1+\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4] \\ J_{22} &= \frac{1}{4}[-(1-\xi)y_1 - (1+\xi)y_2 + (1+\xi)y_3 + (1-\xi)y_4] \end{aligned}$$

Q4-J

FE matrices for Q4 element

$$\left\{ \begin{array}{c} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{array} \right\} = \mathbf{J}^{-1} \left\{ \begin{array}{c} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{array} \right\}$$

Q-J3

These are the terms in the **B** matrix

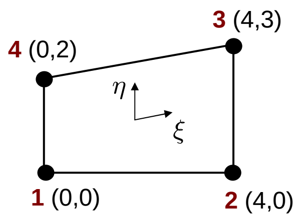
where

$$\mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \left[\begin{array}{cc} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{array} \right] \quad \text{and} \quad \det \mathbf{J} = J_{11}J_{22} - J_{12}J_{21}$$

Q-J4

Example Q4-3

1. Find $\mathbf{J}$
2. What is $\mathbf{J}$ at $(\xi, \eta) = (0, 0)$
3. Find $\mathbf{J}^{-1}$
4. What is $\mathbf{J}^{-1}$ at $(\xi, \eta) = (0, 0)$
5. Evaluate the $\mathbf{B}$ matrix at $(\xi, \eta) = (0, 0)$



1. Using Q4-J

$$\begin{aligned}
J_{11} &= \frac{1}{4}[-(1-\eta) \times 0 + (1-\eta) \times 4 + (1+\eta) \times 4 - (1+\eta) \times 0] \\
J_{12} &= \frac{1}{4}[-(1-\eta) \times 0 + (1-\eta) \times 0 + (1+\eta) \times 3 - (1+\eta) \times 2] \\
J_{21} &= \frac{1}{4}[-(1-\xi) \times 0 - (1+\xi) \times 4 + (1+\xi) \times 4 + (1-\xi) \times 0] \\
J_{22} &= \frac{1}{4}[-(1-\xi) \times 0 - (1+\xi) \times 0 + (1+\xi) \times 3 + (1-\xi) \times 2]
\end{aligned}$$

$$\boxed{\mathbf{J} = \frac{1}{4} \left[\begin{array}{cc} 8 & 1+\eta \\ 0 & 5+\xi \end{array} \right]}$$

Example Q4-3 (cont.)

2. Substituting $(\xi, \eta) = (0, 0)$ in $\mathbf{J} = \frac{1}{4}\begin{bmatrix} 8 & 1+\eta \\ 0 & 5+\xi \end{bmatrix}$

$$\mathbf{J} = \begin{bmatrix} 2 & 1/4 \\ 0 & 5/4 \end{bmatrix}$$

Note: If $\mathbf{J}$ is required only at a specific point, then it is quicker to first substitute for ξ and η in $\partial N_i/\partial \xi$ and ξ and η in $\partial N_i/\partial \eta$ and then calculate $\mathbf{J}$

3. $\mathbf{J}^{-1}$

$$\det \mathbf{J} = \frac{8}{4}\frac{(5+\xi)}{4} = \frac{5+\xi}{2}$$

$$\mathbf{J}^{-1} = \frac{2}{(5+\xi)}\frac{1}{4}\begin{bmatrix} 5+\xi & -(1+\eta) \\ 0 & 8 \end{bmatrix} = \frac{1}{2(5+\xi)}\begin{bmatrix} 5+\xi & -(1+\eta) \\ 0 & 8 \end{bmatrix}$$

4. Substituting $(\xi, \eta) = (0, 0)$ in $\mathbf{J}^{-1}$ or inverting $\mathbf{J}$ at $(\xi, \eta) = (0, 0)$

$$\mathbf{J}^{-1} = \begin{bmatrix} 0.5 & -0.1 \\ 0 & 0.8 \end{bmatrix}$$

Example Q4-3 (cont.)

5. Using $\mathbf{J}^{-1}$ at $(\xi, \eta) = (0, 0)$, the terms of the $\mathbf{B}$ matrix at $(\xi, \eta) = (0, 0)$ can be evaluated.
Consider partial derivatives w.r.t N_1. Using Q-J3

$$\underbrace{\left\{ \begin{array}{c} \frac{\partial N_1}{\partial x} \\ \frac{\partial N_1}{\partial y} \end{array} \right\}}_{\text{at } (\xi,\eta)=(0,0)} = \mathbf{J}^{-1}\left\{ \begin{array}{c} \frac{\partial N_1}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} \end{array} \right\} = \underbrace{\begin{bmatrix} 0.5 & -0.1 \\ 0 & 0.8 \end{bmatrix}}_{\text{at } (\xi,\eta)=(0,0)} \underbrace{\left\{ \begin{array}{c} -\frac{1}{4} \\ -\frac{1}{4} \end{array} \right\}}_{\text{at } (\xi,\eta)=(0,0)} = \left\{ \begin{array}{c} -0.1 \\ -0.2 \end{array} \right\}$$

Similarly

$$\left\{ \begin{array}{c} \frac{\partial N_2}{\partial x} \\ \frac{\partial N_2}{\partial y} \end{array} \right\} = \begin{bmatrix} 0.5 & -0.1 \\ 0 & 0.8 \end{bmatrix}\left\{ \begin{array}{c} 0.25 \\ -0.25 \end{array} \right\} = \left\{ \begin{array}{c} 0.15 \\ -0.2 \end{array} \right\}$$

$$\left\{ \begin{array}{c} \frac{\partial N_3}{\partial x} \\ \frac{\partial N_3}{\partial y} \end{array} \right\} = \begin{bmatrix} 0.5 & -0.1 \\ 0 & 0.8 \end{bmatrix}\left\{ \begin{array}{c} 0.25 \\ 0.25 \end{array} \right\} = \left\{ \begin{array}{c} 0.1 \\ 0.2 \end{array} \right\}$$

$$\left\{ \begin{array}{c} \frac{\partial N_4}{\partial x} \\ \frac{\partial N_4}{\partial y} \end{array} \right\} = \begin{bmatrix} 0.5 & -0.1 \\ 0 & 0.8 \end{bmatrix}\left\{ \begin{array}{c} -0.25 \\ 0.25 \end{array} \right\} = \left\{ \begin{array}{c} -0.15 \\ 0.2 \end{array} \right\}$$

Example Q4-3 (cont.)

Now we have the $\mathbf{B}$ matrix as

$$\mathbf{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix}$$

$$= \begin{bmatrix} -0.1 & 0 & 0.15 & 0 & 0.1 & 0 & -0.15 & 0 \\ 0 & -0.2 & 0 & -0.2 & 0 & 0.2 & 0 & 0.2 \\ -0.2 & -0.1 & -0.2 & 0.15 & 0.2 & 0.1 & 0.2 & -0.15 \end{bmatrix}$$

FE matrices for Q4 element

Now we are fairly close to the evaluation of the following two key matrices

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{D}\,\mathbf{B}\,dV \quad (13) \qquad \mathbf{p}_b = \int_V \mathbf{N}^T \mathbf{b} dV \quad (14)$$

Element stiffness matrix

Equivalent nodal loads matrix

For a 2D element of thickness t we can write the above equations as

$$\mathbf{K} = t\int_A \mathbf{B}^T \mathbf{D}\,\mathbf{B} dA \qquad \mathbf{p}_b = t\int_A \mathbf{N}^T \mathbf{b} dA$$

It is very convenient to work out these area integrals using natural coordinates – in terms of ξ, η the integration is over a rectangle which is much easier. We have already seen that all terms except dA can be expressed in terms of ξ, η. It can be shown that dA can be expressed in terms of ξ, η as

$$dA = \det \mathbf{J} d\xi d\eta$$

FE matrices for Q4 element

Thus

$$\mathbf{K} = t\int_{-1}^{1}\int_{-1}^{1}\mathbf{B}^T(\xi,\eta)\mathbf{D}\ \mathbf{B}(\xi,\eta)\ \det\mathbf{J}(\xi,\eta)d\xi d\eta$$

or simply

$$\mathbf{K} = t\int_{-1}^{1}\int_{-1}^{1}\mathbf{B}^T\mathbf{D}\ \mathbf{B}\ \det\mathbf{J}d\xi d\eta$$

$$\begin{aligned}\mathbf{p}_b &= t\int_{-1}^{1}\int_{-1}^{1}\mathbf{N}^T(\xi,\eta)\mathbf{b}(\xi,\eta)\det\mathbf{J}(\xi,\eta)d\xi d\eta \\ &= t\int_{-1}^{1}\int_{-1}^{1}\mathbf{N}^T\mathbf{b}\det\mathbf{J}d\xi d\eta\end{aligned}$$

Even with the simplifications, above integrals are not easy to evaluate in closed form (for most cases). We, therefore, need to employ numerical integration.

Numerical integration – Q elements

For numerical integration FE codes employ Gaussion integration or Gaussian quadrature. This has distinct advantages over Newton Cotes formulae.

In this an integral of the form

$$I = \int_{-1}^{1}\int_{-1}^{1}f(\xi,\eta)\det\mathbf{J}(\xi,\eta)d\xi d\eta$$

is converted to a summation as

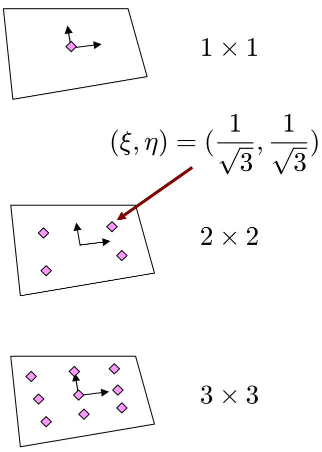
$$I = \sum_{k=1}^{n}\sum_{j=1}^{n}H_jH_kf(\xi_j,\eta_k)\det\mathbf{J}(\xi_j,\eta_k)$$

Thus numerical integration requires that the value of the function to be integrated be

- ★ evaluated at specific points
- ★ multiplied by weights H_jH_k and
- ★ summed up

Numerical integration – Q elements

Weights and location of points for 2D Gaussian quadrature



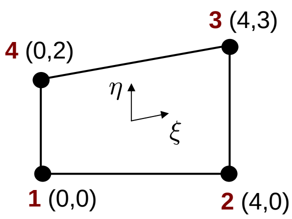
n × n	p	ξ_j	η_k	H_j	H_k
1 × 1	1	0	0	2	2
2 × 2	3	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	1	1
		$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$		
		$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$		
		$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$		
3 × 3	5	$-\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$	$\frac{5}{9}$
		$-\sqrt{\frac{3}{5}}$	0	$\frac{5}{9}$	$\frac{8}{9}$
		$-\sqrt{\frac{3}{5}}$	$\sqrt{\frac{3}{5}}$	$\frac{5}{9}$	$\frac{5}{9}$
		0	$-\sqrt{\frac{3}{5}}$	$\frac{8}{9}$	$\frac{5}{9}$
		0	0	$\frac{8}{9}$	$\frac{8}{9}$
		0	$\sqrt{\frac{3}{5}}$	$\frac{8}{9}$	$\frac{5}{9}$
		$\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$	$\frac{5}{9}$
		$\sqrt{\frac{3}{5}}$	0	$\frac{5}{9}$	$\frac{8}{9}$
		$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{3}{5}}$	$\frac{5}{9}$	$\frac{5}{9}$

Example Q4-4

Use 1×1 Gaussian integration to evaluate

$$I = \int_{-1}^{1}\int_{-1}^{1} \det \mathbf{J}(\xi, \eta) d\xi d\eta$$

for the Q4 element shown.



Using 1×1 integration and converting the integral to a summation we get

$$I = \sum_{k=1}^{1}\sum_{j=1}^{1} H_j H_k \det \mathbf{J}(\xi_j, \eta_k) = H_1 H_1 \det \mathbf{J}(\xi_1, \eta_1)$$

To find the above integral all we need is the value of $\det \mathbf{J}$ at $(\xi, \eta) = (0, 0)$ and we know that the weighting factor for 1 point integration is 2. Recall we found $\det \mathbf{J}$ in Example Q4-3 for this element as

$$\det \mathbf{J} = \frac{5 + \xi}{2} \qquad \text{at } (\xi, \eta) = (0, 0) \qquad \det \mathbf{J} = \frac{5}{2} = 2.5$$

So $\quad I = 2 \times 2 \times 2.5 = 10$

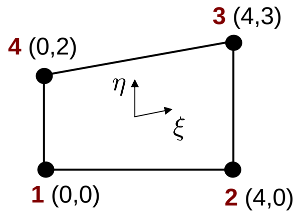
Recall $dA = \det \mathbf{J} d\xi d\eta$.
Therefore the above integral gives the area of the element

Example Q4-5

Use 2×2 Gaussian integration to evaluate

$$I = \int_{-1}^{1}\int_{-1}^{1} \det \mathbf{J}(\xi, \eta) d\xi d\eta$$

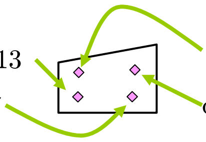
for the Q4 element shown.
What does the above integral give?



$$\det \mathbf{J} = \frac{5+\xi}{2}$$

$\det \mathbf{J}(-1/\sqrt{3}, -1/\sqrt{3}) = 2.2113$

$\det \mathbf{J}(1/\sqrt{3}, -1/\sqrt{3}) = 2.7887$



$\det \mathbf{J}(-1/\sqrt{3}, 1/\sqrt{3}) = 2.2113$

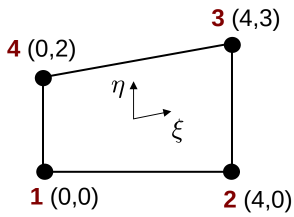
$\det \mathbf{J}(1/\sqrt{3}, 1/\sqrt{3}) = 2.7887$

Using 2×2 integration and converting the integral to a summation we get

$$\begin{aligned} I &= \sum_{k=1}^{2}\sum_{j=1}^{2} H_j H_k \det \mathbf{J}(\xi_j, \eta_k) \\ &= H_1 H_1 \det \mathbf{J}(\xi_1, \eta_1) + H_1 H_2 \det \mathbf{J}(\xi_1, \eta_2) \\ &\quad + H_2 H_1 \det \mathbf{J}(\xi_2, \eta_1) + H_2 H_2 \det \mathbf{J}(\xi_2, \eta_2) \\ &= 1 \times 1 \times 2.2113 + 1 \times 1 \times 2.7887 + 1 \times 1 \times 2.2113 + 1 \times 1 \times 2.7887 \\ &= 10 \end{aligned}$$

Example Q4-6

Find the k_{33} and k_{67} terms of the element stiffness matrix $\mathbf{K}$ for the Q4 element shown.
Assume it is a plane strain element with $E = 3 \times 10^7$, $\nu = 0.3$ and $t = 1$.
Use 1×1 Gaussian integration



$$\mathbf{K} = t\int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^T(\xi, \eta)\mathbf{D}\ \mathbf{B}(\xi, \eta)\ \det \mathbf{J}(\xi, \eta) d\xi d\eta$$

Using 1×1 integration and converting the integral to a summation we get

$$\begin{aligned} \mathbf{K} &= t\sum_{k=1}^{1}\sum_{j=1}^{1} H_j H_k \mathbf{B}^T(\xi_j, \eta_k)\mathbf{D}\ \mathbf{B}(\xi_j, \eta_k)\ \det \mathbf{J}(\xi_j, \eta_k) \\ &= 1 \times 2 \times 2 \times \mathbf{B}^T(0,0)\mathbf{D}\ \mathbf{B}(0,0)\ \det \mathbf{J}(0,0) \end{aligned}$$

Example Q4-6 (cont.)

For plane strain $\mathbf{D}$ is given by

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 40384616 & 17307694 & 0 \\ 17307694 & 40384616 & 0 \\ 0 & 0 & 11538462 \end{bmatrix}$$

From Example Q4-3

$\det \mathbf{J} = \dfrac{5+\xi}{2}$ therefore $\det \mathbf{J}(0,0) = \dfrac{5}{2} = 2.5$

and

$$\mathbf{B}(0,0) = \begin{bmatrix} -0.1 & 0 & 0.15 & 0 & 0.1 & 0 & -0.15 & 0 \\ 0 & -0.2 & 0 & -0.2 & 0 & 0.2 & 0 & 0.2 \\ -0.2 & -0.1 & -0.2 & 0.15 & 0.2 & 0.1 & 0.2 & -0.15 \end{bmatrix}$$

Example Q4-6 (cont.)

$$\mathbf{K} = 1 \times 2 \times 2 \times \mathbf{B}^T(0,0)\mathbf{D}\ \mathbf{B}(0,0)\ \det \mathbf{J}(0,0)$$

Using the $\mathbf{B}$ evaluated at $(\xi,\eta) = (0.0, 0.0)$ in Example Q4-3

$$\mathbf{K} = 4 \times \begin{bmatrix} -0.1 & 0 & -0.2 \\ 0 & -0.2 & 0.1 \\ \boxed{0.15 \quad 0 \quad -0.2} \\ 0 & -0.2 & 0.15 \\ 0.1 & 0 & 0.2 \\ \boxed{0 \quad 0.2 \quad 0.1} \\ -0.15 & 0 & 0.2 \\ 0 & 0.2 & -0.15 \end{bmatrix} \begin{bmatrix} 40384616 & 17307694 & 0 \\ 17307694 & 40384616 & 0 \\ 0 & 0 & 11538462 \end{bmatrix}$$

$$\begin{bmatrix} -0.1 & 0 & 0.15 & 0 & 0.1 & 0 & -0.15 & 0 \\ 0 & -0.2 & 0 & -0.2 & 0 & 0.2 & 0 & 0.2 \\ -0.2 & -0.1 & -0.2 & 0.15 & 0.2 & 0.1 & 0.2 & -0.15 \end{bmatrix} \times 2.5$$

$\mathbf{B}$ matrix terms required for k_{33}

$\mathbf{B}$ matrix terms required for k_{67}

Example Q4-6 (cont.)

$$
\begin{aligned}
k_{33} &= 10 \times \begin{bmatrix} 0.15 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} 40384616 & 17307694 & 0 \\ 17307694 & 40384616 & 0 \\ 0 & 0 & 11538462 \end{bmatrix} \begin{bmatrix} 0.15 \\ 0 \\ -0.2 \end{bmatrix} \\
&= 10 \times \begin{bmatrix} 6057692.4 & 2596154.1 & -2307692.4 \end{bmatrix} \begin{bmatrix} 0.15 \\ 0 \\ -0.2 \end{bmatrix} \\
&= 13701923.4
\end{aligned}
$$

$$
\begin{aligned}
k_{67} &= 10 \times \begin{bmatrix} 0 & 0.2 & 0.1 \end{bmatrix} \begin{bmatrix} 40384616 & 17307694 & 0 \\ 17307694 & 40384616 & 0 \\ 0 & 0 & 11538462 \end{bmatrix} \begin{bmatrix} -0.15 \\ 0 \\ 0.2 \end{bmatrix} \\
&= 10 \times \begin{bmatrix} 3461538.8 & 8076923.2 & 1153846.2 \end{bmatrix} \begin{bmatrix} -0.15 \\ 0 \\ 0.2 \end{bmatrix} \\
&= -2884615.8
\end{aligned}
$$

Example Q4-6 (cont.)

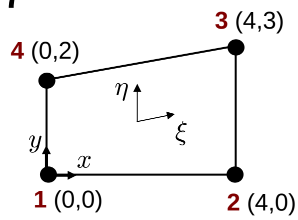
The complete stiffness matrix using one point Guass integration is obtained as

$$
\mathbf{K} = \begin{bmatrix}
0.865E7 & 0.577E7 & -0.144E7 & 0 & -0.865E7 & -0.577E7 & 0.144E7 & 0 \\
0.577E7 & 0.173E8 & -0.288E7 & 0.144E8 & -0.577E7 & -0.173E8 & 0.288E7 & -0.1442308E8 \\
-0.144E7 & -0.288E7 & 0.137E8 & -0.866E7 & 0.144E7 & 0.288E7 & -0.137E8 & 0.865E7 \\
0 & 0.144E8 & -0.865E7 & 0.188E8 & 0 & -0.144E8 & 0.865E7 & -0.188E8 \\
-0.865E7 & -0.577E7 & 0.144E7 & 0 & 0.865E7 & 0.577E7 & -0.144E7 & 0 \\
-0.577E7 & -0.173E8 & 0.288E7 & -0.144E8 & 0.577E7 & 0.173E8 & -0.288E7 & 0.144E8 \\
0.144E7 & 0.288E7 & -0.137E8 & 0.865E7 & -0.144E7 & -0.288E7 & 0.137E8 & -0.865E7 \\
0 & -0.144E8 & 0.865E7 & -0.187E8 & 0 & 0.144E8 & -0.865E7 & 0.187E8
\end{bmatrix}
$$

One point integration for quadrilateral elements has problems.

Example Q4-7

Find equivalent nodal loads matrix due to gravity loads (acting downwards) for the Q4 element shown using
(a) 1×1 numerical integration and
(b) without using numerical integration.
The weight density of the material of the element is 24000 N/m^3.
Assume $t = 1$.



The equivalent nodal loads are given by

$$\mathbf{p}_b = t\int_{-1}^{1}\int_{-1}^{1}\mathbf{N}^T\mathbf{b}\det\mathbf{J}\,d\xi d\eta$$

where

$$\mathbf{b} = \left\{\begin{array}{c} b_x \\ b_y \end{array}\right\} = \left\{\begin{array}{c} 0 \\ -24000 \end{array}\right\}$$

and

$$\mathbf{N}^T = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \\ N_4 & 0 \\ 0 & N_4 \end{bmatrix}$$

Example Q4-7 (cont.)

$$\mathbf{N}^T\mathbf{b} = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \\ N_4 & 0 \\ 0 & N_4 \end{bmatrix}\left\{\begin{array}{c} 0 \\ -24000 \end{array}\right\} = \left\{\begin{array}{c} 0 \\ -24000N_1 \\ 0 \\ -24000N_2 \\ 0 \\ -24000N_3 \\ 0 \\ -24000N_4 \end{array}\right\}$$

$$\mathbf{p}_b = t\int_{-1}^{1}\int_{-1}^{1}\left\{\begin{array}{c} 0 \\ -24000N_1 \\ 0 \\ -24000N_2 \\ 0 \\ -24000N_3 \\ 0 \\ -24000N_4 \end{array}\right\}\det\mathbf{J}\,d\xi d\eta$$

Each term can be integrated separately

Example Q4-7 (cont.)

(a) Using single point integration and noting that
$t = 1$
$\det \mathbf{J}(0,0) = 2.5$
weighting factors $H_j = H_k = 2$
and $N_1(0,0) = N_2(0,0) = N_3(0,0) = N_4(0,0) = 1/4$ we get

$$\mathbf{p}_b = 1 \times 2 \times 2 \begin{Bmatrix} 0 \\ -24000 \times \frac{1}{4} \\ 0 \\ -24000 \times \frac{1}{4} \\ 0 \\ -24000 \times \frac{1}{4} \\ 0 \\ -24000 \times \frac{1}{4} \end{Bmatrix} 2.5 = \begin{Bmatrix} 0 \\ -60000 \\ 0 \\ -60000 \\ 0 \\ -60000 \\ 0 \\ -60000 \end{Bmatrix}$$

The total weight of the element is equally divided amongst 4 nodes

Example Q4-7 (cont.)

(b) Without using numerical integration and noting that
$t = 1$
$\det \mathbf{J} = (5 + \xi)/2$ we get

$$\mathbf{p}_b \quad = \quad 1 \times \int_{-1}^{1} \int_{-1}^{1} \begin{Bmatrix} 0 \\ -24000 \times \frac{1}{4}(1-\xi)(1-\eta) \\ 0 \\ -24000 \times \frac{1}{4}(1+\xi)(1-\eta) \\ 0 \\ -24000 \times \frac{1}{4}(1+\xi)(1+\eta) \\ 0 \\ -24000 \times \frac{1}{4}(1-\xi)(1+\eta) \end{Bmatrix} \frac{5+\xi}{2} d\xi d\eta$$

Consider the second term

$$\mathbf{p}_b = \frac{-24000}{4 \times 2} \times \int_{-1}^{1} \int_{-1}^{1} (1-\xi)(1-\eta)(5+\xi) d\xi d\eta$$

Example Q4-7 (cont.)

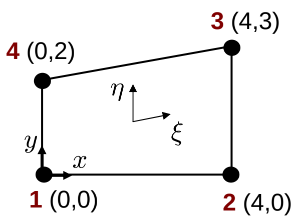
$$
\begin{aligned}
p_{b2} &= \frac{-24000}{4\times 2}\times\int_{-1}^{1}\int_{-1}^{1}(1-\xi)(1-\eta)(5+\xi)d\xi d\eta\\
&= -3000\int_{-1}^{1}\int_{-1}^{1}(-\xi^2-4\xi+5)(1-\eta)d\xi d\eta\\
&= \left[\left[-3000(-\frac{\xi^3}{3}-2\xi^2+5\xi)(\eta-\frac{\eta^2}{2})\right]_{-1}^{1}\right]_{-1}^{1}\\
&= -3000\{(-\frac{1}{3}-2+5)-(\frac{1}{3}-2-5)\}\{(1-\frac{1}{2})-(-1-\frac{1}{2})\}\\
&= -56000
\end{aligned}
$$

Similarly

$$
\begin{aligned}
p_{b4} &= -64000\\
p_{b6} &= -64000\\
p_{b8} &= -56000
\end{aligned}
$$

Example Q4-7 (cont.)

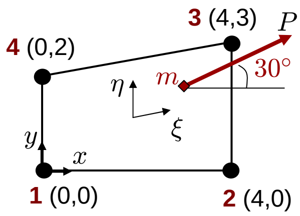
$$
\mathbf{p}_b = \left\{\begin{array}{c} 0\\ -56000\\ 0\\ -64000\\ 0\\ -64000\\ 0\\ -56000 \end{array}\right\}
$$



★ Equivalent nodal loads associated with nodes 2 and 3 are greater
★ The total weight is still 240000

Example Q4-8

A concentrated load P acts at m as shown. The location of m (in terms of ξ and η) is $(\xi, \eta) = (0.5, 0.5)$. Find the equivalent nodal loads vector due to this concentrated force.



The equivalent nodal loads are given by

$$\mathbf{p}_b = t \int_{-1}^{1} \int_{-1}^{1} \mathbf{N}^T \mathbf{b} \det \mathbf{J} \, d\xi d\eta$$

For a concentrated force the above reduces to

$$\mathbf{p}_b = (\mathbf{N}^T)_m (\mathbf{b})_m$$

where

$$(\mathbf{b})_m = \left\{ \begin{array}{c} b_x \\ b_y \end{array} \right\}_m = \left\{ \begin{array}{c} P\cos 30^\circ \\ P\sin 30^\circ \end{array} \right\} = \left\{ \begin{array}{c} 0.866P \\ 0.5P \end{array} \right\}$$

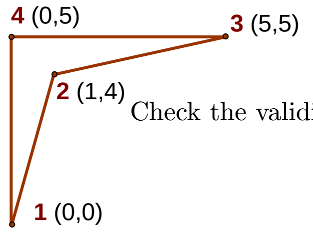
Example Q4-8 (cont.)

$$(\mathbf{N}^T)_m = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \\ N_4 & 0 \\ 0 & N_4 \end{bmatrix}_m = \begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{16} \\ \frac{3}{16} & 0 \\ 0 & \frac{3}{16} \\ \frac{9}{16} & 0 \\ 0 & \frac{9}{16} \\ \frac{3}{16} & 0 \\ 0 & \frac{3}{16} \end{bmatrix}$$

$$\mathbf{p}_b = (\mathbf{N}^T)_m (\mathbf{b})_m = \begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{16} \\ \frac{3}{16} & 0 \\ 0 & \frac{3}{16} \\ \frac{9}{16} & 0 \\ 0 & \frac{9}{16} \\ \frac{3}{16} & 0 \\ 0 & \frac{3}{16} \end{bmatrix} \left\{ \begin{array}{c} 0.866P \\ 0.5P \end{array} \right\} = \frac{P}{16} \left\{ \begin{array}{c} 0.866 \\ 0.5 \\ 2.598 \\ 1.5 \\ 7.794 \\ 4.5 \\ 2.598 \\ 1.5 \end{array} \right\}$$

Example Q4-9

Importance of the Jacobian and mapping



Check the validity of mapping for the following 4-node Q4 element

$$x = \frac{1}{4}[(1-\xi)(1-\eta)0 + (1+\xi)(1-\eta)1 + (1+\xi)(1+\eta)5 + (1-\xi)(1+\eta)0]$$

$$y = \frac{1}{4}[(1-\xi)(1-\eta)0 + (1+\xi)(1-\eta)4 + (1+\xi)(1+\eta)5 + (1-\xi)(1+\eta)5]$$

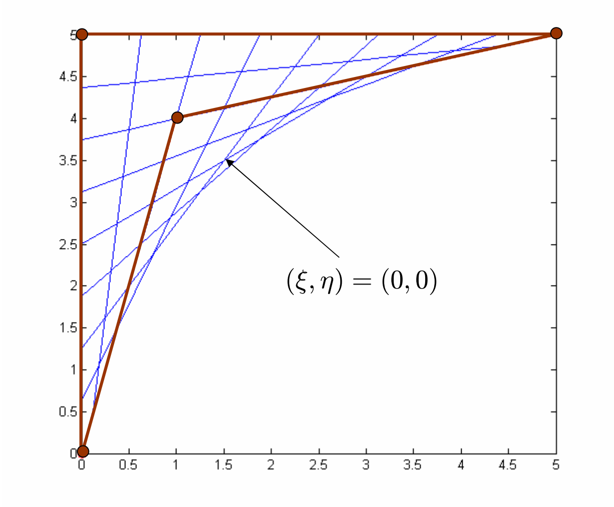
or

$$x = \frac{1}{2}(1+\xi)(3+2\eta) \qquad y = \frac{1}{2}(7+2\xi+3\eta-2\xi\eta)$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(3+2\eta) & \frac{1}{2}(2-2\eta) \\ (1+\xi) & \frac{1}{2}(3-2\xi) \end{bmatrix}$$

$$\begin{aligned} \det \mathbf{J} &= J_{11}J_{22} - J_{12}J_{21} \\ &= \frac{1}{4}(5 - 10\xi + 10\eta) \end{aligned}$$

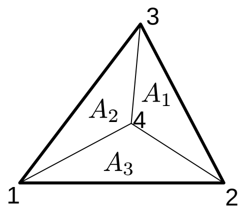
$\det \mathbf{J}$ is negative at $(\eta, \xi) = (0, 1)$. It is zero for $\xi - \eta = 1/2$



Also note several points in the parent (ξ, η) element are mapped outside the physical element

Triangular elements – natural coordinates

- Consider a triangular element with vertices 1, 2 and 3
- Now consider any generic point (say 4) within the element
- The location of the generic point can be uniquely defined by the dimensionless area coordinates

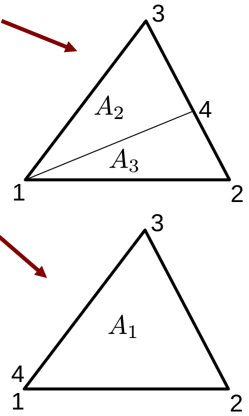


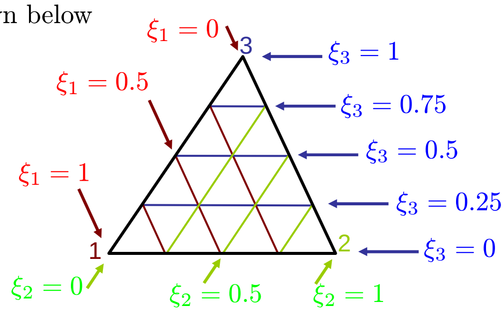
$$\xi_1 = \frac{A_1}{A}; \quad \xi_2 = \frac{A_2}{A}; \quad \xi_3 = \frac{A_3}{A}$$

- Since $A_1 + A_2 + A_3 = A$ we have $\xi_1 + \xi_2 + \xi_3 = 1$
- Thus only two of the three coordinates are independent, since $\xi_3 = 1 - \xi_1 - \xi_2$

Triangular elements – natural coordinates

- Consider generic point 4 located anywhere on line 2-3. Since $A_1 = 0$, therefore $\xi_1 = 0$
- Similarly $\xi_2 = 0$ on line 1-3 and $\xi_3 = 0$ on line 1-2
- If the generic point (4) is coincident with point 1 then $A_1 = A$ and $A_2 = A_3 = 0$. Thus at 1 we have $\xi_1 = 1$ and $\xi_2 = \xi_3 = 0$
- As generic point (4) moves from 1 to 3 or from 1 to 2, ξ_1 goes linearly from 1 to 0.
- Similarly we can examine the variation of ξ_2 and ξ_3 and plot the variation of natural coordinates as shown below





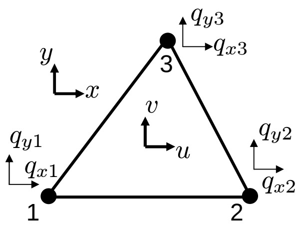
T3 element – shape functions

We can observe that ξ_1, ξ_2 and ξ_3 can function as natural coordinates as well as shape functions – they have a value of unity at the node they represent and zero at other nodes. Thus

$$N_1 = \xi_1; \quad N_2 = \xi_2; \quad N_3 = \xi_3$$

and generic displacements $\mathbf{u}$ can be related to nodal displacements as

$$\mathbf{u} = \left\{ \begin{array}{c} u \\ v \end{array} \right\} = \left[\begin{array}{cccccc} \xi_1 & 0 & \xi_2 & 0 & \xi_3 & 0 \\ 0 & \xi_1 & 0 & \xi_2 & 0 & \xi_3 \end{array} \right] \left\{ \begin{array}{c} q_{x1} \\ q_{y1} \\ q_{x2} \\ q_{y2} \\ q_{x3} \\ q_{y3} \end{array} \right\}$$



or

$$\begin{aligned} u &= \xi_1 q_{x1} + \xi_2 q_{x2} + \xi_3 q_{x3} \\ v &= \xi_1 q_{y1} + \xi_2 q_{y2} + \xi_3 q_{y3} \end{aligned}$$

T3-1

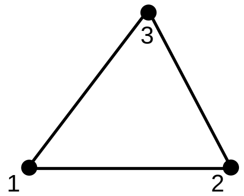
T3 element – isoparametric mapping

It is also easy to see that the natural coordinates can be used to obtain the global coordinates of any generic point as

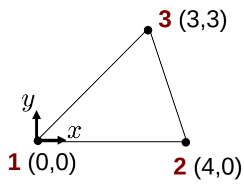
$$
\begin{aligned}
x &= \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 \\
y &= \xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3
\end{aligned}
$$

T3-2

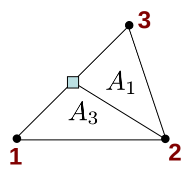
$$
\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} \xi_1 & 0 & \xi_2 & 0 & \xi_3 & 0 \\ 0 & \xi_1 & 0 & \xi_2 & 0 & \xi_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{Bmatrix}
$$



Example T3-1



For the element shown the nodal displacements in the x and y directions are (0.1, 0.2), (0,0.2) and (-0.1, 0.3) for nodes 1, 2 and 3 respectively.
Find the generic displacements at
(a) midpoint of side 1-3
(b) at the centroid of the element



(a) At midpoint of side 1-3
$A_2 = 0$ and $A_1 = A_3$
Thus
$\xi_1 = \xi_3 = 0.5$
Using T3-1

$$
\begin{aligned}
u &= 0.5q_{x1} + 0.5q_{x3} = 0.5 \times 0.1 + 0.5 \times (-0.1) = 0 \\
v &= 0.5q_{y1} + 0.5q_{y3} = 0.5 \times 0.2 + 0.5 \times 0.3 = 0.25
\end{aligned}
$$

Average of nodes 1 and 3

Example T3-1

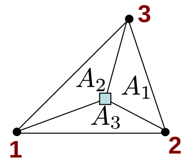
(b) At the centroid

$A_1 = A_2 = A_3$

Thus

$\xi_1 = \xi_2 = \xi_3 = 1/3$

Using T3-1

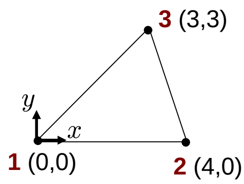


$$
\begin{aligned}
u &= \frac{1}{3}(q_{x1} + q_{x2} + q_{x3}) = \frac{1}{3}(0.1 + 0 - 0.1) = 0 \\
v &= \frac{1}{3}(q_{y1} + q_{y2} + q_{y3}) = \frac{1}{3}(0.2 + 0.2 + 0.3) = 0.233
\end{aligned}
$$

Average of the three nodes

Example T3-2

For the element shown use isoparametric mapping to determine the x and y (or global) coordinates of
(a) midpoint of side 1-3
(b) the centroid of the element



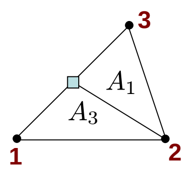
(a) At midpoint of side 1-3

$A_2 = 0$ and $A_1 = A_3$

Thus

$\xi_1 = \xi_3 = 0.5$

Using T3-2



$$
\begin{aligned}
x &= 0.5x_1 + 0.5x_3 = 0.5 \times 0 + 0.5 \times 3 = 1.5 \\
y &= 0.5y_1 + 0.5y_3 = 0.5 \times 0 + 0.5 \times 3 = 1.5
\end{aligned}
$$

Average of nodes 1 and 3

Example T3-2

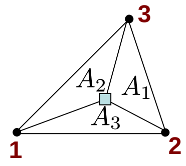
(b) At the centroid

$A_1 = A_2 = A_3$

Thus

$\xi_1 = \xi_2 = \xi_3 = 1/3$

Using T3-2



$$x = \frac{1}{3}(x_1 + x_2 + x_3) = \frac{1}{3}(0 + 4 + 3) = 2.333$$

$$y = \frac{1}{3}(y_1 + y_2 + y_3) = \frac{1}{3}(0 + 0 + 3) = 1$$

Average of the three nodes

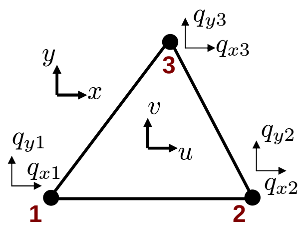
FE matrices for T3 element

Generic displacements

$$\mathbf{u} = \begin{Bmatrix} u \\ v \end{Bmatrix}$$ 1

Body forces

$$\mathbf{b} = \begin{Bmatrix} b_x \\ b_y \end{Bmatrix}$$ 2



Nodal displacements matrix

$$\mathbf{q} = \begin{Bmatrix} q_{x1} \\ q_{y1} \\ \vdots \\ q_{x3} \\ q_{y3} \end{Bmatrix}$$ 3

Nodal loads matrix

$$\mathbf{p} = \begin{Bmatrix} p_{x1} \\ p_{y1} \\ \vdots \\ p_{x3} \\ p_{y3} \end{Bmatrix}$$ 4

FE matrices for T3 element

Relating generic displacements to nodal displacements using shape functions

$$\mathbf{u} = \mathbf{N}\mathbf{q}$$ 5 or

$$\begin{aligned} u &= N_1 q_{x1} + N_2 q_{x2} + N_3 q_{x3} \\ v &= N_1 q_{y1} + N_2 q_{y2} + N_3 q_{y3} \end{aligned}$$ 5

where

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$ 5a

$$\begin{aligned} N_1 &= \xi_1 \\ N_2 &= \xi_2 \\ N_3 &= \xi_3 = 1 - \xi_1 - \xi_2 \end{aligned}$$ T3-N

FE matrices for T3 element

Strain displacement relations

$$\boldsymbol{\epsilon} = \mathbf{L}\mathbf{u}$$ 6 or

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$ 6

or

$$\boldsymbol{\epsilon} = \mathbf{L}\mathbf{N}\mathbf{q}$$ or $$\boldsymbol{\epsilon} = \mathbf{B}\mathbf{q}$$ 7 where $$\mathbf{B} = \mathbf{L}\mathbf{N}$$ 8

$$\begin{aligned} \mathbf{B} &= \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \end{aligned}$$ 8

We need derivatives of N_1, N_2 etc in terms of global coordinates x and y.
We have N_1, N_2 etc in terms of natural coordinates ξ_1 and ξ_2.
Remember ξ_3 is not independent.

FE matrices for T3 element

To obtain partial derivatives of shape functions w.r.t. the the natural coordinates we use the following chain rule of differenciation

$$\frac{\partial N_i}{\partial \xi_1} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi_1} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \xi_1}$$
$$\frac{\partial N_i}{\partial \xi_2} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi_2} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \xi_2}$$

$$\left\{ \begin{array}{c} \frac{\partial N_i}{\partial \xi_1} \\ \frac{\partial N_i}{\partial \xi_2} \end{array} \right\} = \left[\begin{array}{cc} \frac{\partial x}{\partial \xi_1} & \frac{\partial y}{\partial \xi_1} \\ \frac{\partial x}{\partial \xi_2} & \frac{\partial y}{\partial \xi_2} \end{array} \right] \left\{ \begin{array}{c} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{array} \right\} \quad \text{T-J1}$$

No problem evaluating these as shape functions are in terms of natural coordinates

These can be evaluated using the isoparametric concept
Once again this matrix is called the Jacobian matrix and denoted by $\mathbf{J}$
$\mathbf{J}$ contains derivatives of global coordinates w.r.t. natural coordinates

We need these terms for matrix $\mathbf{B}$

FE matrices for T3 element

$$\left\{ \begin{array}{c} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{array} \right\} = \mathbf{J}^{-1} \left\{ \begin{array}{c} \frac{\partial N_i}{\partial \xi_1} \\ \frac{\partial N_i}{\partial \xi_2} \end{array} \right\} \quad \text{T-J2}$$

These are the terms in the **B** matrix

Therefore to evaluate all the terms in the $\mathbf{B}$ matrix:

1. Evaluate $\mathbf{J}$
2. Evaluate $\mathbf{J}^{-1}$
3. Evaluate derivatives of the shape functions w.r.t. to ξ_1 and ξ_2
4. Conduct matrix multiplication as above

Note: We will consider ξ_1 and ξ_2 as independent coordinates.
Thus $\xi_3 = 1 - \xi_1 - \xi_2$

FE matrices for T3 element

Let us first work out the terms of $\mathbf{J}$

Since from T3-2

$$
\begin{aligned}
x &= \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 = \xi_1 x_1 + \xi_2 x_2 + (1 - \xi_1 - \xi_2) x_3 \\
y &= \xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3 = \xi_1 y_1 + \xi_2 y_2 + (1 - \xi_1 - \xi_2) y_3
\end{aligned}
$$

we have

$$
\frac{\partial x}{\partial \xi_1} = x_1 - x_3 \qquad \frac{\partial y}{\partial \xi_1} = y_1 - y_3
$$

$$
\frac{\partial x}{\partial \xi_2} = x_2 - x_3 \qquad \frac{\partial y}{\partial \xi_2} = y_2 - y_3
$$

or

$$
\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial y}{\partial \xi_1} \\ \frac{\partial x}{\partial \xi_2} & \frac{\partial y}{\partial \xi_2} \end{bmatrix} = \begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix}
$$

T-J3

FE matrices for T3 element

$$
\mathbf{J} = \begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix}
$$

T-J3

Now $\mathbf{J}^{-1}$ is given by

$$
\mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_2 - y_3 & y_3 - y_1 \\ x_3 - x_2 & x_1 - x_3 \end{bmatrix}
$$

T-J4

$$
\det \mathbf{J} = (x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3) = 2A
$$

Prove

FE matrices for T3 element

Using T-J2 to find partial derivatives w.r.t. N_1

$$\left\{ \begin{array}{c} \frac{\partial N_1}{\partial x} \\ \frac{\partial N_1}{\partial y} \end{array} \right\} = \left\{ \begin{array}{c} \frac{\partial \xi_1}{\partial x} \\ \frac{\partial \xi_1}{\partial y} \end{array} \right\} = \frac{1}{2A} \left[\begin{array}{cc} y_2 - y_3 & y_3 - y_1 \\ x_3 - x_2 & x_1 - x_3 \end{array} \right] \left\{ \begin{array}{c} \frac{\partial N_1}{\partial \xi_1} \\ \frac{\partial N_1}{\partial \xi_2} \end{array} \right\}$$

Since $\frac{\partial N_1}{\partial \xi_1} = 1; \quad \text{and} \quad \frac{\partial N_1}{\partial \xi_2} = 0$

We get

$$\begin{aligned} \left\{ \begin{array}{c} \frac{\partial N_1}{\partial x} \\ \frac{\partial N_1}{\partial y} \end{array} \right\} &= \frac{1}{2A} \left[\begin{array}{cc} y_2 - y_3 & y_3 - y_1 \\ x_3 - x_2 & x_1 - x_3 \end{array} \right] \left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\} \\ &= \frac{1}{2A} \left\{ \begin{array}{c} y_2 - y_3 \\ x_3 - x_2 \end{array} \right\} \end{aligned}$$

FE matrices for T3 element

Similarly again using T-J2 to find partial derivatives w.r.t. N_2

$$\left\{ \begin{array}{c} \frac{\partial N_2}{\partial x} \\ \frac{\partial N_2}{\partial y} \end{array} \right\} = \left\{ \begin{array}{c} \frac{\partial \xi_2}{\partial x} \\ \frac{\partial \xi_2}{\partial y} \end{array} \right\} = \frac{1}{2A} \left[\begin{array}{cc} y_2 - y_3 & y_3 - y_1 \\ x_3 - x_2 & x_1 - x_3 \end{array} \right] \left\{ \begin{array}{c} \frac{\partial N_2}{\partial \xi_1} \\ \frac{\partial N_2}{\partial \xi_2} \end{array} \right\}$$

Since $\frac{\partial N_2}{\partial \xi_1} = 0; \quad \text{and} \quad \frac{\partial N_2}{\partial \xi_2} = 1$

We get

$$\begin{aligned} \left\{ \begin{array}{c} \frac{\partial N_2}{\partial x} \\ \frac{\partial N_2}{\partial y} \end{array} \right\} &= \frac{1}{2A} \left[\begin{array}{cc} y_2 - y_3 & y_3 - y_1 \\ x_3 - x_2 & x_1 - x_3 \end{array} \right] \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\} \\ &= \frac{1}{2A} \left\{ \begin{array}{c} y_3 - y_1 \\ x_1 - x_3 \end{array} \right\} \end{aligned}$$

FE matrices for T3 element

Similarly again using T-J2 to find partial derivatives w.r.t. N_3

$$\left\{ \begin{array}{c} \frac{\partial N_3}{\partial x} \\ \frac{\partial N_3}{\partial y} \end{array} \right\} = \left\{ \begin{array}{c} \frac{\partial \xi_3}{\partial x} \\ \frac{\partial \xi_3}{\partial y} \end{array} \right\} = \frac{1}{2A} \left[\begin{array}{cc} y_2 - y_3 & y_3 - y_1 \\ x_3 - x_2 & x_1 - x_3 \end{array} \right] \left\{ \begin{array}{c} \frac{\partial N_3}{\partial \xi_1} \\ \frac{\partial N_3}{\partial \xi_2} \end{array} \right\}$$

Since

$$\frac{\partial N_3}{\partial \xi_1} = \frac{\partial (1 - \xi_1 - \xi_2)}{\partial \xi_1} = -1; \quad \text{and} \quad \frac{\partial N_3}{\partial \xi_2} = \frac{\partial (1 - \xi_1 - \xi_2)}{\partial \xi_2} = -1$$

We get

$$\begin{aligned} \left\{ \begin{array}{c} \frac{\partial N_3}{\partial x} \\ \frac{\partial N_3}{\partial y} \end{array} \right\} &= \frac{1}{2A} \left[\begin{array}{cc} y_2 - y_3 & y_3 - y_1 \\ x_3 - x_2 & x_1 - x_3 \end{array} \right] \left\{ \begin{array}{c} -1 \\ -1 \end{array} \right\} \\ &= \frac{1}{2A} \left\{ \begin{array}{c} y_1 - y_2 \\ x_2 - x_1 \end{array} \right\} \end{aligned}$$

FE matrices for T3 element

Now we have all elements of the $\mathbf{B}$ matrix

$$\begin{aligned} \mathbf{B} &= \left[\begin{array}{cccccc} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{array} \right] \\ &= \frac{1}{2A} \left[\begin{array}{cccccc} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2 \end{array} \right] \end{aligned}$$

Thus $\mathbf{B}$ is constant for the T3 element. This implies that strain and stress states are constant withing the element as:

$\boldsymbol{\epsilon} = \mathbf{Bq}$ 7 and $\boldsymbol{\sigma} = \mathbf{D}\,\boldsymbol{\epsilon}$ 9

Therefore the T3 element is also called a Constant Strain Triangle or CST.

FE matrices for T3 element

The stiffness matrix of an element with thickness t is given by

$$\mathbf{K} = t\int_A \mathbf{B}^T\mathbf{D}\,\mathbf{B}\,dA$$

13

Since $\mathbf{B}$ is constant we get

$$\mathbf{K} = t\mathbf{B}^T\mathbf{D}\ \mathbf{B}\int_A dA = t\mathbf{B}^T\mathbf{D}\ \mathbf{B}\,A$$

Thus no integration is required for the evaluation of the stiffness matrix

FE matrices for T3 element

The equivalent nodal loads matrix of an element with thickness t is given by

$$\mathbf{p}_b = t\int_A \mathbf{N}^T\mathbf{b}\,dA$$

14

or

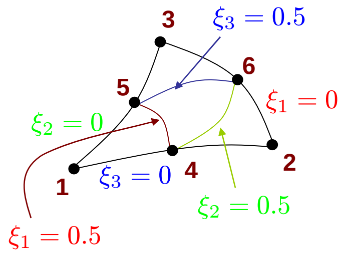
$$\mathbf{p}_b = t\int_A \begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_1 \\ \xi_2 & 0 \\ 0 & \xi_2 \\ \xi_3 & 0 \\ 0 & \xi_3 \end{bmatrix} \mathbf{b}\,dA$$

Thus we need to integrate. However it can be shown for a triangle that

$$\int_A \xi_1^a \xi_2^b \xi_3^c\,dA = \frac{a!\,b!\,c!}{(a+b+c+2)!}2A$$

which simplifies the integration

T6 element – Linear strain triangle



$$
\begin{aligned}
N_1 &= (2\xi_1 - 1)\xi_1 \\
N_2 &= (2\xi_2 - 1)\xi_2 \\
N_3 &= (2\xi_3 - 1)\xi_3 \\
N_4 &= 4\xi_1\xi_2 \\
N_5 &= 4\xi_2\xi_3 \\
N_6 &= 4\xi_1\xi_3
\end{aligned}
$$

Numerical integration for triangular elements

The numerical integration formula for a triangle is

$$I = \int_A f(\xi_1, \xi_2, \xi_3) dA = \frac{1}{2} \sum_{j=1}^{n} W_j f(\xi_1, \xi_2, \xi_3)_j \det \mathbf{J}(\xi_1, \xi_2)_j$$

Numerical integration for triangular elements

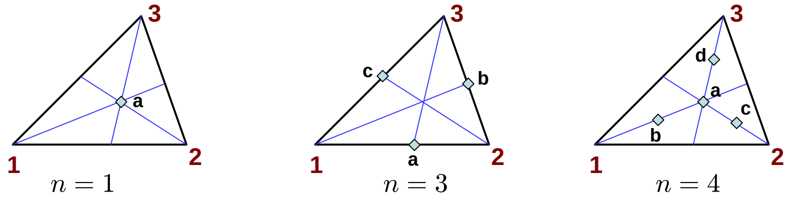
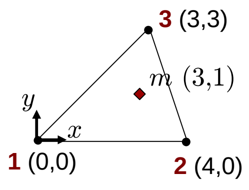


Fig	*n*	order	pts	ξ_1	ξ_2	ξ_3	W_j
1	1	Linear	a	1/3	1/3	1/3	1
2	3	Quadratic	a	1/2	1/2	0	1/3
			b	0	1/2	1/2	1/3
			c	1/2	0	1/2	1/3
3	4	Cubic	a	1/3	1/3	1/3	-27/28
			b	0.6	0.2	0.2	25/48
			c	0.2	0.6	0.2	25/48
			d	0.2	0.2	0.6	25/48

Example T3-3

For the element shown the x-y coordinates of point m are as shown in the figure.
Find the coordinates of m in terms of ξ_1, ξ_2, ξ_3.



From isoparametric mapping we know

$$x = \xi_1 x_1 + \xi_2 x_2 + (1-\xi_1-\xi_2)x_3 = 0\xi_1 + 4\xi_2 + 3(1-\xi_1-\xi_2)$$
$$y = \xi_1 y_1 + \xi_2 y_2 + (1-\xi_1-\xi_2)y_3 = 0\xi_1 + 0\xi_2 + 3(1-\xi_1-\xi_2)$$

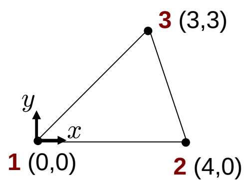
If we substitute the x-y coordinates of m we can then solve for corresponding ξ_1, ξ_2, ξ_3 values

$$3 = 4\xi_2 + 3(1-\xi_1-\xi_2)$$
$$1 = 3(1-\xi_1-\xi_2)$$

or

$$\begin{aligned} 3\xi_1 - \xi_2 &= 0 \\ 3\xi_1 + 3\xi_2 &= 2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \xi_1 &= 1/6 \\ \xi_2 &= 1/2 \\ \xi_3 &= 1/3 \end{aligned}$$

Example T3-4



Find the k_{34} term of the element stiffness matrix $\mathbf{K}$ for the T3 element shown. Assume it is a plane stress element with $E = 2 \times 10^5$, $\nu = 0.25$ and $t = 1$.

$$\mathbf{K} = t\mathbf{B}^T\mathbf{D}\;\mathbf{B}\,A$$

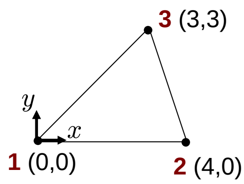
where

$$A = \frac{1}{2} \times 4 \times 3 = 6$$

also recall

$$\begin{aligned} 2A = \det \mathbf{J} &= (x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3) \\ &= (-3) \times (-3) - (1) \times (-3) \\ &= 12 \end{aligned}$$

Example T3-4 (cont.)

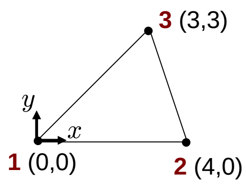


$$\mathbf{K} = t\mathbf{B}^T\mathbf{D}\;\mathbf{B}\,A$$

$$\begin{aligned} \mathbf{B} &= \frac{1}{2A}\begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2 \end{bmatrix} \\ &= \frac{1}{12}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -3 & 0 & 4 \\ -1 & -3 & -3 & 3 & 4 & 0 \end{bmatrix} \end{aligned}$$

$$\mathbf{D} = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 213333 & 53333 & 0 \\ 53333 & 213333 & 0 \\ 0 & 0 & 80000 \end{bmatrix}$$

Example T3-4 (cont.)



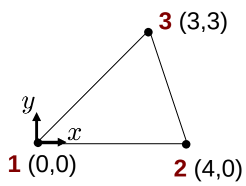
$$\mathbf{K} = t\mathbf{B}^T\mathbf{D}\ \mathbf{B}\,A$$

$$\mathbf{K} = \frac{6}{144}\begin{bmatrix} -3 & 0 & -1 \\ 0 & -1 & -3 \\ 3 & 0 & -3 \\ 0 & -3 & 3 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}\begin{bmatrix} 213333 & 53333 & 0 \\ 53333 & 213333 & 0 \\ 0 & 0 & 80000 \end{bmatrix}$$

$$\times\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -3 & 0 & 4 \\ -1 & -3 & -3 & 3 & 4 & 0 \end{bmatrix}$$

$$k_{34} = -0.5 \times 10^5$$

Example T3-5



Find equivalent nodal loads matrix due to gravity loads (acting downwards) for the T3 element shown.
The weight density of the material of the element is 24000 N/m^3.
Assume $t = 1$.

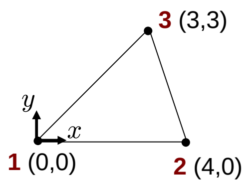
The equivalent nodal loads are given by

$$\mathbf{p}_b = t\int_A \mathbf{N}^T\mathbf{b}dA = t\int_A \begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_1 \\ \xi_2 & 0 \\ 0 & \xi_2 \\ \xi_3 & 0 \\ 0 & \xi_3 \end{bmatrix}\mathbf{b}dA$$

where

$$\mathbf{b} = \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ -24000 \end{Bmatrix}$$

Example T3-5 (cont.)



$$\mathbf{p}_b = 1 \times \int_A \begin{bmatrix} 0 \\ -24000\xi_1 \\ 0 \\ -24000\xi_2 \\ 0 \\ -24000\xi_3 \end{bmatrix} dA$$

Each term can be integrated separately

Recall

$$\int_A \xi_1^a \xi_2^b \xi_3^c dA = \frac{a!\, b!\, c!}{(a+b+c+2)!} 2A$$

Therefore

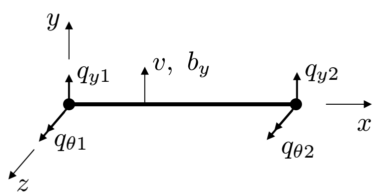
$$p_{b2} = -24000 \int_A \xi_1 dA = -24000 \frac{1!\, 0!\, 0!}{(1+0+0+2)!} \times 2 \times 6 = -48000$$

Similarly $p_{b4} = p_{b6} = -48000$ i.e. the equivalent nodal load at each node is 1/3 the weight of the element

Thin beam elements – Euler Bernoulli

- We now consider a FE basis of a beam element of the kind you have discussed in matrix structural analysis
- We have discussed axial (or truss elements), which could only deform along their length i.e. they could not bend
- In Abaqus we modelled a cantilever using plane stress elements — not ideal for skeletal structural members

Consider a 2-node straight flexural element in which $x - y$ plane is the plane of bending. Let us ignore axial displacements



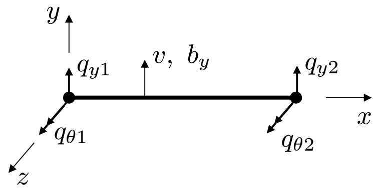
Generic displacements

$$\underset{1\times 1}{\mathbf{u}} = \{v(x)\} = v \qquad (1)$$

Body forces

$$\underset{1\times 1}{\mathbf{b}} = \{b_y(x)\} = b_y \qquad (2)$$

Thin beam elements – Euler Bernoulli



Nodal displacements matrix

$$\underset{4\times 1}{\mathbf{q}} = \begin{Bmatrix} q_{y1} \\ q_{\theta 1} \\ q_{y2} \\ q_{\theta 2} \end{Bmatrix}$$ 3

where

$$q_{\theta 1} = \left(\frac{dv}{dx}\right)_1 ; \qquad q_{\theta 2} = \left(\frac{dv}{dx}\right)_2$$ 3a

are small rotations (or slopes) at nodes 1 and 2

Nodal loads matrix

$$\underset{4\times 1}{\mathbf{p}} = \begin{Bmatrix} p_{y1} \\ M_1 \\ p_{y2} \\ M_2 \end{Bmatrix}$$

force acting in y direction at node 1
moment about z at node 1
force acting in y direction at node 2
moment about z at node 2

4

Thin beam elements – Euler Bernoulli

Relating generic displacements to nodal displacements using shape functions

$$\underset{1\times 1}{\mathbf{u}} = \underset{1\times 4}{\mathbf{N}}\,\underset{4\times 1}{\mathbf{q}}$$ 5

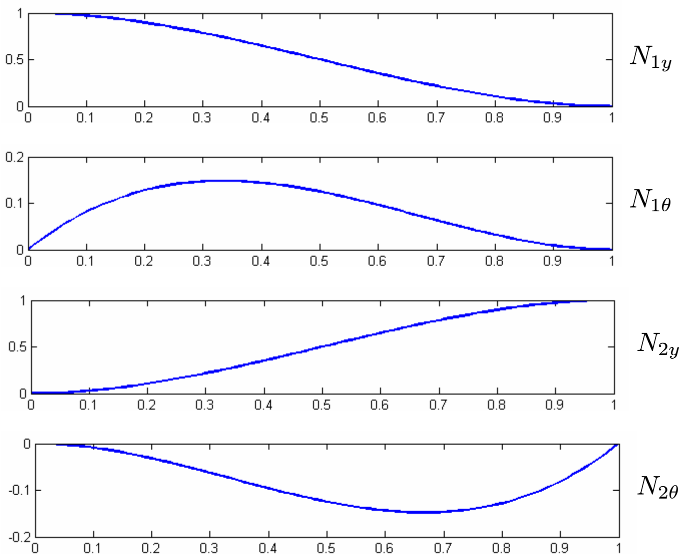
where

$$\mathbf{N} = \begin{bmatrix} N_{1y} & N_{1\theta} & N_{2y} & N_{2\theta} \end{bmatrix}$$ 5a

and

$$\begin{aligned} N_{1y} &= \frac{1}{L^3}(2x^3 - 3x^2L + L^3) \\ N_{1\theta} &= \frac{1}{L^3}(x^3L - 2x^2L^2 + xL^3) \\ N_{2y} &= \frac{1}{L^3}(-2x^3 + 3x^2L) \\ N_{2\theta} &= \frac{1}{L^3}(x^3L - x^2L^2) \end{aligned}$$ 5b

Thin beam elements – Euler Bernoulli



Note: the slope at node 1 is unity

Note: the slope at node 2 is unity

Thin beam elements – Euler Bernoulli

You can also verify that at

$$x = 0; \qquad N_{1y} = 1, \quad N_{1\theta} = 0, \quad N_{2y} = 0, \quad N_{2\theta} = 0$$

and at

$$x = L; \qquad N_{1y} = 0, \quad N_{1\theta} = 0, \quad N_{2y} = 1, \quad N_{2\theta} = 0$$

Also at

$$x = 0; \qquad \frac{dN_{1y}}{dx} = 0, \quad \frac{dN_{1\theta}}{dx} = 1, \quad \frac{dN_{2y}}{dx} = 0, \quad \frac{dN_{2\theta}}{dx} = 0$$

and at

$$x = L; \qquad \frac{dN_{1y}}{dx} = 0, \quad \frac{dN_{1\theta}}{dx} = 0, \quad \frac{dN_{2y}}{dx} = 0, \quad \frac{dN_{2\theta}}{dx} = 1$$

So shape functions or their derivatives have a value of unity for the DOF they represent and zero at other DOFs

Thin beam elements – Euler Bernoulli

Derivation of shape functions

Since there are 4 nodal "displacements, a complete cubic displacement function may be assumed for the flexural element

$$v = c_1 + c_2 x + c_3 x^2 + c_4 x^3 \qquad \text{a}$$

at $\quad x = 0; \quad v = q_{y1} = c_1; \quad \Rightarrow c_1 = q_{y1}$

also at $\quad x = 0; \quad \frac{dv}{dx} = q_{\theta 1} = [c_2 + 2c_3 x + 3c_4 x^2]_{x=0}; \quad \Rightarrow c_2 = q_{\theta 1}$

$$\begin{aligned}
\text{at } x = L; \quad v = q_{y2} &= c_1 + c_2 L + c_3 L^2 + c_4 L^3 \\
&= q_{y1} + q_{\theta 1} L + c_3 L^2 + c_4 L^3 \\
\Rightarrow \quad & c_3 + c_4 L = \frac{1}{L^2}[q_{y2} - q_{y1} - q_{\theta 1} L] \qquad \text{b}
\end{aligned}$$

$$\begin{aligned}
\text{also at } x = L; \quad \frac{dv}{dx} = q_{\theta 2} &= [c_2 + 2c_3 L + 3c_4 L^2] \\
&= q_{\theta 1} + 2c_3 L + 3c_4 L^2 \\
\Rightarrow \quad & c_3 + \frac{3}{2} c_4 L = \frac{1}{2L}[q_{\theta 2} - q_{\theta 1}] \qquad \text{c}
\end{aligned}$$

Thin beam elements – Euler Bernoulli

Solving for c_3 and c_4 using b and c we get

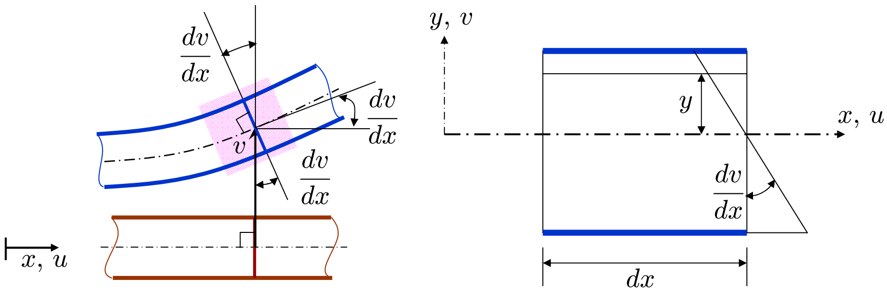
$$\begin{aligned}
c_3 &= -\frac{3q_{y1}}{L^2} - \frac{2q_{\theta 1}}{L} + \frac{3q_{y2}}{L^2} + \frac{q_{\theta 2}}{L} \\
c_4 &= \frac{2q_{y1}}{L^3} + \frac{q_{\theta 1}}{L^2} - \frac{2q_{y2}}{L^3} + \frac{q_{\theta 2}}{L^2}
\end{aligned}$$

Putting values of c_1, c_2, c_3 and c_4 in a

$$\begin{aligned}
v &= q_{y1} + q_{\theta 1} x + [-\frac{3q_{y1}}{L^2} - \frac{2q_{\theta 1}}{L} + \frac{3q_{y2}}{L^2} + \frac{q_{\theta 2}}{L}]x^2 \\
&\quad + [\frac{2q_{y1}}{L^3} + \frac{q_{\theta 1}}{L^2} - \frac{2q_{y2}}{L^3} + \frac{q_{\theta 2}}{L^2}]x^3
\end{aligned}$$

$$v = \frac{1}{L^3}\left[\ (2x^3 - 3x^2 L + L^3) \quad (x^3 L - 2x^2 L^2 + xL^3) \quad (-2x^3 + 3x^2 L) \quad (x^3 L - x^2 L^2)\ \right] \begin{Bmatrix} q_{y1} \\ q_{\theta 1} \\ q_{y2} \\ q_{\theta 2} \end{Bmatrix}$$

Thin beam elements – Euler Bernoulli



Strain displacement relations in general form

$$\boldsymbol{\epsilon} = \mathbf{Lu} \quad (6)$$

Displacement in the x-direction from figures above $u = -y\dfrac{dv}{dx}$ (6a)

So flexural strain

$$\epsilon_x = \frac{du}{dx} = -y\frac{d^2v}{dx^2} = -y\phi \quad (6b)$$

where ϕ represents the curvature and $\phi = \dfrac{d^2v}{dx^2}$ (6c)

Thin beam elements – Euler Bernoulli

Strain displacement relations for this element

$$\underset{1\times 1}{\boldsymbol{\epsilon}} = \epsilon_x = \underset{1\times 1}{\mathbf{L}}\underset{1\times 1}{\mathbf{u}} = \left[-y\frac{d^2}{dx^2}\right]\{v\} \quad (6)$$

$$\boldsymbol{\epsilon} = \mathbf{LNq} \quad \text{or} \quad \underset{1\times 1}{\boldsymbol{\epsilon}} = \underset{1\times 4}{\mathbf{B}}\underset{4\times 1}{\mathbf{q}} \quad (7)$$

where

$$\begin{aligned}\mathbf{B} &= \mathbf{LN}\\ &= -y\frac{d^2}{dx^2}\left[\begin{array}{cccc} N_{1y} & N_{1\theta} & N_{2y} & N_{2\theta}\end{array}\right]\\ &= \frac{-y}{L^3}\left[\begin{array}{cccc} 12x-6L & 6xL-4L^2 & -12x+6L & 6xL-2L^2\end{array}\right]\end{aligned} \quad (8)$$

Thin beam elements – Euler Bernoulli

Stress-strain relations in general

$$\boldsymbol{\sigma} = \mathbf{D}\,\boldsymbol{\epsilon} \quad (9)$$

For this element

$$\boldsymbol{\sigma} = \sigma_x = \mathbf{D}\,\boldsymbol{\epsilon} = E\epsilon_x \quad (9a)$$

$$\boldsymbol{\sigma} = \sigma_x = \mathbf{DBq} = E\mathbf{Bq} \quad (10)$$

$$\begin{aligned}\mathbf{K} &= \int_V \mathbf{B}^T E \mathbf{B} dV \\ &= \int_0^L \int_A \frac{Ey^2}{L^6} \begin{bmatrix} 12x - 6L \\ 6xL - 4L^2 \\ -12x + 6L \\ 6xL - 2L^2 \end{bmatrix} \begin{bmatrix} 12x - 6L & \cdots & \cdots & \cdots \end{bmatrix} dAdx \quad (13)\end{aligned}$$

Note $\int_A y^2 dA = I$ second moment of area of beam cross-section

Thin beam elements – Euler Bernoulli

So stiffness matrix of a beam with constant EI can be obtained by integrating each term as

$$\mathbf{K} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \quad (13a)$$

★ We have ignored axial displacement which can be easily included
★ The matrix is identical to that obtained from direct stiffness approach

Equivalent nodal loads matrix

$$\mathbf{p}_b = \int_V \mathbf{N}^T \mathbf{b} dV = \int_0^L \mathbf{N}^T b_y(x) dx \quad (14)$$

For udl i.e. $b_y(x) = b_y$

Thin beam elements – Euler Bernoulli

Equivalent nodal loads matrix

$$\mathbf{p}_b = \int_V \mathbf{N}^T \mathbf{b} dV = \int_0^L \mathbf{N}^T b_y(x) dx$$

14

For udl i.e. $b_y(x) = b_y$

$$\mathbf{p}_b = b_y \int_0^L \begin{bmatrix} N_{1y} \\ N_{1\theta} \\ N_{2y} \\ N_{2\theta} \end{bmatrix} dx = \frac{b_y L}{12} \begin{bmatrix} 6 \\ L \\ 6 \\ -L \end{bmatrix}$$

Thin beam elements – Euler Bernoulli

★ The assumption that plane sections normal to the neutral axis prior to deformation remain plane and normal to the neutral axis after deformation led us to derive

$$u = -y \frac{dv}{dx}$$

Because the Euler Bernoulli beam theory assumes that $v(x)$ is independent of y and the above equation we get the shear strain γ_{xy} as

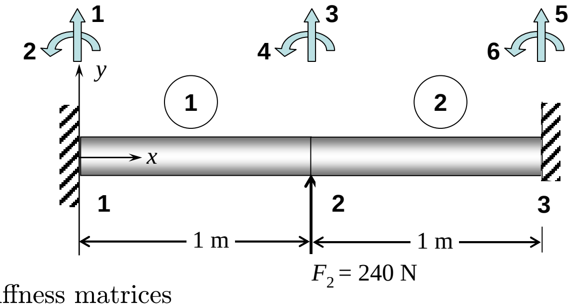
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x} = 0$$

Thus Euler-Bernoulli beam theory predicts zero shear strain.

★ We have only discussed beams aligned in the x-direction. If they are inclined transformations can be applied.

Example – EB1

A beam of length 2 m is clamped at both ends and subjected to a transverse concentrated force of 240 N at the mid-span. Find deflections and slopes at $x = 0.5$, 1.0, and 1.5 m.



Element stiffness matrices

$$\mathbf{K}^1 = 1000 \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \qquad \mathbf{K}^2 = 1000 \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

Superscripts indicate element number

Example – EB1

Structural equations of equilibrium

$$1000 \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ q_{y2} \\ q_{\theta 2} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} p_{y1} \\ M_1 \\ 240 \\ 0 \\ p_{y3} \\ M_3 \end{Bmatrix}$$

- ★ You can see that the free DOFs can be solved for by simply using the 2×2 part of the stiffness matrix in this case – just delete rows and columns corresponding to restrained DOFs
- ★ This is equivalent to deleting rows and columns associated with the fully restrained DOFs

$$\begin{bmatrix} 24000 & 0 \\ 0 & 8000 \end{bmatrix} \begin{Bmatrix} q_{y2} \\ q_{\theta 2} \end{Bmatrix} = \begin{Bmatrix} 240 \\ 0 \end{Bmatrix} \quad \text{which gives} \quad \begin{aligned} q_{y2} &= 0.01 \\ q_{\theta 2} &= 0 \end{aligned}$$

Example – EB1

Deflection and slope at different points can be found using shape functions

Using nodal displacements for element 1 we get for $x = 0.5$

$$\begin{aligned}
\mathbf{u} = v(0.5) = \mathbf{Nq} &= N_{1y}q_{y1} + N_{\theta 1}q_{\theta 1} + N_{2y}q_{y2} + N_{\theta 2}q_{\theta 2} \\
&= N_{1y}(0.5) \times 0 + N_{\theta 1}(0.5) \times 0 + N_{2y} \times 0.01 + N_{\theta 2} \times 0 \\
&= \left(\frac{3 \times 0.5^2}{1^2} - \frac{2 \times 0.5^3}{1^3} \right) 0.01 \\
&= 0.005\,\mathrm{m}
\end{aligned}$$

$$\begin{aligned}
\theta(0.5) = \left(\frac{\partial v}{\partial x}\right)_{0.5} &= \left(\frac{\partial \mathbf{N}}{\partial x}\right)_{0.5} \\
\mathbf{q} &= \frac{\partial N_{y1}}{\partial x} \times 0 + \frac{\partial N_{\theta 1}}{\partial x} \times 0 + \frac{\partial N_{y2}}{\partial x} \times 0.01 + \frac{\partial N_{\theta 2}}{\partial x} \times 0 \\
&= \left(\frac{6x}{1^2} - \frac{6x^2}{1^3} \right)_{x=0.5} \times 0.01 \\
&= 1.5 \times 0.01 \\
&= 0.015\,\mathrm{rad}
\end{aligned}$$

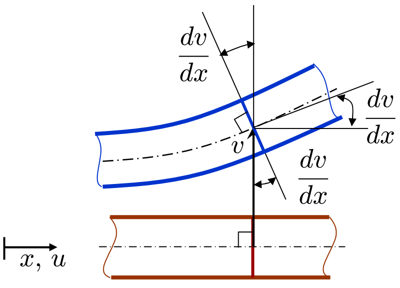
Example – EB1

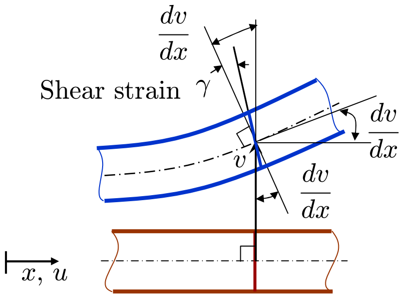
Deflection and slope at $x = 1$ m are already known. Values at $x = 1.5$ can be found using element 2

$$\begin{aligned}
v(1.0) &= 0.01\,\mathrm{m} \\
\theta(1.0) &= 0\,\mathrm{rad} \\
v(1.5) &= 0.005\,\mathrm{m} \\
\theta(1.5) &= -0.015\,\mathrm{rad}
\end{aligned}$$

Solve this problem using symmetry

Thick beam elements – Timoshenko





Euler-Bernoulli

- No shear deformation
- Rotation of the normal

$$\theta = \frac{dv}{dx}$$

Timoshenko

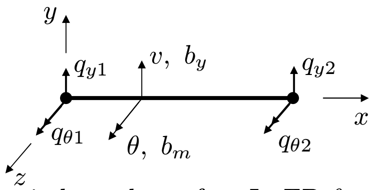
- Shear deformation γ
- Rotation of the normal

$$\theta = \frac{dv}{dx} - \gamma$$

Ukraine commemorative stamp in honour of S. P. Timoshenko.

Thick beam elements – Timoshenko

Consider a 2-node straight flexural element in which $x-y$ plane is the plane of bending. Let us ignore axial displacements



Generic displacements

$$\mathbf{u} = \begin{Bmatrix} v(x) \\ \theta(x) \end{Bmatrix} = \begin{Bmatrix} v \\ \theta \end{Bmatrix}$$ 1

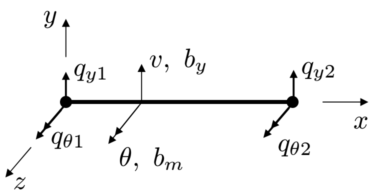
Unlike EB generic rotation is considered to be independent of v. In EB θ can be found from v.

Body forces

$$\underset{2\times 1}{\mathbf{b}} = \begin{Bmatrix} b_y(x) \\ b_m(x) \end{Bmatrix} = \begin{Bmatrix} b_y \\ b_m \end{Bmatrix}$$ 2

where M denotes moment

Thick beam elements – Timoshenko



Nodal displacements matrix

$$\underset{4\times 1}{\mathbf{q}} = \begin{Bmatrix} q_{y1} \\ q_{\theta 1} \\ q_{y2} \\ q_{\theta 2} \end{Bmatrix}$$ 3

Nodal loads matrix

$$\underset{4\times 1}{\mathbf{p}} = \begin{Bmatrix} p_{y1} \\ M_1 \\ p_{y2} \\ M_2 \end{Bmatrix} \begin{matrix} \text{force acting in } y \text{ direction at node 1} \\ \text{moment about } z \text{ at node 1} \\ \text{force acting in } y \text{ direction at node 2} \\ \text{moment about } z \text{ at node 2} \end{matrix}$$ 4

Thick beam elements – Timoshenko

Relating generic displacements to nodal displacements using shape functions

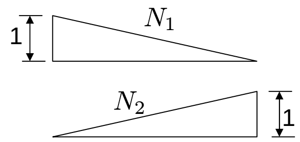
$$\underset{2\times 1}{\mathbf{u}} = \underset{2\times 4}{\mathbf{N}}\,\underset{4\times 1}{\mathbf{q}} \qquad (5)$$

where

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \qquad (5a)$$

$$\begin{aligned} v &= N_1 q_{y1} + N_2 q_{y2} \\ \theta &= N_1 q_{\theta 1} + N_2 q_{\theta 2} \end{aligned}$$

$$N_1 = 1 - \frac{x}{L}$$
$$N_2 = \frac{x}{L}$$

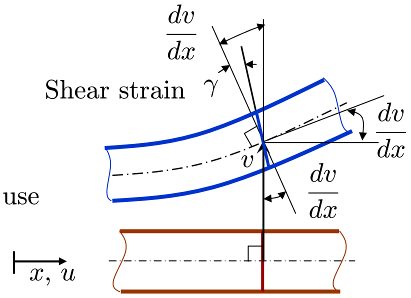


- ★ We used the same shape functions for the axial element
- ★ Thus v and θ vary linearly based on nodal values

Thick beam elements – Timoshenko

Strain displacement relations
in general form

$$\boldsymbol{\epsilon} = \mathbf{L}\mathbf{u} \qquad (6)$$



- ★ Instead to using strain here we will use a *type of strain*.
- ★ This is called *generalised strain.*

$$\left\{ \begin{matrix} \phi \\ \gamma \end{matrix} \right\} = \begin{bmatrix} 0 & \frac{d}{dx} \\ \frac{d}{dx} & -1 \end{bmatrix} \left\{ \begin{matrix} v \\ \theta \end{matrix} \right\} \qquad (6a)$$

where $\phi = \dfrac{d\theta}{dx}$ is the curvature

Thick beam elements – Timoshenko

$$\boldsymbol{\epsilon} = \mathbf{LNq} \quad \text{or} \quad \underset{2\times 1}{\boldsymbol{\epsilon}} = \left\{ \begin{array}{c} \phi \\ \gamma \end{array} \right\} = \mathbf{Bq} = \underset{2\times 4}{\mathbf{B}} \underset{4\times 1}{\left\{ \begin{array}{c} q_{y1} \\ q_{\theta 1} \\ q_{y2} \\ q_{\theta 2} \end{array} \right\}} \quad (7)$$

where

$$\begin{aligned} \mathbf{B} &= \mathbf{LN} \\ &= \left[\begin{array}{cc} 0 & \frac{d}{dx} \\ \frac{d}{dx} & -1 \end{array} \right] \left[\begin{array}{cccc} 1-\frac{x}{L} & 0 & \frac{x}{L} & 0 \\ 0 & 1-\frac{x}{L} & 0 & \frac{x}{L} \end{array} \right] \\ &= \left[\begin{array}{cccc} 0 & -\frac{1}{L} & 0 & \frac{1}{L} \\ -\frac{1}{L} & -1+\frac{x}{L} & \frac{1}{L} & -\frac{x}{L} \end{array} \right] \end{aligned} \quad (8)$$

Thick beam elements – Timoshenko

Stress-strain relations in general

$$\boldsymbol{\sigma} = \mathbf{D}\,\boldsymbol{\epsilon} \quad (9)$$

The *generalised strains*, curvature ϕ and shear strain γ for this element can be related to *generalised stresses* – bending moment M and shear force V

Recall

$$M = EI\phi$$

$$V = \tau A_s = \gamma G A_s = \gamma \frac{GA}{f_s} = S\gamma$$

★ A_s is equivalent shear area and A is full cross-sectional area
★ $S = GA/f_s$
★ $f_s = A/A_s$ depends on cross-sectional shape. For rectangular sections $f_s = 6/5 = 1.2$

Thus we have the generalised stress-strain relationship as

$$\boldsymbol{\sigma} = \left\{ \begin{array}{c} M \\ V \end{array} \right\} = \mathbf{D}\,\boldsymbol{\epsilon} = \left[\begin{array}{cc} EI & 0 \\ 0 & S \end{array} \right] \left\{ \begin{array}{c} \phi \\ \gamma \end{array} \right\} \quad (9a)$$

Thick beam elements – Timoshenko

$$\mathbf{K} = \int_0^L \mathbf{B}^T\mathbf{D}\ \mathbf{B}\,dx$$

13

$$= \int_0^L \begin{bmatrix} 0 & -\frac{1}{L} \\ -\frac{1}{L} & -1+\frac{x}{L} \\ 0 & \frac{1}{L} \\ \frac{1}{L} & -\frac{x}{L} \end{bmatrix} \begin{bmatrix} EI & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{L} & 0 & \frac{1}{L} \\ -\frac{1}{L} & -1+\frac{x}{L} & \frac{1}{L} & -\frac{x}{L} \end{bmatrix} dx$$

$$\mathbf{K} = \int_0^L \begin{bmatrix} \frac{S}{L^2} & \frac{S}{L}\left(1-\frac{x}{L}\right) & -\frac{S}{L^2} & \frac{Sx}{L^2} \\ \frac{S}{L}\left(1-\frac{x}{L}\right) & \frac{EI}{L^2}+S\left(1-\frac{x}{L}\right)^2 & -\frac{S}{L}\left(1-\frac{x}{L}\right) & -\frac{EI}{L^2}+\frac{Sx}{L}\left(1-\frac{x}{L}\right) \\ -\frac{S}{L^2} & -\frac{S}{L}\left(1-\frac{x}{L}\right) & \frac{S}{L^2} & -\frac{Sx}{L^2} \\ \frac{Sx}{L^2} & -\frac{EI}{L^2}+\frac{Sx}{L}\left(1-\frac{x}{L}\right) & -\frac{Sx}{L^2} & \frac{EI}{L^2}+\frac{Sx^2}{L^2} \end{bmatrix} dx$$

13a

Thick beam elements – Timoshenko

Integrating 13a we get

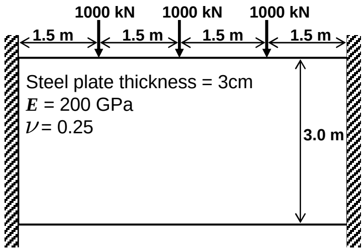
$$\mathbf{K} = \begin{bmatrix} \frac{S}{L} & \frac{S}{2} & -\frac{S}{L} & \frac{S}{2} \\ \frac{S}{2} & \frac{EI}{L}+\frac{SL}{3} & -\frac{S}{2} & -\frac{EI}{L}+\frac{SL}{6} \\ -\frac{S}{L} & -\frac{S}{2} & \frac{S}{L} & -\frac{S}{2} \\ \frac{S}{2} & -\frac{EI}{L}+\frac{SL}{6} & -\frac{S}{2} & \frac{EI}{L}+\frac{SL}{3} \end{bmatrix}$$

Research shows that the above matrix is too stiff and the element displays a phenomenon called shear locking. This can be corrected to a significant extent by reduced (one point) integration which gives

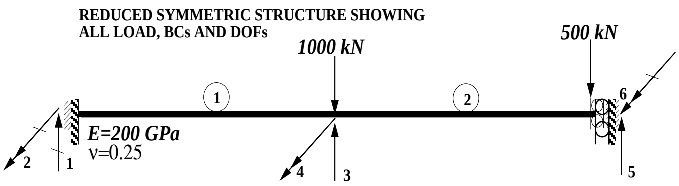
$$\mathbf{K} = \begin{bmatrix} \frac{S}{L} & \frac{S}{2} & -\frac{S}{L} & \frac{S}{2} \\ \frac{S}{2} & \frac{EI}{L}+\frac{SL}{4} & -\frac{S}{2} & -\frac{EI}{L}+\frac{SL}{4} \\ -\frac{S}{L} & -\frac{S}{2} & \frac{S}{L} & -\frac{S}{2} \\ \frac{S}{2} & -\frac{EI}{L}+\frac{SL}{4} & -\frac{S}{2} & \frac{EI}{L}+\frac{SL}{4} \end{bmatrix}$$

Example – TB1

Steel diaphragm (supporting bridge girders) between rigid boundaries



2 beam element model exploiting problem symmetry



Example – TB1

For a Timoshenko beam element (deep beam), the stiffness matrix is

$$\mathbf{K_e} = \begin{pmatrix} \frac{S}{L} & \frac{S}{2} & \frac{-S}{L} & \frac{S}{2} \\ \text{sym.} & \frac{EI}{L} + \frac{SL}{3} & \frac{-S}{2} & -\frac{EI}{L} + \frac{SL}{6} \\ \text{sym.} & \text{sym.} & \frac{S}{L} & \frac{-S}{2} \\ \text{sym.} & \text{sym.} & \text{sym.} & \frac{EI}{L} + \frac{SL}{3} \end{pmatrix}$$

$$\begin{aligned} E &= 200 \times 10^9 \,\mathrm{N/m^2} \\ G &= \frac{E}{2(1+\nu)} = 8 \times 10^{10} \,\mathrm{N/m^2} \\ A &= 0.03 \times 3 = 0.09 \,\mathrm{m^2} \\ S &= \frac{GA}{f_s} = \frac{8 \times 10^{10}}{1.2} = 6 \times 10^9 \,\mathrm{N/m^2} \\ L &= 1.5 \,\mathrm{m} \\ I &= \frac{0.03 \times 3^3}{12} = 0.0675 \,\mathrm{N^4} \end{aligned}$$

Note: Both elements have identical dimensions – so will have identical element stiffness matrices

Example – TB1

The equilibrium equations for the model after assembling the element stiffness matrices and the nodal loads matrices (there are no non-nodal loads)

$$10^9 \begin{pmatrix} 4 & 3 & -4 & 3 & 0 & 0 \\ \text{sym.} & 12 & -3 & -7.5 & 0 & 0 \\ \text{sym.} & \text{sym.} & 8 & 0 & -4 & 3 \\ \text{sym.} & \text{sym.} & \text{sym.} & 24 & -3 & -7.5 \\ \text{sym.} & \text{sym.} & \text{sym.} & \text{sym.} & 4 & -3 \\ \text{sym.} & \text{sym.} & \text{sym.} & \text{sym.} & \text{sym.} & 12 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ q_{y2} \\ q_{\theta 2} \\ q_{y3} \\ 0 \end{pmatrix} = \begin{pmatrix} p_{y1} \\ M_1 \\ -1 \times 10^6 \\ 0 \\ -0.5 \times 10^6 \\ M_3 \end{pmatrix}$$

To solve for displacements corresponding to free DOFs simply delete rows and columns corresponding to restrained DOFs

$$10^9 \begin{pmatrix} 8 & 0 & -4 \\ 0 & 24 & -3 \\ -4 & -3 & 4 \end{pmatrix} \begin{pmatrix} q_{y2} \\ q_{\theta 2} \\ q_{y3} \end{pmatrix} = \begin{pmatrix} -1.0 \times 10^6 \\ 0 \\ -0.5 \times 10^6 \end{pmatrix} \Rightarrow \begin{pmatrix} q_{y2} \\ q_{\theta 2} \\ q_{y3} \end{pmatrix} = \begin{pmatrix} -0.433 \\ -0.077 \\ -0.615 \end{pmatrix} 10^{-3}$$

Note: q_{y2} and q_{y3} are vertical displacements in meters, whiles $q_{\theta 2}$ is rotation in radians

Example – TB1

Once nodal displacements have been determined, the unknown reactions can be found (we only need the shaded part of the stiffness matrix – why?)

$$10^9 \begin{pmatrix} 4 & 3 & -4 & 3 & 0 & 0 \\ \text{sym.} & 12 & -3 & -7.5 & 0 & 0 \\ \text{sym.} & \text{sym.} & 8 & 0 & -4 & 3 \\ \text{sym.} & \text{sym.} & \text{sym.} & 24 & -3 & -7.5 \\ \text{sym.} & \text{sym.} & \text{sym.} & \text{sym.} & 4 & -3 \\ \text{sym.} & \text{sym.} & \text{sym.} & \text{sym.} & \text{sym.} & 12 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ q_{y2} \\ q_{\theta 2} \\ q_{y3} \\ 0 \end{pmatrix} = \begin{pmatrix} p_{y1} \\ M_1 \\ -1 \times 10^6 \\ 0 \\ -0.5 \times 10^6 \\ M_3 \end{pmatrix}$$

Solving for reactions

$$10^9 \begin{pmatrix} -4 & 3 & 0 \\ -3 & -7.5 & 0 \\ 3 & -7.5 & -3 \end{pmatrix} \begin{pmatrix} -0.433 \\ -0.077 \\ -0.615 \end{pmatrix} 10^{-3} = \begin{pmatrix} p_{y1} \\ M_1 \\ M_3 \end{pmatrix} = 10^6 \begin{pmatrix} 1.5 \\ 1.875 \\ 1.125 \end{pmatrix}$$

Note: p_{y1} is the vertical reaction in N (equals total applied force), while M_1 and M_3 are moments in Nm

Solve this example with reduced integration

Finite Element Method for Solids and Structures

Tutorial 1

1. Define and sketch two problems for which plane stress analysis is appropriate.

2. Define and sketch two practical problems for which plane strain analysis is appropriate.

3. Consider the 2-element plane stress idealisation of the beam discussed in the lecture and shown below. Each node has 2 degrees of freedom as discussed but the node numbering has been changed. Degrees of freedom 1 and 2 are associated with node 1; 3 and 4 with node 2 and so on. Work out which elements of the 12×12 global stiffness matrix will be (a) zero; which will have contribution from (b) element 1 only; (c) element 2 only; and (d) both elements 1 and 2. State which elements of the displacement matrix are zero and write down the equivalent nodal loads matrix.

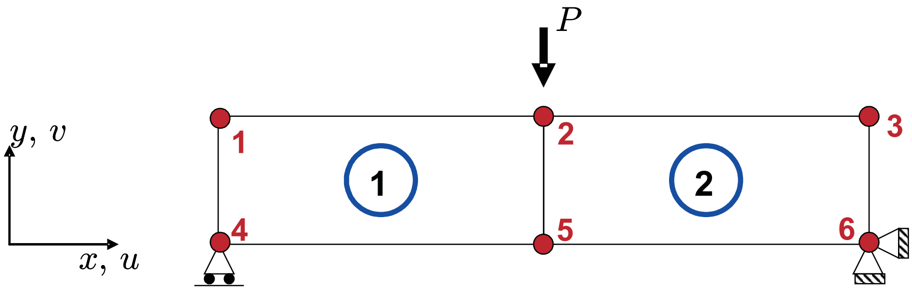


Figure 1: Problem 3

Finite Element Method for Solids and Structures

Tutorial 2

1. Starting from 3D stress-strain relations for isotropic materials in the form $\boldsymbol{\epsilon} = \mathbf{C}\boldsymbol{\sigma}$ derive the $\mathbf{D}$ matrix for (a) plane stress and (b) plane strain.

2. A 3-noded triangular element is shown in Figure 1. The x and y coordinates of the nodes are given in parenthesis next to the node number. It is found that for this element the generic displacements can be interpolated using the following funtions

$$\begin{aligned} u &= N_1 q_{x1} + N_2 q_{x2} + N_3 q_{x3} \\ v &= N_1 q_{y1} + N_2 q_{y2} + N_3 q_{y3} \end{aligned}$$

where (q_{x1}, q_{y1}) are x and y direction displacements of node 1; (q_{x2}, q_{y2}) are x and y direction displacements of node 2; and (q_{x3}, q_{y3}) are x and y direction displacements of node 3 and

$$\begin{aligned} N_1 &= 1 - 0.2x - 0.1y \\ N_2 &= 0.2x - 0.15y \\ N_3 &= 0.25y \end{aligned}$$

(a) Assuming $(q_{x1}, q_{y1}) = (0.1, 0)$; $(q_{x2}, q_{y2}) = (0.3, 0.1)$; and $(q_{x3}, q_{y3}) = (0.1, 0.2)$, check the validity of the expressions for u and v.

(b) Using nodal displacements given in (a) above evaluate the displacements at generic points A and B. Comment whether the evaluated displacements "appear sensible". Also comment on the influence of nodal displacements on the displacement at A and B.

(c) Show that for any given set of nodal displacements the strain and stress matrices are identical at all generic points within the element. Explain why this is so. Find the strain matrix at A and B for the nodal displacements given in (a).

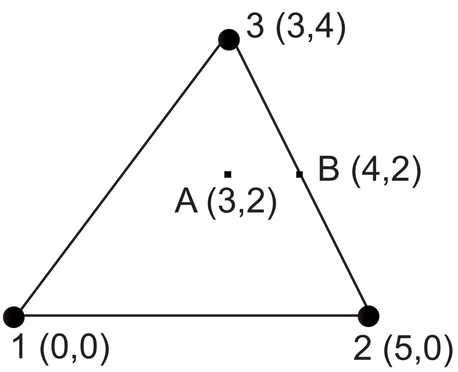


Figure 1: Problem 2

3. A 4-noded rectangular element is shown in Figure 2. The x and y coordinates of the nodes are given in parenthesis next to the node number. It is found that for this element the generic displacements can be interpolated using the following funtions

$$
\begin{aligned}
u &= N_1 q_{x1} + N_2 q_{x2} + N_3 q_{x3} + N_4 q_{x4} \\
v &= N_1 q_{y1} + N_2 q_{y2} + N_3 q_{y3} + N_4 q_{y4}
\end{aligned}
$$

where (q_{xi}, q_{yi}) are x and y direction displacements of node i and

$$
\begin{aligned}
N_1 &= \frac{1}{24}(6-x)(4-y) \\
N_2 &= \frac{1}{24}x(4-y) \\
N_3 &= \frac{1}{24}xy \\
N_4 &= \frac{1}{24}(6-x)y
\end{aligned}
$$

(a) Assuming $(q_{x1}, q_{y1}) = (0.1, 0)$; $(q_{x2}, q_{y2}) = (0.3, 0.1)$; $(q_{x3}, q_{y3}) = (0.1, 0.2)$; and $(q_{x4}, q_{y4}) = (-0.1, -0.2)$, check the validity of the expressions for u and v.

(b) Using nodal displacements given in (a) above evaluate the displacements at generic points A and B. Comment whether the evaluated displacements "appear sensible". Also comment on the influence of nodal displacements on the displacement at A and B.

(c) Show that for any given set of nodal displacements the strain component ϵ_x will not vary with x and the strain component ϵ_y will not vary with y. Explain why this is so. Find the strain matrix at A and B for the nodal displacements given in (a).

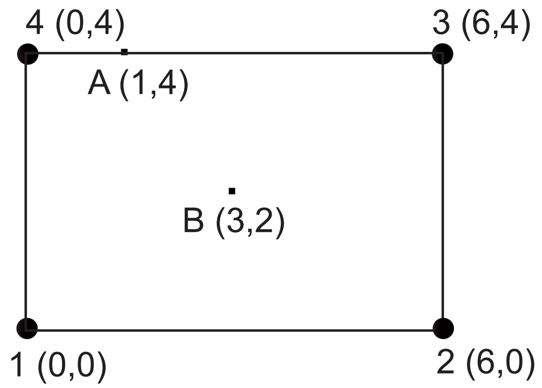


Figure 2: Problem 3

Finite Element Method for Solids and Structures

Tutorial 3

1. For the element shown in Figure 1 write the full form of the following matrices: (a) $\mathbf{u}$; (b) $\mathbf{q}$; (c) $\mathbf{N}$; (d) $\mathbf{L}$; (e) $\mathbf{B}$; (f) $\boldsymbol{\epsilon}$; (g) $\mathbf{D}$; (h) $\boldsymbol{\sigma}$; (i) $\mathbf{K}$; (j) $\mathbf{b}$; and (k) $\mathbf{p_b}$.

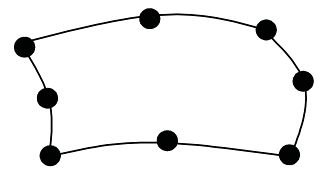


Figure 1: Problem 1

2. Find the stiffness matrix $\mathbf{K}$ of an axial element having quadratic displacement function $u = c_1 + c_2 x + c_3 x^2$ and three nodes as shown below. Assume EA is constant.

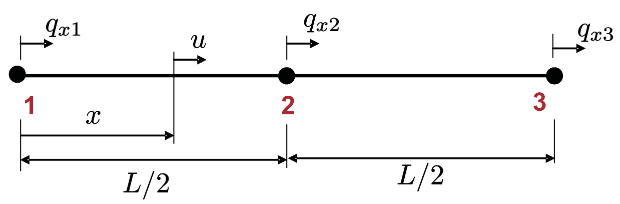


Figure 2: Problem 2

3. For the axial element of Problem 2, evaluate the equivalent nodal loads matrix due to a linearly varying axial force as shown below. What will be the equivalent nodal loads matrix for the special case $b_{x1} = b_{x3} = b_x$. Comment on the values obtained.

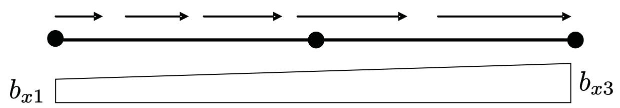


Figure 3: Problem 3

Finite Element Method for Solids and Structures

Tutorial 4

1. A 4-noded plane strain quadrilateral element is shown below. Assume the thickness t is 1 m. The x and y coordinates of nodes 1, 2, 3 and 4 are (1,1), (5,2), (3,4) and (0,3) respectively. The Young's modulus of elasticity for the material of the element is 500 and the Poisson's ratio is zero. All above values are in consistent units.

   (a) Evaluate $\mathbf{J}$ at the geometric centre ($\xi = 0$, $\eta = 0$) of the element.

   (b) Evaluate $\mathbf{J}^{-1}$ at the geometric centre of the element.

   (c) Evaluate the terms $\partial N_1/\partial x$ and $\partial N_1/\partial y$ of the displacement to strain transformation matrix $\mathbf{B}$ at the geometric centre of the element.

   (c) Using the terms evaluated above work out the stiffness matrix terms K_{11} and K_{12} using a single point Gaussian numerical integration.

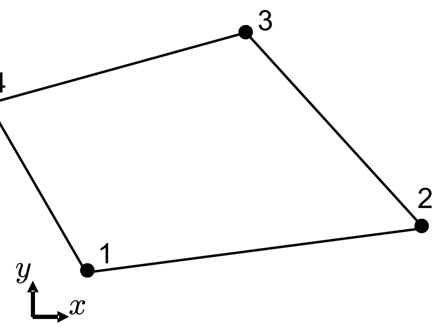


Figure 1: Problem 1

2. The figure below shows a Q8 element with the x and y components P_x and P_y of a concentrated force applied at the location $(\xi, \eta) = (0.25, 0.5)$. Find the equivalent nodal loads at nodes 3 and 4 due to these forces.

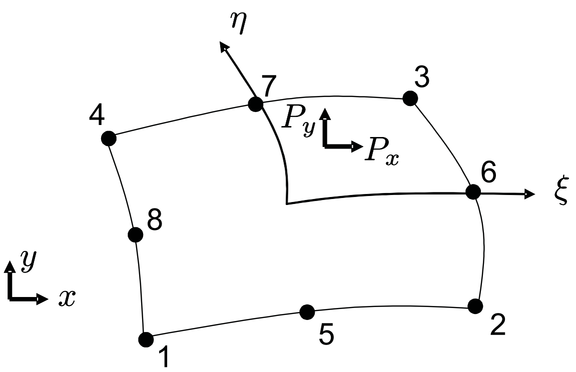


Figure 2: Problem 2

Finite Element Method for Solids and Structures

Tutorial 5

1. A 3-noded plane stress triangular element is shown below. Assume the thickness t is 1 m. The x and y coordinates of nodes 1, 2 and 3 are (0,0), (4,1) and (6,6) respectively. The Young's modulus of elasticity for the material of the element is 500 and the Poisson's ratio is 0.2. All above values are in consistent units.

   (a) Find the x and y coordinates of a point with $\xi_2, \xi_3 = 0.2, 0.3$.

   (b) Find the ξ_1, ξ_2, ξ_3 coordinates of a point with $(x, y) = (5, 4)$.

   (c) Determine the **B** matrix for this element.

   (d) Determine the **D** matrix for this element.

   (e) Work out the stiffness matrix terms K_{11} and K_{16}.

   (f) If the nodal displacement pairs (displacement in x and y directions) for nodes 1, 2 and 3 are (0.0,0.0), (0.2,-0.3) and (0.1,0.1) respectively, what is the state of strain and stress at any generic point in the element?

   (g) A concentrated force P is applied in the x direction at the midpoint of edge 2-3 as shown. Evaluate the equivalent nodal loads due to this concentrated load.

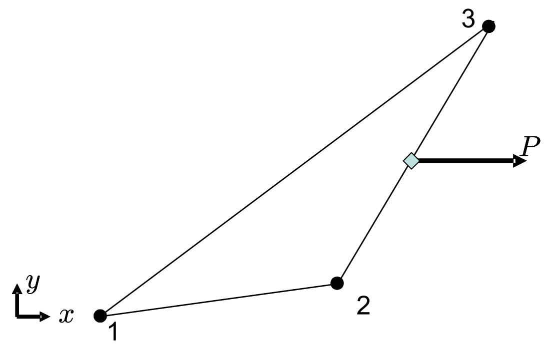


Figure 1: Problem 1

Finite Element Method for Solids and Structures

Tutorial 6

1. Solve example problem EB1 discussed in the class using symmetry.

2. A linraly varying distributed load is applied to an Euler Bernoulli beam element of length L as shown in the figure below. Evaluate the equivalent nodal loads matrix.

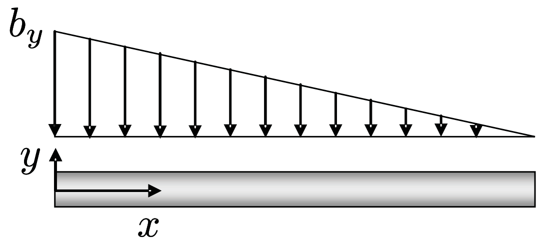


Figure 1: Problem 2

3. Two beam elements are used to model the structure shown below. The beam is clamped at the left end (node 1), supports a load $P = 100$ N at the centre (node 2), and is supported on a pin at the right end (node 3). Elements 1 and 2 are 2-node Euler Bernoulli beam elements, each of length 0.05 m and a flexural rigidity $EI = 0.15\,\mathrm{Nm}^2$.

   (a) Assemble the global stiffness and load matrices for the above system. Apply boundary conditions and solve for displacements (deflections/slopes) corresponding to the free degrees of freedom.

   (b) Write the equations and plot the deformed shape of each element showing deflection and angles at all nodes.

   (c) What are the reactions at the wall and the pin support?

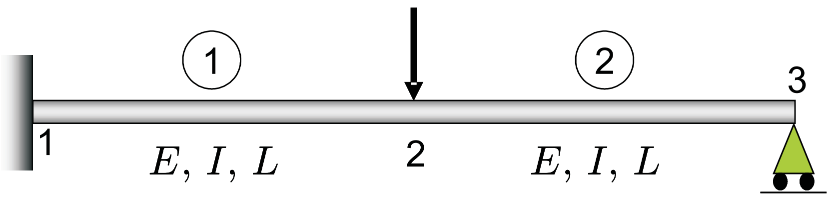


Figure 2: Problem 3

4. Solve example problem TB1 discussed in the class using stiffness matrix obtained using reduced integration.